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Simple derivation of the parameter formulas for motions and transfers

By **A. Schoenflies (Königsberg i. Pr.)**

Translated by D. H. Delphenich

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In what follows, I shall communicate a simple derivation of the parametric representation of motions and transfers. A particular advantage of it consists of the fact that it *exhibits the geometric meaning of the parameters immediately.* I know of such a derivation only for the EULER formulas that are concerned with the rotations around a fixed point; it goes back to DARBOUX ([1]) and is based upon the known representation of the velocity components of the points of a rotating body in terms of the quantities p, q, r. The one that I shall communicate comes from my consideration of a direct route to that objective. I do not know of such a derivation for the parametric representation of the most general motion of RODRIGUES that was generalized by STUDY. The parameters that were employed by STUDY in his comprehensive paper ([2]) have still not found their simplest geometric interpretation, even partially; however, the elementary method that I employed will likewise be given in what follows.

The basic idea that I start with consists of *reducing to operations of period two;* for example, one will obtain a representation of a rotation as a product of two *reversals* or two *reflections*. That proves to be most convenient. In fact, the parameter formulas for the operations of period two can be exhibited very simply and directly. Moreover, they will appear in a different form that illustrates the geometric meaning of the parameters, and this has the further consequence that the formulas for the general operations that are composed from them will also possess the same property.

§ 1.

The EULER formulas.

Let O be the fixed point around which the rotation takes place. Let X, Y, Z and x, y, z be the coordinates of a point in the initial and final positions, when referred to a fixed

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([1]) Cf., a Note in the *Leçons de cinématique* by G. KOENIGS (Paris, 1897), pp. 343.

([2]) "Von den Bewegungen und Umlegungen: (Parts I and II), Mathematische Annalen **34** (1891), 441-566; pp. 526, *et seq.*

coordinate system. As is known, the formulas that connect them then imply that one also directs one's attention to the positions of the coordinate axes before and after the rotation, and indeed, one regards the system X, Y, Z as the initial position and x, y, z as the final position. One then has the known equations ([3]):

$$
\text{(I)} \qquad \begin{cases} x = X\cos(xX) + Y\cos(xY) + Z\cos(xZ), \\ y = X\cos(yX) + Y\cos(yY) + Z\cos(yZ), \\ z = X\cos(zX) + Y\cos(zY) + Z\cos(zZ). \end{cases}
$$

I shall denote the nine cosines by c_{ik}, for brevity.

Now, let $\mathfrak{U}$ be a reversal around an axis u that goes through O, so the axes x, y, z and X, Y, Z go to each other *reciprocally*; it will then follow immediately that:

$$\angle (x\,Y) = (X\,y), \qquad \angle (x\,Z) = (X\,z), \qquad \angle (y\,Z) = (Y\,z),$$

so:

$$c_{12} = c_{21}, \qquad c_{13} = c_{31}, \qquad c_{23} = c_{32},$$

which is a result that is obvious from the outset from the symmetric character of the reversal. Moreover, one has:

$$\angle (x\,X) = 2\,(x\,u), \qquad \angle (y\,Y) = 2\,(y\,u), \qquad \angle (z\,Z) = 2\,(z\,u).$$

If one then sets:

$$\cos(x\,u) = \xi, \quad \cos(y\,u) = \eta, \quad \cos(z\,u) = \zeta$$

then it will follow that:

$$
\begin{aligned}
c_{11} &= \cos(2\,x\,u) = 2\xi^2 - 1 = \xi^2 - \eta^2 - \zeta^2, \\
c_{22} &= \cos(2\,y\,u) = 2\eta^2 - 1 = \eta^2 - \zeta^2 - \xi^2, \\
c_{33} &= \cos(2\,z\,u) = 2\zeta^2 - 1 = \zeta^2 - \xi^2 - \eta^2.
\end{aligned}
$$

