"La théorie projective de la relativité," Ann. de l'Inst. Henri Poincaré **5** (1935), 51-88.

The projective theory of relativity

By

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Introduction. – In what follows, one will find a summary of the projective theory of relativity, as it was developed in the publications G. F. III, IV, V, VI, VIII ([1]), upon starting with the idea, which is due to O. VEBLEN and B. HOFFMANN ([2]), of a projective connection that leaves invariant a given quadric in local space. As in those publications, we shall make use of the method of homogeneous coordinates that is due to van DANTZIG ([3])([4]). Meanwhile, the theory and calculations have been simplified greatly by introducing the signature $- - - - +$ for the fundamental projector $\mathcal{G}_{\lambda\kappa}$ directly and employing auto-geodesics, instead of the induced geodesics that were utilized in G. F. III-VI ([5]). We have also employed the RICCI symbolism in its most modern form, as

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([1]) In the text, the following abbreviations will be used:

G. F. III: J. A. SCHOUTEN and D. v. DANTZIG, "Generelle Feldtheorie," Zeit. Phys. **78**(1932), 639-667.

G. F. VI: "On projective connexions and their application to general field-theory," Ann. Math. **34** (1933), 271-312.

G. F. IV: J. A. SCHOUTEN, "Zur generellen Feldtheorie – Ableitung des Impulsenergiestromprojektors aus einem Variationsprinzip," Zeit. Phys. **81** (1933), 129-138.

G. F. V: J. A. SCHOUTEN, "Zur generellen Feldtheorie – Raumzeit und Spinraum," Zeit. Phys. **81** (1933), 405-417.

G. F. VIII: J. A. SCHOUTEN and J. Haantjes, "Autogeodätische Linien und Weltlinien," Zeit. Phys. **89** (1934), 357-369.

([2]) O. VEBLEN and B. HOFFMANN, "Projective relativity," Phys. Rev. **36** (1930), 810-822.
B. HOFFMANN, "Projective relativity and the quantum-field," Phys. Rev. **37** (1931), 88-89.

([3]) D. v. DANTZIG, "Theorie des projektiven Zusammenhangs n-dimensionaler Räume," Math. Ann. **106** (1932), 400-454. "Zur allgemeinen projektiven Differenetialgeometrie, I. Einordnung der Affinegeometrie," II. X_{n+1} mit eingliedriger Gruppe," Proc. Kon. Ned. Akad. Wet. Amsterdam **35** (1932), 524-534, 535-542. "On the general projective differential geometry. III. Projective pointfield-algebra and analysis," Proc. Kon. Ned. Akad. Wet. Amsterdam **37** (1934), 150-155.

([4]) In O. VEBLEN, *Projektive Relativitätstheorie*, Berlin, J. Springer, 1933, one will find the projective theory treated from another viewpoint that makes no use of homogeneous coordinates.

([5]) The consequences of the choice of the other signature $+ - - - +$ are found sketched out briefly in the notes at the bottoms of the pages.

it is presented in the book by J. A. SCHOUTEN and D. J. STRUIK ([1]), which will appear shortly.

It is proved that if the signature $- - - - +$ is given then the projective connection will be determined completely by two geometric coordinates and five physical coordinates, namely:

I. The quadric is invariant.

II. The induced connection is identical to the Riemannian connection.

III. The auto-geodesics are the trajectories of charged particles.

IV. The equation of the trajectories is identical with the equation of the conservation of impulse and energy.

V. The two bivectors $\mathcal{P}_{\mu\lambda}$ and $\mathcal{Q}_{\mu\lambda}$ differ from the electromagnetic bivector only by constant factors.

VI. When the simplest invariant N that one can deduce from that connection – viz., the scalar curvature – is taken to be the universal function in the variational equations, that will provide the equations of gravitation and the second MAXWELL equation without the current term.

VII. The simplest invariant M that one can deduce from the Dirac equation, when it is taken to be the universal function, provides the current term with no additional terms.

When one discards the last condition, there will be an infinitude of possible connections. Among them, one finds the connection that was utilized in the theory of A. EINSTEIN and W. MAYER ([2]) ($\mathcal{P}_{\mu\lambda} = 0$) and the one in the theory of O. VEBLEN and B. HOFFMANN ($\mathcal{P}_{\mu\lambda} = \mathcal{Q}_{\mu\lambda}$), which was employed by W. PAULI ([3]) in his last work. Meanwhile, PAULI's theory utilizes induced geodesics; consequently, it does not satisfy condition IV. (The two types of geodesics are identical for $\mathcal{P}_{\mu\lambda} = 0$.) When one employs the auto-geodesics, all of those theories will satisfy conditions I-VI. Moreover, the additional term in the second MAXWELL equation is small enough that for the moment any experimental verification of it seems absolutely impossible.

([1]) *Einführungen in die neueren Methoden der Differentialgeometrie*, 2nd ed., Noordhoff, Groningen, 1935.

([2]) A. EINSTEIN and W. MAYER, "Einheitliche Theorie von Gravitation und Elektrizität," Sitz., Preuss. Akad. Wiss. Berlin **25** (1931), 541-557.

([3]) W. PAULI, "Ueber die Formulierung der Naturgesetze mit fünf homogenen Koordinates," Ann. Phys. (Leipzig) **18** (1933), 305-372.

I. – **Geometric notions.**

COORDINATES AND LOCAL SPACES. – Let:

$$\xi^h \qquad (h, i, j, \ldots, m = 1, 2, 3, 4)$$

be the coordinates of a four-dimensional space. They can be subjected to all of the transformations of the group $\mathfrak{G}_4$, which consists of all the continuous, several-times differentiable transformations. A space that admits the group $\mathfrak{G}_4$ will be called an X_4 . An arbitrarily-chosen system of ξ^h will be called a *point* of X_4 .

Let:

$$x^\kappa \qquad (\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau = 0, 1, 2, 3, 4)$$

denote the homogeneous coordinates of a four-dimensional space, which can be subjected to the transformations of the group $\mathfrak{H}_5$, which consists of all continuous, several-times differentiable transformations that are *homogeneous of degree one.* A space that admits the group $\mathfrak{H}_5$ will called an H_4 . A system of $x^\kappa \neq 0$ that is chosen arbitrarily, up to a numerical factor, will be denoted by $\lfloor x^\kappa \rfloor$, and will be called a *point* of H_4 . A system of x^κ is called a *marked point* (1) of H_4 . The marked points of H_4 can be represented by the *points* of an auxiliary space X_5 in which the x^κ are special inhomogeneous coordinates. The points of H_4 are represented by the curves in X_5 that take the form: $x^\kappa = \mu\, c^\kappa$, with c^κ = const. The point $x^\kappa = 0$ of X_5 has no representation in H_4 . We make the points of the spaces X_4 and H_4 correspond pair-wise in the following manner: If the coordinate system x^κ is chosen in an arbitrary manner then we take ξ^h to be arbitrary homogeneous functions of degree zero of the x^κ. Obviously, the nature of the relation between ξ^h and x^κ will persist when the x^κ and ξ^h are subjected to transformations that belong to the groups $\mathfrak{H}_5$ and $\mathfrak{G}_4$, respectively. One must remark that despite that identification, the two spaces X_4 and H_4 do not have the same geometric properties, since the groups $\mathfrak{G}_4$ and $\mathfrak{H}_5$ are different.

