"Ueber die in der Wellengleichung verwendeten hyperkomplexen Zahlen," Proc. Kon. Akad. Wet. Amst. **32** (1929), 105-108.

On the hypercomplex numbers that are employed in the wave equation

By **J. A. SCHOUTEN**

(Communicated by Prof. **P. Ehrenfest** at the meeting on February 23, 1929)

Translated by D. H. Delphenich

In his paper "A symmetrical treatment of the wave equation" ([1]), A. S. EDDINGTON showed that the well-known asymmetry in the DIRAC equations can be extended when one starts with the theory of matrices. Whether that method, which has, in the meantime, led to the mathematical establishment of the natural constant $\hbar c / 2\pi e^2$ ([2]), must probably be acknowledged to be a meaningful improvement, nonetheless, it still lacks a secure foundation, since it is constructed upon certain remarkable imaginary "rotations" that are introduced from the outset that serve only to provide the operators $E_1, \ldots, E_3$ with their properties, but are not used again.

Now, it shall be shown in what follows that the imaginary rotations can be acknowledged to be the entire foundation, as long as one takes one step further and begins with the theory of complex number systems. That will show that the smallest number system that can come under question at all is the "primary" system of sixteen units, and that will imply the properties of the operators E, including the possibility of their matrix representation from the known properties of that system.

1. – Let E_1, E_2, E_3, E_4 be four higher complex numbers with the rules of calculation:

$$\left.\begin{aligned} E_i\, E_i &= 1, \\ E_i\, E_j &= -E_j\, E_i, \end{aligned}\right\} \qquad i, j = 1, \ldots, 4, \tag{1}$$

which are subject to the *associative* law, in addition. It will then follow that the sixteen numbers 1, $E_1, \ldots, E_{12} = E_1\, E_2, \ldots, E_{123} = E_1\, E_2\, E_3, \ldots, E_{1234} = E_1\, E_2\, E_3\, E_4$ define a closed, associative system. It is the fourth system in the sequence of so-called "primary systems" ([3]) – viz., systems that contain no invariant subsystems – and we would like to

([1]) Proc. Roy. Soc. London **A 121** (1928), 524-542.
([2]) "The charge of the electron," Proc. Roy. Soc. London **A 121** (1929), 358-369.
([3]) Also called the systems of "sedenions" or "quadriquaternions."

denote to them by U_4 accordingly. The real form of the rules of calculation that emerges from (1) (i.e., the form with real coefficients) will lead back to CLIFFORD.

A second real form, which can nonetheless be derived from the first one by means of complex transformations, is formed from four-rowed matrices by the product rules (more precisely: the mixed second-degree quantities in four dimensions):

$$\sum_j P_i^{\cdot j} Q_j^{\cdot k} = R_i^{\cdot j}, \qquad i, j, k = 1, \ldots, 4. \tag{2}$$

It will then follow from the associative law that the sixteen numbers E can be regarded as four-rowed matrices.

A third real form, which is likewise coupled to the *second one* in a complex way, but with the *first one* in a real way, will arise when one remarks that E_{1234} likewise behaves anticommutatively in regard to E_1, E_2, E_3, and E_4, while $E_{1234}\,E_{1234} = 1$. If one now writes $E_{1234} = E_5$ then one will have:

$$\left.\begin{aligned} E_i\,E_i &= 1, \\ E_i\,E_j &= -E_j\,E_i, \end{aligned}\right\} \qquad i, j = 1, \ldots, 5, \tag{3}$$

which will imply the rules of calculation for the sixteen units 1, E_1, …, $E_{12} = E_1\,E_2$, …

Those rules likewise lead back to CLIFFORD. The remarkable relations between four-dimensional and five-dimensional invariance find their basis in this self-extending property of the four hypercomplex numbers E_1, E_2, E_3, and E_4. A system of five numbers $E_1, \ldots, E_5$, that are subject to the rules (3) shall be called an *orthogonal system.* Obviously, there is no orthogonal system in the system U_4 that contains more than five numbers.

A fourth real form follows from the property of the system U_4 that it is the product of two systems of quaternions. If the 1, λ_1, λ_2, λ_3, on the one hand, and the 1, μ_1, μ_2, μ_3, on the other, are systems of quaternions with the rules of calculation:

$$\left.\begin{array}{ll} \lambda_1\,\lambda_1 = -1, & \mu_1\,\mu_1 = -1, \\ \lambda_1\,\lambda_2 = -\lambda_2\,\lambda_1 = \lambda_3, & \mu_1\,\mu_2 = -\mu_2\,\mu_1 = \mu_3, \\ \quad cycl. & \quad cycl. \end{array}\right\} \tag{4}$$

then 1, λ_i, μ_i, $\lambda_i\,\mu_j = \mu_j\,\lambda_i$, $i, j = 1, 2, 3$ will define a system U_4. It is easy to see that an orthogonal system can be defined perhaps as follows:

$$E_1 = \lambda_1\,\mu_3, \quad E_2 = \lambda_2\,\mu_3, \quad E_3 = \lambda_3\,\mu_3, \quad E_4 = -i\,\mu_1, \quad E_5 = -i\,\mu_2. \tag{5}$$

