"Allgemeinrelativistiche Prinzipien der Kontinuumsmechanik," Ann. Phys. (7) **12** (1964), 372-395.

General-relativistic principles of continuum mechanics

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Abstract

Starting from a comparison of general-relativistic field theory and continuum mechanics, we will consider, above all, so-called conservative systems. These will be defined by the existence of an elastic potential and the absence of a heat current. The basic equations that pertain to these systems will be derived from a variational principle. The variation of the world lines that is thus implied will be clarified intuitively. The entropy balance that relates to general mechanical continua will be formulated.

§ 1. Field theory and continuum mechanics

If we consider a physical field that is described by certain field quantities ψ_A (A = 1, …, N) from a general-relativistic standpoint then we must compute, not only the ψ_A, but also the components $g_{\rho\sigma}$ [1]) of the metric tensor as functions of the spacetime coordinates. Thus, we have at our disposal the field equations:

$$\mathbf{F}^B(\psi_A, g_{\rho\sigma}) = 0, \qquad B = 1, \ldots, N, \tag{1.1}$$

along with the Einstein field equations:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = - \chi T_{\mu\nu}. \tag{1.2}$$

The field equations (1.1) are differential equations that involve the function $\mathbf{F}^B$, as well as the derivatives of the field quantities.

If no other variable quantities appear then one is dealing with matter in a closed system, as described by the ψ_A. If the gravitational field is also produced by this matter alone then the energy-impulse tensor that appears in (1.2) must be a function of the form:

$$T_{\mu\nu} = \mathbf{T}_{\mu\nu}(\psi_A, g_{\rho\sigma}). \tag{1.3}$$

The remaining quantities that appear in (1.2) have their conventional meaning.

The problem that arises in a general-relativistic field theory then consists of the simultaneous solution of (1.1) and (1.2). Thus, the form of $T_{\mu\nu}$ must be known and the necessary condition:

$$T_{\mu\;;\,\nu}^{\;\nu} = 0 \tag{1.4}$$

[1]) Small Greek indices run from 1 to 4, while small Latin ones run from 1 to 3.

must be satisfied [2]). There are no additional requirements if a Lagrangian function:

$$L = \mathbf{L}(\psi_A, g_{\rho\sigma}) \tag{1.5}$$

exists, in such a way that one has:

$$\mathbf{F}^B = \frac{\delta \mathbf{L}\sqrt{-g}}{\delta \psi_B}. \tag{1.6}$$

Equation (1.4) is, in fact, satisfied as a consequence of (1.1), as long as one sets:

$$\sqrt{-g}\, T^{\mu\nu} = -\, 2\frac{\delta \sqrt{-g}\, L}{\delta g_{\mu\nu}}. \tag{1.7}$$

Apparently, the construction of $T^{\mu\nu}$ by this process cannot be dubious.

What form does this field-theoretic aspect take in the situation where we are concerned with phenomenological matter; i.e., with a mechanical continuum? Here, we ultimately seek a congruence of timelike world-lines that represents a "droplet" of our medium. Equivalent to this is the determination of a normalized contravariant vector field u^μ:

$$u^\mu\, u_\mu = -\, 1, \quad ^{3}) \tag{1.8}$$

which is geometrically the tangent field of this congruence and physically – up to the factor c – the four-velocity field.

We carry out a coordinate transformation:

$$a^\chi = A^\chi(x^\lambda) \tag{1.9}$$

in such a way that in the coordinate system a^χ the normalized four-velocity has the form:

$$\bar{u}^\alpha = \lambda\, \delta_4^\alpha\,. \tag{1.10}$$

We equip all of the quantities relative to the coordinate system a^χ with an overbar; $\bar{u}_\beta$ is then computed from:

$$\bar{u}_\beta = g_{\alpha\beta}\bar{u}^\alpha = \lambda\, g_{4\beta}\,.$$

According to (1.8), one must have:

$$\lambda = 1/\sqrt{-g_{44}}\,,$$

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[2]) A semi-colon denotes the covariant derivative with respect to the coordinates, while a comma denotes the ordinary one.

[3]) We attribute a signature of + 2 (+ + + –) to the spacetime manifold.

such that we ultimately have:

$$\bar{u}^\alpha = \frac{\delta_4^\alpha}{\sqrt{-\bar{g}_{44}}}, \qquad \bar{u}_\beta = \frac{g_{4\beta}}{\sqrt{-\bar{g}_{44}}}. \tag{1.10}$$

The a^χ play the role of Lagrangian coordinates and the congruence of world-lines will be described by:

$$a^k = \text{const.} \tag{1.11}$$

The coordinate system a^χ determines a "reference system" in the following way: We call a fictitious mass point with constant spatial coordinates a^χ a "reference point" and call the set of all such reference points that belongs to the coordinate system a^χ a "reference system" [4]). In our case, we speak of a "co-moving reference system." Coordinate systems a'^λ that arise from a^χ through:

$$a'^k = a'^k(a^j), \quad a'^4 = a'^4(a^\chi) \tag{1.12}$$

– i.e., through spatial transformations and a change in the time coordinate – define the same reference system. In them, a representation of the form (1.10) is likewise valid for the normalized four-velocity vector.

We denote the inverse transformations to (1.9) by:

$$x^\mu = \varphi^\mu (a^1, a^2, a^3, a^4). \tag{1.13}$$

By holding a^k fixed, these equations define a parametric representation of a selected world-line of our medium, where a^4 acts as an arbitrary parameter. With the abbreviation:

$$\overset{\circ}{\varphi}{}^\mu = \frac{\partial \varphi^\mu}{\partial a^4} \tag{1.14}$$

this yields u^μ in the x^μ-coordinate system as:

$$u^\mu = \frac{\partial \varphi^\mu}{\partial a^\alpha}\bar{u}^\alpha = \frac{\overset{\circ}{\varphi}{}^\mu}{\sqrt{-g_{\alpha\beta}\,\overset{\circ}{\varphi}{}^\alpha\,\overset{\circ}{\varphi}{}^\beta}}. \tag{1.15}$$

An analogy between continuum mechanics and field theory comes about when we possess field equations of the type (1.1) for u^μ or for the functions a^χ (1.9) in which the ψ_A were identified with any functions. However, the equations that were first presented for a mechanical continuum are relativistic analogues of the pre-relativistic continuity equation for mass and the equations of motion; the latter give a physical expression to the balance of impulse. The relativistic generalization of this combined complex of

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[4]) For this nomenclature, cf., C. Møller, The Theory of Relativity, Oxford, Clarendon Press (1955), § 88.

equations likewise includes the balance of impulse and, due to the equivalence of energy and mass, also the balance of energy. If we do not consider external forces then we deal with statements that are equivalent to eq. (1.4).

One can therefore say that (1.1) and (1.4) will be identical in the case of the mechanical continuum theory. A simultaneous computation of the world-line congruence and the gravitational field with the help of these equations, assumes, however, that $T_{\mu\nu}$ in (1.2) is a functional of the form (1.3), in which one makes the aforementioned identification for the ψ_A .

As a matter of fact, a representation of the energy-impulse tensor in the desired form does not generally exist. On the contrary, we will examine how one can introduce certain physically plausible material equations, by whose aid any representation is achievable. However, these material equations will not relate to $T^{\mu\nu}$ primarily. Rather, in order to characterize the state of motion of the medium $T^{\mu\nu}$ must be connected with u^μ by the introduction of its physical components. The material equations then relate to these physical components.

