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Analysis of motor fields in Cosserat continua

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The motor calculus that was developed by R. v. Mises and published in this journal more than forty years ago is extended to motor analysis. These investigations were stimulated by the study of the vector and tensor fields of the Cosserat continuum, whose smallest elements are rigid bodies. All of the displacement, strain, and stress quantities appear to be differential forms that constitute the six coordinates of a motor in each case. The analysis of such motor fields is the geometry of an affine connection that is based on the parallel displacement of the moment vector. An effective tool for the development of the motor field analysis turned out to be the exterior calculus of alternating differential forms, the knowledge of which is not presupposed in this paper. The concepts of the absolute differential and codifferential of the connection will lead to a complete representation of the motor differential forms in a simple manner, and especially to a complete representation of stresses and couple stresses by stress functions.

1. Introduction

The infinitesimal displacements and rotations of the elements of a classical point-continuum may be described by the vector field $\mathbf{u}(x_1, x_2, x_3)$ in the form of the displacement vector, from which one defines the rotation vector (i.e., the position-dependent infinitesimal mean rotation of the displacement vector field) $\boldsymbol{\varphi} = (1/2)$ rot $\mathbf{u}$. By contrast, in the COSSERAT continuum, whose smallest elements are rigid bodies, there exists an autonomous rotation vector field $\boldsymbol{\varphi}(x_1, x_2, x_3)$ that is generally different from the field (1/2) rot $\mathbf{u}$. (The difference of the two rotation vector fields defines a skew-symmetric part of the deformation tensor $\boldsymbol{\varepsilon}$.) Any point (x_1, x_2, x_3) of the displacement fields of a COSSERAT is therefore associated with a vector pair $\boldsymbol{\varphi}$, $\boldsymbol{u}$, and the infinitesimal deformation state will be described by the tensor pair [**1**]:

$$\chi_{\alpha\beta} = \partial_\alpha \varphi_\beta , \quad \chi_{\alpha\beta} = \partial_\alpha u_\beta - \varepsilon_{\alpha\beta\lambda} \varphi_\lambda . \tag{1.1}$$

(We employ Cartesian coordinates; $\varepsilon_{\alpha\beta\lambda}$ is the RICCI tensor that is alternating in all three indices.)

A force system on a rigid body may be reduced to a point O – i.e., it may be represented by a unit force $\mathbf{K}$ whose line of action goes through O and by a moment vector $\mathbf{M}_0$. The infinitesimal displacement field of a rigid body is determined by the rotation vector $\boldsymbol{\varphi}$ and displacement vector $\boldsymbol{u}$ and $\boldsymbol{u}_0$ at the point O.

One calls such vector pairs $\boldsymbol{\varphi}_0$, $\boldsymbol{u}_0$ and $\mathbf{K}$, $\mathbf{M}_0$ that are associated with a vector pairs "screws," "rods," or "motors;" the pair $\mathbf{K}$, $\mathbf{M}_0$ is also called a "dyname." In 1924, in Band 4 of this Zeitschrift, R. v. Mises [**2**] developed the analogy between vector calculus and motor calculus, defined a scalar and motor product, and treated the calculus of motor dyadics. Today, after a span of more than forty years, one must admit that this motor calculus is made use of in only a few cases; it has almost been forgotten. The reader of the current publications is advised to browse through the work of v. MISES. In it, the consequent use of the "reduced" scalar product in the multiplication of the 6×6 motor matrices leads to an unconventional matrix calculus that complicates the lecture considerably, which is, however, somewhat mitigated by the fact that in the last decade processes have been developed for dealing with transition matrices in the dynamics of frameworks (in which the motor algebra has also been used). The motor calculus of v. MISES is actually only a motor algebra with scalar and motor products, and one finds no analogue there of the field operators grad, div, rot of vector analysis. A true motor field was considered only in a different place (pp. 199, eq. 19a, b) in the presentation of the equilibrium conditions of a continuum – today, we call it a COSSERAT continuum – that is endowed with force and moment stresses. In fact, the need for a motor analysis first emerges from the concept of a COSSERAT continuum with its motor fields. The objective of this paper is to develop such a thing. The starting point is the knowledge that the vectorial and tensorial equations of the statics and kinematics of the COSSERAT continuum, which always appear in pairs, can be fruitfully described by the introduction of the six operators Grad, Div, Grad*, Div*, Rot* in such a way that all calculations would be simple and intuitive. However, the author first succeeded in finding the basis for this empirically-discovered calculus by perusing the recent literature on alternating differential forms. How they are connected with the motor calculus will be established in the next sections.

2. The parallel translation of moment vectors as an affine connection on the motor field

Let a motor field:

$$\mathbf{v}(\mathbf{x}) = \begin{bmatrix} \overset{1}{\mathbf{v}}(\mathbf{x}) \\ \overset{2}{\mathbf{v}}(\mathbf{x}) \end{bmatrix} \tag{2.1}$$

be given on the three-dimensional Euclidian space E^3, where $\overset{1}{\mathbf{v}}$ is the first vector of the motor $\mathbf{v}$ and $\overset{2}{\mathbf{v}}$ is the second vector, and, as we shall establish, the moment vector of $\mathbf{v}$. In order to stimulate the discussion that follows, we would like to suggest that $\overset{1}{\mathbf{v}}$ is the infinitesimal rotation vector and $\overset{2}{\mathbf{v}}$ is the likewise infinitesimal translation vector of a COSSERAT continuum. If the state of displacement in the neighborhood $\mathbf{x} + d\mathbf{x}$ of the field point $\mathbf{x}$ is that of a rigid body then one has:

(2.2) $$\mathbf{v}(\mathbf{x} + d\mathbf{x}) = \begin{bmatrix} \overset{1}{\mathbf{v}}(\mathbf{x}+d\mathbf{x}) \\ \overset{2}{\mathbf{v}}(\mathbf{x}+d\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \overset{1}{\mathbf{v}}(\mathbf{x}) \\ \overset{2}{\mathbf{v}}(\mathbf{x}) + \overset{1}{\mathbf{v}}(\mathbf{x}) \times d\mathbf{x} \end{bmatrix}.$$

Obviously, no deformation of the continuum in the neighborhood of **x** is linked with this particular state of displacement, and we will be inclined to regard the motors **v**(**x**) in (2.1) and **v**(**x** + *d***x**) in (2.2) as "kinematically equivalent" or "equal."

