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# Gravity in multiply-extended Gaussian and Riemannian spaces

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I presented the fundamental law and some propositions about the essential properties of gravity in triply-extended Gaussian spaces in these Nachrichten on 13 July 1870. As I have now learned, Dirichlet also addressed that topic recently during his visit to Berlin. He discussed it with his friends but did not publish his investigations.

Exhibiting the laws for the fictitious forces in such spaces, of which the one that surrounds us is only a special case, means to us, on the one hand, that we have to make a better decision in regard to the natural forms of the laws for the forces that are known to us, but it also means that the investigation of such a more general law will offer us a glimpse into the realm of pure analysis by means of new tools that are to be extended in a way that is similar to what has been done many times in the study of the known forces of nature. That hope has already been fulfilled in one case by Kronecker's paper "Ueber Systeme von Funktionen mehrer Variabeln." The property of gravity in multiply-extended planar spaces has given rise to the introduction of the concept of "characteristic of a system of functions" that has proved so fruitful in analysis. I would like to add the following propositions to the one that I presented before:

**Proposition I.**

*Let $\mathfrak{R}_n$, $R_n$, and $P_n$ be regions of space that can overlap or intersect arbitrarily. Let $x_1, \ldots, x_\nu, \ldots, x_n$ be the n rectilinear rectangular coordinates of a point in the spatial element $dR_n$ in an n-fold extended planar space while $\xi_1, \ldots, \xi_\nu, \ldots, \xi_n$ are the coordinates of a point in the spatial element $dP_n$. Let $a_1, \ldots, a_\nu, \ldots, a_n$ be the coordinates of a point in the element $dR_n$ or also a point in the element $d\mathfrak{R}_{n-1}$, which belongs to an $(n-1)$-fold extended spatial form $\mathfrak{R}_{n-1}$ that bounds the spatial region $\mathfrak{R}_n$. Let $m(\ldots, x_\nu, \ldots)$ and $\mu(\ldots, \xi_\nu, \ldots)$ be functions of the coordinates for points in the spatial regions $R_n$ and $P_n$, resp. Let r denote the positive value of $\sqrt{\sum (x_\nu - a_\nu - \xi_\nu)^2}$, while $d\mathfrak{N}$ is the normal to the spatial form $d\mathfrak{R}_{n-1}$ at the point $\ldots, a_\nu, \ldots$ on the side where the spatial region $\mathfrak{R}_n$ is found. If the function $\Pi$ has the same meaning that Gauss gave it then:*

$$K(n) = 2\,(4\pi)^{(n-1)/2} \cdot \frac{\Pi\left(\dfrac{n-1}{2}\right)}{\Pi\,(n-1)} \qquad \textit{for odd } n,$$

$$K(n) = 2\,(4\pi)^{n/2} \cdot \frac{1}{\Pi\left(\dfrac{n-2}{2}\right)} \qquad \textit{for even } n,$$

*and if one sets:*

$$\iint m(x_1, x_2)\,\mu(\xi_1, \xi_2) \log\frac{1}{r} \cdot dR_2\, dP_2 = \Phi$$

*for n* = 2, *while:*

$$\iint \frac{m(\ldots, x_\nu, \ldots) \cdot \mu(\ldots, \xi_\nu, \ldots)}{(n-2)\, r^{n-2}}\, dR_n\, dP_n = \Phi$$

*for n* > 2 *then one will have:*

$$\int \frac{\partial\Phi}{\partial\mathfrak{R}}\, d\mathfrak{R}_{n-1} = K(n) \iint m(\ldots, x_\nu, \ldots) \cdot \mu(\ldots, \xi_\nu, \ldots)\, dR'_n\, dP'_n\,,$$

*in which the integral with respect to* $d\mathfrak{R}_{n-1}$ *extends over the entire space form* $\mathfrak{R}_{n-1}$ *that bounds the spatial region* $\mathfrak{R}_n$ .