§ 2. The physical components of the energy-impulse tensor

The aforementioned connection between $T^{\mu\nu}$ and u^μ was first presented by C. Eckart [5]) in order to formulate a special relativistic thermodynamics of irreversible processes for simple fluids. This Ansatz was taken up by G. A. Kluitenberg [6]) [7]) [8]), who generalized this theory to multi-component fluids. It can, with no further restrictions, be applied to the treatment of an arbitrary mechanical medium. By means of this Ansatz, the symmetric tensor $s_{\mu\nu}$ is defined by:

$$s_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu , \tag{2.1}$$

and has the following properties:

$$u^\mu s_{\mu\nu} = 0, \qquad\qquad s^\mu_\lambda s^\lambda_\nu = s^\mu_\nu . \tag{2.2}$$

As for the physical components of $T^{\mu\nu}$, we point out the scalar w, the four-vector w^α, and the symmetric tensor $w^{\alpha\beta}$, which are defined as follows:

$$w \equiv T^{\rho\sigma} u_\rho u_\sigma , \tag{2.3}$$

$$w^\alpha \equiv - s^\alpha_\rho T^{\rho\sigma} u_\sigma , \tag{2.4}$$

$$w^{\alpha\beta} \equiv s^\alpha_\rho s^\beta_\sigma T^{\rho\sigma} . \tag{2.5}$$

There then exists the identity:

[5]) C. Eckart, Phys. Rev. **58**, 919 (1940).
[6]) G. A. Kluitenberg, S. R. de Groot, and P. Mazur, Physica **19**, 689 (1953).
[7]) G. A. Kluitenberg, S. R. de Groot, and P. Mazur, Physica **19**, 1079 (1953).
[8]) G. A. Kluitenberg and S. R. de Groot, Physica **20**, 199 (1954).

$$T^{\mu\nu} \equiv w\, u^\mu\, u^\nu + w^\mu\, u^\nu + w^\nu\, u^\mu + w^{\mu\nu} . \tag{2.6}$$

We can shed light on the physical interpretation of this equation when we consider it in a local inertial rest system that we indicate by the index 0. In it, one has:

$$g_{0\mu\nu} = \eta_{\mu\nu} , \qquad u_0^\mu = \delta_4^\mu , \qquad u_{0\mu} = -\, \delta_\mu^4 ,$$
$$s_{0\mu\nu} = s_{0\mu}^\nu = s_0{}^{\mu\nu} = \mathrm{Diag}(1\ 1\ 1\ 0),$$

and therefore:

$$T_0^{\mu\nu} = \left(\begin{array}{c|c} w_0^{jk} & w_0^k \\ \hline w_0^k & w \end{array}\right). \tag{2.7}$$

Whereas w naturally represents the energy density in the local inertial rest system, one will interpret $-\, w^{\alpha\beta}$ as the stress tensor, and finally:

$$q^\alpha = c\, w^\alpha$$

as the heat current, since no other macroscopic energy current can exist in any rest system.

Since:

$$w^{\alpha\beta}\, u_\beta = 0, \qquad w^\alpha\, u_\alpha = 0, \tag{2.8}$$

and due to (1.8), the 13 quantities included in $T^{\mu\nu}$ are algebraically independent. One thus needs nine more equations, in addition to (1.4). This number will increase when it becomes necessary to introduce further physical quantities in the formulation of relations of this type – for example, temperature in the discussion of w^α.

Since the inertial rest system has only a local meaning, it is appropriate to base the further considerations on the co-moving reference system. In it, one has:

$$s_{\alpha\beta} = g_{\alpha\beta} + \bar{u}_\alpha \bar{u}_\beta = g_{\alpha\beta} - \frac{g_{4\alpha} g_{4\beta}}{g_{44}} ; \tag{2.9}$$

hence:

$$\rho_{\alpha 4} = 0. \tag{2.10}$$

Furthermore:

$$s_{ik} = g_{ik} - \frac{g_{i4} g_{k4}}{g_{44}} = \gamma_{ik} , \tag{2.11}$$

is the three-dimensional tensor, which, according to:

$$dl^2 = \gamma_{ik}\, da^i\, da^k,$$

defines the spatial distance dl between neighboring reference points [9]). One has:

$$\gamma_{ik}\, g^{jk} = \delta_i^j \tag{2.12}$$

The form (2.11) of γ_{ik} is reproduced naturally under the transformations (1.12). In general, the property of a covariant tensor having no 4-components in a coordinate system a^χ is conserved under the transformations (1.12) for which the spatial components transform according to the rules of three-dimensional tensors. The latter statement is also true for the spatial components of a contravariant tensor, whether or not the former property is valid.

We further denote:

$$s_\alpha^\beta = \delta_\alpha^\beta + \bar{u}_\alpha \bar{u}^\beta = \delta_\alpha^\beta - \frac{g_{4\alpha}\delta_4^\alpha}{g_{44}} \tag{2.13}$$

hence:

$$s_4^\beta = 0, \qquad s_i^k = \delta_i^k . \tag{2.14}$$

For w^{ik} and w^k this yields:

$$\bar{w}^{ik} = \bar{T}^{ik}, \tag{2.15}$$

$$\bar{w}^k = -\,\bar{T}^{k\gamma}\bar{u}_\gamma . \tag{2.16}$$

The relation (2.15) that is true in our reference system agrees with the same one in the local inertial rest system. One further deduces that the corresponding covariant components are obtained by manipulating the indices using γ_{ik} :

$$\bar{w}_{ik} = \gamma_{ir}\gamma_{ks}\bar{w}^{rs}, \tag{2.17}$$

$$\bar{w}_k = \gamma_{kj}\bar{w}^j . \tag{2.18}$$

Finally, one can write:

$$w^{\mu\nu} = s_\chi^\mu s_\rho^\nu \frac{\partial\varphi^\chi}{\partial a^r}\frac{\partial\varphi^\rho}{\partial a^s}\bar{w}^{rs}, \tag{2.19}$$

$$w^\alpha = s_\chi^\alpha \frac{\partial\varphi^\chi}{\partial a^k}\bar{w}^k \tag{2.20}$$

in a coordinate system x^μ that is linked to a^χ by (1.13).

All of these relations make it easier for us to see in $\bar{w}^{ik}$ and $\bar{w}^k$ the independent physical quantities in which to formulate the material equations.

[9]) See loc. cit. [4]), § 89 and L. D. Landau and E. M. Lifschitz. Feldtheorie (German translation) Berlin (1963), § 84.

§ 3. Conservative systems

The mechanical continua for which the analogy with field theory mentioned in the first paragraph is valid may be called conservative systems. As a first requirement that we place on them, we assume that no heat current needs to be considered; i.e., $w^\alpha = 0$.

Furthermore, the stresses shall be of an "elastic" nature. Thus, we require an explanation of the concept of "deformation." In the general theory of relativity it is connected with certain difficulties in such a way that J. L. Synge [10]) has conceived of a theory of elasticity that is based, not on the deformation itself, but on its velocity of variation. However, C. B. Rayner [11]) has recently developed a theory in which the deformation is defined by comparing the actual metric with a second one. Relative to it, the true congruence of world lines shall describe a rigid motion in the sense of Born and Rosen [12]).

Whereas in the pre-relativistic theory of elasticity one assumes different world-lines for the deformed and rigid comparison bodies, nonetheless, the same (Euclidian) metric will be used, while in the relativistic theory one must use the same world lines, but different metrics. It seems to me that the latter procedure can also be understood intuitively when one introduces simply a "body-fixed" coordinate system in place of the "space-fixed" one. Instead of saying that from the standpoint of the former system the points of the body move relative to each other under a deformation (i.e., their coordinate distances change), one can say that from the standpoint of the latter system these points can be regarded as at rest relative to each other (constant coordinate distance), whereas the metric changes temporally.

In order to mathematically formulate the thoughts of Rayner, one introduces a symmetric tensor $\overset{*}{s}_{\mu\nu}$, which shall have the coordinates:

$$\overset{*}{s}_{\mu\nu} = \left(\begin{array}{c|c} \overset{*}{\gamma}_{ik} & 0 \\ \hline 0 & 0 \end{array}\right) \tag{3.1}$$

in the co-moving reference system, such that in general one has:

$$\overset{*}{s}_{\mu\nu} u^\nu = 0. \tag{3.2}$$

The second metric will now be defined by:

$$\overset{*}{g}_{\mu\nu} = \overset{*}{s}_{\mu\nu} - u_\mu \, u_\nu \, . \tag{3.3}$$

One then has:

$$\overset{*}{u}_\mu \equiv \overset{*}{g}_{\mu\nu} u^\nu = u_\mu \tag{3.4}$$

and:

$$\overset{*}{g}_{\mu 4} = g_{\mu 4} \, . \tag{3.5}$$

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[10]) J. L. Synge, Math Z. **72**, 82 (1959).
[11]) C. B. Rayner, Proc. Roy. Soc. London **272**, 44 (1963).
[12]) N. Rosen, Phys. Rev. **71**, 54 (1947).

