The focal theory of linear ray congruences

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Jolles gave the focal theory of linear ray congruences and some theorems on the principal axis cylindroid, which is closely connected with it, in volume 63 of the Mathematischen Annalen ([1]). His investigations are carried out in a purely synthetic way, and the proofs thus often present difficulties that can be overcome only by circumstantial, over-extended devices, especially when one treats imaginary structures. However, if we seek to present Jolles's results by means of an analytical method then that will show that not only is the transition to imaginary structures possible, with no further assumptions, but that the proofs will assume a much simpler and more intuitive form. In what follows, we therefore must first seek an analytical representation of a congruence that will define the foundation for the simplest-possible development of focal theory. However, whereas Jolles defined the focal paraboloid to be a surface that is enveloped by the so-called "middle planes" of the linear congruence, we will start from the rotational ruled family that is contained in the congruence, which consists of rotational axes that determine the focal paraboloid and define it as the geometric locus of these rotational axes. The polar properties of the focal paraboloid will allow us to pursue some general considerations about the polar systems for which one of the linear congruences is invariant, and we will, in turn, deal with the search for an analytical representation of "primary" and "secondary" polar spaces, as Jolles called these systems in another place ([2]). All of the polar properties of the focal paraboloid can be easily verified as special cases of this more general theorem. When we then conversely consider the paraboloid and the involutions of its ruling rays to be given, we will be led to the theory of "confocal" linear ray congruences and to some theorems that are connected with the principal axis cylindroid and the relationships between the cylindroid and the focal paraboloid. Finally, the results that were found will be carried over into the special case of the "rotational" linear congruence.

I. General properties of a linear ray congruence.

We imagine that a linear ray congruence is given by the intersection of two conjugate (null-invariant) ray complexes. Two complexes (A and B) are represented analytically by the equations:

(1) A: $$(a, p) \equiv a_{34}\,p_{12} + a_{24}\,p_{31} + a_{14}\,p_{23} + a_{23}\,p_{14} + a_{31}\,p_{24} + a_{12}\,p_{34} = 0,$$

(2) B: $$(b, p) \equiv b_{34}\,p_{12} + b_{24}\,p_{31} + b_{14}\,p_{23} + b_{23}\,p_{14} + b_{31}\,p_{24} + b_{12}\,p_{34} = 0,$$

([1]) Some important focal properties can also be found already in: R. Sturm: *Die Gebilde ersten und zweiten Grades der Lineiengeometrie*, Leipzig, 1892, Part 1, § 121.

The main results of Jolles are cited in Reye's *Geometrie der Lage*, 4th ed., 2nd section, page 287.

([2]) St. Jolles, "Primäre und Sekundäre polare Räume einer linearen Strahlenkongruenz," Journal für reine und angewandte Mathematik, Band 134, Heft 1.

where the quantities $a_{ik} = a_{ki}$ ($b_{ik} = b_{ki}$, resp.) are constants and the $p_{ik} = -p_{ki}$ mean the line coordinates of its rays. They will be called *conjugate* ([1]) when one of them is associated with the other one, and we assert:

The necessary and sufficient analytical condition for the conjugate position of two linear ray complexes is:

(3) $$\boxed{(a,b) = a_{12}b_{34} + a_{23}b_{14} + a_{31}b_{24} + a_{14}b_{23} + a_{24}b_{31} + a_{34}b_{12} = 0.}$$

In order to prove this, we consider two lines that will be given by their line coordinates p_{ik} and q_{ik} . If we now assume that they are associated with each other by the complex A then we will get ([2]):

(4) $$\rho\, q_{ik} = (a, p)\, a_{ik} - a \cdot p_{ik} ,$$

where ρ is a proportionality factor and $a = a_{12}\, a_{34} + a_{23}\, a_{14} + a_{31}\, a_{24} \neq 0$. If we further assume that the line p_{ik} belongs to complex B, but not to complex A, then:

(5) $$(b, p) = 0; \qquad (a, p) \neq 0.$$

If we then multiply the six equations that are included in (4) in sequence by $b_{l,m}$ (where l, m are the variations of the numbers 1, 2, 3, 4 that are "complementary" to i, k) and add them then that will give:

(6) $$\rho\, (b, q) = (a, p) \cdot (b, a) - a\, (b, p),$$

where the last term will vanish, due to (5). If we now demand that the line q_{ik} must likewise be contained in the complex B, but not in A, so:

$$(b, q) = 0, \qquad (a, q) \neq 0,$$

then it will follow from (6) that:

$$(a, b) = 0.$$

Q. E. D.

A *null correlation* is defined by the two complexes A and B whose null rays comprise the rays of the complex in question:

A	B
(7a) $\rho\, x_i = \sum_{k=1}^{4} a_{ik} u_k$,	$\sigma x'_k = \sum_{l=1}^{4} b_{kl} u'_k$,

([1]) Reye, *Geometrie der Lage II*, 4th ed., lecture 18.
([2]) Staude, *Analytische Geometrie I*, 1910, pp. 469, *et seq.*

(7b) $$\rho\, u_i = \sum_{k=1}^{4} \alpha_{ik} u_k\,, \qquad \Big| \qquad \sigma u'_k = \sum_{l=1}^{4} \beta_{kl} u'_k\,,$$

$$(\alpha_{12} = a_{34},\ \alpha_{23} = a_{14},\ \text{etc.}) \qquad (\beta_{12} = b_{34},\ \beta_{23} = b_{14},\ \text{etc.})$$

For both correlations, a point will lie in its polar plane, and any plane will go through its pole. If we now seek the polar plane π to a point P that is associated with it *by the correlation A* and then construct the pole P' that corresponds to the plane π *according to B* then we will have defined a *collineation* ($P \to P'$). Two points P and P' will then correspond to each other collinearly when they are associated with the same plane π as the polar plane under the two correlations (A and B). We will obtain the analytical expression for this collinear relationship when set $u_k = u'_k$ in (7) and substitute the value of u'_k from the right-hand side of equation (7b) in the left-hand side of equation (7a). Thus:

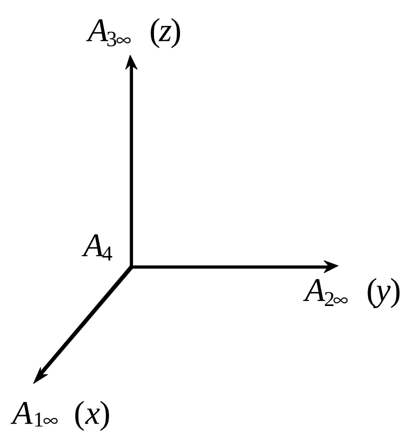
(8a) $$\lambda\, x_i = \sum_{k,l=1}^{4} \beta_{kl}\, a_{ik}\, x'_l$$

and analogously, it will follow for the coordinates of two corresponding planes that:

(8b) $$\lambda\, u_i = \sum_{k,l=1}^{4} \alpha_{kl}\, b_{ik}\, u'_l\,.$$

If we assume that the complexes A and B are conjugate then this collineation will be an involution; i.e., each element will correspond to the other one in a double way. Analytically, this can be show quite easily when one switches a_{ik} with b_{ik} (α_{ik} with β_{ik}, resp.) in (8) and considers the condition (3).

The double rays of an involutory collineation – i.e., the connecting lines of any two corresponding points (lines of intersection of two corresponding planes, resp.) – *are the rays that are common to the complexes A and B. They will then define the linear ray congruence that is represented by A and B.*



However, we will arrive at far simpler equations, as will be shown, when we base our investigations upon a congruence C_1^1 that is defined by the following two complexes:

$$\alpha)\ \boxed{p_{23} + \alpha\, p_{14} = 0,} \qquad \beta)\ \boxed{p_{13} + \beta\, p_{14} = 0,}$$

where α and β are constant quantities. With this specialization of the complex equations, we have, in fact, arrived at the following result: We can assume that the coordinate tetrahedron to which the quantities p_{ik} are referred is such that one of its faces (say, $A_1\,A_2\,A_3$, when we denote the vertices by A_i) goes to infinity, and the remaining ones are perpendicular to each other. The edges of the tetrahedron that do not lie in the plane at infinity will thus define a rectangular coordinate cross $A_4\,A_{1\infty}$; $A_4\,A_{2\infty}$; $A_4\,A_{3\infty}$, whose axes we will refer to as the x, y, and z axes, respectively, for the sake of convenience in our terminology. Equations (7) then immediately show that the point A_4 is associated with the plane $A_2\,A_3\,A_4$ ($x_1 = 0$) through the complex α), while the plane $A_1\,A_3\,A_4$ ($x_2 = 0$) will correspond to it in the complex β). It is known that one has:

	p_{23}	p_{31}	p_{12}	p_{14}	p_{24}	p_{34}
A_2A_3	1	0	0	0	0	0
A_1A_3	0	1	0	0	0	0
A_1A_2	0	0	1	0	0	0
A_1A_4	0	0	0	1	0	0
A_2A_4	0	0	0	0	1	0
A_3A_4	0	0	0	0	0	1

for the coordinates of the edges of the tetrahedron.

The edges A_1A_2 and A_3A_4 are contained in the congruence C_1^1, since their coordinates satisfy equations α) and β). By contrast, it follows from equations (4) that the lines A_2A_3 and A_1A_4 are conjugate in complex α), while A_1A_3 and A_2A_4 are conjugate in complex β). The coordinate axes are, moreover, the three symmetry axes of the congruence. Equations α) and β), which we can also write in the form:

$$\alpha)\qquad x_2x_3' - x_3x_2' + \alpha\,(x_1x_4' - x_4x_1') = 0,$$

$$\beta)\qquad x_1x_3' - x_3x_1' + \beta\,(x_2x_4' - x_4x_2') = 0,$$

will then remain valid when we first replace the quantities x_3, x_3' and x_1, x_1', and then x_2, x_2' and x_1, x_1', and finally x_2, x_2' and x_3, x_3', with their negative values.

With the choice of the complexes α) and β), equations (8) for the involutory collineation that is given by the congruence now go to:

$$(9)\qquad x_1 = x_2', \qquad \beta\,x_2 = \alpha\,x_1', \qquad x_3 = \alpha\,x_4, \qquad \beta\,x_4 = x_3',$$

$$(10)\qquad \beta\,u_1 = \alpha\,u_2', \qquad u_2 = \alpha\,u_1', \qquad \beta\,u_3 = u_4', \qquad u_4 = \alpha\,u_3',$$

when we omit an overall proportionality factor. The plane at infinity ($x_4 = 0$) is, as is obvious, associated with the plane $x_3 = 0$, which is the "alignment plane" (*Fluchtebene*) of the congruence.