A second example: Any system of forces on a rigid body will be "reduced" to the point **x** and will be represented there by the force vector $\overset{1}{\mathbf{v}}$ and the moment vector $\overset{2}{\mathbf{v}}$, and therefore, by the dyname (2.1). A reduction at the point **x** + *d***x** is given by (2.2). (2.10) and (2.2) are "statically equivalent" representations of the same dyname.

This concept of the equivalence of motors that is imprinted on rigid bodies defines a "parallel translation," in the sense of differential geometry. In the six-dimensional vector space V^6 (which we also call *motor space*), the actual position-dependent change in the motor field will be described by the absolute differential of the translation:

(2.3) $$D\mathbf{v}(\mathbf{x}) = \begin{bmatrix} \overset{1}{\mathbf{v}}(\mathbf{x}+d\mathbf{x}) - \overset{1}{\mathbf{v}}(\mathbf{x}) \\ \overset{2}{\mathbf{v}}(\mathbf{x}+d\mathbf{x}) - [\overset{2}{\mathbf{v}}(\mathbf{x}) + \overset{1}{\mathbf{v}}(\mathbf{x}) \times d\mathbf{x}] \end{bmatrix}$$

or:

(2.4) $$D\mathbf{v} = \begin{bmatrix} d\,\overset{1}{\mathbf{v}} \\ d\,\overset{2}{\mathbf{v}} + d\mathbf{x} \times \overset{1}{\mathbf{v}} \end{bmatrix}.$$

We regard a field with $D\mathbf{v} = 0$ as "constant;" all motors of the field are "parallel." In our examples, $D\mathbf{v}$ is a measure of the deformation of the continuum or the deviation from the equilibrium state in the infinitesimal neighborhood of the field point **x**. While preserving the Cartesian coordinates x_1, x_2, x_3 of E^3, we give (2.4) the usual form from differential geometry of an affine (linear) connection in V^6:

(2.5) $$\boxed{Dv^r = dv^r + \Gamma^r_{\beta l} dx^\beta v^l}.$$

In this, we have enumerated the coordinates of the vectors $\overset{1}{\mathbf{v}}$ and $\overset{2}{\mathbf{v}}$ from 1 to 6. Furthermore, in further calculation, the Latin indices run from 1 to 6 and the Greek indices, from 1 to 3. One thus has:

(2.6) $$\begin{cases} \overset{1}{\mathbf{v}} = (v^r) = (v^\rho) & \text{for } r = \rho = 1, 2, 3, \\ \overset{2}{\mathbf{v}} = (v^r) = (v^{\rho+3}) & \text{for } r = 4, 5, 6;\ \rho = 1, 2, 3. \end{cases}$$

We contrast this contravariant indexing with the covariant one:

(2.7) $$Dv_r = dv_r - \Gamma^l_{\beta r} dx^\beta v_l \,,$$

with:

(2.8) $$\begin{cases} \overset{2}{\mathbf{v}} = (v_r) = (v_\rho) \quad \text{for } r = \rho = 1, 2, 3, \\ \overset{1}{\mathbf{v}} = (v_r) = (v_{\rho+3}) \ \text{ for } r = 4, 5, 6; \rho = 1, 2, 3. \end{cases}$$

The sequence of vectors $\overset{1}{\mathbf{v}}$ and $\overset{2}{\mathbf{v}}$ in (2.8) has been switched with respect to (2.6)

There exists a mechanically meaningful scalar product in V^6: the *work product*.

Force motor · displacement motor = force · displacement + moment · rotation,

so:

(2.9) $$v^r\, u_r = \overset{1}{\mathbf{v}} \circ \overset{2}{\mathbf{u}} + \overset{2}{\mathbf{v}} \circ \overset{1}{\mathbf{u}} = v^\rho\, u_\rho + v^{\rho+3}\, u_{\rho+3},$$

is invariant under parallel translation:

$$D(v^r\, u_r) = u_r\, Dv^r + v_r\, Du^r = d(v^r\, u_r) + (\Gamma^r_{\beta l} v^l u_r - \Gamma^l_{\beta r} u_l v^r)\, dx^\beta = d(v^r\, u_r).$$

We associate the scalar product (2.9) with the symmetric bilinear form:

(2.10) $$a_{rs}\, v^r\, u^s, \qquad a_{rs} = a_{sr},$$

by which V^6 takes on a metric. A comparison of (2.10) with (2.9) and (2.8) with (2.6) yields:

(2.11) $$a_{rs}\, v^r\, u^s = a_{\rho,\sigma+3}\, v^\sigma\, u^{\rho+3} + a_{\rho+3,\sigma}\, v^{\rho+3}\, u^\sigma,$$

such that:

(2.12) $$a_{rs} = a_{sr} = 0 \quad \text{up to} \quad a_{\rho,\sigma+3} = \delta_{\rho\sigma}; \quad a_{\rho+3,\sigma} = \delta_{\rho\sigma}\,.$$

In matrix form:

(2.13) $$(a_{rs}) = \left[\begin{array}{c|c} 0 & E \\ \hline E & 0 \end{array}\right].$$

Since det $E = 1$, one has det $a_{rs} = -1$, so the metric is indefinite. a_{rs} can be brought into the form:

(2.14) $$a_{rs} = \varepsilon_r\, \delta_{rs}\,,$$

with $\varepsilon_r = +1$ for $r = 1, 2, 3$ and $\varepsilon_r = -1$ for $r = 4, 5, 6$ by an orthogonal transformation in V^6. This brief consideration of the geometric structure of V^6 will suffice here.