The demand that the motion described by u^μ shall be "rigid" with respect to $\overset{*}{g}_{\mu\nu}$ was posed by N. Rosen as:

$$u_\nu\, u_{\mu\|\chi}\, u^\chi + u_{\mu|}\, u_{\nu\|\chi}\, u^\chi + u_{\mu\|\nu} + u_{\nu\|\mu} = 0, \tag{3.6}$$

in which we have denoted the covariant derivative by means of $\overset{*}{g}_{\mu\nu}$ with a "‖".

The assumption that we encountered here, when it is applied to an examination that was carried out by E. T. Newman and A. I. Janis [13]), allows one to replace (3.6) with the equivalent, and quite intuitive, condition:

$$\frac{\partial \overset{*}{\gamma}_{ik}}{\partial a^4} = 0. \tag{3.7}$$

From now on, we shall define the "deformation tensor," in the sense of Rayner, as follows:

$$\bar{e}_{ik} \equiv \gamma_{ik} - \overset{*}{\gamma}_{ik}\,. \tag{3.8}$$

Finally, we require the existence of an elastic potential Φ for our conservative system since it allows one to obtain the stresses $-\bar{w}^{ik}$ by differentiating it by $\bar{e}_{ik}$. In this case, we denote the four-dimensional stress tensor by:

$$-\, w^{\alpha\beta}_{\text{const.}} = S_{\alpha\beta} \tag{3.9}$$

in order to characterize it in general cases, in particular. In light of the classical analogy, however, we single out the following Ansatz:

$$w = \rho\Phi, \tag{3.10}$$

$$\bar{S}^{ik} = 2\,\rho\,\frac{\partial\Phi}{\partial e_{ik}}\,. \tag{3.11}$$

ρ is an analogue of the pre-relativistic mass density, and therefore Φ specifies the potential energy. Here, it shall depend, apart from on $\bar{e}_{ik}$, on material quantities that are defined in a co-moving system. We express this by an explicit dependence of a^χ on Φ.

Now, there is no place in the theory of relativity for a mass density along with an energy density w. Rather, one must introduce a new definition. A particularly closely related path seem to us to be the following one: We start with the pre-relativistic connection between the mass density ρ and specific volume v. We now define the latter as a consequence of following through with the comparison between the actual and associated rigid motion, being the ratio of the volume, when measured in the actual metric γ_{ik}, to the associated one, which is then an infinitesimal amount of matter that a world line takes on between a^k and $a^k + da^k$. If γ and $\overset{*}{\gamma}$ denote the determinants of γ_{ik} and $\overset{*}{\gamma}_{ik}$ then the mathematical formulation of this definition reads:

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[13]) E. T. Newman and A. I. Janis, Phys. Rev. **116**, 1610 (1959).

$$\frac{1}{\rho} \equiv v \equiv \frac{\sqrt{\gamma}}{\sqrt{\gamma^*}}. \tag{3.12}$$

If one imagines that one has [14]):

$$\sqrt{-g} = \sqrt{-g_{44}}\sqrt{\gamma} \tag{3.13}$$

for the determinant g of $g_{\mu\nu}$ then one recognizes that $\sqrt{\gamma}$ and $\sqrt{\gamma^*}$ behave like spatial tensor densities under (1.12):

$$\sqrt{\gamma} = \sqrt{\gamma'}\left|\frac{\partial a'}{\partial a}\right|_{\text{spatial}}. \tag{3.14}$$

Furthermore, one has the following analogue of the pre-relativistic conservation law for mass, namely, the equation of continuity:

$$(\rho\, u^\mu)_{;\,\mu} = 0. \tag{3.15}$$

In the co-moving system it is:

$$(\rho \bar{u}^\mu)_{;\mu} = \frac{1}{\sqrt{-\bar{g}}}\left(\sqrt{-\bar{g}}\;\rho\,\bar{u}^\mu\right)_{,\mu} = \frac{1}{\sqrt{-\bar{g}}}\left(\sqrt{-\gamma^*}\right)_{,4} = 0.$$

The definition (3.12) has the corollary that ρ and v do not have the well-known physical dimensions. If one wishes to remedy this without giving up the equation of continuity (3.15) then one must obviously replace $\sqrt{\gamma^*}$ with a function $f(a^1, a^2, a^3)$. The formula:

$$\frac{1}{\rho} = v = \frac{\sqrt{\gamma}}{f(a^1, a^2, a^3)} \tag{3.12a}$$

is identical with the corresponding expression in the theories of Eckart and Kluitenberg.

We remark that we arrive at the special case of the ideal fluid when we assume:

$$\Phi_{\text{fl.}} = \Phi(v). \tag{3.16}$$

With:

$$\frac{\partial \Phi_{\text{fl.}}}{\partial v} = -p \tag{3.17}$$

one obtains:

$$\bar{S}^{ik}_{\text{fl.}} = -p g^{ik}. \tag{3.18}$$

According to (2.19), it follows by a simple computation:

$$w^{\mu\nu}_{\text{fl.}} = p\, s^{\mu\nu}. \tag{3.19}$$

[14]) See, loc. cit. [4]) Appendix 8.

Henceforth, we write the energy-impulse tensor of our conservative system as:

$$T^{\alpha\beta} = w\, u^\alpha\, u^\beta - S^{\alpha\beta}. \tag{3.20}$$

Due to the requirement on $T^{\alpha\beta}$ that it be divergence-free, along with (2.9), one then has:

$$T^{\alpha\beta}{}_{;\beta} u_\alpha = -w u^\beta{}_{;\beta} - w_{,\beta} u^\beta + S^{\alpha\beta} u_{\alpha;\beta} = 0. \tag{3.21}$$

On the other hand, if one computes $w_{,\beta}\, u^\beta$ from (3.10), while taking (3.11) into account, then one finds (cf., Appendix A):

$$w_{,\beta}\, u^\beta = -w u^\beta_{;\beta} + S^{\alpha\beta} u_{\alpha;\beta} + \rho\, \bar{u}^4 \left(\frac{\partial\Phi}{\partial a^4}\right)_{\text{expl.}}. \tag{3.22}$$

One must then have:

$$\left(\frac{\partial\Phi}{\partial a^4}\right)_{\text{expl.}} = 0. \tag{3.23}$$

The material quantities that enter into Φ must therefore be "constant" with respect to the co-moving system, a condition that is very intuitive.

If it is fulfilled then the relation (3.21) is strictly valid; i.e., independently of whether (1.4) is true. In order to understand this, one imagines that in the previous situation only the gravitational field and the world-line congruence are assumed to be unknown. However, the determination of the latter requires three spacetime functions.

The strong specialization that the conservative systems are based upon now leads, in fact, to the aforementioned analogy with field theory. Namely, we can substitute the expressions (3.10), (3.11) in (3.20), while taking (3.9) into account. If we then write:

$$\gamma_{ik} = \frac{\partial\varphi^\rho}{\partial a^i}\frac{\partial\varphi^\sigma}{\partial a^k} s_{\rho\sigma},$$

introduce the expression (1.15) for u^μ everywhere, and finally express the $\partial\varphi^\rho / \partial a^\lambda$ in terms of $\partial a^\mu / \partial\varphi^\nu$, then one has:

$$T^{\alpha\beta} = \mathbf{T}^{\alpha\beta}(g_{\rho\sigma}, a^\chi),$$

from which the analogy with field is established.