In order to find the guiding lines of C_1^1, we consider the pencil of complexes that is defined by the complexes α) and β), and whose common rays define the congruence C_1^1. It is represented by the equation:

$$p_{23} + \alpha p_{14} + \lambda (p_{13} + \beta p_{24}) = 0.$$

Two special complexes are included in this pencil whose parameters λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 = \alpha / \beta$ (1). It follows that these special complexes have the equations:

(12)
$$\begin{cases} p_{23} + \alpha p_{14} + \sqrt{\dfrac{\alpha}{\beta}}(p_{13} + \beta p_{24}) = 0, \\ p_{23} + \alpha p_{14} - \sqrt{\dfrac{\alpha}{\beta}}(p_{13} + \beta p_{24}) = 0, \end{cases}$$

and its guiding lines are the guiding lines of C_1^1, whose line coordinates can be obtained immediately from (12):

(13)
$$p_{13} : p_{23} : p_{31} : p_{14} : p_{24} : p_{34} = \begin{cases} 0 : \alpha : +\sqrt{\alpha\beta} : 1 : -\sqrt{\dfrac{\alpha}{\beta}} : 0, \\ 0 : \alpha : -\sqrt{\alpha\beta} : 1 : +\sqrt{\dfrac{\alpha}{\beta}} : 0. \end{cases}$$

The equations of the two guiding lines will then read:

(14)
$$\begin{cases} x_3 = +\sqrt{\alpha\beta}, \quad x_2 = +\sqrt{\dfrac{\alpha}{\beta}}x_1, \\ x_3 = -\sqrt{\alpha\beta}, \quad x_2 = -\sqrt{\dfrac{\alpha}{\beta}}x_1. \end{cases}$$

They will then intersect the z-axis at right angles, and the angle of their orthogonal projections onto the xy-plane will be bisected by the (x and y) coordinate axes. They are $\left\{\begin{matrix}\text{real} \\ \text{imaginary} \\ \text{coincident}\end{matrix}\right\}$ according to whether $\alpha \cdot \beta$ is > 0, < 0, or = 0, respectively. Thus:

The linear ray congruence C_1^1 is elliptic or hyperbolic, according to whether the quantities α and β have different or equal signs, respectively, and parabolic when α or β vanishes.

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(1) Cf., Clebsch-Lindemann. *Vorlesungen über Geometrie II*, 1, page 58.

II. The focal paraboloid.

In order to find the focal properties of our congruence C_1^1, we now first ask what the second-order ruled surfaces are that have one ruled family that consists of rays of the congruence. A second-order ruled surface is generally defined by three skew lines, which will be given by their line coordinates q_{ik}, r_{ik}, s_{ik}. All of the points of a fourth line (p_{ik}) will then belong to the surface when it simultaneously cuts the lines q, r, s. The analytical conditions for that read:

$$\begin{aligned}
q_{34}\,p_{12} + q_{24}\,p_{31} + q_{14}\,p_{23} + q_{12}\,p_{34} + q_{31}\,p_{24} + q_{23}\,p_{14} &= 0,\\
r_{34}\,p_{12} + r_{24}\,p_{31} + r_{14}\,p_{23} + r_{12}\,p_{34} + r_{31}\,p_{24} + r_{23}\,p_{14} &= 0,\\
s_{34}\,p_{12} + s_{24}\,p_{31} + s_{14}\,p_{23} + s_{12}\,p_{34} + s_{31}\,p_{24} + s_{23}\,p_{14} &= 0.
\end{aligned}$$

If we add the identities that the coordinates x_i of any point that belongs to a line (p_{ik}) must satisfy, namely:

$$\begin{aligned}
x_2\,p_{34} - x_3\,p_{24} + x_4\,p_{23} &= 0,\\
x_3\,p_{14} - x_1\,p_{34} + x_4\,p_{31} &= 0,\\
x_1\,p_{24} - x_2\,p_{14} + x_4\,p_{12} &= 0,
\end{aligned}$$

to these then we will have six independent, homogeneous equations between the six quantities p_{ik}. If we eliminate them then the desired ruled surface will be represented by an equation in running point coordinates x_i whose coefficients will be composed from the line coordinates of the three given lines. The result of the elimination is the determinant of the six equations, which, from a theorem of the theory of determinants, must vanish:

$$\begin{vmatrix}
q_{23} & q_{34} & q_{24} & q_{12} & q_{31} & q_{14}\\
r_{23} & r_{34} & r_{24} & r_{12} & r_{31} & r_{14}\\
s_{23} & s_{34} & s_{24} & s_{12} & s_{31} & s_{14}\\
0 & 0 & 0 & x_2 & -x_3 & x_4\\
x_3 & 0 & x_4 & -x_1 & 0 & 0\\
-x_2 & x_4 & 0 & 0 & x_1 & 0
\end{vmatrix} = 0$$

We will obviously obtain a third-degree equation; one can then eliminate x_4 as a factor, since it cannot actually be zero. If we develop the determinant and set:

$$|\,q_{23}\;q_{34}\;q_{24}\,|, \text{ etc.} \qquad \text{for} \qquad \begin{vmatrix} q_{23} & q_{34} & q_{24}\\ r_{23} & r_{34} & r_{24}\\ s_{23} & s_{34} & s_{24}\end{vmatrix}, \text{ etc.,}$$

to abbreviate, then the equation will reduce to:

$$
(1) \qquad \begin{cases}
\quad |\, q_{23}\, q_{34}\, q_{24} \,| \cdot x_1^2 + |\, q_{34}\, q_{31}\, q_{14} \,| \cdot x_2^2 + \\
+ |\, q_{12}\, q_{24}\, q_{14} \,| \cdot x_3^2 + |\, q_{23}\, q_{12}\, q_{31} \,| \cdot x_4^2 \\
+ \{ \underline{|\, q_{34}\, q_{23}\, q_{14} \,|} + \underline{|\, q_{34}\, q_{24}\, q_{31} \,|} \} \cdot x_1\, x_2 \\
+ \{ \underline{|\, q_{23}\, q_{24}\, q_{14} \,|} + |\, q_{34}\, q_{24}\, q_{12} \,| \} \cdot x_1\, x_3 \\
+ \{ |\, q_{31}\, q_{24}\, q_{14} \,| + |\, q_{34}\, q_{12}\, q_{14} \,| \} \cdot x_2\, x_3 \\
+ \{ \underline{|\, q_{23}\, q_{24}\, q_{31} \,|} + |\, q_{23}\, q_{34}\, q_{12} \,| \} \cdot x_1\, x_4 \\
+ \{ |\, q_{34}\, q_{12}\, q_{31} \,| + \underline{|\, q_{23}\, q_{31}\, q_{14} \,|} \} \cdot x_2\, x_4 \\
+ \{ \underline{|\, q_{23}\, q_{12}\, q_{14} \,|} + \underline{|\, q_{12}\, q_{24}\, q_{31} \,|} \} \cdot x_3\, x_4 = 0.
\end{cases}
$$

If we now assume that the rectilinear surface that is represented in this way contains a ruled family whose rays belong to the congruence C_1^1 – e.g., the ruled family that is determined by the lines *q*, *r*, *s* – then one will have the equation:

$$
(2) \qquad \begin{cases}
q_{23} = -\alpha q_{14}, & q_{31} = \beta q_{24}, \\
r_{23} = -\alpha r_{14}, & r_{31} = \beta r_{24}, \\
s_{23} = -\alpha s_{14}, & s_{31} = \beta s_{24}.
\end{cases}
$$

However, one will then have:

$$|\, q_{24}\, q_{31}\, q_{14} \,| = \beta \,|\, q_{24}\, q_{24}\, q_{14} \,| \equiv 0,$$

since two columns of the determinant are equal to each other. All of the *underlined* determinants in (1) will likewise vanish. Furthermore, as a result of (2), the following relations will exist between the remaining coefficients:

$$
|\, q_{23}\, q_{34}\, q_{24} \,| = -\frac{\alpha}{\beta}\, |\, q_{34}\, q_{31}\, q_{14} \,|, \qquad |\, q_{34}\, q_{24}\, q_{12} \,| = -\frac{1}{\beta}\, |\, q_{34}\, q_{12}\, q_{31} \,|,
$$
$$
|\, q_{12}\, q_{24}\, q_{14} \,| = -\frac{1}{\alpha\beta}\, |\, q_{23}\, q_{12}\, q_{31} \,|, \qquad |\, q_{23}\, q_{34}\, q_{12} \,| = -\alpha \,|\, q_{34}\, q_{12}\, q_{14} \,|.
$$

We then get the result:

A second-order rectilinear surface, one of whose ruled families consists of rays of the linear congruence C_1^1, is represented by the equation:

$$
(3) \qquad \sum_{i,k=1}^{4} a_{ik}\, x_i\, x_k = 0,
$$

where the following relations exist between the coefficients a_{ik} :

$$\text{(4)} \qquad \begin{cases} \beta a_{11} = -\alpha a_{22}, & \alpha\beta a_{33} = -a_{44}, & \beta a_{13} = -\alpha_{34}, \\ a_{14} = -\alpha a_{23}, & a_{12} = 0, & a_{34} = 0. \end{cases}$$

If we demand, in addition, that this ruled surface that is "included in the congruence C_1^1" must be a surface of rotation – and thus, a one-sheeted hyperboloid of rotation – then we will have two cases to distinguish: Namely, since one can have $a_{12} = 0$ in equation (3), a_{23} and a_{13} cannot both be non-zero for the case of a surface of rotation ([1]). If we then take:

1) $a_{23} = 0$

then we will have $a_{13}^2 = (a_{11} - a_{22})\,(a_{33} - a_{22})$, in addition. As a result of (4), we will then also have $a_{14} = 0$, and with hindsight of (4) we will obtain the values:

$$f : g : h = (a_{11} - a_{22}) : 0 : a_{13} = (\alpha + \beta) : 0 : \frac{a_{24}}{a_{22}}$$

for the direction cosines for the rotational axis of our surface of rotation. Moreover, one gets:

$$x_1^0 = 0, \qquad x_2^0 = -\frac{a_{24}}{a_{22}}, \qquad x_3^0 = 0, \qquad x_4^0 = 1$$

for the coordinates of the center of the surface. If:

2) $a_{13} = 0$

then it will follow from (4) that $a_{24} = 0$, and one will have, in addition:

$$a_{23}^2 = (a_{22} - a_{11})\,(a_{33} - a_{11}).$$

In this case, the direction cosines of the rotational axis will read:

$$f : g : h = 0 : (a_{22} - a_{11}) : a_{23} = 0 : (\alpha + \beta) : \frac{a_{14}}{a_{11}},$$

and the coordinates of the center of the surface will be:

$$x_1^0 = -\frac{a_{24}}{a_{22}}, \qquad x_2^0 = 0, \qquad x_3^0 = 0, \qquad x_4^0 = 1,$$

([1]) See: Schur, *Analytische Geometrie,* 2nd ed., pp. 221.

so the center of our surface of rotation will lie on either the *x*-axis or the *y*-axis, and the rotational axis will be perpendicular to the coordinate axes in question. We have thus found:

There are two families of surfaces of rotation such that one ruled family includes the linear congruence C_1^1. *The rotational axis of the first family cuts the x-axis perpendicularly, and that of the second family cuts the y-axis perpendicularly. The equations of the rotational axis are:*

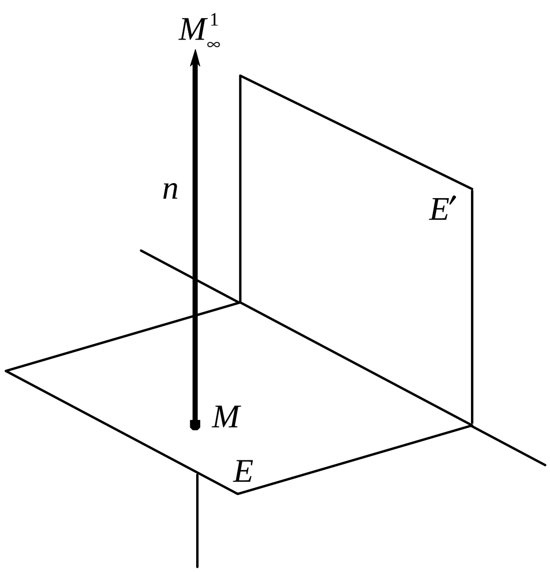
$$
(5) \qquad \begin{cases} 1) \ \boxed{x_1 = -\rho} & f:g:h = 0:(\alpha+\beta):\rho \quad \text{or} \quad \boxed{\dfrac{x_2}{x_3} = \dfrac{\alpha+\beta}{\rho},} \\[2ex] 2) \ \boxed{x_2 = -\sigma} & f:g:h = (\alpha+\beta):0:\sigma \quad \text{or} \quad \boxed{\dfrac{x_1}{x_3} = \dfrac{\alpha+\beta}{\sigma}.} \end{cases}
$$

If we now eliminate the parameters ρ or σ from this then we will obtain a second-degree surface as the geometric locus of these rotational axes:

$$
(6) \qquad \boxed{x_1 \cdot x_2 = -(\alpha+\beta) x_3 \cdot x_4,}
$$

and indeed, an equilateral, hyperbolic paraboloid that will have the *x* and *y* axes as its guiding lines. Following Jolles, we call it the *focal paraboloid* and its ruling rays, the *focal ruling rays* of the congruence, and thus get the theorem:

The ruling rays of the focal paraboloid, and only these rays, are the rotational axes of a second-degree surface of rotation, one of whose ruled families is included in the congruence C_1^1.