Since $a_{rs} = a^{rs}$, as one easily confirms, for this metric the covariant coordinates of a motor can be very easily converted into contravariant ones; e.g.:

(2.15) $$v^\rho = a^{\rho,\,\sigma+3}\, v_{\sigma+3} = v_{\rho+3}\,, \qquad v_\rho = a_{\rho,\sigma+3}\, v^{\sigma+3} = v^{\rho+3}.$$

For the determination of the translation quantities $\Gamma^r_{\beta l}$, we compare, e.g.:

(2.16) $$Dv^\rho = dv^\rho, \qquad Dv^{\rho+3} = dv^{\rho+3} + \Gamma^{\rho+3}_{\beta\lambda}\; dx^\beta\, v^\lambda,$$

with (2.4):

(2.17) $$D\mathbf{v} = \begin{bmatrix} dv^\rho \\ dv_\rho + e_{\rho\beta\lambda} dx^\beta v^\lambda \end{bmatrix},$$

in which one must observe (2.6) and (2.15). Just like (2.15), the second equation of (2.16) becomes:

(2.18) $$Dv_\rho = dv_\rho + a_{\rho,\, \sigma+3} \cdot \Gamma^{\sigma+3}_{\beta\lambda}\, dx^\beta\, v^\lambda,$$

and upon comparing this with (2.17), one finds:

$$a_{\rho,\, \sigma+3} \cdot \Gamma^{\sigma+3}_{\beta\lambda} = e_{\rho\beta\lambda}$$

or:

(2.19) $$\boxed{\Gamma^{\sigma+3}_{\beta\lambda} = a^{\sigma+3,\rho} e_{\rho\beta\lambda}}\,.$$

All other $\Gamma^r_{\beta l}$ are zero. The use of the covariant form (2.7) of the absolute differential, along with (2.19), likewise leads to (2.17).

The absolute differential of the affine connection in V^6 is given by (2.5), (2.19), and:

(2.20) $$\begin{cases} \overset{1}{\mathbf{v}} = (v^r) = (v^\rho) \quad \text{for } r = \rho = 1, 2, 3, \\ \overset{2}{\mathbf{v}} = (v^r) = (v^{\rho+3}) \;\; \text{for } r = 4, 5, 6; \rho = 1, 2, 3, \\ \quad = (v_\rho). \end{cases}$$

As a result of (2.5), a motor field in V^6 possesses the covariant derivative:

(2.21) $$\nabla_\beta\, v^r = \partial_\beta\, v^r + \Gamma^r_{\beta\lambda} v^\lambda ,$$

in which, from (2.12), (2.19):

$$\Gamma^\sigma_{\beta\lambda} = 0 \qquad \text{and} \qquad \Gamma^{\rho+3}_{\beta\lambda} = e_{\rho\beta\lambda}\, .$$

3. The differential forms of the Cosserat continuum and the alternating calculus.

The deformation state of the continuum will be described by the tensor pair $E_\beta^{\cdot r}$:

(3.1) $$\begin{cases} E_{\beta}^{\cdot\rho} = \chi_{\beta}^{\cdot\rho} & \text{(distortion)},\\ E_{\beta\rho} = \varepsilon_{\beta\rho} & \text{(length and angle deformation, relative rotation)}.\end{cases}$$

$E_{\beta}^{\cdot r}$ is the covariant derivative of the displacement motor:

(3.2) $$\begin{cases} v^{\rho} = \varphi^{\rho} & \text{(rotation vector)},\\ v_{\rho} = u_{\rho} & \text{(displacement vector)}.\end{cases}$$

According to (2.20) and (2.21), we obtain:

(3.3) $$\begin{cases} E_{\beta}^{\cdot\rho} = \chi_{\beta}^{\cdot\rho} = \partial_{\beta}\varphi^{\rho},\\ E_{\beta\rho} = \varepsilon_{\beta\rho} = \partial_{\beta}u_{\rho} - e_{\rho\lambda\beta}\varphi^{\lambda}.\end{cases}$$

We denote the differential forms of first degree $Dv^r = E_{\beta}^{\cdot r}\,dx^{\beta}$, which is the absolute differential of the displacement motor, by:

(3.4) $$\varepsilon^r = E_{\beta}^{\cdot r}\,dx^{\beta},$$

and call them the *differential forms of the deformation.* The absolute differential $D\varepsilon^r$ will give the compatibility conditions for the deformation state (cf. 4.1). We shall make a few brief explanations of the required calculus and then refer to the literature [3, 4, 5, 6, 7].

The differential $d\overset{1}{\omega}$ of a differential form of first degree $\overset{1}{\omega} = a_{\beta}\,dx^{\beta}$ is defined by:

$$d(dx^{\beta}) = 0, \qquad d\overset{1}{\omega} = (da_{\beta})\,dx^{\beta} = \partial_{\alpha}a_{\beta}\,dx^{\alpha} \wedge dx^{\beta}.$$

In this, ^ (wedge) is the sign of "exterior" multiplication. One has:

$$dx^{\alpha} \wedge dx^{\beta} = -\,dx^{\beta} \wedge dx^{\alpha},$$

and naturally $dx^{\alpha} \wedge dx^{\alpha} = 0$. The volume element in E^3 is $dV = dx^1 \wedge dx^2 \wedge dx^3$ and the vectorial surface element is $\mathbf{df} = (dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2) = (dF_1, dF_2, dF_3)$, such that one has:

$$dF_{\alpha} = \tfrac{1}{2}\, e_{\alpha\beta\gamma}\,dx^{\alpha} \wedge dx^{\beta}.$$