Finally, we mention that the theory that was conceived here can be regarded as, in a certain sense, a general-relativistic analogue of the one that G. Herglotz [15]) formulated for the special-relativistic theory of deformable media. In place of the "rest deformations" of Herglotz, we deal with the deformations (3.8) that are defined in the co-moving system.

[15]) G. Herglotz, Ann. Physik **36**, 491 (1911).

§ 4. The balance of entropy

One arrives at the balance of entropy most simply when one, in the spirit of the ordinary thermodynamics of irreversible processes for a non-conservative system, retains the relations (3.10) and (3.11), although Φ is now regarded as the specific internal energy. It will depend on, not only the $\bar{e}_{ik}$, but also the specific entropy η in the co-moving system, as well as systems composed of matter with suitably defined concentration gradients $c^{(k)}$. (Indices in parentheses denote the type of matter in each.) We thus write:

$$\Phi = \Phi(\bar{e}_{ik}, \eta, c^{(1)}, \ldots, c^{(k)}, \ldots, c^{(n)}). \tag{4.1}$$

The differentiations in (3.11) are thus now to be understood as carried out for constant η and $c^{(k)}$. The temperature T and the chemical potential $\mu^{(k)}$ are then defined by:

$$\frac{\partial \Phi}{\partial \eta} = T, \qquad \frac{\partial \Phi}{\partial c^{(k)}} = \mu^{(k)}. \tag{4.2}$$

As in the aforementioned special-relativistic theories of irreversible thermodynamics (loc. cit. [5-8])), we also work only with a scalar temperature quantity. We append an argument of M. Strauss [16]) that addresses the transformation of temperature. Likewise, the $c^{(k)}$ will be treated as scalar quantities.

If one now recalls the computations that were carried out in Appendix A then one obtains:

$$\begin{aligned} w_{,\beta}\, u^{\beta} + w\, u^{\beta}{}_{;\beta} &= S^{\alpha\beta}\, u_{\alpha;\beta} + \rho\, T\, \eta_{,\beta}\, u^{\beta} + \sum_{(k)} \rho\, \mu^{(k)}\, c^{(k)}{}_{,\beta}\, u^{\beta} \\ &= S^{\alpha\beta}\, u_{\alpha;\beta} + \rho\, T\, \eta_{,\beta}\, u^{\beta} + \sum_{(k)} \rho\, \mu^{(k)}\, c^{(k)}{}_{,\beta}\, u^{\beta}. \\ &\left(\frac{d\eta}{ds} \equiv \eta_{,\beta} u_{\beta} \right). \end{aligned} \tag{4.3}$$

Thus, we assume that the remarks of the previous paragraphs correspond to the assumption that Φ cannot depend upon other material quantities beyond a^1, a^2, a^3, but not a^4.

On the other hand, we consider the unabridged expression (2.6) for $T^{\alpha\beta}$. By assumption, we also bring an external electromagnetic field into consideration, although our medium shall possess no polarization and magnetization. With the electromagnetic field strength tensor $F^{\alpha\beta}$ and the four-current density j^{α} one then has:

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[16]) M. Strauss, Z. Naturforsch. **17**a, 827 (1962). The argument we mentioned may be found on pp. 845 and relates to the fact that whereas the transformation formulas for spatial and temporal distances in special relativity follow from the fact that these quantities are projections of four-vectors there, nevertheless, an analogous conclusion cannot be established for temperature.

$$T^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta} j_\beta \,,$$

and:

$$\begin{aligned} T^{\alpha\beta}{}_{;\beta}\, u_\alpha &= u_\alpha F^{\alpha\beta} j_\beta \,, \\ &= - w_{,\beta}\, u^\beta - w\, u^\beta{}_{;\beta} - w^\alpha u_{\alpha;\beta} u^\beta - w^\beta{}_{;\beta} - w^{\alpha\beta} u_{\alpha;\beta} \,. \end{aligned} \tag{4.4}$$

We thus have (2.8), and therefore consider:

$$(w^\alpha u^\beta + w^\beta u^\alpha)_{;\beta}\, u_\alpha = - w^\beta{}_{;\beta} - w^\alpha u_{\alpha;\beta} u^\beta \,.$$

The combination of (4.3) and (4.4) then yields:

$$\rho \frac{d\eta}{ds} = -\frac{1}{T}\left\{ \sum_{(k)} \mu^{(k)} (\rho\, c^{(k)} u^\beta)_{;\beta} + w^\beta{}_{;\beta} + w^\alpha \frac{Du_\alpha}{ds} - j_\beta F^{\beta\gamma} u_\gamma + (S^{\alpha\beta} + w^{\alpha\beta}) u_{\alpha;\beta} \right\}$$

$$\left(\frac{Du_\alpha}{ds} \equiv u_{\alpha;\beta} u^\beta \right).$$

This balance of entropy can likewise be brought into the form derived by von Kluitenberg, (loc. cit. [6]) eq. (5.1) and loc. cit [8]) eq. (6.12), when one, in the spirit of his work, introduces the "mass current vector" $m^{(k)\nu}$ for the k^{th} material, as well as the vector $I^{(k)\nu}$ of the relative mass current for this material, which are linked by way of:

$$m^{(k)\nu} = I^{(k)\nu} + \rho\, c^{(k)}\, u^\nu \,. \tag{4.6}$$

Naturally, these equations do not succeed in defining the new quantities. They show that the form of the balance of entropy that was derived does not depend upon precise definitions when only (4.6) is true, as well as showing that in the presence of a chemical reaction with the stoichiometric coefficients $r^{(k)}$ the balance equation:

$$m^{(k)\nu}{}_{;\beta} = \nu^{(k)}\, Q \tag{4.7}$$

is fulfilled, in which Q is an analogue of the rate of reaction. If one then writes:

$$j_\nu = \sum_{(k)} e^{(k)} m^{(\nu)}{}_\nu \tag{4.8}$$

and:

$$\sum_{(k)} \mu^{(k)} \nu^{(k)} = -A. \tag{4.9}$$

(A = Affinity), (4.5) then transforms into:

$$\rho \frac{d\eta}{ds} = -\frac{1}{T}\left\{ w^\beta{}_{;\beta} - \sum_{(k)} \mu^{(k)}\, I^{(k)\beta}{}_{;\beta} + w^\alpha \frac{Du_\alpha}{ds} - \sum_{(k)} e^{(k)}\, I^{(k)\beta} F_{\beta\gamma} u^\gamma + (S^{\alpha\beta} + w^{\alpha\beta}) u_{\alpha;\beta} - AQ \right\}$$

$$=\left(\frac{w^{\beta}}{T}-\sum_{(k)}\frac{\mu^{(k)}\,I^{(k)\beta}}{T}\right)_{;\beta}+w^{\beta}\left(-\frac{T_{,\beta}}{T^{2}}-\frac{1}{T}\frac{Du_{\beta}}{ds}\right)+\sum_{(k)}I^{(k)\beta}\left[e^{(k)}\frac{F_{\beta\gamma}u^{\gamma}}{T}-\left(\frac{\mu^{(k)}}{T}\right)_{;\beta}\right]$$
$$-\frac{1}{T}(S^{\alpha\beta}+w^{\alpha\beta})u_{\alpha;\beta}+\frac{A}{T}Q\,. \tag{4.10}$$

The first term is the divergence of the energy current, while the remaining ones collectively represent the entropy production.

In a conservative system ($w^{\alpha} = F_{\alpha\beta} = S^{\alpha\beta} + w^{\alpha\beta} = 0$), one has, from (4.5):

$$T\frac{d\eta}{ds}+\sum_{(k)}\mu^{(k)}c^{(k)}{}_{,\beta}u^{\beta}=0 \qquad \text{(conservative system).} \tag{4.11}$$

The change in the entropy then results from diffusion alone. Whether or not the constancy of entropy and all concentrations in that case is to be physically expected, we would nonetheless like to retain the general expression (4.11).