With that, we have already found the EULER formulas for the c_{ii}. Since $c_{23} = c_{32}$, one further has:

$$c_{11} = c_{22} \cdot c_{33} - c_{23}^2;$$

with consideration given to the fact that $\xi^2 + \eta^2 + \zeta^2 = 1$, one will then have:

$$
\begin{aligned}
c_{23}^2 &= (\xi^2 - \eta^2 + \zeta^2)(\xi^2 + \eta^2 - \zeta^2) - (\xi^2 - \eta^2 - \zeta^2)(\xi^2 + \eta^2 + \zeta^2) \\
&= \xi^4 - (\eta^2 - \zeta^2)^2 - \xi^4 + (\eta^2 - \zeta^2)^2 = 4\,\eta^2\zeta^2.
\end{aligned}
$$

One will then obtain:

$$c_{23} = 2\,\varepsilon_{23}\,\eta\,\zeta, \qquad c_{31} = 2\,\varepsilon_{31}\,\zeta\,\xi, \qquad c_{12} = 2\,\varepsilon_{12}\,\xi\,\eta,$$

([3]) Cf., e.g., KLEIN-SOMMERFELD, *Über die Theorie des Kreisels*, Heft I (Leipzig, Teubner, 1897), pp. 16.

in which the ε_{ik} have the values ± 1. These values are yet-to-be-determined. It next follows that one must have $\varepsilon_{23} = \varepsilon_{31} = \varepsilon_{12}$ on the basis of symmetry; we denote the common sign by ε. Now, the determinant of the substitution has the value + 1; as a result, one will get:

$$\begin{aligned} 1 &= (2\xi^2 - 1)(2\eta^2 - 1)(2\zeta^2 - 1) + 16\,\varepsilon^2\,\xi^2\,\eta^2\,\zeta^2 \\ &\quad - 4\eta^2\zeta^2(2\xi^2 - 1) - 4\zeta^2\xi^2(2\eta^2 - 1) - 4\xi^2\eta^2(2\zeta^2 - 1)\,. \end{aligned}$$

It further follows from this directly that:

$$1 = 16\xi^2\zeta^2(\varepsilon^2 - 1) + 2(\xi^2 + \eta^2 + \zeta^2) - 1,$$

so finally one will have $\varepsilon^2 - 1 = 0$; i.e., $\varepsilon = + 1$.

With that, the formulas for the reversal have been found, *including the meaning of the parameters.* The nine cosines have the values:

$$\text{(II)} \qquad \left\{ \begin{array}{ccc} \xi^2 - \eta^2 - \zeta^2, & 2\xi\eta, & 2\xi\zeta, \\ 2\xi\eta, & \eta^2 - \zeta^2 - \xi^2, & 2\eta\zeta, \\ 2\xi\zeta, & 2\eta\zeta, & \zeta^2 - \xi^2 - \eta^2, \end{array} \right.$$

and one has:

$$\xi = \cos(x\,u), \quad \eta = \cos(y\,u), \quad \zeta = \cos(z\,u).$$

Now, let $\mathfrak{U}_1$ be a second reversal around the axis u_1, so the formulas:

$$\text{(III)} \qquad \left\{ \begin{aligned} x_1 &= c'_{11}X + c'_{12}Y + c'_{13}Z, \\ y_1 &= c'_{12}X + c'_{22}Y + c'_{23}Z, \\ z_1 &= c'_{13}X + c'_{23}Y + c'_{33}Z, \end{aligned} \right.$$

will exist between the coordinates X, Y, Z of the initial position and the coordinates x_1, y_1, z_1 of the final position, and the nine cosines are represented by the formulas:

$$\text{(IV)} \qquad \left\{ \begin{array}{ccc} \xi_1^2 - \eta_1^2 - \zeta_1^2, & 2\xi_1\eta_1, & 2\xi_1\zeta_1, \\ 2\xi_1\eta_1, & \eta_1^2 - \zeta_1^2 - \xi_1^2, & 2\eta_1\zeta_1, \\ 2\xi_1\zeta_1, & 2\eta_1\zeta_1, & \zeta_1^2 - \xi_1^2 - \eta_1^2, \end{array} \right.$$

which are analogous to (II), for:

$$\xi_1 = \cos(x\,u_1), \qquad \eta_1 = \cos(u\,u_1), \qquad \zeta_1 = \cos(z\,u_1).$$