However, before we go further into the consideration of this surface of rotation that is included in the congruence, we would like to derive the equation of the focal paraboloid in another way that will lead us to more interesting focal properties.

Any ray of the linear congruence C_1^1 is the carrier of a point involution and a plane involution of the gathered (*geschart*) involutory space that is determined by C_1^1. The latter contains two conjugate planes E and E', in general, that are perpendicular to each other and which we, following Jolles, call *middle planes.* Each of these middle planes intersects the ray n of the congruence that is perpendicular to it at the midpoint M of its point involution; i.e., the point that is associated with the infinitely distant point M' of the ray n. n is then parallel to E', and its infinitely-distant intersection with E' is the point M' that is conjugate to M. The midpoints of all rays of the congruence lie in the plane ($x_3 = 0$) that is associated with the plane at infinity, and thus in the alignment plane of the linear congruence. If we now seek to determine the geometric locus of all middle planes of C_1^1 then we will have to subject the coordinates u_i and u_i' of E and E', first to equations (10), pp. 4, but then, since E is perpendicular to E', also to the orthogonality condition:

$$u_1 u_1' + u_2 u_2' + u_3 u_3' = 0.$$

However, the latter goes to:

(7) $$(\alpha + \beta)\, u_1\, u_2 + u_2\, u_4 = 0$$

with the help of (10), pp. 4.

The coordinates of the middle plane then satisfy a second-degree equation; we then obtain a surface of class two that will be enveloped by the middle planes of the congruence. If we replace the plane coordinates in (7) with point coordinates then that will show that this surface is identical with the focal paraboloid that was found on pp. 9. In fact, we get the equation:

(6) $$\boxed{x_1 x_2 + (\alpha + \beta) x_3 x_4 = 0.}$$

One then has the theorem:

The focal paraboloid is enveloped by the middle planes of the congruence (i.e., the middle planes of their ∞^2 rays). Its vertex plane is the locus of midpoints of the rays of the congruence.

Conversely, one will have:

Each plane through a focal ruling ray is a middle plane of the linear congruence; i.e., it is the perpendicular to the plane that is associated with it.

If we then draw an arbitrary plane through – e.g., the focal ruling ray (see pp. 9):

(7) $$x_1 = -\rho, \qquad \rho x_2 = (\alpha + \beta)\, x_3\,,$$

then its equation will read:

(8) $$x_1 - \mu\, \rho\, x_2 + \mu\, (\alpha + \beta)\, x_3 + \rho\, x_4 = 0$$

From equation (10), pp. 4, we then get:

(9) $$-\mu\, \rho\, x_1 + \frac{\beta}{\alpha} x_2 + \frac{\rho}{\alpha} x_3 + \mu\beta\, (\alpha + \beta)\, x_4 = 0$$

for the plane that is conjugate to it, and see from it that orthogonality condition:

$$u_1 u_1' + u_2 u_2' + u_3 u_3' = 0$$

for the two planes will be fulfilled, so it is, in fact, the middle plane.

As a tangential plane to the focal paraboloid, any middle plane of both focal ruled families will contain one ray of each. If we now associate the two rays that will be cut out of the *same* focal ruled family by two conjugate middle planes with each other as corresponding then we will obtain an involutory pairing in each focal ruled family and, with Jolles, call these two involutions the *focal involutions* of the linear congruence. Since all focal ruled families cut the x (y, resp.) axis perpendicularly, we will find the analytic expression for this involution when we look for the point of intersection of two conjugate middle planes with the x (y, resp.) axis. The coordinates of that point will follow immediately from the foregoing two equations:

$$x_1 = -\rho, \qquad x_1' = \frac{\beta(\alpha+\beta)}{\rho},$$

or

$$x_2 = -\sigma, \qquad x_2' = \frac{\beta(\alpha+\beta)}{\sigma},$$

resp.

By eliminating the parameter ρ (σ, resp.), one will obtain:

$$\text{(10)} \qquad \boxed{x_1 \cdot x_1' = -\beta(\alpha+\beta),} \qquad \boxed{x_2 \cdot x_2' = -\alpha(\alpha+\beta)}$$

for the two focal involutions. The infinitely-distant ray of the one family will then correspond to the x-axis, while the infinitely-distant ray in the other family will be associated with the y-axis. The (real or imaginary) double rays of the focal involution are called the *focal axes* of the congruence. One then has $x_1 = x_1'$ ($x_2 = x_2'$, resp.) for them. As a result, they will have the parameter:

$$\text{(11)} \qquad \bar{\rho} = \pm\sqrt{-\beta(\alpha+\beta)} \qquad (\bar{\sigma} = \pm\sqrt{-\alpha(\alpha+\beta)}\text{, resp.}),$$

and their equations will be:

$$\text{(12)} \qquad \begin{cases} x_1 = \mp\bar{\rho}, & \dfrac{x_2}{x_3} = \pm\dfrac{\alpha+\beta}{\bar{\rho}}, \\[2ex] x_2 = \mp\bar{\sigma}, & \dfrac{x_1}{x_3} = \pm\dfrac{\alpha+\beta}{\bar{\sigma}}, \text{ resp.} \end{cases}$$

One can calculate their line coordinates from this. They will be:

$$p_{23} : p_{31} : p_{12} : p_{14} : p_{24} : p_{34} = \begin{cases} 0 : (\alpha+\beta) : \pm\sqrt{-\beta(\alpha+\beta)} : 0 : 1 : \pm\sqrt{\dfrac{\beta}{\alpha+\beta}}, \\[2ex] \pm\sqrt{-\alpha(\alpha+\beta)} : 0 : (\alpha+\beta) : \pm\sqrt{\dfrac{\alpha+\beta}{-\alpha}} : 0 : 1. \end{cases}$$

From the foregoing, two associated planes, one of which goes through the focal axis, will intersect orthogonally in that focal axis. In fact, for two such planes, one will fulfill the orthogonality conditions:

$$x_1 - \mu\,\bar{\rho}\,x_2 - (\alpha + \beta)\,\mu\,x_3 - \bar{\rho}\,x_4 = 0$$

and

$$\alpha\,(\alpha + \beta)\,\mu\,x_1 + \bar{\rho}\,x_2 + \frac{1}{\alpha+\beta}\,x_3 - \frac{\bar{\rho}}{\alpha(\alpha+\beta)\mu}\,x_4 = 0,$$

as one will verify immediately. With that, we have proved:

A linear ray congruence contains two pairs of (real or imaginary) focal axes. Their plane involutions are circular.

It still remains for us to decide when the focal axes are real and when they are imaginary. We found that the congruence C_1^1 is elliptic or hyperbolic according to whether α and β have equal or different signs. Equations (11) and (12) will then yield the following table:

Congruence C_1^1	Focal involutions		Focal axes	α and β
	Family 1	Family 2		
hyperbolic	elliptic	elliptic	2 pairs, conj. imaginary	equal signs
elliptic	hyperbolic	elliptic	1 pair real, 1 pair imaginary	different signs

that is:

In the case of the hyperbolic congruence, the focal involutions are both elliptic, so the focal axes are all imaginary, while in the case of the elliptic congruence, the one focal ruled family is then hyperbolically-paired involutorily, so it will contain two real focal axes, while the involution of the other will be elliptic, so it will contain no real double element.

If we return to the truncated examination on pp 9 then we will next have to visualize once more the equations of the surface of rotation that is contained in the congruence C_1^1. We found two families of surfaces of rotation, so two kinds of second-degree equations, whose coefficients satisfied the following conditions:

(1) $\beta\,a_{11} = -\,\alpha\,a_{22}$, $\alpha\beta\,a_{33} = -\,a_{44}$, $\beta a_{13} = -\,a_{24}$, $a_{12} = 0$,
$a_{34} = 0$, $a_{14} = 0$, $a_{14} = 0$, $a_{13}^2 = (a_{11} - a_{22})\,(a_{33} - a_{22})$,

(2) $\quad \beta\, a_{11} = -\,\alpha\, a_{22}\,, \qquad \alpha\beta\, a_{33} = -\,a_{44}, \qquad a_{14} = -\,\alpha\, a_{23}, \qquad a_{12} = 0,$

$\quad a_{34} = 0, \qquad a_{13} = 0, \qquad a_{24} = 0, \qquad a_{23}^2 = (a_{22} - a_{11})\,(a_{33} - a_{22}).$

If we would now like to answer the question of when these surfaces are real and when they are imaginary then we can apply the theorem:

The second-order surface that is represented by the equation $\sum_{ik} a_{ik} x_i x_k = 0$ *contains no real points if and only if:*

1) *The determinant* $|\,a_{ik}\,|$ *and all sub-determinants of order two* $a_{ii}\,a_{kk} - a_{ik}^2$ *are positive,*

and

2) *The diagonal terms* a_{ii} *and their sub-determinants* A_{ii} *have equal signs.*

We next treat the first family of the surface of rotation, for which, conditions 1) are valid. With the help of these conditions, the determinant of the a_{ik} can be represented as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{vmatrix} = \beta^2\,(a_{11}\,a_{33} - a_{13}^2)^2 > 0.$$

It is positive, since one is dealing with rectilinear surfaces. In the examination of second-order sub-determinants $a_{ii}\,a_{kk} - a_{ik}^2$, we distinguish the two cases of hyperbolic and elliptic congruences:

I. If C_1^1 is hyperbolic then $\alpha \cdot \beta > 0$, and these sub-determinants are not all positive. For example:

$$a_{ii}\,a_{kk} - a_{ik}^2\,,$$

which will go to $-\,\alpha\,/\,\beta\; a_{22}^2$ as a result of conditions 1) above, is then certainly negative. Our surfaces of rotation of the first family will always be real then in the case of hyperbolic congruences.