§ 5. Variation of the world lines

Since we presented a far-reaching analogy with field theory at the end of the third paragraph, it is to be expected that a variational principle also exists. Thus, the variation of the action integral relative to $g_{\mu\nu}$ shall yield the Einstein eqs. (1.2) and the variation of the world-lines shall yield the equations of motion; viz., in our case (1.4). The Lagrange function shall, corresponding to the general schema, depend upon the a^{χ} and their derivatives with respect to the x^{μ}, as well as on $g_{\nu\mu}$. A procedure for the variation of world lines was developed by A. H. Taub [17]) and refined by V. Fock [18]). This procedure relates to ideal fluids and seems to overlook the close connection with the variational procedure that is employed in field theories. We will therefore see that this is not the case, but that one is dealing with a straightforward modification of the calculations. Thus, we will likewise have the general case of conservative systems in mind.

As an intuitive conception of the variational procedure we think of the world-lines of our medium as infinitesimally distorted rubber bands, in which we retain the naming of material world-point of the medium by means of the coordinates a^{χ}. Therefore, this entire coordinate system will be distorted, such that, as before, the world-lines of a droplet will be described by a^k = const., a^4 = variable. A material world-point of the medium that was at the point x^{μ} in the x^{μ}-coordinate system before the distortion will then be at an infinitesimally neighboring point with the coordinates $x^{\mu} + \xi^{\mu}$. Moreover, the form of the transformations (1.13), which lead from the co-moving system to the x^{μ}-coordinate system, will change. One will then have:

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[17]) A. H. Taub, Phys. Rev. **94**, 1468 (1954).
[18]) V. Fock, Theorie von Raum, Zeit, und Gravitation (German translation), Berlin (1960) §§ 47, 48.

$$x^\mu + \xi^\mu = \tilde{\varphi}^\mu(a^\chi)\,, \qquad (5.1)$$
$$\xi^\mu = \tilde{\varphi}^\mu(a^\chi) - \varphi^\mu(a^\chi).$$

In order to investigate how these physical properties of the medium behave under these distortions of the world lines, we think of one such property as represented by a function of the fundamental field quantities, namely, $g_{\rho\sigma}$, a^χ, and their derivatives. Therefore, the latter may be expressed by:

$$\varphi^\nu_\lambda \equiv \frac{\partial \varphi^\nu}{\partial a^\lambda}\,. \qquad (5.2)$$

We thus write for the property χ:

$$\chi = \zeta(g_{\rho\sigma}(x^\mu), \varphi^\nu_\lambda, a^\chi) \qquad (5.3)$$
$$= \chi(x^\mu, \varphi^\nu_\lambda, a^\chi)\,.$$

Now, if the material point a^χ carries the property (5.3) before the distortion then after the distortion of the world-lines, one will have:

$$\tilde{\chi} = \chi(x^\mu + \xi^\mu, \tilde{\varphi}^\nu_\lambda, a^\chi) \qquad (5.4)$$
$$= \chi(x^\mu, \varphi^\nu_\lambda, a^\chi) + \left(\frac{\partial \chi}{\partial x^\mu}\right)_{\text{expl.}} \xi^\mu + \frac{\partial \chi}{\partial \varphi^\nu_\lambda}(\tilde{\varphi}^\nu_\lambda - \varphi^\nu_\lambda)\,.$$

Thus, according to (5.1):

$$\tilde{\varphi}^\nu_\lambda - \varphi^\nu_\lambda = \delta^* \varphi^\nu_\lambda = \frac{\partial}{\partial a^\lambda}\xi^\nu = \xi^\nu{}_{,\sigma}\varphi^\sigma_\lambda\,, \qquad (5.5)$$

$$\delta^* \chi = \tilde{\chi} - \chi = \left(\frac{\partial \chi}{\partial x^\mu}\right)_{\text{expl.}} \xi^\mu + \frac{\partial \chi}{\partial \varphi^\nu_\lambda}\varphi^\sigma_\lambda \xi^\nu{}_{,\sigma} \qquad (5.6)$$

represents the (infinitesimal) difference between the changed quality at the point $x^\mu + \xi^\mu$ and the unchanged quality at the point x^μ. However, one finds the difference between the changed quality and the original one by means of the variation of the world-lines, both of them at one and the same world-point x^μ to be:

$$\delta\chi = \delta^*\chi - \left(\frac{\partial \chi}{\partial x^\rho}\right)_{\text{total}} \xi^\rho\,. \qquad (5.7)$$

In this:

$$\left(\frac{\partial \chi}{\partial x^\rho}\right)_{\text{total}} = \left(\frac{\partial \chi}{\partial x^\rho}\right)_{\text{expl.}} + \frac{\partial \chi}{\partial \varphi^\nu_\lambda}\frac{\partial \varphi^\nu_\lambda}{\partial x^\rho} + \frac{\partial \chi}{\partial a^\chi}\frac{\partial a^\chi}{\partial x^\rho}\,. \qquad (5.8)$$

The operator δ then acts on the φ^ν_λ and the explicit a^χ. If one thinks of the former as being further expressed in terms of the $\partial a^\rho / \partial x^\mu$ then one can also say that δ acts directly upon the desired functions (and their derivatives) that describe the congruence of world-lines. The "variation of the world-lines" thus has principally the same character as the variational procedure that is used in field theory.

We now compute some variations that will be used later. One has:

$$\begin{aligned}\overset{*}{\delta} g_{\alpha\beta} &= \delta^*(\varphi^\lambda_\alpha \varphi^\chi_\beta g_{\lambda\chi}) \\ &= g_{\lambda\sigma}(\varphi^\lambda_\alpha \varphi^\chi_\beta + \varphi^\lambda_\beta \varphi^\chi_\alpha)\xi^\sigma{}_{;\chi}\,;\end{aligned} \tag{5.9}$$

(cf., Appendix B)
in particular, one then has:

$$\overset{*}{\delta} g_{44} = 2g_{\lambda\sigma}\,\overset{\circ}{\varphi}{}^\lambda\,\overset{\circ}{\varphi}{}^\chi\,\xi^\sigma{}_{;\chi}\,, \tag{5.10}$$

and, as a result:

$$\begin{aligned}\delta^*\bar{u}^4 &= \delta^*(1/\sqrt{-g_{44}}) \\ &= \bar{u}^4 u_\sigma u^\chi \xi^\sigma{}_{;\chi} + u^\chi \xi^\beta{}_{,\chi}\,.\end{aligned} \tag{5.11}$$

It follows that:

$$\begin{aligned}\overset{*}{\delta} u^\beta &= \delta^*(\overset{\circ}{\varphi}{}^\beta\,\bar{u}^4) \\ &= u^\beta u_\sigma\, u^\chi\, \xi^\sigma{}_{;\chi} + u^\chi\, \xi^\beta{}_{;\chi}\end{aligned} \tag{5.12}$$

and:

$$\begin{aligned}\delta u^\beta &= \overset{*}{\delta} u^\beta - u^\beta{}_{,\sigma}\,\xi^\sigma \\ &= u^\beta u_\sigma\, u^\chi\, \xi^\sigma{}_{;\chi} + u^\chi\, \xi^\beta{}_{;\chi} - u^\beta{}_{;\sigma}\,\xi^\sigma\,.\end{aligned} \tag{5.13}$$

One sees that $\overset{*}{\delta} u^\beta$, as the difference between two vectors at different points, does not represent a vector, but δu^β, as the difference of two vectors at the same point, is a vector.