Let us now press on and identify the points of X_4 and H_4 with the points of the physical space-time. The group $\mathfrak{H}_5$ will then permit us to introduce some new geometric notions into physics that are not known in the geometry of X_4, which is based upon the usual theory of relativity. Each point of space-time can now be denoted by either ξ^h or by $\lfloor x^\kappa \rfloor$.

The $d\xi^h$ transform linearly under $\mathfrak{G}_4$ as follows:

$$(1) \qquad d\xi^{h'} = A_h^{h'} d\xi^h , \qquad A_h^{h'} = \frac{\partial \xi^{h'}}{\partial \xi^h} .$$

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(1) In the publications that were cited in footnotes (1) and (3) on page 1, point = Ort = spot, and marked point = Punkt = point.

For each point of H_4, we introduce affine Cartesian coordinates into E_4 (viz., ordinary affine geometric space) that will be denoted by Ξ^h, and whose transformation law is:

$$\Xi^{h'} = A_h^{h'} \, \Xi^h . \tag{2}$$

That space is the *affine local space* or the *local* E_4 at the point of H_4 in question. A system of Ξ^h is a corresponding *point* of the local E_4. The point ξ^h of X_4 can be identified with the point $\Xi^h = 0$ of the local E_4; we call it the *contact point.*

In the same fashion, the dx^κ transform linearly under $\mathfrak{H}_5$:

$$dx^{\kappa'} = \mathcal{A}_\kappa^{\kappa'} dx^\kappa , \qquad \mathcal{A}_\kappa^{\kappa'} = \frac{\partial x^{\kappa'}}{\partial x^\kappa} , \tag{3}$$

which will permit us to introduce a P_4 (viz., an ordinary projective geometric space) at each point of H_4 that has *ordinary* homogeneous coordinates X^κ with the transformation law:

$$X^{\kappa} = \mathcal{A}_\kappa^{\kappa'} X^\kappa . \tag{4}$$

That space is the *local projective space* or the local P_4 of the point of H_4 in question. A system of X^κ is a *marked point* of the local P_4. Naturally, up to now, it has been impossible to represent the local E_4 on the local P_4, since the E_4 contains a hyperplane at infinity that does not exist in P_4.

GEOMETRICAL OBJECTS IN X_4. – A *geometric object* of X_4 is a set that consists of a well-defined number of components that are functions of ξ^h that are defined in a region of X_4 (which can reduced to just one point), and transform under $\mathfrak{G}_4$ in such a manner that the new components will depend upon the untransformed components uniquely and the transformation law of the ξ^h. Here are some examples:

1. Scalar ([1]):

$$\overset{(h')}{p} = \overset{(h)}{p} .$$

2. Contravariant vector:

$$v^{h'} = A_h^{h'} v^h .$$

3. Covariant vector:

$$w_{i'} = A_{i'}^{i} \, w_i .$$

4. Affinor; for example ([2]):

$$v_{j'i'}^{\cdot\cdot h'} = A_{h\,j'i'}^{h'ji} \, v_{ji}^{\cdot\cdot h} .$$

([1]) When the components of a geometric object have no indices, sometimes we will add one in parentheses above the principal letter in order to denote the reference system.

([2]) For the sake of brevity, we shall not repeat the letter *A*.

5. *A*-density (scalar) [1] of weight $\mathfrak{k}$:

$$\overset{(h')}{\mathfrak{P}} = \Delta^{-\mathfrak{k}} \overset{(h)}{\mathfrak{P}}, \qquad \Delta = \det\,(A_h^{h'})\,.$$

6. Affinor-density, of weight $\mathfrak{k}$; for example:

$$\mathfrak{W}_{j'i'}^{\cdot\cdot h'} = \Delta^{-\mathfrak{k}} A_{j'i'h}^{jih'} \mathfrak{W}_{ji}^{\cdot\cdot h}\,.$$

Obviously, all the objects in 1-6 are special cases of affinor-densities. For the affinors whose *valence* (i.e., the number of indices) is greater than unity, one can transform each of the indices independently of the other ones and thus obtain some *intermediary components;* for example:

(5) $$Q_{j\cdot i}^{\cdot\cdot h'} = A_h^{h'} Q_{j\cdot i}^{\cdot\cdot h} = A_{ji}^{j'i'} Q_{j'\cdot i'}^{\cdot\cdot h'}\,.$$

Obviously, the $A_h^{h'}$ and $A_{h'}^{h}$ are the intermediary components of the *identity affinor* [2]*:*

(6) $$A_i^h \triangleq \delta_i^h\,.$$

Each of the objects 2-6, when defined at a point of X_4, can be represented by a geometric figure in the local E_4 of that point. For example, one can represent the contravariant vector v^h by the point $\Xi^h = v^h$, the covariant vector w_i by the hyperplane $w_i\,\Xi^i = 1$, the tensor (= symmetric affinor) g_{ij} by the quadric $g_{ij}\,\Xi^i\,\Xi^j = 1$, etc. It is remarkable to confirm that, from (1), the linear element $d\xi^h$ is a *vector*. Hence, every direction that passes through a point ξ^h of X_4 is represented in a bijective manner by a direction in the local E_4 at ξ^h, and the infinitesimal region that surrounds the point ξ^h in X_4 will be represented in a bijective manner in the region around the point of contact in E_4, up to second-order infinitesimals.

In the E_4 at each point of X_4 , there are four contravariant vectors $\underset{i}{e}^h$ and four covariant vectors $\overset{h}{e}_i$ that belong to the coordinate system ξ^h and which are defined by the equation:

(7) $$\underset{i}{e}^h \triangleq \delta_i^j\,, \qquad \overset{h}{e}_i \triangleq \delta_i^h\,.$$

[1] It is well-known that an "*A*-density" of weight one is nothing but an abbreviated expression for a covariant *n*-vector (alternating affinor of valence *n*). We shall take Gothic letters for the principal letters of densities.

[2] The symbol $\triangleq$ means that the equality is valid only for the coordinate systems for which it is essentially written.

These vectors are *coordinate vectors*. They constitute the *local frame* of ξ^h in the local E_4 . Each coordinate system ξ^h, $\xi^{h'}$, etc., has its proper system of local frames, which will be denoted by (h), (h') from now on.