One now lets the two reversals $\mathfrak{U}$ and $\mathfrak{U}_1$ take place in succession. The resulting motion is equivalent to a rotation $\mathfrak{A}$ through an angle of rotation $\omega = 2(u, u_1)$ around an axis a that is normal to u and u_1. The formulas that correspond to it are obtained when we replace the X, Y, Z in (III) with the values x, y, z. If we write the nine formulas in the form:

$$\begin{aligned} x' &= C_{11} X + C_{12} Y + C_{13} Z, \\ y' &= C_{21} X + C_{22} Y + C_{23} Z, \\ z' &= C_{31} X + C_{32} Y + C_{33} Z \end{aligned}$$

then, since $c_{ik} = c_{ki}$ and $c'_{ik} = c'_{ki}$, it will follow directly that:

$$C_{ik} = c'_{i1} c_{k1} + c'_{i2} c_{k2} + c'_{i3} c_{k3} .$$

We now set:

$$\xi\xi_1 + \eta\eta_1 + \zeta\zeta_1 = D,$$
$$\eta\zeta_1 - \zeta\,\eta_1 = A, \qquad \zeta\xi_1 - \xi\,\zeta_1 = B, \qquad \xi\eta_1 - \eta\,\xi_1 = C,$$

and obtain the known values of the C_{ik} in terms of A, B, C, D directly. One requires only the simplest conversion for that. It is:

$$\begin{aligned} C_{11} &= c'_{11} c_{11} + c'_{12} c_{12} + c'_{13} c_{13} \\ &= (\xi^2 - \eta^2 - \zeta^2)(\xi_1^2 - \eta_1^2 - \zeta_1^2) + 4\xi\xi_1\eta\eta_1 + 4\xi\xi_1\zeta\zeta_1 . \end{aligned}$$

If we add the vanishing quantity $2\eta\eta_1\zeta\zeta_1 - 2\eta\eta_1\zeta\zeta_1$ then it follow immediately that:

$$\begin{aligned} C_{11} &= (\xi\xi_1 + \eta\eta_1 + \zeta\zeta_1)^2 + (\eta\zeta_1 - \zeta\,\eta_1)^2 - (\xi\eta_1 - \eta\,\xi_1)^2 - (\zeta\xi_1 - \xi\,\zeta_1)^2 \\ &= D^2 + A^2 - B^2 - C^2. \end{aligned}$$

Moreover, one will have:

$$\begin{aligned} C_{12} &= c'_{11} c_{21} + c'_{12} c_{22} + c'_{13} c_{23} \\ &= (\xi_1^2 - \eta_1^2 - \zeta_1^2)\, 2\xi\eta + (\eta^2 - \zeta^2 - \xi^2)\, 2\xi_1\eta_1 + 4\xi_1\eta\zeta\,\zeta_1 . \end{aligned}$$

If we again add the quantity $2\xi\eta_1\zeta\zeta_1 - 2\xi\eta_1\zeta\zeta_1$ on the right then it will follow by a simple conversion that:

$$\begin{aligned} C_{12} &= 2[(\xi\xi_1 + \eta\eta_1 + \zeta\zeta_1)(\xi\eta_1 - \eta\,\xi_1) - (\zeta\xi_1 - \xi\,\zeta_1)(\eta\zeta_1 - \zeta\,\eta_1)] \\ &= 2(AB - CD). \end{aligned}$$

