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The stress functions of a dyname

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With 2 figures

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Summary: The tensor of stress functions is investigated for a bar that is loaded with a dyname (force screw). Connections with Volterra's "Theory of distortions" are discussed.

1. Introduction

W. Günther [**1**] has recently shown how the outer surface forces of a three-dimensional stress state are connected with the tensor of the stress functions. The six components of the dyname of the outer surface forces of a surface patch that is bounded by the surface curve C will be represented by functionals of C that include linear forms of the six stress functions and their first derivatives.

Simpler connections exist for a two-dimensional stress state. The dyname of the boundary forces on a curve segment that links the points A and B will possess only three components that now appear as functionals of the point-pair AB. One will find the corresponding formulas in ([**1**], (3.9)) (*).

In Cartesian coordinates, they take the form:

$$K_1 = \left[\frac{\partial\Phi}{\partial x_2}\right]_A^B, \qquad K_2 = \left[-\frac{\partial\Phi}{\partial x_1}\right]_A^B, \qquad M_0 = \left[\Phi - x_1\frac{\partial\Phi}{\partial x_1} - x_2\frac{\partial\Phi}{\partial x_2}\right]_A^B. \tag{1.1}$$

Φ is the single stress function of the two-dimensional stress state – viz., the Airy stress function – and the stress tensor has the form:

$$S_{ik} = -\frac{\partial^2\Phi}{\partial x_i\,\partial x_k} + \delta_{ik}\cdot\Delta\Phi, \tag{1.2}$$

in which, δ_{ik} means the *Kronecker* symbol and Δ means the *Laplacian* operator.

In stress-free domains of the continuum, $\Phi(x_1, x_2)$ will be a linear function, namely, the null stress function:

$$\Phi^0 = a + \omega_1 x_2 - \omega_2 x_1 , \tag{1.3}$$

(*) In contrast to [**1**], we have allowed ourselves an inessential change of sign and denoted the moment relative to the origin of the coordinate system by M_0 .

with the constants a, ω_1, ω_2 .

The dyname (1.1) is reduced at the origin of the (x_1, x_2) coordinate system (Fig. 1). Its moment with respect to a point $P(x_1, x_2)$ in the plane is:

$$M = M_0 + K_1\, x_2 - K_2\, x_1 \,. \tag{1.4}$$

Suppose that a thin rod that is acted upon by this dyname lies along the x_2-axis of the coordinate system (Fig. 2).

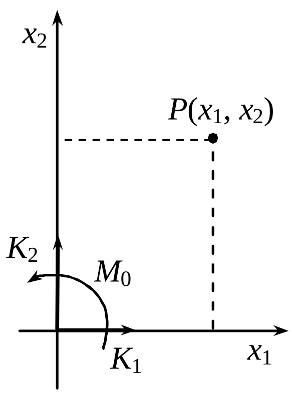

Figure 1.

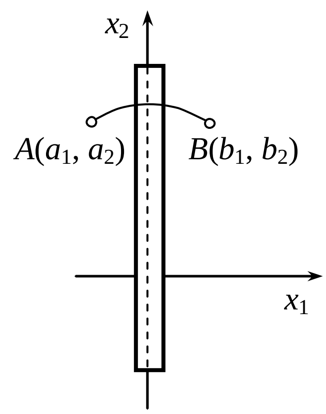

Figure 2.

The two half-planes to the left and right of the rod are stress-free. If we choose a, ω_1, and ω_2 to be zero in the left half-plane then their values in the right half-plane will be established by (1.1). One will have:

$$K_1 = \omega_1, \qquad K_2 = \omega_2 \,, \tag{1.5}$$

and

$$M_0 = \Phi^0(b_1, b_2) + b_1\,\omega_2 - b_2\,\omega_1 = a, \tag{1.6}$$

such that Φ^0 will have the form:

$$\Phi^0 = M_0 + K_1\, x_2 - K_2\, x_1 \tag{1.7}$$

in the right half-plane, so, from (1.4), it will be identical with M. Upon passing the x_2-axis, Φ will jump from zero to $M_0 + K_1\, x_2$ by the magnitude of the moment of the dyname at x_2 .

Naturally, we are still at liberty to assign non-zero values to the three constants of Φ^0 arbitrarily in the left half-plane. The components of the dyname will then be expressed in terms of the differences between the constants a, ω_1, and ω_2 in the right and left half-planes. We would like to call (1.7), which is obviously a singular solution of (1.2), the *stress function of a planar dyname.* Corresponding arguments shall be presented for just the spatial case.

2. Notations and summary of the required formulas

We shall link our notations closely to *Günther's* paper [**1**], although we would like to work in Cartesian coordinates throughout.