GEOMETRIC OBJECTS in H_4 . – In what follows, we will imagine only geometric objects in H_4 with components that are *homogeneous* functions of the x^κ that are defined in a region of the auxiliary H_4 that can, moreover, reduce to one of the curves in X_5 that are given by $x^\kappa = \mu\, c^\kappa$. These objects are defined with respect to the group $\mathfrak{H}_5$ in the same way that the objects of X_4 are defined with respect to the group $\mathfrak{G}_4$. Hence, the transformations must be defined with respect to *two* groups, namely, the group $\mathfrak{H}_5$ of *coordinate* transformations, and the group $\mathfrak{F}$ of *marked point* transformations:

(8) $$'x^\kappa = \rho\, x^\kappa.$$

Here are some examples:

1. Scalar:

$$\mathfrak{H}_5 : \qquad \overset{(\kappa')}{p} = \overset{(\kappa)}{p} ,$$
$$\mathfrak{F} : \qquad '\overset{(\kappa)}{p} = \overset{(\kappa)}{p} .$$

2. Marked point:

$$\mathfrak{H}_5 : \qquad v^{\kappa'} = \mathcal{A}_\kappa^{\kappa'} v^\kappa ,$$
$$\mathfrak{F} : \qquad 'v^\kappa = \rho\, v^\kappa.$$

The corresponding point is denoted by $\lfloor v^k \rfloor$.

3. Marked hyperplane:

$$\mathfrak{H}_5 : \qquad w_{\lambda'} = \mathcal{A}_{\lambda'}^{\lambda}\, w_\lambda ,$$
$$\mathfrak{F} : \qquad 'w_\lambda = \rho^{-1}\, w_\lambda .$$

The corresponding hyperplane is denoted by $\lfloor w_\lambda \rfloor$.

4. Projector ([1]); for example:

$$\mathfrak{H}_5 : \qquad v^{\kappa'}_{\cdot\lambda'\mu'} = \mathcal{A}^{\kappa'\lambda\mu}_{\cdot\kappa\lambda'\mu'}\, v^{\kappa}_{\cdot\lambda\mu} ,$$
$$\mathfrak{F} : \qquad 'v^{\kappa}_{\cdot\lambda\mu} = \rho^{-1}\, v^{\kappa}_{\cdot\lambda\mu} .$$

([1]) The letter $\mathcal{A}$ will be written only once, to abbreviate.

5′. *P*-density (scalar) (1) of weight $\mathfrak{k}$:

$$\mathfrak{H}_5: \qquad \overset{(\kappa')}{\mathfrak{h}} = \Delta^{-\mathfrak{k}} \overset{(\kappa)}{\mathfrak{h}},$$

$$\mathfrak{F}: \qquad {}'\overset{(\kappa)}{\mathfrak{h}} = \rho^{-(n+1)\,\mathfrak{k}} \overset{(\kappa)}{\mathfrak{h}}.$$

6. Projector-density of weight $\mathfrak{k}$; for example:

$$\mathfrak{H}_5: \qquad \mathfrak{B}^{\kappa'}_{\cdot\lambda'\mu'} = \Delta^{-\mathfrak{k}} \mathcal{A}^{\kappa'\lambda\mu}_{\kappa\lambda'\mu'} \mathfrak{B}^{\kappa}_{\cdot\lambda\mu},$$

$$\mathfrak{F}: \qquad {}'\mathfrak{B}^{\kappa}_{\cdot\lambda\mu} = \rho^{-(n+1)\,\mathfrak{k}-1} \mathfrak{B}^{\kappa}_{\cdot\lambda\mu}.$$

Obviously, all of the objects 1′-6′ are special cases of projector-densities. The *intermediary components* are deduced from them as in X_4. As before, the $\mathcal{A}^{\kappa'}_{\kappa}$ and $\mathcal{A}^{\kappa}_{\kappa'}$ are the intermediary components of the identity projector:

$$\mathcal{A}^{\kappa}_{\lambda} \triangleq \delta^{\kappa}_{\lambda}. \tag{9}$$

In that presentation, we almost always employ projector-densities for which the excess, which is defined (for holonomic systems) by: excess = degree – contravariant valence + covariant valence + $\mathfrak{k}$ $(n + 1)$, is equal to zero (2).

Each of the objects 2′-6′ that are defined at a point of H_4 [i.e., along a curve of the system ($x^\kappa = \mu\, c^\kappa$) in X_5] is represented by a marked geometric figure in the local P_4 at that point. For example, the marked point v^κ is represented by the marked point $X^\kappa = v^\kappa$, and if one introduces the hyperplane coordinates U_λ into P_4 (which are well-known from ordinary projective geometry) then w_λ will be represented by the marked hyperplane $U_\lambda = w_\lambda$. It is remarkable to conform that *the x^κ themselves are the components of a marked point of* P_4, since, from EULER's theorem, one has:

$$x^{\kappa'} = \mathcal{A}^{\kappa'}_{\kappa} x^{\kappa}. \tag{10}$$

That property has no correspondent or analogue in the geometry of X_4. In addition, the *linear element dx^κ is not a marked point of the local* P_4, since the transformation of $\mathfrak{F}$:

$$d\,'x^{\kappa} = \rho\,(dx^{\kappa} + x^{\kappa}\, d \log \rho) \tag{11}$$

(1) A "*P*-density" of weight one is nothing but the abbreviated expressions for an alternating covariant projector of valence *n*.

(2) The coordinate points and hyperplanes that are defined on pages 6 and 8 define the sole exception to that rule.

is more complicated than that of a marked point. We use the relation (10) in order to identify a point $\lfloor \underset{\mathrm{o}}{x}^{\kappa} \rfloor$ of H_4 with the point $\lfloor X^{\kappa} \rfloor = \lfloor \underset{\mathrm{o}}{x}^{\kappa} \rfloor$ of the local P_4 at $\underset{\mathrm{o}}{x}^{\kappa}$, which we call the *contact point* of P_4 . The linear element dx^{κ} at $\underset{\mathrm{o}}{x}^{\kappa}$ uniquely fixes a direction in P_4 that passes through the contact point. One will then have a bijective representation of the directions that pass through $\underset{\mathrm{o}}{x}^{\kappa}$ in H_4 and the directions in P_4 that pass through the contact point. Meanwhile, the point $\underset{\mathrm{o}}{x}^{\kappa} + dx^{\kappa}$ is not represented by a fixed point in P_4, but by a variable point that depends upon transformations of $\mathfrak{F}$, and displaces along the line that passes through the contact point that is determined by the preceding direction.