Obviously:

$$dx^1 \wedge dF_1 = dx^2 \wedge dF_2 = dx^3 \wedge dF_3 = dV,$$

and in general:

$$dx^{\alpha} \wedge dF_{\beta} = dx_{\beta} \wedge dF^{\alpha} = \delta_{\beta}^{\alpha}\,dV.$$

Furthermore, we still require that:

$$dx^\beta \wedge dx^\gamma = e^{\alpha\beta\gamma}\, dF_\alpha .$$

Thus, what we calculated above becomes:

$$d\overset{1}{\omega} = e^{\alpha\beta\gamma}\,\partial_\alpha a_\beta\, dF_\gamma = (\text{rot }\mathbf{a})^\gamma\, dF_\gamma .$$

The differential of a form of degree two leads to the divergence map:

$$\overset{2}{\omega} = a^\beta\, dF_\beta, \quad d\overset{2}{\omega} = \partial_\alpha a^\beta\, dF_\gamma = \partial_\alpha a^\beta\, \delta^\alpha_\beta\, dV = \partial_\alpha a^\alpha\, dV = \text{div }\mathbf{a}\; dV.$$

As a result of the alternating calculus, one has $d\overset{3}{\omega} = 0$. The forms of degree zero are defined by:

$$\overset{0}{\omega} = h(x_1, x_2, x_3),$$

such that:

$$d\overset{0}{\omega} = \partial_\alpha h\; dx^\alpha = (\text{grad } h)_\alpha\, dx^\alpha.$$

One of the most important formulas of the calculus $dd\overset{p}{\omega} = 0$. It corresponds to the identity rot grad = 0 for $p = 0$, div rot = 0 for $p = 1$. If $d\overset{p}{\omega} = 0$ then one calls $\overset{p}{\omega}$ "closed, while if $\overset{p}{\omega} = d\overset{p-1}{\omega}$ then $\overset{p}{\omega}$ is "exact." An exact form is closed.

Along with the differential form of degree 1 called the deformation that was defined in (3.4), the differential form of degree 2 that is called the *stress state* plays an essential role:

(3.5) $$\sigma^r = T^{\beta r}\, dF_\beta$$

with:

(3.5a) $$T^{\beta\rho} = \sigma^{\beta\rho} \qquad \text{(force stresses)},$$

(3.5b) $$T^{\beta}_{\cdot\rho} = \mu^{\beta}_{\cdot\rho} \qquad \text{(moment stresses)}.$$

$D\sigma^r$ will yield the equilibrium conditions of the stress state.

We can also always establish later that all of the kinematically and statically meaningful differential forms of the COSSERAT continuum are the coordinates of a motor.

We will now prove the important theorem:

(3.6) $$DD\overset{p}{\omega}{}^r = 0.$$

The $\Gamma^r_{\beta l} dx^\beta = \Omega^r_l$ in (2.5) are differential forms of degree 1. With this abbreviation, we have, from (2.5):

$$D\overset{p}{\omega}{}^r = d\overset{p}{\omega}{}^r + \Omega^r_l \wedge \overset{p}{\omega}{}^l = (\delta^r_s d + \Omega^r_s) \wedge \overset{p}{\omega}{}^s ,$$

and furthermore:

$$D D \overset{p}{\omega}{}^{r} = (\delta^{r}_{s} d + \Omega^{r}_{s}) \wedge \overset{p}{\omega}{}^{s} + \Omega^{r}_{s} \wedge d \overset{p}{\omega}{}^{s} .$$

After multiplying out, while observing the order of multiplication:

$$D D \overset{p}{\omega}{}^{r} = d d \overset{p}{\omega}{}^{r} + \Omega^{r}_{k} \wedge \Omega^{k}_{s} \wedge \overset{p}{\omega}{}^{s} + d (\Omega^{r}_{s} \wedge \overset{p}{\omega}{}^{s}) + \Omega^{r}_{s} \wedge d \overset{p}{\omega}{}^{s} .$$

Now, from the usual calculus, one next has $dd \overset{p}{\omega} = 0$. The second summand is written:

$$(\Gamma^{\rho+3}_{\alpha\chi} \Gamma^{\chi}_{\beta\alpha} + \Gamma^{\rho+3}_{\alpha\chi+3} \Gamma^{\chi+3}_{\beta\alpha}) \; dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma},$$

and vanishes due to the fact that:

$$\Gamma^{\rho+3}_{\alpha\chi+3} = 0, \qquad \Gamma^{\chi}_{\beta\sigma} = 0.$$

Now, from the usual calculus, one has $d\Omega^{r}_{l} = 0$, because the Ω are differential forms of degree 1 with constant coefficients. While observing the order of the differentials dx^{α} that appear, one then has:

$$d (\Omega^{r}_{l} \wedge \overset{p}{\omega}{}^{s}) = - \, \Omega^{r}_{l} \wedge d \overset{p}{\omega}{}^{s} ,$$

from which the proof is complete.