The value of C_{21} is obtained from this by permuting ξ, η, ζ with ξ_1, η_1, ζ_1, resp., so one will have:

$$C_{21} = 2\,(AB + CD),$$

and the remaining C_{ik} will follow by cyclic permutation. One will then arrive at the known matrix:

$$\text{(V)}\qquad \begin{cases} D^2+A^2-B^2-C^2, & 2(AB-CD), & 2(AC+BD), \\ 2(AB+CD), & D^2+B^2-A^2-C^2, & 2(BC-AD), \\ 2(AC-BD), & 2(BC+AD), & D^2+C^2-A^2-B^2, \end{cases}$$

but one also obtains the geometric meaning of the parameters from this immediately. From some elementary formulas of analytic geometry, and when one denotes the direction angles of the rotational axis by α, β, γ and recalls the meaning of ξ, η, ζ and ξ_1, η_1, ζ_1, one will then have immediately that:

$$(\text{V}_a)\qquad \begin{cases} D = \cos(u\,u_1) = \cos\dfrac{\omega}{2}, \\ A = \cos\alpha\sin\dfrac{\omega}{2},\; B = \cos\beta\sin\dfrac{\omega}{2},\; C = \cos\gamma\sin\dfrac{\omega}{2}. \end{cases}$$

In conclusion, we make a remark that concerns the determination of the sign. In order to establish the sense of a rotation in space, it is convenient to give the rotational axis a direction, and thus, to regard it as a half-ray. I then define the positive sense of rotation as would correspond to a right-hand screw whose axis is the axis of rotation. One and the same change of position can then be mediated by a rotation through an angle ω and a rotation through an angle $\omega - 2\pi$ (a negative rotation through the angle $2\pi - \omega$, resp.). A, B, C, D will all change signs under the transition from ω to $\omega - 2\pi$, but the formulas themselves will remain unchanged.

Either of the two directions of a reversal axis can be chosen to be the positive direction; one can then also choose the signs of ξ, η, ζ and ξ_1, η_1, ζ_1 to be opposite. However, should the angles ω and α, β, γ keep their values then that would have to be the case for *both* direction cosines, in such a way that *ABCD* would once more remain unchanged. If that were to happen for only one direction then the angle (uu_1) would go to $\pi - uu_1$, while the rotational axis would change its positive sense, and one would get the opposite values for A, B, C, D.

§ 2.

The transfers about a fixed point.

The formulas for the transfers will be obtained from the ones for the rotations when we combine the rotation $\mathfrak{A}$ with an inversion $\mathfrak{J}$ through the point O. If ω is again the angle of rotation of $\mathfrak{A}$, and a is its axis then $\mathfrak{AJ}$ will represent a *rotational reflection* around the axis a whose plane is normal to a and whose angle has the magnitude $\bar{\omega} = \omega + \pi$. On the other hand, x, y, z will be changed to $-x, -y, -z$ by the inversion. Thus, if the coordinates of the point $\bar{P}$, which emerge from X, Y, Z by means of the operation $\mathfrak{AJ}$, are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$ then one will have immediately:

$$\text{(VI)} \qquad \begin{cases} \bar{x} = -C_{11}X - C_{12}Y - C_{13}Z, \\ \bar{y} = -C_{21}X - C_{22}Y - C_{23}Z, \\ \bar{z} = -C_{31}X - C_{32}Y - C_{33}Z. \end{cases}$$

If one calls the coefficients of the substitution $\bar{c}_{ik}$ then one will have simply:

$$\bar{c}_{ik} = -C_{ik} .$$

The meanings of the quantities A, B, C, D that appear in the C_{ik} are the following ones: Since the angle of rotation $\bar{\omega}$ of $\mathfrak{A}\mathfrak{J}$ has the value $\omega + \pi$, it will follow that:

$$D = \cos\frac{\omega}{2} = \sin\frac{\bar{\omega}}{2},$$
$$A = -\cos\alpha\cos\frac{\bar{\omega}}{2}, \quad B = -\cos\beta\cos\frac{\bar{\omega}}{2}, \quad C = -\cos\gamma\cos\frac{\bar{\omega}}{2}.$$