If one uses the contact point then one can now represent the objects 2′-6′ (whose excesses equal zero) by means of a figure that is composed of points (and not by *marked* points) in P_4 ([1]). For example, the marked point v^{κ} can be represented by the set of two *points:*

(12)
$$\lfloor x^{\kappa} \rfloor, \qquad \lfloor x^{\kappa} + v^{\kappa} \rfloor,$$

and the marked hyperplane w_{λ} , by the homography:

(13)
$$X^{\kappa} \to (A_{\lambda}^{\kappa} - x^{\kappa} w_{\lambda}) X^{\lambda} .$$

One will obtain a simple, purely-geometric representation in the local P_4 of all the objects in 2′-6′ by that method.

At each point of H_4, there are five marked points $\underset{\lambda}{e}^{\kappa}$ and five marked hyperplanes $\overset{\kappa}{e}_{\lambda}$, which belong to the coordinate system x^{κ} and are defined by the equations:

(14)
$$\underset{\lambda}{e}^{\kappa} \triangleq \delta_{\lambda}^{\nu}, \qquad \overset{\kappa}{e}_{\lambda} \triangleq \delta_{\lambda}^{\kappa} .$$

They are the *coordinate points* and *hyperplanes* ([2]). They constitute the *local frame* of x^{κ} in the local P_4 . Each system of coordinates x^{κ}, $x^{\kappa'}$, etc. has its proper system of local frames, which will be denoted in the text by (κ), (κ'), etc.

There exist geometric objects that carry indices h, i, …, as well as indices κ, λ, … They are the *junction objects*, which belong to X_4, as well as to H_4 . The simplest example is the projector-affinor:

(15)
$$g_{\lambda}^{h} = \partial_{\lambda}\, \xi^{h} .$$

One can deduce an invariant quantity $\mathfrak{a}$ from a given $\mathcal{E}_{\lambda}^{h}$ by means of the equation:

([1]) Cf., D. v. DANTZIG, *loc. cit.*, 1934.
([2]) Their excesses are not equal to zero.

(16) $$\mathcal{E}^{[1}_{[0} \cdots \mathcal{E}^{4]}_{3} \mathcal{A}^{\kappa}_{4]} = \mathfrak{a}\, x^{\kappa},$$

which is a "*p*-density" of weight + 1 and an "*A*-density" of weight – 1. One then deduces that the *P*-densities and the *A*-densities have *the same geometric significance.*

The object uniquely affixes a contravariant vector in E_4 :

(17) $$'v^h = \mathcal{E}^h_\lambda\, v^\lambda$$

to each marked point $\mathcal{E}^h_\lambda$ of P_4 and a marked hyperplane of P_4:

(18) $$'w_\lambda = \mathcal{E}^h_\lambda\, w_h$$

to each covariant vector of E_4 , but it does not determine a bijective representation.

ANHOLONOMIC FRAMES IN X_4 . – If one is given vector fields v^h, w_i in X_4 then one can introduce other components by the equations:

(19) $$v^{h'} = A^{h'}_h v^h, \qquad w_{i'} = A^i_{i'} w_i, \qquad \det(A^{h'}_h) \neq 0, \quad A^h_{j'} A^{j'}_j = A^h_i,$$

in which the 16 parameters $A^{h'}_h$ are arbitrary functions of the ξ^h. In general, the new components of $d\xi^h$:

(20) $$(d\xi)^{h'} = A^{h'}_h d\xi^h$$

are not exact differentials of the *n* functions $\xi^{h'}$. In order for this exceptional case to present itself, it is necessary and sufficient that the $\overset{A}{\Omega}{}^{h'}_{j'i'}$ that are defined by the equations:

(21) $$\overset{A}{\Omega}{}^{h'}_{j'i'} \triangleq A^{ji}_{j'i'}\, \partial_{[j} A^{h'}_{i]} \qquad [(h) \text{ holonomic}]$$

must be identically zero.

$\overset{A}{\Omega}{}^{h'}_{j'i'}$ is called the *object of affine anholonomy*. If the $(d\xi)^{h'}$ are deduced from a coordinate system $\xi^{h'}$ then the frame (h') that is defined by:

(22) $$\underset{i'}{e}^{h'} \triangleq \delta^{h'}_{i'}, \qquad \overset{h'}{e}_{i'} \triangleq \delta^{h'}_{i'}$$

will be called *holonomic*, and in the contrary case, *anholonomic*. Hence, the components of the object $\overset{A}{\Omega}$ are zero for each holonomic frame, and there are non-zero components for each anholonomic frame. In physics, after introducing a metric in X_4, one generally prefers to utilize a system of anholonomic frames that is composed of real vectors that are orthogonal and have length or duration equal to 1.

ANHOLONOMIC FRAMES IN H_4. – In the same fashion, one can introduce other components of the fields of marked points v^κ and marked hyperplanes w_λ in H_4 with the equations:

(23) $$v^{\kappa'} = \mathcal{A}_\kappa^{\kappa'} v^\kappa, \quad w_{\lambda'} = \mathcal{A}_{\lambda'}^{\lambda} w_\lambda, \quad \det(\mathcal{A}_{\lambda'}^{\lambda}) \neq 0, \quad \mathcal{A}_{\mu'}^{\kappa} A_{\lambda}^{\mu'} = A_\lambda^\kappa,$$

in which the $\mathcal{A}_\kappa^{\kappa'}$ are chosen arbitrarily and have degree $-\mathfrak{g}$ in x^κ. *The new components* $v^{\kappa'}$ *have degree* $(1-\mathfrak{g})$. Since the marked points of the new frame:

(24) $$\underset{\lambda'}{e}^{\kappa'} \triangleq \delta_{\lambda'}^{\kappa'}, \qquad \overset{\kappa'}{e}_{\lambda'} \triangleq \delta_{\lambda'}^{\kappa'}$$

have components with respect to the system (κ) that have degrees equal to $+\mathfrak{g}$ or $-\mathfrak{g}$, respectively:

(25) $$\underset{\lambda'}{e}^{\kappa'} \triangleq \mathcal{A}_{\lambda'}^{\kappa'}, \qquad \overset{\kappa'}{e}_{\lambda'} \triangleq \mathcal{A}_{\lambda'}^{\kappa'},$$

we call the system of frames (κ') *an anholonomic system of degree* $\mathfrak{g}$ (1).