4. The compatibility condition for the deformation state

It reads:

$$D\varepsilon^{r} = DDv^{r} = 0.$$

From (3.3), it becomes:

(4.1) $$D\varepsilon^{r} = d(E^{\cdot r}_{\beta}) \, dx^{\beta} + \Gamma^{r}_{\alpha\lambda} E^{\cdot r}_{\beta} \, dx^{\alpha} \wedge dx^{\beta} = (\partial_{\alpha} E^{\cdot r}_{\beta} + \Gamma^{r}_{\alpha\lambda} \chi^{\cdot \lambda}_{\beta}) \, dx^{\alpha} \wedge dx^{\beta} ,$$

or:

(4.1a) $$D\varepsilon^{\rho} = \partial_{\alpha} \chi^{\cdot \rho}_{\beta} \, dx^{\alpha} \wedge dx^{\beta} ,$$

(4.1b) $$D\varepsilon^{\rho} = (\partial_{\alpha} \varepsilon_{\beta\rho} + e_{\alpha\lambda\rho} \chi^{\cdot \lambda}_{\beta}) \, dx^{\alpha} \wedge dx^{\beta} ,$$

and with:

$$dx^{\alpha} \wedge dx^{\beta} = e^{\mu\alpha\beta} \, dF_{\mu} ,$$

(4.2a) $$D\varepsilon^{\rho} = e^{\mu\alpha\beta} \, \partial_{\alpha} \chi^{\cdot \rho}_{\beta} \; dF_{\mu} ,$$

(4.2b) $$D\varepsilon^{\rho} = (e^{\mu\alpha\beta} \partial_{\alpha} \varepsilon_{\beta\rho} + \delta^{\mu}_{\rho} \chi^{\cdot \beta}_{\beta} - \chi^{\cdot \mu}_{\rho}) \, dF_{\mu} .$$

With the χ and ε that were defined in (3.3), $D\varepsilon^{\rho}$ and $D\varepsilon_{\rho}$ are identically, because $\varepsilon^{r} = Dv^{r}$ is a total differential.

If ε^{r} is not a total differential then:

$$(4.3) \qquad D\varepsilon^{r} = \alpha^{\mu r}\, dF_{\mu}\,, \qquad D\varepsilon^{\rho} = \overset{1}{\alpha}{}^{\mu\rho}\, dF_{\mu}\,, \qquad D\varepsilon_{\rho} = \overset{2}{\alpha}{}^{\mu}_{\cdot\rho}\; dF_{\mu}$$

are non-zero. However, one now has $DD\varepsilon^{r} = 0$. Carrying out this calculation yields:

$$(4.4a) \qquad DD\varepsilon^{\rho} = \partial_{\beta}\,\overset{1}{\alpha}{}^{\mu\rho} dx^{\beta}\, dF_{\mu} \;=\; \partial_{\beta}\,\overset{1}{\alpha}{}^{\mu\rho}\delta^{\beta}_{\mu}\, dV = \partial_{\mu}\,\overset{1}{\alpha}{}^{\mu\rho}\, dV = 0,$$

$$(4.4b) \qquad DD\varepsilon_{\rho} = (\partial_{\beta}\,\overset{2}{\alpha}{}^{\mu}_{\cdot\rho} + e_{\beta\lambda\rho}\,\overset{1}{\alpha}{}^{\mu\lambda})\, dx^{\beta}\wedge dF_{\mu} \;=\; (\partial_{\beta}\,\overset{2}{\alpha}{}^{\mu}_{\cdot\rho} + e_{\rho\mu\lambda}\,\overset{1}{\alpha}{}^{\mu\lambda})\, dV = 0.$$

We recall eq. (4.1b) and (4.3), which we would like to write as:

$$(4.5) \qquad D\varepsilon_{\rho} = \Gamma'_{\alpha\beta\rho} dx^{\alpha}\wedge dx^{\beta}$$

or:

$$(4.6) \qquad D\varepsilon_{\rho} = e^{\mu\alpha\beta}\Gamma'_{\alpha\beta\rho} dF_{\mu} \;=\; \overset{2}{\alpha}{}^{\mu}_{\cdot\rho}\; dF_{\mu}\,.$$

The continuum theory of dislocations and proper stresses [**8**] will be described by an affine connection in the RIEMANN space R^3 whose translation quantities are:

$$(4.7) \qquad \Gamma^{*}_{\alpha\beta\rho} = \partial_{\alpha}\,\varepsilon_{\beta\rho} + e_{\beta\rho\lambda}\;\chi_{\alpha}^{\cdot\lambda}\,.$$

Now, as one quickly convinces oneself:

$$(4.8) \qquad e^{\mu\alpha\beta}\;\Gamma^{*}_{\alpha\beta\rho} = e^{\mu\alpha\beta}\;\Gamma'_{\alpha\beta\rho} \;=\; \overset{2}{\alpha}{}^{\mu}_{\cdot\rho}\,,$$

such that $\overset{2}{\alpha}$ appears to be the “dislocation density,” as the anti-symmetric part of the connection Γ^{*}, as well as Γ'. Thus, the deformation tensor $\varepsilon_{\beta\rho}$ in (4.7) was assumed to be symmetric: It gives the metric of R^3. Furthermore, the $\Gamma'_{\alpha\beta\rho}$ itself does not enter into consideration, but only its skew-symmetric (in the first two indices) part. Günther [**1**] has already exhibited the fact that dislocation theory uses an incompatible COSSERAT continuum as its geometric model. Our differential-geometric considerations, which are based upon the parallel translation of the moment vector and tailored to the COSSERAT continuum from now on, seem to possess the advantage of simplicity when compared to the differential-geometric representation of dislocation theory. However, beyond that, they likewise simplify the basic equations of the statics of the COSSERAT continuum, as will be shown in the next paragraph.

5. Equilibrium conditions and stress functions

Let the continuum be impressed with volume forces and moments, which we summarize in the motor X^r:

(5.1) $$\begin{cases} X^r = X^\rho & \text{for } r = \rho = 1,2,3, \\ X^r = Y_\rho & \text{for } r = 4,5,6; \rho = 1,2,3. \end{cases}$$

With the differential forms σ^r of the stress states (3.5), the equilibrium conditions read:

(5.2) $$D\sigma^r + X^r\, dV = 0.$$

The calculation runs completely analogous to (4.3) and (4.4). The result is:

(5.2a) $$D\sigma^r + X^r\, dV = (\partial_\beta\, \sigma^{\beta\rho} + X^\rho)\, dV = 0,$$
(5.2b) $$D\sigma_\rho + Y_\rho\, dV = (\partial_\beta\, \mu^{\beta}_{\cdot\rho} + e_{\rho\mu\lambda}\, \sigma^{\mu\lambda} + Y_\rho)\, dV = 0.$$

For $X^r = 0$, one has:

(5.3) $$D\sigma^r = 0.$$

σ^r is now a closed differential form of degree 2. One can seek to represent it as an exact form:

(5.4) $$\sigma^r = D(S_\beta^{\cdot r} dx^\beta)\,.$$

$S_\beta^{\cdot r}$ is the tensor pair of 18 stress functions. Obviously, $S_\beta^{\cdot r}$ is analogous to the $E_\beta^{\cdot r}$ in (3.4), such that the development of the right-hand side of (5.4) runs parallel to that of formulas (4.1) to (4.2).