*The integrals with respect to* $dR_n$ *and* $dP_n$ *extend over the entire spatial regions* $R_n$ *and* $P_n$ *, respectively, but the integrals with respect to* $dR'_n$ *and* $dP'_n$ *extend over only all of the pairs of elements* $dR'_n$ *at the point* $x_1, \ldots, x_\nu, \ldots, x_n$ *and* $dP'_n$ *at the point* $\xi_1, \ldots, \xi_\nu, \ldots, \xi_n$ *, resp., that lie in the spatial regions* $R_n$ and $P_n$*, respectively, and the coordinates of a point that lies in the spatial region* $\mathfrak{R}_n$ *are given by* $x_1 - \xi_1, \ldots, x_\nu - \xi_\nu, \ldots, x_n - \xi_n$ *. If the masses m and* $\mu$ *are not continuously-distributed throughout the spatial regions* $R_n$ *and* $P_n$ *, resp., but over less-multiply-extended forms* $R_{n-n'}$ *and* $P_{n-\nu'}$ *, resp., or concentrated at points then integrals with respect to* $dR_{n-n'}$*,* $dR'_{n-n'}$ *and* $dP_{n-\nu'}$*,* $dP'_{n-\nu'}$ *, resp., will enter in place of the* $dR_n\, dR'_n$ *and* $dP_n\, dP'_n$ *, resp., or also finite sums.*

Lipschitz has also published investigations into the study of motion in multiply-extended non-planar spaces in his treatises on the subject of homogeneous forms of differentials.

The article that I published in the previous volume of these Nachrichten on multiply-extended Gaussian and Riemannian spaces contains the tools for proving the following propositions for gravity in such spaces.

**Proposition II.**

*If r means the distance between the mass-particles m and* $\mu$*, as measured with an arbitrary unit of length, and* $\sqrt{-1}\,/\,\varepsilon$ *denotes the length for a Gaussian space, while* $1\,/\,\varepsilon$ *denotes the length for a Riemannian one, which depends upon the special nature of the space and is measured with* *the unit that is based upon the determination of the remaining lengths, which can be considered to be the absolute unit of length that is peculiar to the space, and if* $\Pi$ *and* $K(n)$ *have the same meaning as above and one sets:*

$$\sum_{\nu=0}^{(n-3)/2} 2^{n-2\nu-3}\,\frac{\Pi(2\nu)}{\Pi(n-2)}\cdot\frac{\Pi\left(\frac{n-3}{2}\right)}{\Pi(\nu)}\cdot\frac{\Pi\left(\frac{n-3}{3}\right)}{\Pi(\nu)}\cdot\varepsilon^{n-2}\cos\varepsilon r\cdot\sin\varepsilon r^{-2\nu-1} = w_n(r) \qquad \textit{for odd } n,$$

*while*

$$2^{2-n}\,\frac{\Pi(n-2)}{\Pi\left(\frac{n-2}{2}\right)\cdot\Pi\left(\frac{n-2}{2}\right)}\,\varepsilon^{n-2}\log\left(\tfrac{1}{2}\varepsilon\cot\tfrac{1}{2}\varepsilon r\right)$$

*for even n*

$$+\sum_{\nu=0}^{(n-4)/2} 2^{2\nu-n+2}\,\frac{\Pi(n-2)}{\Pi(2\nu+1)}\cdot\frac{\Pi(\nu)}{\Pi\left(\frac{n-2}{2}\right)}\cdot\frac{\Pi(\nu)}{\Pi\left(\frac{n-2}{2}\right)}\cdot\varepsilon^{n-2}\cos\varepsilon r\cdot\sin\varepsilon r^{-2\nu-2} = w_n(r)$$

*then*

$$m\,\mu\,w_n(r)$$

*will be the potential function for the attraction between the positively-taken mass-particles m and* $\mu$ *in the n-fold extended homogeneous spaces.*

**Proposition III.**

*If V means the potential function for the action that is exerted upon a unit mass that is found at a point, which shall be considered to originate from masses that are distributed throughout space in any way and are regarded as positive or negative according to whether the action is one of attraction or repulsion, resp., then one will have:*

$$V = \sum_m m\cdot w_n(r),$$

*when r denotes the distance from the mass-particle m to the variable point that determines the function V. The total mass that is found in any spatial region* $R_n$ *that is bounded completely, but only simply, by the* $(n-1)$*-fold extended spatial form* $R_{n-1}$ *will be represented by:*

$$\frac{1}{K(n)}\int\frac{\partial V}{\partial N}\,dR_{n-1},$$

*in which* $K(n)$ *has the meaning that it was given above, dN means the normal to the element* $dR_{n-1}$ *of the* $(n-1)$*-fold extended spatial form* $R_{n-1}$ *, which is raised on the side of the spatial region* $R_n$ *that it bounds, and the integral extends over the entire boundary of the spatial region* $R_n$ *.*