Furthermore, one easily sees that:

$$\begin{aligned}\delta^*\bar{u}_k &= \delta^*(\bar{g}_{4k}\bar{u}^4) \\ &= \left(u^\chi s_{\lambda\sigma}\varphi^\lambda_k + u_\sigma \varphi^\chi_k\right)\xi^\sigma{}_{;\chi}\,,\end{aligned} \tag{5.14}$$

from which it then follows that:

$$\delta^*(\bar{u}_k \bar{u}^4) = \bar{u}^4 \varphi^\lambda_k (u^\chi s_{\lambda\sigma} + u_\sigma s^\sigma_\lambda)\xi^\sigma{}_{;\chi}\,. \tag{5.15}$$

With the help of (5.9) and (5.14), this yields:

$$\begin{aligned}\overset{*}{\delta}\gamma_{ik} &= \delta^*(\bar{g}_{ik} + \bar{u}_i\,\bar{u}_k) \\ &= s_{\lambda\sigma}s^\chi_\mu\left(\varphi^\lambda_k \varphi^\mu_i + \varphi^\mu_k \varphi^\lambda_i\right)\xi^\sigma{}_{;\chi}\end{aligned} \tag{5.16}$$

(cf., Appendix C).

If we contract this expression with $\bar{g}^{ik}$ and we consider that:

$$\bar{g}^{ik}\varphi^\lambda_i \varphi^\mu_k = g^{\lambda\mu} - \text{terms proportional to } \overset{\circ}{\varphi}{}^\mu \text{ or } \overset{\circ}{\varphi}{}^\lambda$$

then we obtain, since $s_{\lambda\sigma} s_{\mu}^{\chi} \left(\overset{\circ}{\varphi}{}^{\mu} \text{ or } \overset{\circ}{\varphi}{}^{\lambda} \right) = 0$:

$$\bar{g}^{ik} \delta^* \gamma_{\overline{ik}} = 2 s_{\lambda\sigma} s_{\mu}^{\chi} g^{\lambda\mu} \xi^{\sigma}{}_{;\chi} = 2 s_{\sigma}^{\chi} \xi^{\sigma}{}_{;\chi} \tag{5.17}$$

(cf., Appendix C).

§ 6. The variational principle for conservative systems

We apply the variation that was described in the previous paragraph in order to derive the equations of motion (1.4) for conservative systems from a variational principle. One has:

$$\delta \int \sqrt{-g}\, L\, d^4x = \int \sqrt{-g}\, \delta L\, d^4x = 0, \tag{6.1}$$

in which ξ^{μ} shall vanish on the boundary surface of an arbitrary domain of integration.

Next, we choose:

$$L = w = \rho\, \Phi = \frac{f(a^k)}{\sqrt{\gamma}} \Phi(\bar{e}_{ik}, a^k)\,. \tag{6.2}$$

One then has:

$$\delta L = \delta w = \overset{*}{\delta} w - w_{,\sigma}\, \xi^{\sigma}, \tag{6.3}$$

$$\overset{*}{\delta} w = \rho\, \overset{*}{\delta} \Phi + \Phi \overset{*}{\delta} \rho, \tag{6.4}$$

$$\begin{aligned} \rho\, \overset{*}{\delta} \Phi &= \rho \frac{\partial \Phi}{\partial \bar{e}_{ik}} \delta^* \gamma_{ik} \\ &= \frac{1}{2} \bar{S}^{ik} s_{\lambda\sigma} s_{\mu}^{\chi} \left(\varphi_k^{\lambda} \varphi_i^{\mu} + \varphi_k^{\mu} \varphi_i^{\lambda} \right) \xi^{\sigma}{}_{;\chi} \\ &= -\, w_{\sigma}^{\chi} \xi^{\sigma}{}_{;\chi}\,, \end{aligned} \tag{6.5}$$

in which we have taken (3.11), (5.16), (2.19), and (3.9) into consideration. In addition, we compute, while considering (5.17):

$$\begin{aligned} \Phi\, \overset{*}{\delta} \rho &= -\, \frac{\Phi\, f(a^j)}{2\left(\sqrt{\gamma}\right)^3} \frac{\partial \gamma}{\partial \gamma_{ik}} \delta^* \gamma_{ik} \\ &= -\, \frac{\Phi\, f(a^j)}{\left(2\sqrt{\gamma}\right)^3} \gamma \bar{g}^{ik} \delta^* \gamma_{ik} \\ &= -\, \rho\, \Phi\, s_{\sigma}^{\chi} \xi^{\sigma}{}_{;\chi} = - w\, s_{\sigma}^{\chi} \xi^{\sigma}{}_{;\chi}\,. \end{aligned} \tag{6.6}$$

If we put all of these advances together then what results is:

$$0 = -\int \sqrt{-g} \left\{ \left(w_{\sigma}^{\chi} + w\, s_{\sigma}^{\chi} \right) \xi^{\sigma}{}_{;\chi} + \left(\delta_{\sigma}^{\chi} w \right)_{;\chi} \xi^{\sigma} \right\} d^4x\,, \tag{6.7}$$

or, after partial integration:

$$0 = \int \sqrt{-g} \left\{ w_\sigma^\chi + w\left(s_\sigma^\chi - \delta_\sigma^\chi \right) \right\}_{;\chi} \xi^\sigma d^4 x \,. \tag{6.8}$$

We thus obtain, in fact, the equations of motion for conservative systems:

$$\left(w_\sigma^\chi + w u_\alpha u^\chi \right)_{;\chi} = T_{\sigma;\chi}^\chi = 0.$$

Naturally, ideal fluids are included as a special case in (3.16) through (3.19).

For the sake of heuristics, we assume that Φ has a dependency upon $\bar{u}_k \bar{u}^4$, in addition to its dependency upon $\bar{e}_{ik}$. Then, from (5.15), the right-hand side of (6.5) acquires a supplementary term:

$$\rho \frac{\partial \Phi}{\partial(\bar{u}_k \bar{u}^4)} \bar{u}^4 \varphi_k^\lambda \left(u^\chi s_{\lambda\sigma} + u_\sigma s_\lambda^\chi \right) \xi^\sigma{}_{;\chi} \,.$$

If we can set the heat current equal to:

$$\bar{w}^k = - \, \rho \frac{\partial \Phi}{\partial(\bar{u}_k \bar{u}^4)} \bar{u}^4$$

then, from (2.20), this supplementary term assumes the form:

$$-\,(u^\chi w_\sigma + u_\sigma w^\chi)\, \xi^\sigma{}_{;\,\chi} \,,$$

and we arrive at the final result that the tensor (2.6) is divergence-free. Thus, we can ascribe no true physical sense to the form of the heat current that we chose here.

Instead of it, we seek to extend the variational principle in such a way that the relation (4.11) that is demanded of conservative systems can be preserved. Next, we consider a single-species system such that one demands only that:

$$\frac{d\eta}{ds} = \eta_{,\beta}\, u^\beta = 0,$$

i.e.:

$$\eta = \eta(a^1, a^2, a^3).$$

This requirement is implicit in the choice of the Lagrangian function (6.2).

Henceforth, we replace the quantity Φ in (6.2), which corresponds to the specific internal energy, with:

$$\Psi(\bar{e}_{ik}, T) = \Phi - T\,\eta. \tag{6.9}$$

This function is the analogue of the specific internal free energy. One then has:

$$\frac{\partial\Psi}{\partial\overline{e}_{ik}} = \frac{1}{2\rho}\overline{S}^{ik}, \tag{6.10}$$

$$\frac{\partial\Psi}{\partial T} \equiv \Psi_T = -\,\eta. \tag{6.11}$$

From the research of Taub, however, we do not work directly with T, but with a scalar quantity α, which is connected with T by:

$$T = \alpha_{,\,\mu}\, u^{\mu}. \tag{6.12}$$

The action integral is now:

$$I = \int \sqrt{-g}\,\rho\,\Psi(\overline{e}_{ik}, \alpha_{,\mu}u^{\mu})\,d^4x\,. \tag{6.13}$$

We next vary it relative to α for fixed world-lines and obtain:

$$0 = \delta_\alpha I = \int \sqrt{-g}\,\rho\,\Psi_T\,u^{\mu}\,\delta_\alpha\,\alpha_{,\mu}\,d^4x = -\int \sqrt{-g}\,(g\,\Psi_T\,u^{\mu})_{;\mu}\delta_\alpha\,\alpha\,d^4x\,; \tag{6.14}$$

hence:

$$-\,(\rho\,\Psi_T\,u^{\mu})_{;\,\mu} = \rho\,u^{\mu}\,\eta_{,\,\mu} = 0, \tag{6.15}$$

as we desired.