6. Summary of the results up to now:

(6.1)	displacement motor:	v^r,
(6.2)	deformation motor:	$\varepsilon^r = Dv^r$,
(6.3)	stress motor:	$\sigma^r = T^{\beta r}\, dF_\beta$,
(6.4)	motor of stress functions:	$s^r = S_\beta^{\cdot r} dx^\beta$,
(6.5)	compatibility of the deformation state:	$D\varepsilon^r = 0$,
(6.6)	equilibrium of the stress state (in the absence of volume forces and moments):	$D\sigma^r = 0$,
(6.7)	satisfaction of the equilibrium conditions (6.6) by stress functions (6.4):	$\sigma^r = Ds^r$.

7. Introduction of differential operators

The coordinates of a motor $\overset{p}{\boldsymbol{\omega}}$ are all differential forms of degree p (p = 0, 1, 2, 3). The application of the absolute differential raises the degree p by one step:

(7.1) $$D\overset{p}{\boldsymbol{\omega}} = \overset{p+1}{\boldsymbol{\omega}}.$$

Furthermore, one has:

(7.2) $$DD\overset{p}{\boldsymbol{\omega}} = 0.$$

For the differential operators (7.1), we introduce operators in the following way:

(7.3) $p = 0$: $\quad \text{Grad}\ \overset{0}{\boldsymbol{\omega}} = \overset{1}{\boldsymbol{\omega}},$

(7.4) $p = 1$: $\quad \text{Rot}\ \overset{1}{\boldsymbol{\omega}} = \overset{2}{\boldsymbol{\omega}},$

(7.5) $p = 2$: $\quad \text{Div}\ \overset{2}{\boldsymbol{\omega}} = \overset{3}{\boldsymbol{\omega}}.$

Thus, (7.2) may be written:

(7.6), (7.7) $\quad \text{Rot Grad}\ \overset{0}{\boldsymbol{\omega}} = 0, \qquad \text{Div Rot}\ \overset{1}{\boldsymbol{\omega}} = 0.$

From section **6**, we carry over the idea that:

$\boldsymbol{\varepsilon} = \text{Grad}\ \mathbf{v}$, from (6.2), $\quad \text{Rot}\ \boldsymbol{\varepsilon} = 0$, from (6.5).

Furthermore:

$\text{Div}\ \boldsymbol{\sigma} = 0$, from (6.6), $\quad \boldsymbol{\sigma} = \text{Rot}\ \mathbf{s}$, from (6.7).

8. The absolute codifferential

It is well-known that one needs to introduce formulas for potentials and vector potentials, in which the differential operators grad, div, and rot are related to the LAPLACIAN operator $\Delta = -\,\partial_\alpha\,\partial_\alpha$. Such formulas appear in the calculus of alternating differential forms combined into the single equation:

(8.1) $$d\delta\overset{p}{\omega} + \delta d\overset{p}{\omega} = -\,\Delta\overset{p}{\omega}.$$

Let us briefly explain the codifferential δ that enters into this. Since we have restricted ourselves to the Euclidian space E^3 with Cartesian coordinates (in the other case, δ is a very complicated differential operator), the type of indexing is irrelevant. We arrange that only differentials with upper indices shall appear in the differential forms – e.g., dx^α, dF^μ. In E^3, $\delta\overset{p}{\omega}$ is defined by:

(8.2) $$\delta\overset{p}{\omega} = (-1)^p * d * \overset{p}{\omega}.$$

In turn, the star operator * in this needs to be clarified: * associates any differential form $\overset{p}{\omega}$ with an adjoint form of degree 3 – p that will be written $*\overset{p}{\omega}$. Thus, a form of degree p – 1 is on the right-hand side of (8.2); the operator d lowers the degree of a form by 1. The following definition of the star operator will suffice for E^3:

(8.3) $$*(a_\beta\, dx^\beta) = a_\beta\, dF^\beta,$$
(8.4) $$*(a_\beta\, dF^\beta) = a_\beta\, dx^\beta,$$
(8.5) $$*h = h\, dV,$$
(8.6) $$*(h\, dV) = h.$$

From (8.3) to (8.6), it follows that:

(8.7) $$**\overset{p}{\omega} = \overset{p}{\omega}.$$

Since $d(*\,h) = 0$, one also has $\delta h = 0$.