II. If C_1^1 is elliptic then the one focal ruled family will be hyperbolic, while the other one will be elliptic and involutorily paired. We accordingly distinguish two sub-cases:

a) Let the focal ruled family that the rotational axes of our first surface family comprise (they are the ones that cut the *y*-axis orthogonally) be elliptic and involutorily paired. One will then have (cf., pp. 11):

$$-\,\alpha\,(\alpha + \beta) < 0, \qquad\qquad -\,\beta\,(\alpha + \beta) > 0,$$

and one will obtain:

$$a_{11}\,a_{22} = -\frac{\alpha}{\beta}a_{22}^2 > 0, \qquad a_{22}\,a_{33} = -\frac{\beta(\alpha+\beta)}{(\alpha+\beta)^2}a_{13}^2 + a_{22}^2 < 0,$$

$$a_{33}\,a_{44} = -\,\alpha\beta\,a_{33}^2 > 0, \qquad a_{11}\,a_{33} - a_{13}^2 > 0,$$

since one has:

$$a_{11}\,a_{33} = -\frac{\alpha}{\alpha+\beta}a_{13}^2 = -\frac{\alpha}{\beta}a_{22}^2 ;$$

one will then have:

$$\left|\frac{\alpha+\beta}{\alpha}a_{11}a_{33}\right| - a_{13}^2 > 0$$

for the sub-determinants in question and, *a fortiori*, the foregoing inequality.

$$a_{11}\,a_{44} = -\,\alpha\beta\,a_{11}\,a_{33} > 0,$$
$$a_{22}\,a_{44} - a_{24}^2 = \beta^2\,(a_{11}\,a_{33} - a_{13}^2) > 0.$$

Furthermore, the following expressions have equal signs:

$$a_{11} \quad \text{and} \quad a_{22}\,a_{33}\,a_{44} - a_{33}\,a_{24}^2 = a_{33}\,(a_{22}\,a_{44} - a_{24}^2),$$

since one has:

$$a_{22}\,a_{44} - a_{24}^2 > 0 \quad \text{and} \quad a_{11}\,a_{33} > 0,$$

and

$$a_{22} \quad \text{and} \quad a_{11}\,a_{33}\,a_{44} - a_{44}\,a_{13}^2 = a_{44}\,(a_{11}\,a_{33} - a_{13}^2),$$

also have equal signs, on analogous grounds, as well as:

$$a_{22} \quad \text{and} \quad a_{11}\,(a_{22}\,a_{44} - a_{24}^2),$$
$$a_{44} \quad \text{and} \quad a_{22}\,(a_{11}\,a_{33} - a_{13}^2).$$

In case II.a), all of the surfaces are then imaginary.

b) However, if we assume that the focal ruled family that cuts the *y*-axis is hyperbolic and involutorily-paired, so we assume that:

$$\alpha\,(\alpha+\beta) < 0, \qquad \beta\,(\alpha+\beta) > 0,$$

then the sub-determinants:

$$a_{11}\,a_{22} - a_{12}^2 = -\frac{\beta}{\alpha}a_{22}^2 \qquad \text{and} \qquad a_{33}\,a_{44} - a_{34}^2 = -\alpha\beta\,a_{44}^2$$

will indeed always be positive, so the quantities:

$$a_{11}\,a_{33} - a_{13}^2 \quad \text{and} \quad a_{22}\,a_{44} - a_{24}^2$$

can also assume negative values. In fact, with the help of conditions 1) on pp. 13, both expressions go to:

$$a_{11}\,a_{33} - a_{13}^2 = -\frac{1}{\beta(\alpha+\beta)}(a_{24}^2 + \alpha(\alpha+\beta)\,a_{22}^2)\,,$$

and this expression will be $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ according to whether $a_{24}^2 + \alpha(\alpha+\beta)\,a_{22}^2$ is less than 0 or greater than it, resp. Our surfaces are then $\left\{\begin{matrix}\text{imaginary}\\\text{real}\end{matrix}\right\}$ according to whether $a_{24}^2 / a_{22}^2 < -\,\alpha\,(\alpha+\beta)$ or $> -\,\alpha\,(\alpha+\beta)$, resp., or:

$$\frac{a_{24}}{a_{22}} \begin{matrix}<\\>\end{matrix} \pm\sqrt{-\alpha(\alpha+\beta)}\,, \text{ resp.}$$

On the left-hand side of this inequality, however, one finds precisely the value of the y-coordinate of the surface midpoint (cf., pp. 8), while the root on the right-hand side gives the distance to the coordinate origin, at which the two focal axes meet the y-axis. (cf., pp. 11) Our surfaces of rotation are then real when the rotational axis cuts the y-axis outside of the focal axis and imaginary when it cuts the focal axis inside the focal axes, assuming that the still-remaining demands of the theorem on pp. 13 can be likewise fulfilled. However, this will actually lead to the same result. Namely, one will find that:

$$a_{11}\,a_{44} > 0, \qquad a_{22}\,a_{33} > 0$$

if the surface is imaginary. However, this is the case, *a fortiori*, as long as:

$$a_{11}\,a_{33} - a_{31}^2 > 0, \qquad a_{22}\,a_{44} - a_{24}^2 > 0, \text{ resp.}$$

The remaining conditions, namely, that the following quantities have equal signs:

$$\begin{array}{lcl} a_{11} & \text{and} & a_{33}\,(a_{22}\,a_{44} - a_{24}^2), \\ a_{22} & \text{and} & a_{44}\,(a_{11}\,a_{33} - a_{31}^2), \\ a_{33} & \text{and} & a_{11}\,(a_{22}\,a_{44} - a_{24}^2), \\ a_{44} & \text{and} & a_{22}\,(a_{11}\,a_{33} - a_{31}^2), \end{array}$$

will lead to the same inequality that was posed above.

If we also perform the same investigations that we carried out for the first family of surfaces of rotation with the second family then we will now have to apply condition 2 on pp. 13, instead of conditions 1), and in that way we will get the corresponding results:

We can thus summarize all of this in the theorem:

Each generator of the focal paraboloid of a hyperbolic, linear ray congruence that lies at finite points is the rotational axis of a real second-degree surface of rotation such that one family of the congruence is contained in it. For elliptic congruences, the surfaces of rotation whose rotational axes belong to elliptic, involutory, focal ruled families will be imaginary, while the ones whose rotational axes define hyperbolic, involutory, focal ruled families will decompose into real and imaginary ones. They are real when their rotational axes cut outside the focal axes of the guiding lines of the paraboloid in question and imaginary when they cut inside the focal axes of the guiding lines.

III. The second-degree ruled surface that is linked with a focal ruled family.

We will obtain two further families of second-degree surfaces whose generators are most closely connected with the theory of the focal paraboloid by the following consideration: Any two mutually-associated rays of a focal ruled family are the axes of two projective pencils of planes whose planes are associated with each other by C_1^1 and intersect at right angles in each ray of the congruence. The two pencils thus generate an orthogonal second-degree ruled surface, one of whose ruled families is contained in C_1^1; with Jolles, it is said to be *linked* to the respective ruled family of the focal paraboloid.

If we would like to find its equation then we image that we are given two associated rays of the focal ruling that, from pp. 9, will be represented by the equations:

1) $$x_1 - \rho = 0, \qquad \rho\, x_2 + (\alpha + \beta)\, x_3 = 0,$$

2) $$x_1 + \frac{\beta(\alpha+\beta)}{\rho} = 0, \quad -\frac{\beta(\alpha+\beta)}{\rho}\, x_2 + (\alpha + \beta)\, x_3 = 0.$$

Thus, if:

(1) $$x_1 - \mu\,\rho\, x_2 - (\alpha + \beta)\, x_3 - \rho\, x_4 = 0$$

is the equation of a plane that goes through the ray 1) then we will get:

(2) $$x_1 + \mu'\beta\,\frac{\alpha+\beta}{\rho}\, x_2 - \mu'(\alpha + \beta)\, x_3 + \frac{\beta(\alpha+\beta)}{\rho}\, x_4 = 0$$

for the conjugate plane that includes the ray 2). With the help of conditions (10) on pp. 4, this will yield:

$$\mu' = -\frac{1}{\mu(\alpha+\beta)\cdot\alpha}$$

for the parameter μ'. If we now eliminate the quantity μ from (1) and (2) then we will get:

$$\alpha\rho\, x_1^2 - \beta\rho\, x_2^2 + (\alpha+\beta)\rho x_3^2 - \alpha\beta(\alpha+\beta)\rho x_4^2$$
$$+ [\alpha\beta(\alpha+\beta) - \alpha\rho^2]\, x_1\, x_1 + [\rho^2 - \beta(\alpha+\beta)]\, x_2\, x_3 = 0$$

as the equation of the desired orthogonal ruled surface, or:

$$\rho[\alpha\, x_1^2 - \beta x_2^2 + (\alpha+\beta)x_3^2 - \alpha\beta(\alpha+\beta)x_4^2] + [\rho^2 - \beta(\alpha+\beta)](x_2\, x_3 - \alpha\, x_1\, x_4) = 0. \tag{3}$$

If we set:

$$2\lambda = \frac{\rho^2 - \beta(\alpha+\beta)}{\rho}, \tag{4}$$

to abbreviate, then equation (3) will go to:

$$\boxed{\alpha x_1^2 - \beta x_2^2 + (\alpha+\beta)x_3^2 - \alpha\beta(\alpha+\beta)x_4^2 + 2\lambda(x_2 x_3 - \alpha x_1 x_4) = 0.} \tag{5}$$

We can conclude from this that:

All of the orthogonal ruled surfaces that are linked with a focal ruled family define an F^2-pencil.

In a completely analogous way, we will get the equation of the F^2-pencil that is linked to the other focal ruled family:

$$\boxed{\alpha x_1^2 - \beta x_2^2 - (\alpha+\beta)x_3^2 + \alpha\beta(\alpha+\beta)x_4^2 + 2\mu(x_1 x_3 - \beta x_2 x_4) = 0.} \tag{6}$$

Here, we have set:

$$2\mu = \frac{\alpha(\alpha+\beta) - \sigma^2}{\sigma}, \tag{7}$$

where σ is deduced from (5).