For a pure reflection, in particular, one will have $\bar{\omega} = 0$, so $D = 0$, and one will have the naturally symmetric matrix for the coefficients $\bar{c}_{ik}$:

$$\text{(VII)} \qquad \begin{cases} -\xi^2 + \eta^2 + \zeta^2, & -2\xi\eta, & -2\xi\zeta, \\ -2\xi\eta, & \xi^2 - \eta^2 + \zeta^2, & -2\eta\zeta; \\ -2\xi\zeta, & -2\eta\zeta, & \xi^2 + \eta^2 - \zeta^2, \end{cases}$$

which contains the negative values of the matrix (II). In it, ξ, η, ζ are the direction cosines of the altitude to the reflecting plane. Naturally, they can all be chosen to have the opposite signs, as well.

§ 3.

Rotations in space.

We next treat the case of a rotation $\mathfrak{A}$; once more, let a be its axis, and let ω be the angle of rotation. The geometric constants that establish the rotation $\mathfrak{A}$ in addition to the direction of a and the angle ω are obviously the direction and length of the altitude $OL = l$ that goes from the coordinate origin O to the axis a. In fact, the parameters of the rotation can be represented in terms of them; that implies yet a second geometric interpretation for it. We will also find this to again be true in the general cases.

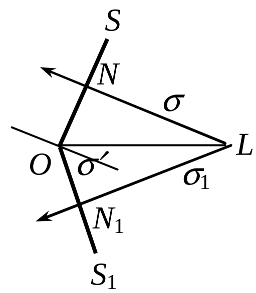

Figure 1.

This time, we derive the formulas for the rotation $\mathfrak{A}$ in such a way that $\mathfrak{A}$ is regarded as the product of two reflections $\mathfrak{S}$ and $\mathfrak{S}_1$ whose planes σ and σ_1, resp., go through a. However, we would like to choose them in a certain way, and indeed, such that they subtend an angle of $\omega / 4$ with the altitude OL in the manner that is given in Figure 1 ([4]). The problem will then reduce to the case of a simple reflection; we shall solve it for the reflection $\mathfrak{S}$ through the plane σ.

For that, we drop the altitude ON from O to σ and extend it to S. Let $\mathfrak{T}$ be the translation that takes O to S. If a plane $\sigma' \parallel \sigma$ were then drawn through O, and if $\mathfrak{S}'$ is the associated reflection then one would have:

$$\mathfrak{S} = \mathfrak{S}'\mathfrak{T},$$

and indeed one should understand $\mathfrak{S}'\mathfrak{T}$ to mean that one first performs the reflection $\mathfrak{S}'$ and then the translation $\mathfrak{T}$. Now, if X, Y, Z are once more the coordinates of a point in the initial position, while $\bar{x}$, $\bar{y}$, $\bar{z}$ are the ones that emerge from them by the reflection $\mathfrak{S}'$, and one finally takes the point $\bar{x}$, $\bar{y}$, $\bar{z}$ to x, y, z by the translation $\mathfrak{T}$ then, from § 2, one will have:

$$\begin{aligned} \bar{x} &= \bar{c}_{11}X + \bar{c}_{12}Y + \bar{c}_{13}Z , \\ \bar{y} &= \bar{c}_{12}X + \bar{c}_{22}Y + \bar{c}_{23}Z , \\ \bar{z} &= \bar{c}_{13}X + \bar{c}_{23}Y + \bar{c}_{33}Z , \end{aligned}$$

in which the $\bar{c}_{ik}$ are given by the matrix VII, and ξ, η, ζ mean the direction cosines of OS in them. When one sets $OS = 2s$, it will further follow immediately that:

$$\begin{aligned} x &= \bar{x} + 2s \cos (x\, s) = \bar{x} + 2s\xi, \\ y &= \bar{y} + 2s \cos (y\, s) = \bar{y} + 2s\eta, \\ z &= \bar{z} + 2s \cos (z\, s) = \bar{z} + 2s\,\zeta, \end{aligned}$$

([4]) Figure 1 depicts the section of the spatial figure by a plane through O that is perpendicular to a.

such that finally it results that the reflection $\mathfrak{S}$ through the plane σ is:

$$\text{(VIII)} \qquad \begin{cases} x = 2s\xi - c_{11}X - c_{12}Y - c_{13}Z, \\ y = 2s\eta - c_{12}X - c_{22}Y - c_{23}Z, \\ z = 2s\zeta - c_{13}X - c_{23}Y - c_{33}Z, \end{cases}$$

in which the c_{ik} are now given by the matrix (II).