**Proposition III** [*sic*]**.**

*If one determines the position of a point by any sort of rectangular curvilinear coordinates* $\eta_1, \ldots, \eta_\nu, \ldots, \eta_n$ *and one denotes any two infinitely-small changes of position of that point by* $d\eta_1, \ldots, d\eta_\nu, \ldots, d\eta_n$ *and* $\delta\eta_1, \ldots, \delta\eta_\nu, \ldots, \delta\eta_n$ *then the product of the lengths of the shortest lines that are drawn from the first position of that point to any two neighboring positions with each other and the cosine of the angle that those lines subtend will have the form:*

$$\sum_{\nu=1}^{n} \eta'_\nu \, \eta'_\nu \, d\eta_\nu \, \delta\eta_\nu \,,$$

*in which* $\eta'_1, \ldots, \eta'_\nu, \ldots, \eta'_n$ *are positive functions of only* $\eta_1, \ldots, \eta_\nu, \ldots, \eta_n$.

**Proposition IV.**

*If the unit mass to which the potential V refers is found at the point that is determined by the coordinates* $\eta_1, \ldots, \eta_\nu, \ldots, \eta_n$*, as in the previous proposition, and if the active mass at that location is distributed continuously in n-fold extended space then the density of the mass there will be equal to:*

$$-\frac{1}{K(n)} \frac{1}{\eta'} \sum_{\nu=1}^{n} \frac{\partial}{\partial \eta_\nu} \left( \frac{\eta'}{\eta'_\nu \eta'_\nu} \frac{\partial V}{\partial \eta_\nu} \right),$$

*in which* $\eta$ *is set equal to the product* $\eta'_1 \cdot \eta'_2 \ldots \eta'_n$.

**Proposition VI** [*sic*].

*If the mass is condensed into an* $(n-1)$*-fold extended spatial form* $R_{n-1}$*, and if the mass varies continuously within that form then:*

$$\frac{1}{K(n)} \left( \frac{\partial V}{\partial N} \right)_1 + \frac{1}{K(n)} \left( \frac{\partial V}{\partial N} \right)_2$$

*will be the density at that location on the form* $R_{n-1}$*, on both sides of which the normals* $dN_1$ *and* $dN_2$ *to the spatial form* $dR_{n-1}$ *are raised.*

**Proposition VII.**

*If the mass is condensed into an* $(n-\nu)$*-fold extended spatial form* $R_{n-\nu}$*, and if the mass varies continuously inside of that form then the limiting values of:*

$$\frac{2\pi}{K(n)} \cdot \frac{V}{\log \frac{1}{N}} \qquad \textit{for } n - \nu = n - 2$$

*and*

$$\frac{K(\nu)}{K(n)} (\nu - 2)\, N^{\nu-2}\, V \qquad \textit{for } n - \nu < n - 2$$

*as n goes to zero will be the density at the location on the form $R_{n-\nu}$ at which the normal N to $dR_{n-\nu}$ is drawn to an external point that is infinitely-close to the form and at which the potential function V is determined.*

**Proposition VIII.**

*Let $R_n$ denote a certain spatial region, and let $dR_n$ denote its spatial element, which includes the point $\eta_1, \ldots, \eta_\nu, \ldots, \eta_n$. Furthermore, let $R_{n-1}$ denote the* $(n-1)$*-fold space form that bounds the spatial region $dR_n$, while $dR_{n-1}$ is an element of that form, and dN denotes a normal to the bounded spatial region $R_n$ that is raised to $dR_{n-1}$. One then has:*

$$\int \sum_{\nu=1}^{n} \frac{1}{\eta'_\nu\,\eta'_\nu} \frac{\partial U}{\partial \eta_\nu} \frac{\partial V}{\partial \eta_\nu} \cdot dR_n$$

$$= -\int U \frac{\partial V}{\partial N} dR_{n-1} - \int U \frac{1}{\eta'} \sum_{\nu=1}^{n} \frac{\partial}{\partial \eta_\nu} \left( \frac{\eta'}{\eta'_\nu\,\eta'_\nu} \frac{\partial V}{\partial \eta_\nu} \right) \cdot dR_n$$

$$= -\int V \frac{\partial U}{\partial N} dR_{n-1} - \int V \frac{1}{\eta'} \sum_{\nu=1}^{n} \frac{\partial}{\partial \eta_\nu} \left( \frac{\eta'}{\eta'_\nu\,\eta'_\nu} \frac{\partial U}{\partial \eta_\nu} \right) \cdot dR_n ,$$

*when U and V are functions of the coordinates that vary continuously in the space $R_n$ and are such that the integrals over $dR_n$ and $dR_{n-1}$ can assume finite values.*

Suppose that *n* shortest lines start from a point, each of which is normal to all of the other ones, and which shall be called *coordinates axes*. If a shortest line is drawn from the point 0 to a point *x* and the section that goes from the point 0 to the midpoint of the shortest line is projected onto the coordinate axes and the lengths of those projections are denoted by $\frac{1}{2}x_1, \ldots, \frac{1}{2}x_\nu, \ldots, \frac{1}{2}x_n$ then $x_1, \ldots, x_\nu, \ldots, x_n$ shall be called the *rectangular symmetric coordinates* of the point *x*.