In order for the system of frames that is introduced to be *holonomic* – i.e., in order for the $(dx)^{\kappa'}$:

(26) $$(dx)^{\kappa'} = \mathcal{A}_\kappa^{\kappa'} dx^\kappa$$

to be the differentials of the $(n+1)$ functions $x^{\kappa'} = \mathcal{A}_\kappa^{\kappa'} x^\kappa$ – it is necessary that:

(27) $$\overset{P}{\Omega}{}_{\mu'\lambda'}^{\kappa'} \triangleq \mathcal{A}_{\mu'\lambda'}^{\mu\lambda} \partial_{[\mu} \mathcal{A}_{\lambda]}^{\kappa'} \triangleq 0 \qquad [(\kappa) \text{ holonomic}]$$

$\overset{P}{\Omega}{}_{\mu'\lambda'}^{\kappa'}$ is the *projective object of holonomity* (2). However, that condition *does not suffice*, since:

(28) $$\begin{aligned} \mathcal{A}_\kappa^{\kappa'} dx^\kappa &= d(\mathcal{A}_\kappa^{\kappa'} x^\kappa) - x^\kappa dx^\mu \partial_\mu \mathcal{A}_\kappa^{\kappa'} \\ &= d(\mathcal{A}_\kappa^{\kappa'} x^\kappa) - dx^\kappa x^\mu \partial_\kappa \mathcal{A}_\mu^{\kappa'}, \end{aligned}$$

or

(29) $$(1-\mathfrak{g})\,\mathcal{A}_\kappa^{\kappa'} dx^\kappa = d(\mathcal{A}_\kappa^{\kappa'} x^\kappa).$$

One must add the condition $\mathfrak{g} = 0$, *from which, it will follow that the frames of degree* $\neq 0$ *are always anholonomic.* That result is very important, since the use of anholonomic projective frames in projective relativity cannot be avoided without great inconvenience as a result.

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(1) Consequently, for an anholonomic system, one will have:

excess = degree + $(1-\mathfrak{g})$ [– contravariant vector + covariant vector + $\mathfrak{k}\,(n+1)$].

(2) In G. F. III-VI, we wrote $\Omega_{\lambda\mu}^{\kappa}$, instead of $\Omega_{\mu\lambda}^{\kappa}$.

INTRODUCTION OF A HYPERPLANE INTO THE LOCAL P_4 . – When one gives a field of hyperplanes in the local space, one can fix a bijective representation of the local P_4 on the local E_4 by the condition that the given hyperplane must be represented by the hyperplane at infinity.

Let:

$$q_\lambda X^\lambda = 0 \tag{30a}$$

be the equation of the hyperplane, and let:

$$q_\lambda X^\lambda = -\chi, \quad \chi = \text{real and constant.} \tag{30b}$$

The most convenient manner of finding the analytic expression for the representation consists of introducing a system of anholonomic frames (a), (a, b, …, g = 0, 1, 2, 3, 4) in H_4 and adding a system of frames (h) (h, …, m = 1, 2, 3, 4) that are holonomic or anholonomic in X_4 and are determined by the equations:

$$\begin{cases} \mathcal{A}_\lambda^h \triangleq \mathcal{E}_\lambda^h, \\ \mathcal{A}_\lambda^h \triangleq -q_\lambda. \end{cases} \tag{31}$$

The system (a) has degree + 1. Consequently, the degree of the components v^a, w_b of a marked point or a marked hyperplane whose excess is equal to zero *is zero*. The $\mathcal{A}_b^\kappa$ are found by solving the equations:

$$\begin{cases} \mathcal{A}_i^\kappa \mathcal{A}_\kappa^h \triangleq A_\lambda^h, \quad \mathcal{A}_i^\kappa q_\kappa = 0, \\ \mathcal{A}_0^h \triangleq \chi^{-1} x^\kappa. \end{cases} \tag{32}$$

They satisfy:

$$\mathcal{A}_b^\kappa \mathcal{A}_\lambda^b = A_\lambda^\kappa, \qquad \mathcal{A}_\lambda^a \mathcal{A}_b^\lambda = \mathcal{A}_b^a \triangleq \delta_b^a. \tag{33}$$

The components of $\overset{P}{\Omega}$ with respect to (a) are:

$$\overset{P}{\Omega}{}_{cb}^{\;a} \triangleq -\chi^{-1} q_{cb}\, x^a + \mathcal{A}_{cbh}^{jia} \overset{A}{\Omega}{}_{ji}^{\;h}, \qquad q_{\mu\lambda} = \partial_{[\mu}\, q_{\lambda]}\,. \tag{34}$$

The vector v^h of E_4 corresponds in a bijective manner to the marked point:

$$v^\kappa = \mathcal{A}_h^\kappa v^h \tag{35}$$

on the hyperplane $\lfloor q_\lambda \rfloor$, and that point will have components v^h, $v^0 = 0$ with respect to the system (a). The set of *contravariant vectors* of E_4 is then represented by the *marked points* that are situated on $\lfloor q_\lambda \rfloor$ in P_4 . Similarly, the vector w_i of E_4 corresponds bijectively to the marked hyperplane:

$$w_\lambda = \mathcal{A}_\lambda^i w_i \tag{36}$$

pass through the contact point and have components w_i , $w_0 = 0$ with respect to the system (a). The set of *covariant vectors* of E_4 is then represented by the marked hyperplanes that pass through the contact point. The identity affinor A_i^h corresponds bijectively to the projector:

(37)
$$A_\lambda^\kappa = \mathcal{A}_j^\kappa \mathcal{A}_\lambda^j = \mathcal{A}_\lambda^\kappa + \chi^{-1} q_\lambda x^\kappa,$$

which has the components:

(38)
$$A_i^h \triangleq \delta_i^h , \qquad A_0^h \triangleq 0 , \qquad A_i^0 \triangleq 0 , \qquad A_0^0 \triangleq 0$$

with respect to the system (a).

One must further deduce a bijective representation of the *points* of P_4 on the *points* of E_4 . One obtains it by remarking that each vector of E_4 is the radius vector of a well-defined point of E_4 and that each marked point of P_4 on $\lfloor q_\lambda \rfloor$ is the difference of a well-defined marked point of P_4 and x^κ. Call:

(39)
$$p_0 = - q_\lambda p^\lambda$$

the *weight* (1) (from MÖBIUS) of the marked point p^κ; the weight of x^κ will be χ. Now represent the point of E_4 that has v^h for its radius vector by the point $\lfloor p^\kappa \rfloor$ of P_4 that is given by the equation:

(40)
$$\frac{p^\kappa}{p^0} - \frac{x^\kappa}{\chi} = \mathcal{A}_h^\kappa v^h ,$$

in which a marked point of $\lfloor q_\lambda \rfloor$ is written in the form of the difference of two marked points with weight one. If that representation has been fixed then we can *identify* P_4 and E_4, and the addition, as well as subtraction, of points, will reduce to simply the well-known MÖBIUS calculation. It goes without saying that from that identification, the affinors are nothing but the special projectors, namely, the projectors whose components of index zero are all zero with respect to the system (a). In other words, the projectors that are annulled by each transvection with x^κ or q_λ , or furthermore, the projectors that do not change under a transvection with $\mathcal{A}_h^\kappa$. One deduces from that remark that one can now write all of the equations in the affinors of H_4 in terms of either affine frames that have the indices $h, \ldots, m$ or in terms of projective frames that have the indices $\kappa, \ldots, \tau$ or $a, \ldots, g$. The result of the transvection of a projector with the $\mathcal{A}_\lambda^\kappa$ on each index, for example:

(41)
$$P'^{\cdot\lambda\mu}_{\kappa} = \mathcal{A}^{\rho\lambda\mu}_{\kappa\sigma\tau} P^{\cdot\sigma\tau}_{\rho} ,$$

is an *affinor* that one called the *affinorial part* of $P^{\cdot\lambda\mu}_{\kappa}$.