(8.1), when adapted to (8.2), reads:

(8.8) $$(-1)^p\,(*d*d - d*d*)\overset{p}{\omega} = \Delta\overset{p}{\omega}.$$

We take the case of p = 1 as an example.

$$\omega = a_\beta\, dx^\beta, \qquad *\omega = a_\beta\, dF^\beta, \quad d*\omega = \partial_\alpha a_\beta\, dx^\alpha \wedge dF^\beta = \partial_\beta a_\beta\, dV, \qquad *d*\omega = \partial_\beta a_\beta,$$
$$d\omega = \partial_\alpha a_\beta\, dx^\alpha \wedge dx^\beta = e_{\mu\alpha\beta}\,\partial_\alpha a_\beta\, dF^\mu, \qquad *d\omega = e_{\mu\alpha\beta}\,\partial_\alpha a_\beta\, dx^\mu,$$
$$d*d\omega = e_{\mu\alpha\beta}\,\partial_\rho\,\partial_\alpha a_\beta\, dx^\rho \wedge dx^\mu = e_{\sigma\rho\mu}\, e_{\mu\alpha\beta}\,\partial_\rho\,\partial_\alpha a_\beta\, dF^\sigma,$$
$$*d*d\omega = (\partial_\sigma\,\partial_\beta a_\beta - \partial_\alpha\,\partial_\alpha a_\sigma)\, dx^\sigma.$$

Thus, one has:

$$\delta\overset{1}{\omega} = -\,\partial_\beta a_\beta, \qquad d\delta\overset{1}{\omega} = -\,\partial_\sigma\partial_\beta a_\beta\, dx^\sigma,$$

and

$$\delta d\overset{1}{\omega} = (-1)^2\,(\partial_\sigma\,\partial_\alpha a_\alpha - \partial_\alpha\,\partial_\alpha a_\sigma)\, dx^\sigma,$$

so:

$$d\delta\overset{1}{\omega} + \delta d\overset{1}{\omega} = -\,\partial_\alpha\,\partial_\alpha a_\sigma\; dx^\sigma = -\,(\Delta a_\sigma)\, dx^\sigma.$$

With the differential operators subjected to the association:

$$\delta\overset{1}{\omega} \to -\,\text{div}\,\mathbf{a}, \qquad d\delta\overset{1}{\omega} \to -\,\text{grad div}\,\mathbf{a},$$
$$d\overset{1}{\omega} \to \text{rot}\,\mathbf{a}, \qquad \delta d\overset{1}{\omega} \to \text{rot rot}\,\mathbf{a},$$

one obtains the known formula:

$$-\,\text{grad div}\,\mathbf{a} + \text{rot rot}\,\mathbf{a} = -\,\Delta\mathbf{a}.$$

From (8.2), it follows that since $dd\overset{3-p}{\omega} = 0$:

(8.9) $$\delta\delta\overset{p}{\omega} = -(*d*)(*d*)\overset{p}{\omega} = -*dd*\overset{p}{\omega} = 0.$$

We now pose the problem of constructing the codifferential $\mathfrak{D}$ to our absolute differential D that was defined in (2.5) with the goal of arriving at a generalization of (8.1) to the formula:

(8.10) $$(D\mathfrak{D} + \mathfrak{D}D)\overset{p}{\boldsymbol{\omega}} = -\Delta\overset{p}{\boldsymbol{\omega}},$$

in which $\overset{p}{\boldsymbol{\omega}}$ is a motor whose coordinates are differential forms of degree p.

We recall (2.17), which we now write as:

(8.11) $$D\overset{p}{\boldsymbol{\omega}} = \begin{bmatrix} d\overset{p}{\omega}_\rho \\ d\overset{p}{\omega}_{\rho+3} + e_{\rho\beta\lambda}dx^\beta \wedge \overset{p}{\omega}_\lambda \end{bmatrix}.$$

Since we agreed that we would no longer distinguish between covariance and contravariance, the difference between (2.5) and (2.7) is irrelevant, and the codifferential $\mathfrak{D}$ must be found by trial and error. We obtain:

(8.12) $$\mathfrak{D}\overset{p}{\boldsymbol{\omega}} = \begin{bmatrix} \delta\overset{p}{\omega}_\rho \\ \delta\overset{p}{\omega}_{\rho+3} - (-1)^p e_{\rho\beta\lambda} * (dx^\beta \wedge * \overset{p}{\omega}_\lambda) \end{bmatrix}.$$

Corresponding to (8.9), one also has, as one confirms by a simple computation:

(8.13) $$\mathfrak{D}\mathfrak{D}\overset{p}{\boldsymbol{\omega}} = 0.$$

9. The differential operators associated with the absolute codifferential

The application of the codifferential lowers the degree of a differential form by one step:

(9.1) $$\mathfrak{D}\overset{p}{\boldsymbol{\omega}} = \overset{p-1}{\boldsymbol{\omega}}.$$

For these differential operators, we introduce the following operators:

(9.2) $p = 1$: $$-\text{Div} *\overset{1}{\boldsymbol{\omega}} = \overset{0}{\boldsymbol{\omega}},$$

(9.3) $p = 2$: $$\text{Rot} *\overset{2}{\boldsymbol{\omega}} = \overset{1}{\boldsymbol{\omega}},$$

(9.4) $p = 3$: $\quad -\,\text{Grad}\ {*}\overset{3}{\boldsymbol{\omega}} = \overset{2}{\boldsymbol{\omega}},$

so

(9.5) $$\mathfrak{D}\mathfrak{D}\overset{p}{\boldsymbol{\omega}} = 0$$

then gives the identity:

(9.6), (9.7) $$\text{Rot}\ {*}\text{Grad}\ {*}\overset{3}{\boldsymbol{\omega}} = 0, \qquad \text{Div}\ {*}\text{Rot}\ {*}\overset{2}{\boldsymbol{\omega}} = 0.$$

From (8.10), and with the use of the six differential operators (7.3), (7.4), (7.5), and (9.2), (9.3), (9.4), it follows that:

(9.8) $p = 0$: $\quad -\,\text{Div}\ {*}\text{Grad} = -\,\Delta,$

(9.9) $p = 1$: $\quad -\,\text{Grad Div}{*} + \text{Rot}{*}\ \text{Rot} = -\,\Delta,$

(9.10) $p = 2$: $\quad \text{Rot Rot}{*} - \text{Grad}\ {*}\text{Div} = -\,\Delta,$

(9.11) $p = 3$: $\quad -\,\text{Div Grad}{*} = -\,\Delta.$

10. Complete representation of the motorial differential forms of the Cosserat continuum

It is well-known that a field vector can be represented as the sum of a gradient and a rotor. If the vector **a** fulfills the condition div **a** = 0 then **a** cannot generally be represented as **a** = rot **v**. Similarly, one has the case rot **a** = 0, where the representation **a** = grad Φ is not generally complete. In the calculus of differential forms, the question that was posed here reads as follows: Let $\overset{p}{\omega}$ be a closed form, so $d\overset{p}{\omega} = 0$. Under which conditions is $\overset{p}{\omega}$ exact? Thus, when is $\overset{p}{\omega} = d\overset{p-1}{\pi}$? The answer is: Any closed form is exact in a region that is star-shaped when seen from the origin; in regions from which the origin is excluded, this is not true, in general. How does one obtain representations that are also true for cavities? We answer this question for motorial differential forms.