In order to perform the variation of the world-lines, we remark that:

$$\begin{aligned}\delta(\rho\,\Psi) &= \delta^{*}(\rho\,\Psi) - (\rho\,\Psi)_{,\,\sigma}\,\xi^{\sigma}\\ &= -\rho\,\Psi\,s_{\sigma}^{\chi}\,\xi^{\sigma}{}_{;\chi} - w_{\sigma}^{\chi}\,\xi^{\sigma}{}_{;\chi} + \rho\,\Psi_T\,\delta^{*}T - (\rho\,\Psi)_{,\sigma}\xi^{\sigma}\,.\end{aligned} \tag{6.16}$$

Therefore, upon consideration of (5.12), one has:

$$\begin{aligned}\delta^{*}T &= \delta^{*}(\alpha_{,\,\mu}\,u^{\mu})\\ &= \alpha_{,\,\mu\sigma}\;u^{\mu}\,\xi^{\sigma} + T\,u_{\sigma}\;u^{\chi}\,\xi^{\sigma}{}_{;\,\chi} + \alpha_{;\,\mu}\;u^{\mu}\,\xi^{\sigma}{}_{,\,\mu}\\ &= \alpha_{;\,\mu\sigma}\;u^{\mu}\,\xi^{\sigma} + T\,u_{\sigma}\;u^{\chi}\,\xi^{\sigma}{}_{;\,\chi} + \alpha_{,,\mu}\;u^{\mu}\,\xi^{\sigma}{}_{;\,\mu}\;.\end{aligned} \tag{6.17}$$

Thus, one has:

$$\begin{aligned}0 &= \delta I\\ &= -\int\sqrt{-g}\,\Big\{\xi^{\sigma}{}_{;\chi}\Big[w_{\sigma}^{\chi} + \rho\,\Psi s_{\sigma}^{\chi} - \rho\,\Psi_T\,(Tu_{\sigma}u^{\chi} + \alpha_{,\sigma}u^{\chi})\Big]\\ &\qquad + \xi^{\sigma}\Big[\Big(\delta_{\sigma}^{\chi}\rho\,\Psi\Big)_{;\chi} - \rho\,\Psi_T\alpha_{;\chi\sigma}u^{\chi}\Big]\Big\}d^4x\\ &= \int\sqrt{-g}\,\xi^{\sigma}\Big\{\Big[w_{\sigma}^{\chi} - u_{\sigma}u^{\chi}\rho(\Psi - \Psi_T T)\Big]_{;\chi} + \Big[\rho\Psi_T\alpha_{;\chi\sigma}u^{\chi} - (\rho\,\Psi_T\alpha_{,\sigma}u^{\chi})_{;\chi}\Big]\Big\}d^4x\,.\end{aligned} \tag{6.18}$$

However, from (6.15), the latter square bracket vanishes, and one has $\Psi - \Psi_T\,T = \Phi$, such that we are again led back to (6.8).

For the case of multi-species systems, we would like to think of the relation (4.11) as being fulfilled in such a way that Φ depends upon η and the $c^{(k)}$ by way of N independent functions $f^1, f^2, \ldots, f^N$, which are constant along the world-lines. Thus, one now has:

$$\Phi = \Phi(\bar{e}_{ik}, f^1, \cdots, f^K, \cdots, f^N)$$
$$f^K = f^K(\eta, c^{(1)}, \ldots, c^{(k)}, \ldots, c^{(n)}) \tag{6.19}$$
$$0 = \frac{\partial f^K}{\partial a^4}\bar{u}^4 = f^K_{,\mu}\, u^\mu = \sum_{(k)=0}^{(n)} \frac{\partial f^K}{\partial c^{(k)}} c^{(k)}{}_{,\mu} u^\mu \qquad (c^{(0)} \equiv \eta).$$

One then has, in fact:

$$T\frac{d\eta}{ds} + \sum_{(k)=1}^{(n)} \mu^{(k)} c^{(k)}{}_{,\mu} u^\mu = \sum_{(k)=0}^{(n)} \frac{\partial \Phi}{\partial c^{(k)}} c^{(k)}{}_{,\mu} u^\mu \tag{6.20}$$
$$= \sum_{K=1}^{N} \frac{\partial \Phi}{\partial f^K} \sum_{(k)=0}^{(n)} \frac{\partial f^K}{\partial c^{(k)}} c^{(k)}{}_{,\mu} u^\mu = 0.$$

We now write:

$$\frac{\partial \Phi}{\partial f^K} \equiv \lambda_K \tag{6.21}$$
$$\Psi(\bar{e}_{ik}, \lambda_K) = \Psi - \sum_{K=1}^{N} \lambda_K f^K,$$

such that henceforth, in addition to (6.10), we have:

$$\frac{\partial \Phi}{\partial \lambda^K} = - f^K. \tag{6.22}$$

In addition, by analogy with (6.12), we introduce the scalar quantities α_K by way of:

$$\lambda_K = \alpha_{K,\mu}\, u^\mu. \tag{6.23}$$

Now, let our action integral be:

$$J = \int \sqrt{-g}\;\rho\,\Psi(\bar{e}_{ik}, \alpha_{4,\mu} u^\mu, \ldots, \alpha_{N,\mu} u^\mu)\, d^4x, \tag{6.24}$$

so variation relative to the α_K delivers the relations (6.19). One obtains the results of the variation of world-lines when one replaces $\Psi_T\, \overset{*}{\delta} T$ with $\sum_K (\partial\Psi/\partial\lambda_K)\,\delta^*\lambda_K$ in (6.16) and then in (6.18) one replaces $\Psi_T T$ with $\sum_K (\partial\Psi/\partial\lambda_K)\,\lambda_K$, $\Psi_T\alpha_{;\chi\sigma}$ with $\sum_K (\partial\Psi/\partial\lambda_K)\,\alpha_{K;\chi\sigma}$, and $\Psi_T\alpha_{;\chi}$ with $\sum_K (\partial\Psi/\partial\lambda_K)\,\alpha_{K;\chi}$. We then obtain the same result as in the previously-treated special case.

Finally, in order to arrive at the gravitational equations (1.2), we must consider the variation of J relative to $g_{\mu\nu}$ for fixed α_K and unvaried world-lines; i.e., for fixed φ^λ_ν. To that end, we notice:

$$\delta_g \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\, \delta g_{\mu\nu}, \tag{6.25}$$

$$\delta_g u^\sigma = \frac{1}{2} u^\sigma u^\mu u^\nu \,\delta g_{\mu\nu}\,. \tag{6.26}$$

From a simple computation, it then follows that:

$$\begin{aligned}\delta_g \gamma_{ik} &= \varphi_i^\alpha \varphi_k^\beta \delta_g s_{\alpha\beta} \\ &= \varphi_i^\alpha \varphi_k^\beta s_\alpha^\mu s_\beta^\nu \,\delta g_{\mu\nu}\end{aligned} \tag{6.27}$$

By way of similar consequences to the ones that were presented in (5.17) and (6.6), one then obtains:

$$\delta_g \rho = -\frac{1}{2}\,\rho\, s^{\mu\nu}\,\delta g_{\mu\nu}\,. \tag{6.28}$$

Furthermore, one has:

$$\begin{aligned}\delta_g \Psi &= \frac{\partial \Psi}{\partial \bar{e}_{ik}} \delta_g \gamma_{ik} + \sum_K \frac{\partial \Psi}{\partial \lambda_K} \alpha_{K,\mu} \delta_g u^\mu \\ &= \frac{1}{2\rho} \bar{S}^{ik} \varphi_i^\alpha \varphi_k^\beta s_\alpha^\mu s_\beta^\nu \,\delta g_{\mu\nu} - \frac{1}{2} \sum_K f^K \lambda_K u^\mu u^\nu \delta g_{\mu\nu} \\ &= -\frac{1}{2}\left(\frac{w^{\mu\nu}}{\rho} + u^\mu u^\nu \sum_K f^K \lambda_K\right) \delta g_{\mu\nu}\,.\end{aligned} \tag{6.29}$$