(1) That “weight” is nothing but its weight as a density.

INTRODUCTION OF A QUADRIC INTO THE LOCAL P_4 . – Introduce a non-degenerate quadric into the local P_4 by means of the equations:

(42*a*) $$\mathcal{G}_{\lambda\kappa} X^{\lambda} X^{\kappa} = 0$$

and let [1]:

(42*b*) $$\mathcal{G}_{\lambda\kappa} x^{\lambda} x^{\kappa} = -\chi^2, \qquad \mathfrak{G} = \det(\mathcal{G}_{\lambda\kappa}) \neq 0,$$

in which χ^2 is a positive constant [1]. $\mathcal{G}_{\lambda\kappa}$ is called the *fundamental projector*. One can raise and lower the indices by means of $\mathcal{G}_{\lambda\kappa}$ and $\mathcal{G}^{\lambda\kappa}$, which is defined by:

(43) $$\mathcal{G}_{\lambda\mu} \mathcal{G}^{\mu\kappa} = \mathcal{A}_{\lambda}^{\kappa},$$

just as in Riemannian geometry. Introduce the marked point:

(44) $$q^{\lambda} = -\chi x^{\lambda},$$

in such a way that:

(45) $$\begin{cases} \mathcal{G}_{\lambda\kappa} q^{\lambda} q^{\kappa} = -1, \\ \quad q_{\lambda} x^{\lambda} = -\chi, \quad q_{\lambda} q^{\lambda} = -1, \quad q^0 = +1. \end{cases}$$

One introduces the hyperplane at infinity with the aid of q_{λ} , and P_4 is identified with E_4, as we have just explained.

If the point $\lfloor n^{\kappa} \rfloor$ is situated on the quadric then one will have:

(46) $$\left(\frac{n^{\lambda}}{n^0} - q^{\lambda}\right)\left(\frac{n^{\kappa}}{n^0} - q^{\kappa}\right) \mathcal{G}_{\lambda\mu} = -\frac{2}{n^0} n^{\lambda} q_{\lambda} - 1 = +1;$$

The quadric then defines a Euclidian metric for vectors, for which it takes the form of a "sphere" of radius equal to + 1. The intersection of the quadric with the hyperplane $\lfloor q_{\lambda} \rfloor$ is the isotropic "sphere" at infinity.

One can deduce two tensors of rank four from the projector $\mathcal{G}_{\lambda\mu}$:

(47) $$\begin{cases} g_{\lambda\kappa} = \mathcal{G}_{\lambda\kappa} + q_{\lambda} q_{\kappa}, \\ g^{\lambda\kappa} = \mathcal{G}^{\lambda\kappa} + q^{\lambda} q^{\kappa}, \end{cases}$$

which are called the *fundamental tensors*, and which determine the metric of the vectors. One deduces from (47) that:

(48) $$g_{\lambda\rho} g^{\rho\kappa} = A_{\lambda}^{\kappa} = \mathcal{A}_{\lambda}^{\kappa} + q_{\lambda} q^{\kappa}.$$

---

[1] In G. F., III-VI, we have written $\mathcal{G}_{\lambda\kappa} x^{\lambda} x^{\kappa} = -\omega^2$ and left the sign of ω^2 undetermined.

It is obvious that $g_{\lambda\kappa}$ gives rise to a Riemannian geometry in H_4, which we assume to be identified with the Riemannian geometry of space-time. We assume that the signature is $---+$ ([1]) for the latter geometry. The local E_4 becomes an R_4 ; i.e., The components of $\mathcal{G}_{\lambda\mu}$ and $g_{\lambda\mu}$ with respect to the system (a) satisfy the equations:

$$(49)\qquad \begin{cases} \mathcal{G}_{00} = -1, & g_{00} = 0, \\ \mathcal{G}_{0i} = \mathcal{G}_{i0} = 0, & g_{0i} = g_{j0} = 0, \\ \mathcal{G}_{ih} = g_{ih}. \end{cases}$$

The components g_{ih} depend upon the choice of the system (h). If one takes an anholonomic system in X_4 with vectors that have real coordinates and length (or duration) 1 then one will have:

$$(50)\qquad \begin{cases} \mathcal{G}_{11} = g_{11} = \mathcal{G}_{22} = g_{22} = \mathcal{G}_{33} = g_{33} = -1, \\ \mathcal{G}_{44} = g_{44} = +1. \end{cases}$$

Hence, $\mathcal{G}_{\lambda\kappa}$ has signature $----+$ ([2]), and that signature will be invariant under real transformations of the coordinates. The quadric is a "sphere" with a radius of length equal to 1 in the local R_4, which signifies that a real line that passes through the contact point will cut the sphere at two real points when the line can be taken to be the coordinate axis in space, and at two complex points when it can be taken to be the time axis.

PROJECTIVE CONNECTIONS IN H_4 . – The ordinary derivatives $\partial_\mu\, v^\kappa$ with respect to a holonomic or anholonomic system ([3]) do not form the components of a projector. Meanwhile, as in the affine case, one can introduce a covariant derivative by the equation ([4]):

$$(51)\qquad \begin{cases} \nabla_\mu v^\kappa = \partial_\mu v^\kappa + \Pi^\kappa_{\mu\lambda} v^\lambda, \\ \nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Pi^\kappa_{\mu\lambda} w_\kappa, \end{cases}$$

with the following condition for the transformation of the geometric object $\Pi^\kappa_{\mu\lambda}$:

([1]) In G. F. III-VI, we supposed that the signature was $+++-$. We remark that $\mathcal{G}_{\kappa\lambda}$ can be replaced with $g_{\kappa\lambda}$ in equation (46), because:

$$\left(\frac{n^\kappa}{n^0} - q^\kappa\right) q_\kappa = 0.$$

([2]) $+---+$ for $\chi^2 < 0$.

([3]) When (κ) is anholonomic and (κ') is holonomic, ∂_μ is defined by $\partial_\mu = \mathcal{A}^{\mu'}_\mu \partial_{\mu'}$.