Let $\overset{p+1}{\boldsymbol{\psi}}$ be given in

(10.1) $$D\overset{p}{\boldsymbol{\omega}} + \overset{p+1}{\boldsymbol{\psi}} = 0$$

where $\overset{p+1}{\boldsymbol{\psi}}$ shall satisfy the compatibility condition:

(10.2) $$D\overset{p+1}{\boldsymbol{\psi}} = 0.$$

We prove that every $\overset{p}{\boldsymbol{\omega}}$ that fulfills (10.1) may be represented by:

(10.3) $$\overset{p}{\boldsymbol{\omega}} = D\overset{p-1}{\boldsymbol{\pi}} + \mathfrak{D}\overset{p+1}{\boldsymbol{\tau}}.$$

Obviously, due to (10.1), one must have:

(10.4) $$D\overset{p}{\boldsymbol{\omega}} = D\mathfrak{D}\overset{p+1}{\boldsymbol{\tau}} = -\overset{p+1}{\boldsymbol{\psi}}.$$

For the proof of (10.3), we set:

(10.5) $$\overset{p-1}{\boldsymbol{\pi}} = \mathfrak{D}\overset{p}{\boldsymbol{\eta}}, \qquad \overset{p+1}{\boldsymbol{\tau}} = D\overset{p}{\boldsymbol{\eta}},$$

such that from (10.3), one has:

(10.6) $$\overset{p}{\boldsymbol{\omega}} = (D\mathfrak{D}+\mathfrak{D}D)\overset{p}{\boldsymbol{\eta}} = -\Delta\overset{p}{\boldsymbol{\eta}}.$$

From potential theory, one can always determine $\overset{p}{\boldsymbol{\eta}}$ from $\overset{p}{\boldsymbol{\omega}}$ by way of (10.6). The forms $\overset{p-1}{\boldsymbol{\pi}}$ and $\overset{p+1}{\boldsymbol{\tau}}$ can then be calculated from $\overset{p}{\boldsymbol{\eta}}$ using (10.5). All that remains is to establish that $\overset{p+1}{\boldsymbol{\tau}}$ satisfies eq. (10.4). However, from (10.6), it follows that:

(10.7) $$D\overset{p}{\boldsymbol{\omega}} = -D\Delta\overset{p}{\boldsymbol{\eta}} = -\Delta D\overset{p}{\boldsymbol{\eta}} = -\Delta\overset{p+1}{\boldsymbol{\tau}},$$

because the operators D and Δ commute. Furthermore, from (10.5), one has:

(10.8) $$D\overset{p+1}{\boldsymbol{\tau}} = 0,$$

such that:

(10.9) $$(D\mathfrak{D}+\mathfrak{D}D)\overset{p+1}{\boldsymbol{\tau}} = D\mathfrak{D}\overset{p+1}{\boldsymbol{\tau}} = -\Delta\overset{p+1}{\boldsymbol{\tau}}.$$

However, (10.9) and (10.7) confirm that (10.4) is, in fact, fulfilled. The representation (10.3) is therefore proved completely.

We continue to assume that $\overset{p+1}{\boldsymbol{\tau}}$ is the solution of the inhomogeneous problem (10.1) and has to satisfy the POISSON equation (10.7):

(10.10) $$\Delta\overset{p+1}{\boldsymbol{\tau}} - \overset{p+1}{\boldsymbol{\psi}} = 0,$$

with the auxiliary condition that follows from (10.5):

(10.11) $$D\overset{p+1}{\boldsymbol{\tau}} = 0.$$

Of especial significance are the cases of $p = 1$ and $p = 2$ in (10.3). For $p = 2$, we get, in fact, the complete representation of the stress state by stress functions [**9**]:

(10.12) $$\overset{2}{\boldsymbol{\sigma}} = D\overset{1}{\boldsymbol{s}} + \mathfrak{D}\overset{3}{\boldsymbol{\tau}},$$

so:

(10.13) $$\boldsymbol{\sigma} = \text{Rot}\,\boldsymbol{s} - \text{Grad}\,{*\boldsymbol{\tau}},$$

in which, from (10.10) and (5.2), $\boldsymbol{\tau}$ must satisfy the POISSON equation:

(10.14) $$\Delta\boldsymbol{\tau} - \mathbf{X}\,dV = 0,$$

and the auxiliary condition (10.11) is fulfilled identically (the differential of a form of degree 3 vanishes in E^3). The case of $p = 1$ leads to the problem that was announced in (4.3) of calculating the incompatible deformation $\boldsymbol{\varepsilon}$ from a given dislocation density $\boldsymbol{\alpha}$. Here, (10.3) gives the representation:

(10.15) $$\boldsymbol{\varepsilon} = \text{Grad}\,\boldsymbol{v} + \text{Rot}\,{*\boldsymbol{\tau}},$$

and from (10.10) and (10.11), $\boldsymbol{\tau}$ must satisfy the two equations:

(10.16), (10.17) $$\Delta\boldsymbol{\tau} = -\boldsymbol{\alpha}, \qquad \text{Div}\,\boldsymbol{\tau} = 0.$$

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