Putting everything together yields:

$$\begin{aligned}\delta_g J &= -\frac{1}{2} \int \delta g_{\mu\nu} \sqrt{-g} \left\{ \rho \Psi (s^{\mu\nu} - g^{\mu\nu}) + \rho u^\mu u^\nu \sum_K f^K \lambda_K + w^{\mu\nu} \right\} d^4 x \\ &= -\frac{1}{2} \int \delta g_{\mu\nu} \sqrt{-g} (u^\mu u^\nu w + w^{\mu\nu})\, d^4 x\,.\end{aligned} \tag{6.30}$$

Apparently, eq. (1.7) is thus fulfilled. Since, one has:

$$\delta_g \int \sqrt{-g}\, R\, d^4 x = \int \delta g_{\mu\nu} \sqrt{-g}\, (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)\, d^4 x\,,$$

in which R is the Riemannian curvature scalar, the total action integral, whose variation relative to α_K, $g_{\mu\nu}$, and the world-lines gives all of the basic equations of the theory, reads:

$$J_{\text{total}} = \int \sqrt{-g} \left(\rho \Psi + \frac{1}{2\chi} R \right) d^4 x\,. \tag{6.31}$$

Appendix A

One has:

$$w_{,\beta}\, u^{\beta} = \Phi\, \rho_{,\beta}\, u^{\beta} + \rho\, \Phi_{,\beta}\, u^{\beta}.$$

From (3.15):

$$\Phi\, \rho_{,\beta}\, u^{\beta} = -\, w\, u^{\beta}{}_{;\,b}\,.$$

Furthermore:

$$\begin{aligned} \rho\, \Phi_{,\beta}\, u^{\beta} &= \rho \frac{\partial \Phi}{\partial a^4} \bar{u}^4 \\ &= \rho \left(\frac{\partial \Phi}{\partial \bar{e}_{ik}} \frac{\partial \gamma_{ik}}{\partial a^4} + \left(\frac{\partial \Phi}{\partial a^4} \right)_{\text{expl.}} \right) \bar{u}^4 . \end{aligned}$$

In the following summation, vanishing – canceling, resp. – terms will be added. One can conclude:

$$\begin{aligned} \rho \frac{\partial \Phi}{\partial \bar{e}_{ik}} \gamma_{ik,4}\, \bar{u}^4 &= \frac{1}{2} \bar{S}^{ik}\, \bar{s}_{ik,4}\, \bar{u}^4 \\ &= \frac{1}{2} \bar{S}^{\alpha\beta}\, (\bar{s}_{\alpha\beta,4}\, \bar{u}^4 + \bar{s}_{\alpha 4}\, \bar{u}^4{}_{,\beta} + \bar{s}_{4\beta}\, \bar{u}^4{}_{,\alpha}) \\ &= \frac{1}{2} \bar{S}^{\alpha\beta}\, (s_{\alpha\beta,\rho}\, \bar{u}^{\rho} + \bar{s}_{\alpha\rho}\, \bar{u}^{\rho}{}_{,\beta} + \bar{s}_{\rho\beta}\, \bar{u}^{\rho}{}_{,\alpha}) \\ &= \frac{1}{2} S^{\alpha\beta}\, (u_{\alpha;\,\rho}\, u^{\rho}\, u_{\beta} + u_{\beta;\,\rho}\, u^{\rho}\, u_{\alpha} + u_{\alpha;\,\beta} + u_{\beta;\,\alpha}) \\ &= S^{\alpha\beta}\, u_{\alpha;\,\beta}\,. \end{aligned}$$

The combination of all of these results delivers (3.22).

Appendix B

The formula (5.9) can be derived in the following way:

$$\begin{aligned} \delta^{*} \bar{g}_{\alpha\beta} &= \delta^{*}(\varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} g_{\lambda\chi}) = \varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} g_{\lambda\chi,\sigma} \xi^{\sigma} + g_{\lambda\chi} \varphi^{\lambda}_{\alpha} \varphi^{\sigma}_{\beta} \xi^{\chi}{}_{,\sigma} + g_{\lambda\chi} \varphi^{\chi}_{\beta} \varphi^{\sigma}_{\alpha} \xi^{\lambda}{}_{,\sigma} \\ &= g_{\lambda\chi} \varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} \xi^{\sigma}{}_{,\sigma} + g_{\lambda\chi} \varphi^{\chi}_{\beta} \varphi^{\sigma}_{\alpha} \xi^{\lambda}{}_{,\sigma} + \frac{1}{2} \left(\varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} + \varphi^{\lambda}_{\beta} \varphi^{\chi}_{\alpha} \right) \left(g_{\lambda\chi,\sigma} + g_{\lambda\sigma,\chi} - g_{\chi\sigma,\lambda} \right) \xi^{\sigma} \\ &= g_{\lambda\rho} \varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} \xi^{\rho}{}_{,\chi} + g_{\lambda\rho} \varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} \Gamma^{\rho}_{\chi\sigma} \xi^{\sigma} + g_{\lambda\rho} \varphi^{\lambda}_{\beta} \varphi^{\chi}_{\alpha} \xi^{\rho}{}_{,\chi} + g_{\lambda\rho} \varphi^{\lambda}_{\beta} \varphi^{\chi}_{\alpha} \Gamma^{\rho}_{\chi\sigma} \xi^{\sigma} \\ &= g_{\lambda\rho} \left(\varphi^{\lambda}_{\beta} \varphi^{\chi}_{\alpha} + \varphi^{\lambda}_{\alpha} \varphi^{\chi}_{\beta} \right) \xi^{\rho}{}_{;\chi}\,. \end{aligned}$$

Appendix C

In the following computations, which lead up to (5.16), the symbol + ($i \leftrightarrow k$) means that the previous terms are to be added once more with the indices i and k switched.

$$\delta^{*} \gamma_{ik} = \delta^{*} \bar{g}_{ik} + \delta^{*} \bar{u}_i\, \bar{u}_k = \xi^{\sigma}{}_{;\chi} \left\{ \varphi^{\lambda}_{k} \varphi^{\chi}_{i} g_{\lambda\sigma} + \bar{u}_i\, u^{\chi} s_{\lambda\sigma} \varphi^{\lambda}_{k} + \bar{u}_i\, u_{\sigma} \varphi^{\chi}_{k} + (i \leftrightarrow k) \right\}$$

$$
\begin{aligned}
&= \xi^{\sigma}{}_{;\chi}\left\{\varphi_k^{\lambda}\varphi_i^{\chi} g_{\lambda\sigma} + \varphi_k^{\lambda}\varphi_i^{\mu} u_{\mu} u^{\chi} s_{\lambda\sigma} + \varphi_i^{\lambda}\varphi_k^{\mu} u_{\lambda} u_{\sigma} + (i \leftrightarrow k)\right\} \\
&= \xi^{\sigma}{}_{;\chi}\left\{\varphi_k^{\lambda}\varphi_i^{\chi} s_{\lambda\sigma} + s_{\lambda\sigma}\varphi_k^{\lambda}\varphi_i^{\mu} u_{\mu} u^{\chi} + (i \leftrightarrow k)\right\} \\
&= \xi^{\sigma}{}_{;\chi}\left\{s_{\lambda\sigma}\varphi_k^{\lambda}\varphi_i^{\chi}(\delta_{\mu}^{\chi} + u_{\mu} u^{\chi}) + (i \leftrightarrow k)\right\} \\
&= \xi^{\sigma}{}_{;\chi}\left\{s_{\lambda\sigma} s_{\mu}^{\chi}(\varphi_k^{\lambda}\varphi_i^{\mu} + \varphi_i^{\lambda}\varphi_k^{\mu})\right\}.
\end{aligned}
$$

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Received for editing on 9 April, 1963.