**Proposition IX.**

*Let $\mathfrak{R}_n$, $R_n$,* and *$P_n$ be spatial regions, which can overlap or intersect arbitrarily, Let $\alpha_1, \ldots, \alpha_\nu, \ldots, \alpha_n$, $x_1, \ldots, x_\nu, \ldots, x_n$, and $\xi_1, \ldots, \xi_\nu, \ldots, \xi_n$ be the rectangular symmetric coordinates of three points that are each found in one of the three spatial regions $\mathfrak{R}_n$, $R_n$,* and *$P_n$, resp. Let m(…,*

*$x_\nu$, …) and $\mu$ (…, $\xi_\nu$, …) be continuous functions inside the spatial regions $R_n$ and $P_n$ . Let $d\mathfrak{N}$ be the normal to the space form $d\mathfrak{R}_{n-1}$ at the point $\alpha_1$, …, $\alpha_\nu$, …, $\alpha_n$ on the side where the spatial region $\mathfrak{R}_n$ that it bounds is found. Set:*

$$\frac{\sum(\tan\frac{1}{2}\varepsilon\,x_\nu - \tan\frac{1}{2}\varepsilon\,\alpha_\nu - \tan\frac{1}{2}\varepsilon\,\xi_\nu)^2}{\{1+\sum(\tan\frac{1}{2}\varepsilon\,\alpha_\nu)^2\}\{1+\sum(\tan\frac{1}{2}\varepsilon\,x_\nu - \tan\frac{1}{2}\varepsilon\,\xi_\nu)^2\}} = (\sin\tfrac{1}{2}\varepsilon\,r)^2,$$

*in which the summation $\sum$ extends over the numbers $\nu$ = 1, 2, 3, …, n. If one ultimately sets:*

$$\iint m(\ldots,x_\nu,\ldots)\,\mu(\ldots,\xi_\nu,\ldots)\,w_n(r)\,dR_n\,dP_n = W_n(\ldots,\alpha_\nu,\ldots)$$

*then one will have the fundamental equation:*

$$\int \frac{\partial}{\partial\mathfrak{N}} W_n(\ldots,\alpha_\nu,\ldots)\,d\mathfrak{R}_{n-1} = K(n)\iint m(\ldots,x_\nu,\ldots)\,\mu(\ldots,\xi_\nu,\ldots)\,dR'_n\,dP'_n\,,$$

*in which the integral with respect to the element $d\mathfrak{R}_{n-1}$ with the point that is determined by the rectangular symmetric coordinates $\alpha_1$, …, $\alpha_\nu$, …, $\alpha_n$ extends over the entire space form $\mathfrak{R}_{n-1}$ , which bounds the spatial region $\mathfrak{R}_n$ simply, and the integrals with respect to $dR_n$ and $dP_n$ extend over the entire spatial regions $R_n$ and $P_n$ , resp., but the integrals with respect to $dR'_n$ and $dP'_n$ extend over only all of the pairs of elements with $dR'_n$ at the point $x_1$ , … , $x_\nu$, …, $x_n$ and $dP'_n$ at the point $\xi_1$ , … , $\xi_\nu$, …, $\xi_n$ , which lie in the spatial regions $R_n$ and $P_n$ , respectively, and the rectangular symmetric coordinates $\alpha_1$, …, $\alpha_\nu$, …, $\alpha_n$ of any point that lies in the spatial region $\mathfrak{R}_n$ are given by:*

$$\tan\frac{1}{2}\,\varepsilon\,x_\nu - \tan\frac{1}{2}\,\varepsilon\,\xi_\nu = \tan\frac{1}{2}\,\varepsilon\,\alpha_\nu$$

*for $\nu$ = 1, 2, …, n. According to the nature of m and $\mu$, integrals that are extended over less-multiply-extended spatial forms can enter in plane of the integrals with respect to $dR_n$ , $dR'_n$ and $dP_n$ , $dP'_n$ , as well as finite sums.*

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