An analogous system of equations exists for the reflection $\mathfrak{S}_1$ through the plane σ_1. According to the way that we defined the positions of σ and σ_1, we will have $OS = OS_1 = 2s$, so it will follow that:

$$\begin{aligned} x_1 &= 2s\xi_1 - c'_{11}X - c'_{12}Y - c'_{13}Z, \\ y_1 &= 2s\eta_1 - c'_{12}X - c'_{22}Y - c'_{23}Z, \\ z_1 &= 2s\zeta_1 - c'_{13}X - c'_{23}Y - c'_{33}Z, \end{aligned}$$

and, in fact, ξ_1, η_1, ζ_1 will be the direction cosines of OS_1.

Now, in order to get the formulas for the rotation $\mathfrak{A}$, we must once more introduce the values x, y, z from (VIII) in place of X, Y, Z in the last equations. If we write the resulting substitution in the form:

$$\text{(IX)} \qquad \begin{cases} x' = A' + C_{11}X + C_{12}Y + C_{13}Z, \\ y' = B' + C_{21}X + C_{22}Y + C_{23}Z, \\ z' = C' + C_{31}X + C_{32}Y + C_{33}Z \end{cases}$$

then it will be obvious that *the C_{ik} have the values that were derived in* § 1, *and thus, they are composed from the quantities A, B, C, D that were introduced in* § 1 *according to the matrix* (V). However, those quantities have a somewhat different geometric meaning here. Namely, the directions OS and OS_1 that are defined by ξ, η, ζ and ξ_1, η_1, ζ_1 subtend the angle $\pi - \omega / 2$ with each other here; the angle that belongs to them in the right-handed system then has the value $\omega / 2 - \pi$.

According to the remark at the end of § 1, we then find the *opposite* values for A, B, C, D to the ones in § 1 here. However, it is naturally convenient to understand the parameters A, B, C, D to mean the same quantities as before; we must therefore define the calculations of A, B, C, D that must be performed here by the equations:

$$D = -(\xi\xi_1 + \eta\eta_1 + \zeta\zeta_1),$$
$$A = \eta_1\zeta - \eta\zeta_1, \qquad B = \zeta_1\xi - \zeta\,\xi_1, \qquad C = \xi_1\eta - \xi\,\eta_1.$$

The parameters A, B, C, D thus-introduced are then identical to the ones that were employed in § 1.

It is still necessary for us to find the values of A', B', C' now. Now, one has:

$$A' = 2s\xi_1 - 2s\,(c'_{11}\xi + c'_{12}\eta + c'_{13}\zeta)$$

$$= 2s\,[\xi_1 - \xi\,(\xi_1^2 - \eta_1^2 - \zeta_1^2) - 2\xi_1\eta_1\eta - 2\xi_1\zeta_1\zeta]\,.$$

If one now multiplies ξ_1 by $(\xi^2 + \eta^2 + \zeta^2)$ in the bracket and adds $\xi\eta\eta_1 - \xi\eta\eta_1 + \xi\zeta\zeta_1 - \xi\zeta\zeta_1$ to the expression the one will get:

$$\begin{aligned} A' &= 2s\,[(\xi_1 - \xi)\,D - (\eta_1 - \eta)\,D + (\zeta_1 - \zeta)\,B], \\ B' &= 2s\,[(\xi_1 - \xi)\,C + (\eta_1 - \eta)\,D - (\zeta_1 - \zeta)\,A], \\ C' &= 2s\,[-(\xi_1 - \xi)\,B + (\eta_1 - \eta)\,A + (\zeta_1 - \zeta)\,D]\,. \end{aligned}$$