The fourth-rank tensor:

$$T_{ik,lm} = e_{ik\alpha}\, e_{lm\beta}\, S_{\alpha\beta} \tag{2.1}$$

will be defined from the symmetric tensor S_{ik} according to ([**1**], (1.23)). From ([**1**], (1.24)), $T_{ik,lm}$ can be represented as the rotation of a third-rank tensor:

$$T_{ik,lm} = \frac{\partial}{\partial x_i}\gamma_{k,lm} - \frac{\partial}{\partial x_k}\gamma_{i,lm}\,. \tag{2.2}$$

When we introduce the tensor of stress functions F_{ik}, which is likewise a symmetric tensor, we will get:

$$\gamma_{l,ik} = \frac{\partial}{\partial x_i}F_{kl} - \frac{\partial}{\partial x_k}F_{il} \tag{2.3}$$

and the cyclic symmetry:

$$\gamma_{ikl} + \gamma_{kli} + \gamma_{lik} = 0. \tag{2.4}$$

One will then have:

$$T_{ik,lm} = \frac{\partial^2}{\partial x_i\,\partial x_j}F_{km} + \frac{\partial^2}{\partial x_k\,\partial x_m}F_{il} - \frac{\partial^2}{\partial x_i\,\partial x_m}F_{kl} - \frac{\partial^2}{\partial x_k\,\partial x_l}F_{im}\,. \tag{2.5}$$

The six components of a dyname will be denoted by K_l and M_l . From *Günther* [**1**], their connection with the γ_{ikl} and F_{ik} can be represented very simply when one introduces the "skew-symmetric force tensor":

$$K_{lm} = e_{lm\alpha}\, K_\alpha\,. \tag{2.6}$$

The moment vector relative to the origin of the coordinate system has the components:

$$M(0)_l = x_\alpha\, K_{l\alpha}\,. \tag{2.7}$$

From ([**1**], (1.14)), the force vector that is attributed to the element df_{ik} on the outer surface is:

$$dK_{lm} = T_{\alpha\beta,lm}\, df_{\alpha\beta}\,, \tag{2.8}$$

and from (2.7), its moment at the origin is:

$$dM(0)_l = T_{\alpha\beta,l\rho}\, x_\rho\, df_{\alpha\beta}\,. \tag{2.9}$$

As a result of (2.2), dK_{lm} is a total differential. If we give ourselves a closed curve C in a simply-connected part of the body in question then the outer surface integral of dK_{lm}

over an outer surface that is bounded by this curve will be independent of the form of the outer surface. From Stokes's theorem, K_{lm} will be a functional of the curve C:

$$K_{lm} = \oint_C \gamma_{k,lm} dx_k \,. \tag{2.10}$$

However, as a result of (2.3), $dM(0)_l$ will also be a total differential. From (2.2) and (2.4), one will next have:

$$T_{ik,\,l\rho}\, x_\rho = \frac{\partial}{\partial x_i}(x_\rho \gamma_{k,l\rho}) - \frac{\partial}{\partial x_k}(x_\rho \gamma_{i,l\rho}) + \gamma_{i,lk}\,, \tag{2.11}$$

from which, after introducing (2.3) and applying Stokes's theorem, the functional:

$$M(0)_l = \oint_C [F_{kl} + x_\rho \gamma_{k,l\rho}] dx_k \tag{2.12}$$

will arise.

(2.10) and (2.12) are, up to an inessential change of sign in F_{ik}, the *Günther* representations that were mentioned in the introduction in the special of Cartesian coordinates.

3. The stress function of a straight rod that occupies space

The rod might now lie on the x_3-axis of our coordinate system, which is embedded in a stress-free continuum.

From (2.5), the tensor of the stress functions F_{ik} is constructed from the singular solutions on the x_3-axis of:

$$T_{ik,\,lm} = 0. \tag{3.1}$$

In ([**1**], (2.1) to (2.7)) and [**2**], it was shown that any solution of (3.1) can be represented as the symmetric gradient tensor:

$$F_{ik}^0 = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right), \tag{3.2}$$

which is called the *tensor of null stress functions*. In the present case, the v_i are regular over all of space, except for the x_3-axis, on which they must possess a singularity that is characteristic of the dyname. According to (2.3), we will get:

$$\gamma_{l,\,ik} = \frac{\partial}{\partial x_l}\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right)\right], \tag{3.3}$$

and we will get the force vector of the dyname from (2.10):

$$K_{lm} = \oint_C \frac{\partial}{\partial x_k}\left[\frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}\right] dx_k = \oint_C d\left[\frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}\right], \tag{3.4}$$

in which the integral is taken over an arbitrary closed curve C that encircles the rod.

The same thing will be true for the circuit integral (2.12), which will take the form:

$$M(0)_l = \oint_C d\left[v_l + x_m \cdot \frac{1}{2}\left(\frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}\right)\right] \tag{3.5}$$

here, after some brief intermediate calculations that shall be omitted here.