For the derivation of equations (5) and (6), we start with two arbitrary focal ruling rays that are characterized by the parameter ρ (σ, resp.), and define the quantities λ (μ, resp.) by relations (4), [(7), resp.]. We can ascribe arbitrary values to them and thus obtain the ∞ surfaces of the pencils (5) and (6). If we ask whether each value of λ (μ, resp.) also belongs to a *real* surface of the pencil then we must seek to arrive again at the generators above by projective pencils of planes, which can indeed yield only real surfaces, from the given surface equation (5) [(6), resp.], and in fact, we will find the axes of the pencils of planes when we obtain real values for the parameter ρ (σ, resp.) from equations (4) [(7), resp.] for given λ (μ, resp.). Equation (4) will read somewhat differently:

$$\rho^2 - 2\lambda\rho - \beta(\alpha + \beta) = 0.$$

This is quadratic in ρ, and thus yields $\left\{\begin{matrix}\text{real} \\ \text{imaginary}\end{matrix}\right\}$ values for ρ according to whether the discriminant:

$$D_1 = \lambda^2 + \beta(\alpha + \beta) \text{ is } > 0 \text{ or } < 0, \text{ resp.}$$

Analogously, it will follow from (7) that σ is $\left\{\begin{matrix}\text{real} \\ \text{imaginary}\end{matrix}\right\}$ according to whether:

$$D_2 = \mu^2 + \alpha(\alpha + \beta) \text{ is } > 0 \text{ or } < 0, \text{ resp.}$$

If $\alpha\beta > 0$ – i.e., if we are dealing with a hyperbolic congruence – then D_1 and D_2 will both be positive, so the roots of the quadratic equations (4) and (7) will always be real. In the case of elliptic congruences ($\alpha\beta < 0$), we will have to imagine that the quantities $\alpha(\alpha + \beta)$ and $\beta(\alpha + \beta)$ always have different signs. Thus, we will take, e.g., $\alpha(\alpha + \beta) > 0$; by contrast, $\beta(\alpha + \beta) < 0$, so ρ will be real, while σ will assume real or imaginary values according to whether:

$$\mu^2 > \text{ or } < |\, \beta(\alpha + \beta) \,| .$$

If then follows from this, with the help of the argument in pp. 11, that:

In the case of hyperbolic congruences, the surfaces that are linked with the focal ruled family will always be real. If the congruence is elliptic then the pencil that is linked with the hyperbolic, involutory, focal ruled family – and only that pencil – can contain imaginary surfaces. Namely, its surfaces will real when:

$$\boxed{\lambda^2 > |\beta(\alpha+\beta)| \quad [\mu^2 > |\alpha(\alpha+\beta)|, \text{resp.}]}$$

and imaginary when:

$$\boxed{\lambda^2 < |\beta(\alpha+\beta)| \quad [\mu^2 > |\alpha(\alpha+\beta)|, \text{resp.}]}$$

The surfaces of each of the two pencils will degenerate twice into a ray, namely, the (real or imaginary) focal axes of the focal ruled family that the pencil in question is linked to. The focal axes that belong to the first focal ruled family are contained in all surfaces of the pencil that is linked to the other ruled family, so they will define a part of the basic curve of this pencil. In addition, all surfaces of both pencils will go through the guiding lines of the congruence, since one of their ruled families consists of rays of the congruence. The basic curve of each of the F^2 pencils that are linked to a focal ruled family then decomposes into the (real or imaginary) guiding lines of the congruence and the (real or imaginary) focal axes that belong to the other focal ruled family. The ruled rays of all surfaces of the pencil that are not contained in C_1^1 will then cut the two focal axes that belong to the basic curve of this pencil, and will thus define a linear congruence F_1^1 that we can say is "linked" to the focal ruled family in question. Analogously, a second linear congruence $\mathfrak{F}_1^1$ will arise that is linked with the other focal ruled family, and, following Jolles, both of them shall be called the *focal congruences* of C_1^1. The guiding lines of a focal congruence that is linked with a focal ruled family are the two focal axes that belong to the other focal ruled family. The focal congruence in question will be hyperbolic or elliptic according to whether they are real or imaginary, so:

Congruence	Focal involutions		F_1^1, linked with the first ruled family	$\mathfrak{F}_1^1$, linked with the second ruled family
	First focal ruled family	Second focal ruled family		
hyperbolic	elliptic	elliptic	elliptic	elliptic
elliptic	hyperbolic	elliptic	elliptic	hyperbolic

Each of the two focal ruled families of a linear ray congruence C_1^1 is linked with a linear ray congruence F_1^1 ($\mathfrak{F}_1^1$, resp.). F_1^1 and $\mathfrak{F}_1^1$ are called the focal congruences of C_1^1. If C_1^1 is hyperbolic then the two focal congruences will be elliptic; if C_1^1 is elliptic then the focal congruence that is linked with the hyperbolic, involutory focal ruled family will be elliptic, and the one that is linked with the elliptic, involutory, focal ruled families will be hyperbolic.

In order to represent the focal congruences analytically, we start with the line coordinates of the focal axes that were found on pp. 11:

$$p_{23} : p_{31} : p_{12} : p_{14} : p_{24} : p_{34} =$$

$$\begin{cases} 0 : \beta(\alpha+\beta) : \pm(\alpha+\beta)\sqrt{-\beta(\alpha+\beta)} : 0 : (\alpha+\beta) : \sqrt{-\beta(\alpha+\beta)} \\ -\alpha(\alpha+\beta) : 0 : \pm(\alpha+\beta)\sqrt{-\alpha(\alpha+\beta)} : (\alpha+\beta) : 0 : \sqrt{-\alpha(\alpha+\beta)} \end{cases}$$

and the exhibit the equations for the four special complexes that have the focal axes for their guiding lines. They read:

$$\begin{cases} (\alpha+\beta)\,p_{31} + \sqrt{-\beta(\alpha+\beta)}\,p_{12} + (\alpha+\beta)\sqrt{-\beta(\alpha+\beta)}\,p_{34} + \beta(\alpha+\beta)\,p_{24} = 0, \\ (\alpha+\beta)\,p_{31} - \sqrt{-\beta(\alpha+\beta)}\,p_{12} + (\alpha+\beta)\sqrt{-\beta(\alpha+\beta)}\,p_{34} + \beta(\alpha+\beta)\,p_{24} = 0, \end{cases}$$

$$\begin{cases} (\alpha+\beta)\,p_{23} - \sqrt{-\alpha(\alpha+\beta)}\,p_{12} + (\alpha+\beta)\sqrt{-\alpha(\alpha+\beta)}\,p_{34} - \alpha(\alpha+\beta)\,p_{24} = 0, \\ (\alpha+\beta)\,p_{23} + \sqrt{-\alpha(\alpha+\beta)}\,p_{12} - (\alpha+\beta)\sqrt{-\alpha(\alpha+\beta)}\,p_{34} - \alpha(\alpha+\beta)\,p_{24} = 0. \end{cases}$$

One of the two focal congruences will then be represented by either of these two complexes.

IV. Polar systems of a linear ray congruence of the first and second kind. ([1])

In order to obtain information about the polar properties of the focal paraboloid and the ruled surfaces that are linked to any focal ruled family (Chap. III), we turn to the consideration of second-degree surfaces in this chapter that the linear congruence C_1^1 will go to under their polar associations; i.e., we demand that each ray of the congruence is again assigned to a ray of the congruence as the reciprocal polar relative to such a surface. To that end, we next prove a theorem that will give us the basis for a simpler analytical association. Namely, we assert:

If the linear ray congruence C_1^1 is taken to itself under the polar system of a second-degree surface F^2 then that surface will be invariant under the involutory collineation that is defined by C_1^1; i.e., a point of F^2 will again be associated with a point of that surface by C_1^1.

In order to prove this, we write $P = \mathfrak{F}\pi$ and $s' = \mathfrak{F}s$, when π is the polar plane of the point P (s' is the reciprocal polar of s, resp.) relative to F^2; moreover, let $P = \mathfrak{C}P'$, so P' corresponds to the point P under the collineation that is defined by C_1^1. Now, if this involutory collineation $\mathfrak{C}$ is characterized among all involutory collineations that convert C_1^1 into itself by the fact that it *takes each ray of C_1^1 to itself* (cf., pp. 3) then the transformation $\mathfrak{F}\mathfrak{C}\mathfrak{F}$ will also be a collineation that is characterized in that way. However, if the congruence C_1^1 is converted into itself by the polarity $\mathfrak{F}$ then the transformation $\mathfrak{F}\mathfrak{C}\mathfrak{F}$ will also be a collineation that is characterized in that way. Since $s' = \mathfrak{F}s$ is again a ray of the congruence, one will then have the identity:

$$\mathfrak{C}\mathfrak{F}s \equiv \mathfrak{F}s. \tag{1}$$

Thus:

$$\mathfrak{F}\mathfrak{C}\mathfrak{F} = \mathfrak{F}\mathfrak{F}s = s,$$

and it will follow from this that:

([1]) Cf., pp. 1, rem. ([2]) and also: *Kippels*, "Involutorische Regelscharen, etc." Inaugural Disseration, Strassburg, 1904.

$$\mathfrak{F}\mathfrak{C}\mathfrak{F} = \mathfrak{C} \qquad \text{or} \qquad \mathfrak{F}\mathfrak{C} = \mathfrak{C}\mathfrak{F}.$$

If P and P' are henceforth the poles of two planes π and π' that are associated with each other by C_1^1, so in our notation:

(2), (3), (4) $$P = \mathfrak{F}\pi, \qquad P' = \mathfrak{F}\pi', \qquad \pi' = \mathfrak{C}\,\pi,$$

then we will also have:

$$P' = \mathfrak{F}\pi' = \mathfrak{F}\mathfrak{C}\pi = \mathfrak{C}\mathfrak{F}\pi = \mathfrak{C}P,$$

or

(5) $$P' = \mathfrak{C}P;$$

i.e.:

A pole and its polar plane under F^2 are again associated with a pole and its polar plane under the collineation $\mathfrak{C}$.

The validity of our assertion above now follows from this immediately. Namely, if the point P lies on the surface F then $\pi = \mathfrak{F}P$ will be the tangential plane to it. However, the same thing must also be true for the corresponding elements $P' = \mathfrak{C}P$ and $\pi' = \mathfrak{C}\pi$, since the plane π' contains the point P', and in addition, from the foregoing, P' and π' will be a pole and polar relative to F^2.

The converse of our theorem can be proved in an entirely similar way:

If a second-degree surface F^2 is invariant under the association of the linear congruence C_1^1 then C_1^1 will go to itself under the polar system that is defined by F^2.

We can now treat our original problem in a somewhat different way. Namely, instead of looking for surfaces such that the congruence is invariant under their polar system, using our theorem (pp. 20), we can now ask what the surfaces are that go to themselves under the involutory collineation of C_1^1.

In order to express this latter demand, we replace the surface point x in the general equation of a second-degree surface:

(6) $$\sum_{i,k=1}^{4} a_{ik}x_i x_k = 0$$

with the point x' that is associated with by C_1^1 – i.e., by equations (9) on pp. 4 – whose coordinates must therefore likewise satisfy equation (6). One thus gets:

(7) $$\sum_{i,k=1}^{4} a_{ik}x_i' x_k' = 0,$$

or, from (9), pp. 4:

$$(8)\quad \begin{cases} \dfrac{\alpha^2}{\beta^2} a_{22} x_1^2 + a_{11} x_2^2 + \dfrac{1}{\beta^2} a_{44} x_3^2 + \alpha^2 a_{33} x_4^2 + 2a_{12} \dfrac{\alpha}{\beta} x_1 x_2 \\ +2\alpha a_{13} x_2 x_4 + 2\dfrac{1}{\beta} a_{14} x_2 x_3 + 2\dfrac{\alpha^2}{\beta} x_1 x_4 + 2\dfrac{\alpha}{\beta^2} a_{24} x_1 x_3 + 2\dfrac{\alpha}{\beta} a_{34} x_3 x_4 = 0. \end{cases}$$

By comparing the coefficients in equations (7) and (8), we will then obtain the following conditions between the quantities a_{ik} that determine the surface:

$$\rho\, a_{11} = \frac{\alpha^2}{\beta^2} a_{22}, \qquad \rho\, a_{22} = a_{11}, \qquad \rho\, a_{33} = \frac{1}{\beta^2} a_{44}, \qquad \rho\, a_{44} = \alpha^2 a_{33},$$

$$\rho\, a_{23} = \frac{1}{\beta} a_{24}, \qquad \rho\, a_{14} = \frac{\alpha^2}{\beta} a_{23}, \qquad \rho\, a_{24} = \alpha\, a_{13}, \qquad \rho\, a_{13} = \frac{\alpha}{\beta^2} a_{24},$$

$$\rho\, a_{12} = \frac{\alpha}{\beta} a_{12}, \qquad \rho\, a_{34} = \frac{\alpha}{\beta} a_{34}.$$

However, it follows from this that $\rho^2 = \alpha^2 / \beta^2$, or $\rho = \pm\, \alpha / \beta$, such that we will then get two different systems of conditions – and thus two different systems of second-degree surfaces – according to whether ρ is positive or negative. Namely, we will get:

I. For negative ρ:

$$\boxed{\begin{gathered} \beta a_{11} = -\alpha a_{22}, \quad \alpha\beta a_{33} = -a_{44}, \quad a_{24} = -\beta a_{13}, \\ a_{14} = -\alpha a_{23}, \quad a_{12} = 0, \quad a_{34} = 0. \end{gathered}}$$

II. For positive ρ:

$$\boxed{\beta a_{11} = \alpha a_{22}, \quad \alpha\beta a_{33} = a_{44}, \quad a_{24} = \beta a_{13}, \quad a_{14} = \alpha a_{23}.}$$

In the first case, we have six conditions between ten homogeneous quantities a_{ik}, so we will get a three-fold manifold of surfaces, and in the second case, there are four conditions, so there will be ∞^5 surfaces. All of these surfaces will go to themselves under the collineation $\mathfrak{C}$, and their polar systems will also transform into themselves as a result of C_1^1. We thus have the theorem ([1]):

There are two systems of second-degree surfaces for which a linear ray congruence is apolar, and indeed there will be ∞^3 *surfaces of the one kind and* ∞^5 *surfaces of the other kind. The coefficients of their equations will satisfy the conditions* I (II, *resp.*)

([1]) Cf., v. Staudt, *Beiträge zur Geometrie der Lage,* Nuremberg, 1856. 1. Heft, nos. 105 and 109.