If one now sets:

$$2s\,(\xi_1 - \xi) = A_1\,, \qquad 2s\,(\eta_1 - \eta) = B_1\,, \qquad 2s\,(\zeta_1 - \zeta) = C_1$$

then it will ultimately follow that:

$$\text{(X)} \qquad \left\{ \begin{aligned} A' &= A_1 D - B_1 C + C_1 B, \\ B' &= A_1 C + B_1 D - C_1 A, \\ C' &= -A_1 B + B_1 A + C_1 D, \end{aligned} \right.$$

and (except for a sign) these are the formulas that STUDY obtained for the case of a simple rotation ([5]).

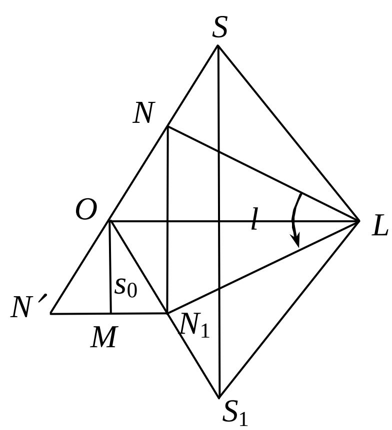

Figure 2.

The geometric meaning of the parameters A_1, B_1, C_1 is obtained as follows: In the rectangle ONN_1L of Fig. 1, we lengthen ON along itself to N' (Fig. 2) and drop the altitude OM to N_1N' from O; x_0, y_0, z_0 are then the coordinates of M, so we have:

$$x_0 = s\,\frac{\xi_1 - \xi}{2}\,, \quad y_0 = s\,\frac{\eta_1 - \eta}{2}\,, \quad z_0 = s\,\frac{\zeta_1 - \zeta}{2}\,.$$

([5]) Cf., pp. 528 in the paper that was cited in footnote ([2]). One must set $\beta_0 = 0$ in the formulas (5) for a rotation.

One now sets $OM = s_0$ and denotes the direction angles of OM by α_0, β_0, γ_0, so one will have:

$$A_1 = 4x_0 = 4s_0 \cos \alpha_0 , \qquad B_1 = 4y_0 = 4s_0 \cos \beta_0 , \qquad C_1 = 4z_0 = 4s_0 \cos \gamma_0 .$$

If one finally connects S with S_1 and L then one will have:

$$SS_1 = 2NN_1 = 4s_0 ,$$

and therefore:

(XI) $$A_1 = SS_1 \cos \alpha_0 , \qquad B_1 = SS_1 \cos \beta_0 , \qquad C_1 = SS_1 \cos \gamma_0 ,$$

such that the parameters A_1, B_1, C_1 will be the projections of SS_1 onto the axes. The segment SS_1 can be characterized in two ways: Namely, since $SLS_1 = \omega$, it follows directly from the triangle LSS_1 that:

$$SS_1 = 2LS \sin\frac{\omega}{2} = 2l \sin\frac{\omega}{2} .$$

This is the meaning that was mentioned at the beginning of this paragraph. A second one is based upon the fact that the path that the point S describes as a result of the rotation $\mathfrak{A}$ is a circular arc that goes through O; therefore, SS_1 *represents the chord of the points whose path includes the starting points and bisects it;* A_1, B_1, C_1 *are the projections of that chord.*

Since the directions of the axis a and the chord SS_1 are perpendicular to each other, the relation:

(XII) $$AA_1 + BB_1 + CC_1 = 0$$

will then follow.

§ 4.

Screwing motions.