The integrands of our circuit integrals are total differentials. The integrals will thus be non-zero in general only when the vector field of v_i is multi-valued. Such a vector field can, however, be defined very simply from the moment field of the dyname. The moment vector at any point $P(x_1, x_2, x_3)$ in space is:

$$M_l = M(0)_l - x_\alpha K_{l\alpha} . \tag{3.6}$$

In the field $\bar{v}_l = M_l$, one will now have:

$$\frac{1}{2}\left(\frac{\partial \bar{v}_m}{\partial x_l} - \frac{\partial \bar{v}_l}{\partial x_m}\right) = \frac{1}{2}(-K_{lm} + K_{ml}) = K_{lm} . \tag{3.7}$$

We will thus obtain the desired multi-valued field when we set:

$$v_l = M_l \cdot \frac{\varphi}{2\pi} = M_l \cdot \frac{1}{2\pi} \cdot \arctan \frac{x_2}{x_1} . \tag{3.8}$$

The derivatives of arctan x_2 / x_1 are single-valued functions, such that we will now have:

$$\frac{1}{2}\left(\frac{\partial v_m}{\partial x_l} - \frac{\partial v_l}{\partial x_m}\right) = K_{lm} \cdot \frac{\varphi}{2\pi} + \text{single-valued function.} \tag{3.9}$$

One immediately sees that the circuit integrals (3.4) and (3.5) will yield the required components of the dyname. The tensor F_{ik}^0 of stress functions can now be calculated with the help of (3.2) and (3.8). Its components will be single-valued in all space, although they will the singular on the x_3-axis. We will refrain from calculating them here.

4. The analogy with Volterra's distortions

One can establish an analogy between the null stress tensor:

$$F_{ik}^{0} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \tag{4.1}$$

and the distortion tensor:

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_l} + \frac{\partial u_k}{\partial x_i}\right) \tag{4.2}$$

of a continuum.

If we pursue this analogy further for the example that was just treated then we must first establish that the distortion field of our continuum is, in fact, single-valued, even though the associated displacement field is multi-valued.

Volterra treated the theory of such distortion fields and displacement fields in his celebrated treatise "Sur l'équilibre des corps élastiques multiplement connexes" [**3**].

A thick-walled hollow cylinder of elastic material stands on the (x_1, x_2) plane in such a way that its figure axis coincides with the x_3-axis. A plane of intersection through the x_3-axis cuts it on one side in such a way that it becomes only simply-connected. If a "distortion" is now present, and indeed one that is such that the plane of the one edge of the cut is fixed, while the other edge is subjected to a translation and a rotation, in such a way that the displacement of its points is:

$$u_l = a_l - x_\alpha\, \omega_{l\alpha}\,. \tag{4.3}$$

The three components of the vector a_l are the components of the translation, and the three components of the skew-symmetric tensor ω_{lm} are those of the rotation. Thus, the second cut plane will be displaced like a rigid body. One subsequently establishes the double connection of the cylinder again when one welds both edges of the cut together, with the addition or removal of material. The deformations are everywhere single-valued in the proper stress state that now prevails. It is the calculation of the displacements from the deformations that first gives one an insight into the nature of the distortions that were performed.

The multi-valued vector field that we gave in (3.8) cannot immediately be regarded as the displacement field of the distorted cylinder. It is indeed a kinematically possible displacement field of a continuum. However, the deformations that are calculated from it for an elastic body by using Hooke's law will lead to stresses that satisfy the equilibrium conditions inside the body. In other words, the displacements of an elastic body will have to satisfy differential equations whose solutions are to be found amongst the biharmonic functions. They themselves will be composed of harmonic functions and products of linear and harmonic functions. Now, since the simplest multi-valued harmonic function is the imaginary part of the complex function $\ln(x_1 + ix_2)$, *Volterra* made the Ansatz:

$$u_l = (a_l - x_\alpha\, \omega_{l\alpha}) \cdot \arctan x_2 / x_1 + (c_{l0} + c_{l\alpha}\, x_\alpha) \cdot \ln\sqrt{x_1^2 + x_2^2} \tag{4.4}$$

and determined the still-unspecified constants c_{l0} and $c_{l\alpha}$ in such a way that the differential equations of the displacements u_l of the elastic body would be fulfilled. One thus has the following analogy in connection with these considerations: The deformations ε_{ik} of a distorted body can be regarded as the stress functions F_{ik}^0 of the dyname whose components $M(0)_l$ and K_{lm} agree with the components a_l and $\omega_{l\alpha}$ of the distortion that corresponds to it.

5. Summary

The stress functions of a rod that was acted upon by a dyname were examined on the basis of the connection between the tensor of stress functions and the dyname of the outer surface forces that was given by *Günther* [**1**]. Its tensor could be interpreted as the deformation tensor of a continuum that was subjected to a *Volterra* distortion, for which the six components of the distortion agreed with the corresponding components of the dyname.

References

[1] *W. Günther*, "Spannungsfunktionen und Verträglichkeitsbedingungen der Kontinuums-mechanik," Abh. d. Braunschw. Wiss. Ges. **6** (1954).

[2] *H. Schaefer*, "Die Spannungsfunktionen des räumlichen Kontinuums und des elastischen Kōrpers," ZAMM **33**, 10/11 (1953).

[3] *V. Volterra*, "Sur l'équilibre des corps élastiques multiplement connexes," Ann. de l'École Normale, Paris (3) **24** (1907), 401-517.