A comparison of this with the results on page 7 will teach us that:

The surfaces of system I are ruled surfaces, one of whose ruled families consists of the rays of the congruence, while the rays of the guiding family will be associated pairwise with each other by C_1^1.

For the investigation of the surfaces of systems of type II, we consider the determination of the coefficients a_{ik}, which can be represented as follows with the help of the conditions II:

$$| a_{ik} | = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= (\beta a_{11} a_{33} - \beta a_{12}^2 - \alpha a_{23}^2 - a_{12} a_{34})^2 - \alpha\beta (a_{12} a_{33} - \frac{1}{\alpha} a_{11} a_{34} - 2 a_{13} a_{23})^2.$$

This be negative only when $\alpha\beta > 0$; i.e., when C_1^1 is hyperbolic. We conclude from this that:

In the case of an elliptic congruence, the surfaces of system II are second-order ruled surfaces, and in the case of a hyperbolic congruence they are then $\left\{ \begin{matrix} \textit{rectilinear} \\ \textit{not rectilinear} \end{matrix} \right\}$ *according to whether:*

$$\boxed{(\beta a_{11} a_{33} - \beta a_{12}^2 - \alpha a_{21}^2 - a_{12} a_{34})^2 \mathop{}^{>}_{<} |\alpha\beta| (a_{12} a_{33} + \frac{1}{\alpha} a_{11} a_{34} - 2 a_{13} a_{23})^2.}$$

If a surface of system II is rectilinear, and the points P_1 and P_2, with the coordinates x_i and y_i, lie on a ruling ray then they will be associated with two points P_1' and P_2' (with coordinates x_i' and y_i', resp.) by C_1^1, which, from the foregoing, must lie on the surface, and, as can be shown, whose connecting line will likewise belong to the surface. With the help of equations (9), pp. 4, in which x_i and x_i' (y_i and y_i', resp.) must be the coordinates of associated points, the condition for that, viz.:

$$\sum_{i=1}^{4} (a_{i1} x_1' + a_{i2} x_2' + a_{i3} x_3' + a_{i3} x_3') y_i' = 0,$$

will then go to:

$$\sum_{i=1}^{4} (a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i3} x_3) y_i = 0,$$

which will then be fulfilled when, as we have assumed, the line $\overline{P_1P_2}$ is a ruling ray of the surface. Since two lines that are associated with each other by C_1^1 do not intersect, in general ([1]), $\overline{P_1'P_2'}$ will belong to the same ruled family as $\overline{P_1P_2}$. We thus obtain an involutory pairing in both ruled families of the surface. Thus:

The rectilinear surfaces of system II are involutory ruled surfaces.

The double elements of these involutions are rays of C_1^1. Thus, at most four rays of the congruence belong to any surface of system II.

Since a ruled surface consists of rays of the congruence for the surfaces of system I, the guiding lines of C_1^1 will be included in all of these surfaces. Each of the guiding lines is then polar to itself in a polar system of the first kind. Analytically, with the help of the equations that are true for the line coordinates (p_{ik} and p'_{ik}) of two polar reciprocals, this will also easily show that:

$$\rho p'_{14} = \alpha_{11} p_{23} + \alpha_{12} p_{31} + \alpha_{13} p_{12} + \alpha_{14} p_{14} + \alpha_{15} p_{24} + \alpha_{16} p_{34}, \text{ etc.} \tag{9}$$

Here, the α_{ik} mean the second-degree sub-determinants of the quantities a_{ik}, which we have subjected to conditions I. If we then apply equations (9) to a polar system of the second kind, so we then require that the a_{ik} must satisfy conditions II, then that will show that the guiding lines in the polar system of the second kind are reciprocal polars. We will then have:

In a polar system of the first kind, each of the guiding lines of C_1^1 will be transformed into itself, while in a polar system of the second kind, they will then go to each other reciprocally.

If two points P and P' are conjugate to each other relative to a second-degree surface then, as is known, their coordinates (viz., x_i and x'_i) will fulfill the equation:

$$\sum_{i=1}^{4} (a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4)\, x'_i = 0. \tag{10}$$

Thus, if the x'_i mean the coordinates of the point that is associated with the point P by the congruence C_1^1 then we can substitute the values of x'_i from eq. (9), pp. 4, in it and get:

([1]) In fact, they will intersect only when one of them – and consequently, the other one, as well – is incident with one of the guiding lines of C_1^1, and indeed, at the point of intersection of these guiding lines. However, the fact that a surface of system II does not contain the guiding lines will be proved below.

$$a_{12}x_1^2 + \frac{\beta}{\alpha}a_{12}x_2^2 + \alpha a_{34}x_3^2 + \beta a_{34}x_4^2 + \left(a_{11} + \frac{\alpha}{\beta}a_{22}\right)x_1x_2 + \left(\alpha a_{33} + \frac{1}{\beta}a_{44}\right)x_3x_4$$
$$+\left(a_{13} + \frac{1}{\beta}a_{24}\right)x_2x_3 + \left(a_{14} + \alpha a_{23}\right)x_2x_4 + \left(\frac{\alpha}{\beta}a_{23} + \frac{1}{\beta}a_{14}\right)x_1x_3 + \left(\frac{\alpha}{\beta}a_{24} + \alpha a_{13}\right)x_1x_4 = 0.$$

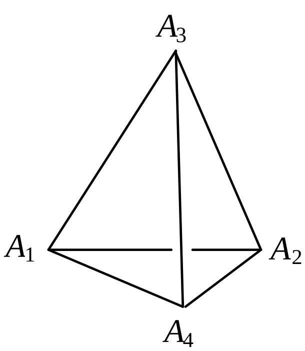
This equation is, however, fulfilled identically only when the a_{ik} satisfy the conditions I; i.e., the points that are associated with each other by C_1^1 are conjugate only in a polar system of the first kind. The proof for conjugate planes takes an analogous form, such that we have shown:

The polar systems of the first kind contain the point and plane involutions that are provoked by C_1^1 among its rays, but the polar systems of the second kind do not contain these involutions.

If the two ruled families of a rectilinear surface of system II (R_2) are paired involutorily then any two associated rays of one ruled family will be the axes of two projective pencils of planes whose homologous planes will be likewise associated with each other by C_1^1. If, for the sake of ease of representation, we draw the plane R_2 in such a way that the edges of the coordinate tetrahedron are contained in it, which is indeed always possible by a coordinate transformation, then its equation will read:

$$\boxed{R_2: \quad a_{12}x_1x_2 + a_{34}x_3x_4 = 0.}$$

In fact, the coefficients fulfill conditions II. Since the vertices A_1 and A_2 (A_3 and A_4, resp.) correspond to each other in C_1^1, the edges $A_1\,A_3$ and $A_2\,A_4$ will lie on associated lines of the same ruled family, which we can thus choose to be the axes of the two projective pencils of planes. The equations of those pencils will then read:



(11), (11a) $\quad x_1 + \lambda\, x_3 = 0, \qquad x_2 + \mu\, x_4 = 0.$

They will be projective when one has:

(12) $$\alpha\lambda = \mu,$$

and will then generate a second-order ruled surface (R_1) whose equations we will obtain by eliminating λ and μ from (11), (11a), and (12):

$$\boxed{R_1: \quad x_2x_3 - \alpha x_1x_4 = 0.}$$

We can proceed with the lines $A_2\,A_3$ and $A_1\,A_4$, which belong to the other ruled family of our surface R_2 and are likewise associated with each other by C_1^1, in a completely analogous way. We will then obtain the two projective pencils of planes:

$$x_1 + \beta\lambda\, x_4 = 0, \qquad x_2 + \lambda\, x_3 = 0,$$

and what they will generate is the ruled surface:

$$\boxed{R_1': \quad x_1 x_4 - \beta x_2 x_4 = 0.}$$

However, what is true for the rays $A_1\,A_3$ and $A_2\,A_4$ ($A_2\,A_3$ and $A_1\,A_4$, resp.) will be true for any arbitrary pair of associated lines that belong to the same ruled family of R_2 . For any two rays of a ruled family, we will obtain a surface R_1 (R_1', resp.), and can thus say that the totality of these surfaces is *linked* with the ruled family of R_2 in question. All surfaces that are linked with a ruled family will belong to the system of the first kind and consequently will have the (real or imaginary) guiding lines of C_1^1 in common. Moreover, they will all go through the two rays of the congruence that are contained in the other ruled family. Therefore, all of them will have the faces of a skew tetrahedron in common and consequently will define a pencil in F^2. Briefly:

Any ruled family of a rectilinear surface R_2 of system II is linked with a pencil of surfaces of system I that cuts R_2 in two associated rays. Its basic curves consist of the guiding lines of C_1^1 and the two rays of the congruence that belong to the guiding family of R_2 .

If we consider the polar properties of R_2 then that will lead us to further relationships between a surface R_2 and the two pencils that are linked with it. A point P^0 (x_i^0) is associated with a plane π^0 by the surface R_2 that has the coordinates:

$$\rho u_1^0 = a_{12}\, x_2^0, \quad \rho u_2^0 = a_{12}\, x_1^0, \quad \rho u_3^0 = a_{34}\, x_4^0, \quad \rho u_4^0 = a_{34}\, x_3^0. \tag{13}$$

Now, if the point P^0 lies on one of the surfaces R_1 or R_1' that are linked with the ruled families of R_2 then it can be shown that the plane π^0 is the tangential plane to that surface. In fact, the equations of R_1 (R_1', resp.) in plane coordinates read:

$$\begin{aligned} R_1: &\quad \alpha\, u_2\, u_3 - u_1\, u_4 = 0, \\ R_1': &\quad \beta\, u_1\, u_3 - u_2\, u_4 = 0. \end{aligned}$$

If we substitute the values of u_i^0 from (13) into this then it will follow that:

$$x_2^0 x_3^0 - \alpha x_1^0 x_4^0 = 0, \qquad x_1^0 x_3^0 - \beta x_2^0 x_4^0 = 0, \text{ resp.,}$$

from which, the validity of the assertion above can be seen immediately. The surfaces R_1 and R_1' are then apolar for the polar system that is defined by R_2. One can likewise prove that R_2 goes to itself under the polar systems of R_1 and R_1', and that R_1 does too under the polar system of R_1', and conversely, such that one has the theorem:

If two pencils of surfaces of the first kind are linked with a ruled surface of the second kind R_2 then the surfaces of these pencils will be reciprocally polar-invariant to each other. Likewise, the polar system of the second kind that is defined by R_2 will transform any surface of the two pencils into itself, and conversely.