Conversely, for any orthogonal system, one can choose two systems λ (μ, resp.) such that they satisfy ([1]) the equations (5):

([1]) One will get Dirac's σ and ρ when one sets: $\sigma_i = i\,\lambda_i$, $\rho_i = i\,\mu_i$.

$$\left.\begin{array}{lll} \mu_1 = i\,E_4, & \mu_2 = i\,E_5, & \mu_3 = i\,E_{45}, \\ \lambda_1 = -E_{23}, & \lambda_2 = -E_{31}, & \lambda_3 = -E_{12}. \end{array}\right\} \tag{6}$$

2. – The matrices that correspond to $E_1, \ldots, E_5$ are roots of the identity matrix. As a result, they have only linear elementary divisors, and each of them can be put into diagonal form with the help of suitable coordinate transformations such that either + 1, – 1, + 1, – 1 or + 1, + 1, – 1, – 1 will appear in the diagonal. However, any of the five numbers E can be written as the difference of a product and its inverse, e.g.:

$$E_1 = \tfrac{1}{2}(E_1\,E_{12} - E_{12}\,E_1), \tag{7}$$

from which, the known theorem that the trace is zero will emerge. Hence, only matrices with the elementary divisors $(\lambda - 1)$, $(\lambda - 1)$, $(\lambda + 1)$, $(\lambda + 1)$ will remain. It will follow from the theory of elementary divisors that all mixed quantities of second degree can be converted into each other by linear transformations, and that will guarantee that when one has any number whose matrix possesses the elementary divisors $(\lambda - 1)$, $(\lambda - 1)$, $(\lambda + 1)$, $(\lambda + 1)$, one can start with an orthogonal system as the first number, and one can always define an orthogonal system that contains that number.

Similar results can be derived for the numbers λ and μ. The elementary divisors of the matrices of the numbers that belong to that triple of numbers are $(\lambda - i)$, $(\lambda - i)$, $(\lambda + i)$, $(\lambda + i)$, and it will emerge from this that when one has any arbitrary number whose matrix possesses those elementary divisors, one can start with a triple of numbers as the first number, and one can always define a triple that contains that number and a second triple that is associated with it in such a way that the rules of calculation (4) are valid.

3. – If one poses the problem of finding how many orthogonal systems exist that contain E_5 then one should first remark that $E'_1, \ldots, E'_4, E_5$; $E'_j = i\,E_5\,E_j$, $j = 1, \ldots, 4$ define such a system. That result goes back to EDDINGTON; however, for him, those two systems are the only ones possible (up to a change of sign), which is connected with the fact that his "perpendicular sets" of five numbers are defined in terms of the imaginary rotations that were mentioned to begin with, and do not therefore overlap precisely with our orthogonal systems. One easily reckons that the general form of the desired system (up to a change of sign) will read:

$$\alpha E_1 + \ldots + \beta\,E'_1, \ldots, \alpha E_4 + \ldots + \beta\,E'_4,\ \alpha^2 + \beta^2 = 1. \tag{8}$$

If we ignore the change of sign then there will be ∞^1 real systems that contain E_5, and they will be pair-wise associated with each other (viz., α, β with $-\beta, \alpha$).

Each of those systems – e.g., $E_1, \ldots, E_5$ – is invariant under orthogonal transformations of the five operators. One multiplies a linear equation in the E – e.g.:

$$\left(\sum_i t_i E_i\right)\psi = 0, \qquad i = 1, \ldots, 5, \tag{9}$$

which exhibits five-dimensional invariance in that sense, times E_5 then that will yield the equation:

$$\left(\sum_i t_i E_i' + t_5\right)\psi = 0, \quad i = 1, \ldots, 0, \tag{10}$$

which is equivalent to (9) and possesses four-dimensional invariance. This important result of EDDINGTON is then independent of his definition of the "perpendicular sets," which was based upon the imaginary rotations, and is purely a consequence of the properties of the number system U_4 .

Any homogeneous form in the $E_1, \ldots, E_5$ can be written as an inhomogeneous quadratic form in $E_1', \ldots, E_4'$ ([1]):

$$\sum_{i,j} t_{ij} E_i E_j = \sum_{a,b} t_{ab} E_a' E_b' + i \sum_a (t_{a5} - t_{5a}) E_a' + t_{55}\,, \ldots \tag{11}$$

4. – In conclusion, we shall discuss the question of which numbers can assume the form $\sum_i t_i E_i$, $i = 1, \ldots, 5$, to begin with. It follows from the possibility of orthogonal transformations of the E that such a number will differ from a number whose matrix possesses the elementary divisors $(\lambda - 1)$, $(\lambda - 1)$, $(\lambda + 1)$, $(\lambda + 1)$ by only an ordinary numerical factor. The desired numbers are then the ones whose matrices possess linear elementary divisors and a characteristic equation with two doubly-counted opposite roots.

___

([1]) I must thank Herrn D. van Dantzig for this remark.