([4]) We shall write $\Pi^\kappa_{\mu\lambda}$, instead of $\Pi^\kappa_{\lambda\mu}$, as we did in G. F. III-VI-VIII, in accord with the more modern notations that are employed in the book that is cited in footnote ([1]) on page 2. Hence, the $S_{\mu\lambda}^{\cdot\cdot\kappa}$ in (53) is written $S_{\lambda\mu}^{\cdot\cdot\kappa}$ in G. F. III-VI-VIII.

(52*a*) $\mathfrak{H}_5$: $\Pi_{\mu'\lambda'}^{\kappa'} = \mathcal{A}_{\kappa\mu'\lambda'}^{\kappa'\mu\lambda}\,\Pi_{\mu\lambda}^{\kappa} + \mathcal{A}_{\kappa}^{\kappa'}\,\partial_{\mu'}\mathcal{A}_{\lambda'}^{\kappa}$,

which is a condition that is valid for all holonomic or anholonomic systems ([1]), and:

(52*b*) $\mathfrak{F}$: $'\Pi_{\mu\lambda}^{\kappa} = \rho^{-1}\,\Pi_{\mu\lambda}^{\kappa}$.

One sees that the excess of the covariant derivative of a projector whose excess is zero is likewise equal to zero. As in affine geometry:

$$S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} \equiv \Pi_{[\mu\lambda]}^{\;\;\kappa} + \overset{P}{\Omega}{}_{\mu\lambda}^{\;\;\kappa} \overset{h}{=} \Pi_{[\mu\lambda]}^{\;\;\kappa} \tag{53}$$

is a projector; when $S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = 0$, one says that the connection is *symmetric*. Meanwhile, we encounter two other projectors here that one does not find in affine geometry, namely:

$$\begin{cases} \mathcal{P}_{\cdots\lambda}^{\kappa} = x^{\mu}\Pi_{\mu\lambda}^{\kappa} + \mathcal{A}_{\lambda}^{\kappa}, \\ \mathcal{Q}_{\cdots\lambda}^{\kappa} = \Pi_{\mu\lambda}^{\kappa}x^{\lambda} + \mathcal{A}_{\lambda}^{\kappa} = \nabla_{\mu}x^{\kappa} = \chi\nabla_{\mu}q^{\kappa}. \end{cases} \tag{54}$$

Another remarkable difference is that, in general, there is no *covariant differential*, because dx^{κ} is not a marked point, and in turn, $dx^{\mu}\nabla_{\mu}P_{\cdots\nu}^{\kappa\lambda}$, for example, will not be a projector, in general. Since the covariant differential does not exist, there is no longer parallel displacement (or pseudo-parallelism), because that displacement is defined precisely by annulling the covariant differential. One can prove that the covariant differential exists only when $\mathcal{P}_{\cdots\lambda}^{\kappa} = 0$, but that case is discarded in projective relativity, because the projector $\mathcal{P}_{\cdots\lambda}^{\kappa}$ cannot be zero. On the contrary, it plays a very important role in all of the theory.

Nevertheless, there exist *geodesics*. Recall that dx^{κ} fixes a direction in the local P_4, but does not correspond to a fixed point in the line that passes through the contact point in that direction. It is always possible then to define a covariant differential in a given direction, up to a constant factor, when one knows a particular point along the line besides $\lfloor x^{\kappa} \rfloor$. Let $\lfloor r^{\kappa} \rfloor$ be the point in question; dx^{κ} can be written in only one manner, in the form:

$$dx^{\kappa} = \varepsilon\, x^{\kappa} + \eta\, r^{\kappa}, \tag{55}$$

in which ε and η are infinitesimals that are not scalars, but transform in a well-defined manner for the transformations of the group $\mathfrak{F}$. One can then take the component $\eta\, r^{\kappa}$ to

([1]) Since anholonomic frames will be employed most frequently, we shall give most of the formulas in their general form here, in such a fashion that they can also be employed in the anholonomic case. In the formulas that are valid only for holonomic systems, one will write $\overset{h}{=}$, instead of =.

be the linear element and define pseudo-parallelism with respect to the point $\lfloor r^\kappa \rfloor$ by the covariant relation:

$$r^{\kappa} \nabla_{\mu} p^{\kappa} = 0. \tag{56}$$

Hence, a projective connection does not determine just one parallel displacement for each direction, but an infinitude of displacements that depend upon the choice of point $\lfloor r^\kappa \rfloor$. Thus, if p^κ is a marked point in the local P_4 of a given $\lfloor \underset{\circ}{x}^\kappa \rfloor$ then one will necessarily know the point $\lfloor p^\kappa \rfloor$ itself, and one can define the geodesic that starts from $\lfloor \underset{\circ}{x}^\kappa \rfloor$ in the direction of $\lfloor x^\kappa \rfloor$ to $\lfloor p^\kappa \rfloor$ by the equation ([1]):

$$p^{\mu} \nabla_{\mu} p^{\kappa} = 0. \tag{57}$$

Indeed, one deduces the value $p^\kappa + dp^\kappa$ at $\underset{\circ}{x} + dx^\kappa$ from that equation:

$$\begin{cases} dp^{\kappa} = dx^{\mu}\partial_{\mu} p^{\kappa} = \varepsilon\, x^{\mu}\partial_{\mu} p^{\kappa} + \eta\, p^{\mu}\partial_{\mu} p^{\kappa} \\ \qquad\qquad\qquad\; = \varepsilon\, p^{\kappa} - \eta\, p^{\mu} p^{\lambda} \Pi^{\kappa}_{\mu\lambda}, \end{cases} \tag{58}$$

and that process, when repeated *ad infinitum*, will successively yield all of the elements of the desired geodesic. The geodesics thus-obtained are called *auto-geodesics*, since their construction is based upon the use of the point $\lfloor p^\kappa \rfloor$ itself. It goes without saying that ∞^1 auto-geodesics pass through each given point in each direction, and that the form of a such a curve depends upon only the situation of the point $\lfloor p^\kappa \rfloor$ in the local P_4 at $\underset{\circ}{x}^\kappa$.

When one is given a quadric in the local P_4, one can take the point r^κ on the polar hyperplane $\lfloor q_\kappa \rfloor$ to $\lfloor x^\kappa \rfloor$.