All of the results of this chapter can now be carried over, with no further analysis, to the focal paraboloid and the orthogonal ruled surfaces that are linked to it:

The focal paraboloid of C_1^1 is a rectilinear surface of the second system that is linked with the two pencils of surfaces of system I.

Each ruled surface that is linked with a focal ruled family is apolar in the polar system that is defined by the focal paraboloid, and conversely, the paraboloid will be transformed into itself by the polar system of that ruled surface.

The polar system of the first kind of any orthogonal ruled surface that is linked with a focal ruled family transforms every surface that is linked with the other focal ruled family into itself.

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V. Confocal, linear ray congruences.

(The principal axis cylindroid)

Since we derived the equation of the focal paraboloid from the properties of a linear ray congruence in Chapter II and determined its focal involutions, we would now lie to address the converse problem and investigate whether a linear ray congruence is defined by an equilateral hyperbolic paraboloid and the two involutions of its ruling rays.

Therefore, let an equilateral hyperbolic paraboloid be given, whose equation we can assume has the form:

(1) $$\boxed{x_1 \cdot x_2 = -2c\, x_3 x_4,}$$

and, in addition, let the involutory pairing of the one ruled family be given by the equation:

(2) $$x_1 \cdot x_1' = \rho,$$

where the infinitely-distant ray of the family is associated with the guiding line of the paraboloid that belongs to the same ruled family, as it must be. However, from what was done on pp. 11, *et seq.*, the involution of the other ruled family will then be determined. We can then change these two involutions only by giving the quantity ρ different values, and there will be, accordingly, ∞^1 linear ray cognruences with the same focal paraboloid. Following Jolles, we call them *confocal*. The middle planes of confocal linear congruences coincide; they are the tangential planes to their common focal paraboloid, and we will then obtain a ray of the congruence when we intersect two normal tangential planes. Analytically, we represent this as follows: We take two ruling rays that are conjugate under the involution $x_1 x_1' = \rho$:

$$1)\qquad x_1 = x_1^0, \qquad x_2 x_1^0 = -2c\, x_3,$$

$$2)\qquad x_1 = \frac{\rho}{x_1^0}, \qquad x_2 \frac{\rho}{x_1^0} = -2c\, x_3,$$

and draw the plane through this:

$$1)\qquad x_1 - \mu\, x_2 - \mu \frac{2c}{x_1^0} x_3 - x_1^0 x_4 = 0,$$

$$2)\qquad x_1 - \mu' x_2 - \mu' \frac{2c x_1^0}{\rho} x_3 - \frac{\rho}{x_1^0} x_4 = 0.$$

Should these planes be perpendicular to each other then one would need to have:

$$1 + \mu\mu'\left(1 + \frac{4c^2}{\rho}\right) = 0 \qquad \text{or} \qquad \mu' = -\frac{\rho}{(\rho + 4c^2)\mu}.$$

We then obtain a ray of the congruence as the intersection of the planes:

$$1)\qquad x_1 - \mu\, x_2 - \mu \frac{2c}{x_1^0} x_3 - x_1^0 x_4 = 0,$$

$$2)\qquad x_1 + \frac{\rho}{(\rho + 4c^2)\mu} x_2 + \frac{2c x_1^0}{(\rho + 4c^2)\mu} x_3 - \frac{\rho}{x_1^0} x_4 = 0.$$

The desired ray then has the line coordinates:

$$p_{14} = \begin{vmatrix} u_2 & u_2' \\ u_3 & u_3' \end{vmatrix} = \frac{2c(\rho - x_1^{0\,2})}{x_1^0(\rho + 4c^2)}, \qquad p_{34} = \frac{2\rho + 4c^2}{\mu(\rho + 4c^2)},$$

$$p_{31} = \frac{\mu\rho}{x_1^0} + \frac{\rho x_1^0}{\mu(\rho + 4c^2)}, \qquad p_{12} = \frac{2\mu c\rho}{x_2^{0\,2}} + \frac{2c x_1^{0\,2}}{\mu(\rho + 4c^2)},$$

$$p_{24} = -\frac{2\mu c}{x_2^0} + \frac{2cx_1^0}{\mu(\rho + 4c^2)}, \qquad p_{23} = \frac{x_1^{0\,2} - \rho}{x_1^0}.$$

However, the linear relations:

$$\boxed{p_{23} + \frac{\rho + 4c^2}{2c} p_{14} = 0, \quad p_{31} + \frac{\rho}{2c} p_{24} = 0,} \tag{3}$$

exist between them, such that we then obtain each of the confocal ray congruences with the common focal paraboloid (1) as the intersection of two linear complexes that are represented by equations (3). Among these ∞^1 linear ray congruences, one finds two parabolic ones; for them, the parameter ρ will assume the values:

1) $\rho_1 = 0$, 2) $\rho_1 = -4c^2$.

Their guiding lines will be the y and x axes, resp., in which the two focal axes of the focal ruled family in question will coincide.

The totality of all linear ray congruences with the same focal paraboloid (1) defines a quadratic complex whose equation arises by eliminating ρ from (3):

$$\boxed{p_{23}p_{24} + p_{13}p_{14} + 2c\, p_{14}p_{24} = 0.} \tag{4}$$

The complex sends a ray cone through each point (y_i) whose equation in running coordinates reads:

$$x_1^2 y_3 y_4 + x_2^2 y_3 y_4 + 2c\, x_4^2 y_1 y_2 + 2c\, x_1 x_2 y_4^2 - x_1 x_3 y_1 y_4 - x_1 x_4 (y_1 y_3 + 2c\, y_2 y_4)$$
$$- x_2 x_4 (2c\, y_1 y_4 + y_2 y_3) - x_2 x_3 y_2 y_4 + x_3 x_4 (y_1^2 + y_2^2) = 0.$$

If we now investigate when this cone decomposes into two planes – i.e., we ask what the Kummer singularity surface is – then we will obtain the following condition for the coordinates of the vertex of that cone:

$$2c\, y_1 y_2 y_4^4 - (y_2^2 + y_1^2) y_3 y_4^3 = 0.$$

Here, one can omit y_4 as a factor. The Kummer singularity surface will then consist of the infinitely-distant plane $y_4 = 0$ and a third-degree surface:

$$y_3 = 2c \frac{y_1 y_2 y_4}{y_1^2 + y_2^2} \tag{5}$$

that is the *principal axis cylindroid* of our confocal ray congruences, which we would like to briefly go into.

As is known, one understands the principal axis cylindroid of a linear ray congruence $p_{23} + \alpha\, p_{14} = 0$, $p_{13} + \beta\, p_{34} = 0$, where $\alpha = \dfrac{\rho + 4c^2}{2c}$, $\beta = -\dfrac{\rho}{2c}$, ρ = const., to mean the locus of the principal axes of the pencil of complexes that is given by the equation:

(6) $$p_{23} + \alpha\, p_{14} + \lambda\,(p_{13} + \beta\, p_{34}) = 0.$$

In this, the principal axis of a complex is defined to be the line that is conjugate to an infinitely-distant line that is orthogonal to it relative to the complex. If we would then like to derive the equation of the principal axis cylindroid from this definition then we would next have to present the orthogonality conditions between the line coordinates q_{ik} of a principal axis and the coordinates p_{12}, p_{23}, p_{31}, 0, 0, 0 of an infinitely-distant line (p_{ik}). Since the quantities p_{12}, p_{23}, p_{31} are the position cosines of the parallel planes that go through the line p_{ik}, these conditions will read:

(7) $$p_{12} : p_{23} : p_{31} = q_{34} : q_{14} : q_{24}\,.$$

The line that is conjugate to the line p_{ik} relative to the pencil of complexes (6) will then have the coordinates (cf., (4), pp. 2):

(8) $$\begin{cases} \rho q_{12} = -(\alpha - \lambda^2\beta)\, p_{12}, & \rho q_{12} = p_{23} + \lambda p_{13}, \\ \rho q_{23} = (\alpha p_{13} + \lambda\beta\, p_{23})\lambda, & \rho q_{24} = -\lambda(p_{23} + \lambda p_{13}), \\ \rho q_{31} = \alpha p_{13} + \lambda\beta\, p_{23}, & \rho q_{34} = 0. \end{cases}$$

Should this cross the infinitely-distant line p_{ik} at right angles then it would from (7) that:

(9) $$q_{12} = p_{12} = 0 \qquad \text{and} \qquad p_{23}\, q_{24} = p_{31}\, q_{14}\,,$$

or, when we express the p_{ik} in terms of q_{ik} using (8):

(10) $$-\lambda\, q_{31}\, q_{24} + \alpha\, q_{14}\, q_{24} + q_{31}\, q_{14} - \lambda\beta\, q_{14}^2 = 0.$$

This equation is then satisfied by the coordinates of all principal axes of the pencil (6). Now, if we would like to represent its geometric locus by an equation in point coordinates then we would have to eliminate the parameter λ by means of the relation:

$$q_{23} = \lambda\, q_{31},$$

which follows from (8) and then replace the line coordinates with point coordinates, in such a way that we let any principal axis go through two points (x_i and x_i') and write the determinant $\begin{vmatrix} x_i & x_i' \\ x_k & x_k' \end{vmatrix}$ for q_{ik}. However, since, from (8) and (9), the quantities q_{12} and q_{34} vanish, all of the principal axes will cut the z-axis perpendicularly, and we can thus choose the two points 0, 0, x_3, 1 and x_1, x_2, x_3, 1 for its determination. If we substitute this into equation (10) then we will obtain:

$$x_3 = (\alpha + \beta)\,\frac{x_1 x_2 x_4}{x_1^2 + x_2^2} = 2c\,\frac{x_1 x_2 x_4}{x_1^2 + x_2^2} \tag{11}$$

for the desired equation of the cylindroid, as we found for a component of the Kummer singularity surface above.

Equation (11), which we have derived for the principal axis cylindroid of any arbitrary one of the confocal ray congruences (3), pp. 29, is independent of the parameter ρ, which is characteristic of each individual one of the confocal congruences. We can then say:

Confocal, linear ray congruences have the same principal axis cylindroid,

and conversely:

Linear ray congruences with the same principal axis cylindroid are confocal.