A screwing motion will arise when we compose the rotation $\mathfrak{A}$ with a slide $\mathfrak{T}$ of length τ along the axis a. If x'', y'', z'' are the coordinates of the final position then one will have:

$$x'' = x' + \tau \cos \alpha, \qquad y'' = y' + \tau \cos \beta, \qquad z'' = z' + \tau \cos \gamma,$$

in which x', y', z' are given by the equations (IX). Only the values of A', B', C' that appear in these equations will be changed by that, and indeed by $\tau \cos \alpha$, $\tau \cos \beta$, $\tau \cos \gamma$. If one now introduces the quantity:

$$D_1 = \frac{\tau}{\sin\frac{\omega}{2}}$$

as a new parameter, and recalls the values of A, B, C, D that were contained in § 1 then one will ultimately get the values of A', B', C' that correspond to the screwing motion in the form:

$$\text{(XIII)}\qquad \begin{cases} A' = A_1 D - B_1 C + C_1 B + D_1 A, \\ B' = A_1 C + B_1 D - C_1 A + D_1 B, \\ C' = -A_1 B + B_1 A + C_1 D + D_1 C. \end{cases}$$

These are precisely STUDY's *formulas* ([6]); they will go to them when we replace D, A, B, C with $-\alpha_0$, α_1, α_2, α_3 and D_1, A_1, B_1, C_1 with $2\beta_0$, $2\beta_1$, $2\beta_2$, $2\beta_3$. Nevertheless, the parameters that are employed here are not identical with STUDY's quantities. That is based upon the fact that one can vary the parameters in many ways as a result of the relation (XII).

We next introduce quantities A_2, B_2, C_2, D_2 by setting:

$$\text{(XIV)}\qquad \begin{cases} A_2 = A_1 + \tau\cos\alpha = A_1 + D_1 A, \\ B_2 = B_1 + \tau\cos\beta = B_1 + D_1 B, \\ C_2 = C_1 + \tau\cos\gamma = C_1 + D_1 C, \\ D_2 = D_1 - \tau\cot\dfrac{\omega}{2} = D_1 - D_1 D = \tau\dfrac{1-\cos\omega/2}{\sin\omega/2}. \end{cases}$$

As one immediately recognizes, formulas (XIII) also remain valid for the parameters A, B, C, D and A_2, B_2, C_2, D_2 . The geometric meanings of the parameters A_2, B_2, C_2 can again be recognized immediately; as in § 3, *they represent the projections of the chords of the points whose path goes through the starting point and bisects it.* However, these parameters are not those of STUDY, either; the relation that exists between them then reads:

$$\text{(XV)}\qquad AA_2 + BB_2 + CC_2 = D_2\,(1 + D) = \tau\sin\frac{\omega}{2},$$

while the STUDY relation possesses the form:

$$\text{(XV}_a\text{)}\qquad AA_2 + BB_2 + CC_2 + DD_2 = 0.$$

In order to introduce parameters A_2, B_2, C_2, D_2 that satisfy this relation, one observes that the equations:

$$\text{(XIV}_a\text{)}\qquad \begin{cases} A_2 = A_1 + \mu D_1 A, \\ B_2 = B_1 + \mu D_1 B, \\ C_2 = C_1 + \mu D_1 C, \\ D_2 = D_1 - \mu D_1 D \end{cases}$$

([6]) They are formulas (5) for a_{10}, a_{20}, a_{30} on pp. 528.

also have the property that for an arbitrary μ equations (XIII) will go to themselves; in addition, however, in addition, however, the value of μ can also be determined in such a way that the relation (XV_a) is fulfilled; one then gets:

$$0 = D + \mu (1 - 2D'),$$

which then implies that:

$$\mu = \frac{D}{2D^2 - 1} = \frac{\cos \omega / 2}{\cos \omega}, \qquad D_2 = -\frac{\tau \sin \omega / 2}{\cos \omega}.$$

We shall not go into further details here. Likewise, I shall content myself with the remark that the most general spatial transfers can be represented in the way that was described in § 2, and that one can also derive STUDY formulas for them of the aforementioned kind ([7]).

Königsberg i. Pr., 11 November 1909.

A. SCHOENFLIES

([7]) Details will soon appear in a Königsberger Dissertation.