If the principal axis cylindroid is given then we can arrive at the associated focal paraboloid by the following construction:

We construct the normal plane to the line of intersection of a tangential plane to the cylindroid:

$$(c\,x_2' x_4' - x_1' x_2')\,x_1 + (c\,x_1' x_4' - x_2' x_3')\,x_2 - \frac{x_1'^2 + x_2'^2}{2}\,x_3 + c\,x_1' x_2' x_4 = 0$$

with the xy-plane. Its equation is:

$$(c\,x_2' x_4' - x_1' x_2')\,x_1 + (c\,x_1' x_4' - x_2' x_3')\,x_2 - \mu x_3 + c\,x_1' x_2' x_4 = 0,$$

where

$$\mu = -\,2\,\frac{(c\,x_2' x_4' - x_1' x_2')^2 + (c\,x_1' x_4' - x_2' x_3')^2}{x_1'^2 + x_2'^2}\cdot x_1' x_2',$$

and we assert that this plane is a tangential plane to the focal paraboloid, so its coordinates will satisfy the equation:

$$u_1\,u_2 = -\,\frac{1}{2c}\,u_3\,u_4\,.$$

However, this can, in fact, be easily verified when one considers only the fact that the quantities x_i' represent a point of the cylindroid, so it will satisfy equation (11). One can likewise show that a plane that is erected perpendicular to a tangential plane to the paraboloid at its line of intersection with the xy-plane will contact the cylindroid (11). One then has the theorem:

If normal planes are erected on the lines of intersection of the tangential planes to a cylindroid (11) with the xy-plane then they will envelope an equilateral paraboloid (1).

If normal planes are erected on the line of intersection of tangential planes to an equilateral paraboloid (1) with its vertex planes then they will envelope a cylindroid.

Following Jolles, the equilateral paraboloid (1) and the cylindroid (11) are thus called *orthogonally linked* to each other. The vertex rays and the principal axis of the paraboloid are the symmetry axes of the cylindroid that is orthogonally linked with it. Both surfaces have the x and y axis in common; however, except for them, no ruling ray of the cylindroid will meet the paraboloid that is orthogonally linked with at real points. In fact, such a ruling ray will be represented by:

$$\rho\, x_1 = x_2 \qquad \text{and} \qquad x_3 = \frac{2c\rho}{1+\rho^2}\, x_4\,,$$

so one must substitute x_2 and x_3 from this in equation (1) if one is to find its point of intersection with the paraboloid. One will then get:

$$x_1^2 = -\frac{4c^2}{1+\rho^2}\,,$$

so one will get an imaginary value for x_1 .

A further relation between the paraboloid (1) and the cylindroid (11) follows from the polar properties of paraboloids: Namely, if we seek to find the reciprocal polars of a ruling ray of the cylindroid relative to the paraboloid then we must first represent this ruling ray in line coordinates:

$$x_2 = \rho\, x_1\,, \qquad x_3 = \frac{2c\rho}{1+\rho^2}\, x_4\,,$$

and obtain:

$$p_{24} : p_{31} : p_{12} : p_{14} : p_{24} : p_{34} = 2c\rho^2 : -2c\rho : 0 : (1+\rho^2)\,\rho : 0.$$

From equations (9), pp. 29, its polar reciprocal relative to the paraboloid (1) will then be:

$$\begin{aligned} p'_{23} : p'_{31} : p'_{12} : p'_{14} : p'_{24} : p'_{34} &= p_{14} : -p_{24} : 0 : p_{23} : -p_{31} : 0 \\ &= (1+\rho^2) : -\rho\,(1+\rho^2) : 0 : 2c\rho^2 : 2c\rho : 0, \end{aligned}$$

or

$$\rho\, x_1 = -x_2\,, \qquad x_3 = -\frac{2c\rho}{1+\rho^2}\, x_4\,.$$

We thus once more obtain a ruling ray of the cylindroid, and indeed the one that arises from the first one by a reflection in the x or y axes. Thus:

The cylindroid will go to itself in the polar system of the paraboloid that is orthogonally linked to it in such a way that two of its ruling rays that are reflected into *each other in one of the symmetry axes (viz., x and y) will be polar reciprocal in this polar system.*

Moreover, if we start with the paraboloid then we can arrive at the cylindroid that is orthogonally linked with it by the following argument. Namely, we assert:

The focal axes of an equilateral paraboloid (which are incident with its principal axis) define the ruled family of the cylindroid that is orthogonally linked with it.

In order to prove this, we imagine that the coordinate system to which the paraboloid $x_1\, x_2 = -\,2c\, x_3\, x_4$ is related is rotated around the z-axis through the angle $\alpha = \varphi / 2$. Its equation will then go to:

$$x_1'^2 \sin\varphi + 2x_1'\, x_2' \cos\varphi - x_2'^2 \sin\varphi = -\,4c\, x_3'\, x_4', \tag{12}$$

and a tangential plane will be represented by:

$$(\mathfrak{x}_1 \sin\varphi + \mathfrak{x}_2 \cos\varphi)\, x_1' + (\mathfrak{x}_1 \cos\varphi - \mathfrak{x}_2 \sin\varphi)\, x_2' + 2c\, \mathfrak{x}_4\, x_3' + 2c\, \mathfrak{x}_3\, x_4' = 0, \tag{13}$$

where the $\mathfrak{x}_i$ mean the coordinates of the contact point relative to the rotated system. The plane (13) will be perpendicular to the coordinate plane $(y'\, z')$ when one has:

$$\mathfrak{x}_1 \sin\varphi + \mathfrak{x}_2 \cos\varphi = 0. \tag{14}$$

Now, all of the tangential planes to the paraboloid that are perpendicular to the $y'z'$-plane will envelop the cylinder:

$$u_2'^2 = \frac{u_3'\, u_4'}{\sin\varphi} \qquad \text{or} \qquad x_2'^2 = 4c \sin\varphi\; x_3'\, x_4'; \tag{15}$$

the equation:

$$(\mathfrak{x}_1 \cos\varphi - \mathfrak{x}_2 \sin\varphi)^2 = \frac{4c\, \mathfrak{x}_3\, \mathfrak{x}_4}{\sin\varphi}$$

will then go to:

$$\mathfrak{x}_1^2 \sin\varphi + 2\mathfrak{x}_1\mathfrak{x}_2 \cos\varphi - \mathfrak{x}_2^2 \sin\varphi = -\,4c\, \mathfrak{x}_3\, \mathfrak{x}_4,$$

with the help of the condition (14). It will then be true when the point with the coordinates $\mathfrak{x}_i$ lies on the paraboloid. However, the focal axis of the cylinder (15) is incident with the principal axis (viz., the z-axis) of the paraboloid. Namely, its equations are:

$$x_2' = 0 \qquad \text{and} \qquad x_3' = c \sin\varphi\; x_4', \tag{16}$$

and we will thus obtain all of the focal axes of the paraboloid that are incident with the z-axis when we give the angle φ all possible values. However, since $\varphi = 2\alpha$, equation (16) will go to:

$$x_2' = 0, \qquad x_3' = c \sin 2\alpha\, x_4'.$$

If we now once more return to our original coordinate system then that will give:

$$(17) \qquad x_1 \sin \alpha - x_2 \cos \alpha = 0, \qquad x_3 = 2c \sin \alpha \cos \alpha\, x_4 ,$$

and we will obtain the geometric locus of all focal axes when we eliminate α from them; it will then follow that:

$$x_3 = 2c \frac{x_1 x_2 x_4}{x_1^2 + x_2^2} ,$$

so in fact, as we asserted in our theorem, this will be the equation of the cylindroid that is orthogonally linked with the given paraboloid.

VI. The rotational, linear ray congruence.

If we make the special assumption in the equations:

$$p_{23} + \alpha p_{14} = 0, \qquad p_{13} + \beta p_{24} = 0,$$

which define our linear congruences, that:

$$\alpha + \beta = 0$$

then they will go to:

$$(1) \qquad p_{23} + \alpha p_{14} = 0, \qquad p_{13} - \alpha p_{24} = 0,$$

and we will obtain the case of the *rotational* linear ray congruence R_1^1. Since α and $\beta = -\alpha$ have different signs, R_1^1 *will be elliptic;* the focal axes will coincide with the z-axis.

The rotational linear congruence goes to itself under a rotation around the z-axis; any ray in the xy-plane that cuts the z-axis will be a symmetry axis of R_1^1.

In fact, if we rotate the coordinate system around z through an angle α then we will get:

$$x_1' = b\, x_1 + a\, x_2 , \qquad x_2' = -a\, x_1 + b\, x_2 , \qquad x_3' = x_3 , \qquad x_1' = x_4$$

when we set $a = \sin \alpha$ and $b = \cos \alpha$. Equations (1):

$$x_2\, x_3 - x_3\, y_2 + \alpha\,(x_1\, y_4 - x_4\, y_1) = 0,$$
$$x_1\, x_3 - x_3\, y_1 - \alpha\,(x_2\, y_4 - x_4\, y_2) = 0,$$

will then go to:

$$(-ax_1' + bx_2')\,y_3' - (-ay_1' + by_2')\,x_3' + (bx_1' + ax_2')\,y_4' - \alpha(by_1' + ay_2')\,x_4' = 0,$$

$$(bx_1 + ax_2)\,y_3' - (by_1' + ay_2')\,x_3' - \alpha(-ax_1' + bx_2')\,y_4' - \alpha(-ay_1' + by_2')\,x_4' = 0.$$

If we multiply the first of these equations by a, the second one by b, and add (subtract, resp.) them then it will follow that:

$$x_2' y_3' - x_3' y_2' + \alpha\,(x_1' y_4' - x_4' y_1') = 0,$$
$$x_1' y_3' - x_3' y_1' - \alpha\,(x_2' y_4' - x_4' y_2') = 0,$$

or

(2) $$p_{23}' + \alpha\, p_{14}' = 0, \qquad p_{13}' - \alpha\, p_{24}' = 0,$$

where the p_{ik}' are now referred to the rotated system. However, one concludes from (2), just as on pp. 4, that the x', y', and z' axes will be symmetry axes for R_1^1, and since the equations prove to be independent of the angle α, this will be true for any line that is erected in the xy-plane perpendicular to z.

However, the rays of this pencil of rays also define the ruling rays of the principal axis cylindroid that belongs to R_1^1. In fact, the equations of the cylindroid, which can also be written in the form:

$$\frac{x_1}{x_2} = \cot\varphi, \qquad x_3 = \frac{\alpha+\beta}{2}\sin\varphi\cos\varphi\, x_4\,,$$

will go to

$$\frac{x_2}{x_1} = \tan\varphi, \qquad x_3 = 0$$

for $\alpha = -\beta$.

From the foregoing, the ruling rays of the principal axis cylindroid are all symmetry axes for R_1^1 then, and any two that are mutually perpendicular will be the axes for two mutually null-invariant ray complexes.

The focal paraboloid of R_1^1:

$$u_1\, u_2\,(\alpha + \beta) = u_3\, u_4 = 0$$

decomposes into two pencils of planes:

1) $u_4 = 0$, 2) $u_3 = 0$,

whose midpoints lie at the coordinate origin and at the infinitely-distant point on the z-axis. They will be collinear when we assign planes and rays in the two pencils that correspond to each other, which will correspond under the involutory collineation of R_1^1, and have the z-axis in common. Moreover, from what was done in Chapter II, the z-axis will be the single rotational axis of a surface of rotation, one of whose ruled families is contained in R_1^1, and indeed these surfaces will have the equation:

$$a_{11}(x_1^2 + x_2^2) + a_{33}(x_3^2 + \alpha^2 x_4^2) = 0.$$

They define an F^2-pencil with z as its common rotational axis. The polar systems of these surfaces are the only rotational polar systems of the first kind of R_1^1.

In general, the following equation will be true for surfaces of system I (cf., chap. IV):

$$a_{11}(x_1^2 + x_2^2) + a_{33}(x_3^2 + \alpha^2 x_4^2) + 2a_{13}(x_1 x_3 - a^2 x_2 x_4) + 2a_{23}(x_2 x_3 - a^2 x_1 x_4) = 0.$$

As a result, all of these surfaces will have circular sections that are parallel to the xy-plane.