Since:

$$\begin{cases} (dx)^{\kappa} = A^{\kappa}_{\lambda}(dx)^{\lambda} - q_{\lambda}(dx)^{\lambda} q^{\kappa} \\ \qquad\quad = \mathcal{A}^{\kappa}_{\lambda}\mathcal{E}^{h}_{\lambda}(dx)^{\lambda} - q_{\lambda}(dx)^{\lambda} q^{\kappa} \\ \qquad\quad = \mathcal{A}^{\kappa}_{h}(dx)^{h} - q_{\lambda}(dx)^{\lambda} q^{\kappa}, \end{cases} \tag{59}$$

one can take the linear vectorial element $(d\xi)^h$ in X_4, which is written:

$$(d\,\acute{x})^{\kappa} = \mathcal{E}^{\kappa}_{h}(d\xi)^{h} \tag{60}$$

in homogeneous coordinates, instead of $(dx)^\kappa$, and the geodesic equation becomes:

---

[1] D. v. Dantzig, *loc. cit.*, Math. Ann. (1932) called a field $\lfloor p^\kappa \rfloor$ that satisfies equation (57) a "geodetic position field," and geodesics that are defined by [(57), *loc. cit.*, Amst. Proc. (1932), pp. 532] "pseudo-geodetic lines."

(61) $$(d\,\acute{x})^{\mu}\,\nabla_{\mu}\,p^{\mu} = 0.$$

We call those curves *induced geodesics* because they are, in some way, "induced" by the quadric ([1]). The other geodesics – viz., the auto-geodesics – always exist, even in the case in which the connection does not even depend upon a quadric.

RIEMANNIAN CONNECTIONS IN PROJECTIVE COORDINATES. – Introduce a fundamental projector $\mathcal{G}_{\lambda\mu}$, and suppose that one has deduced the fundamental tensor $g_{\lambda\mu}$, as well as the components g_{ij} with respect to the system (a) that is adjoint to the systems of holonomic or anholonomic frames (h) of X_4 . With respect to the system (h), the Riemannian connection of the fundamental tensor g_{ih} is given by the equations ([2]):

(62) $$\begin{cases} \overset{R}{\nabla}_j\, w_i = \partial_j w_i - \Gamma^h_{ji} w_h, \\ \overset{R}{\nabla}_j\, v^h = \partial_j v^h + \Gamma^h_{ji} v^i, \end{cases}$$

in which:

(63) $$\Gamma^h_{ji} = \tfrac{1}{2} g^{hk}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ji}) - \overset{A}{\Omega}{}^{\,h}_{ji} + g_{jk}\,\overset{A}{\Omega}{}^{\,k}_{il} + g_{ik}\,g^{hl}\,\overset{A}{\Omega}{}^{\,k}_{jl}.$$

That connection is a connection on *vectors;* i.e., on marked points that are situated in $\lfloor q_{\lambda} \rfloor$.

We deduce a projective connection from it that we denote by $\overset{R}{\nabla}_{\mu}$ by demanding that the covariant derivatives of the vectors v^{κ} and w_{λ} must be identical to the Riemannian derivative. With respect to the (a), one will have:

(64) $$\begin{cases} \overset{R}{\nabla}_j\, v^h \triangleq \partial_j v^h + \overset{R}{\Pi}{}^{\,h}_{ji} v^i = \partial_j v^h + \Gamma^h_{ji} v^i, \\ \overset{R}{\nabla}_0\, v^h \triangleq q^b \partial_b v^h + \overset{R}{\Pi}{}^{\,h}_{0i} v^i = \overset{R}{\Pi}{}^{\,h}_{0i} v^i = 0, \\ \overset{R}{\nabla}_j\, v^0 \triangleq \overset{R}{\Pi}{}^{\,0}_{ji} v^i = 0, \\ \overset{R}{\nabla}_0\, v^0 \triangleq \overset{R}{\Pi}{}^{\,0}_{0j} v^j = 0, \end{cases}$$

and analogous equations for $\overset{R}{\nabla}_c\, w_b$, so one will deduce that:

([1]) Those are the geodesics that used in G. F. III and G. F. VI, and which also appear in the communications of O. Veblen (1933) and W. Pauli (1933). The auto-geodesics that we employ here are the same as the geodesics of the five-dimensional relativity theory of T. Kaluza and O. Klein.

([2]) When (h) is anholonomic and (h') is holonomic, ∂_j is defined by $\partial_j = A^{j'}_j \partial_{j'}$.

(65)
$$\begin{cases} \overset{R}{\Pi}{}^{h}_{ji} \triangleq \Gamma^{h}_{ji}, & \overset{R}{\Pi}{}^{h}_{0i} \triangleq 0, \quad \overset{R}{\Pi}{}^{h}_{j0} \triangleq 0, \\ \overset{R}{\Pi}{}^{0}_{ji} \triangleq 0, & \overset{R}{\Pi}{}^{h}_{00} \triangleq 0, \quad \overset{R}{\Pi}{}^{h}_{0i} \triangleq 0. \end{cases}$$

One sees that $\overset{R}{\Pi}{}^{0}_{00}$ and $\overset{R}{\Pi}{}^{0}_{j0}$ are not further determined. Nevertheless, (34) and (53) show that:

(66)
$$\overset{R}{S}{}_{ji}^{\cdot\cdot\cdot 0} \triangleq -q_{ji}, \qquad q_{\lambda\mu} = \partial_{[\mu}\, q_{\lambda]}\ ;$$

i.e., that the Riemannian connection in homogeneous coordinates *can never be symmetric when $q_{\lambda\mu}$ is non-zero.* In order to fix the undetermined parameters, it suffices to give $\overset{R}{\nabla}_{\mu}\, q^{\kappa}$ or $\overset{R}{\nabla}_{\mu}\, q_{\lambda}$. If:

(67)
$$\overset{R}{\nabla}_{\mu}\, q^{\kappa} = \mathcal{V}_{\mu}^{\cdot\cdot\kappa}, \quad \overset{R}{\nabla}_{\mu}\, q_{\lambda} = \mathcal{U}_{\mu\lambda}$$

then one will have:

(68)
$$\begin{cases} \overset{R}{\Pi}{}^{0}_{00} \triangleq \mathcal{V}_{0}^{\cdot 0} \triangleq \mathcal{U}_{00}, \\ \overset{R}{\Pi}{}^{0}_{j0} \triangleq \mathcal{V}_{j}^{\cdot 0} \triangleq \mathcal{U}_{j0}, \\ \mathcal{U}_{c}^{\cdot h} \triangleq 0, \quad \mathcal{V}_{ci} \triangleq 0. \end{cases}$$

Now adopt the simplest hypothesis, under which:

(69)
$$\overset{R}{\nabla}_{\mu}\, q^{\kappa} = 0, \qquad \overset{R}{\nabla}_{\mu}\, q_{\lambda} = 0.$$

In this case:

(70)
$$\overset{R}{\Pi}{}^{a}_{cb} \triangleq \mathcal{A}^{jia}_{cbh}\, \Gamma^{h}_{ji},$$

so, upon applying formulas (34) and (53):

(71)
$$\begin{aligned} \overset{R}{S}{}_{cb}^{\cdot\cdot\cdot a} &\triangleq \mathcal{A}^{jia}_{cbh}\, \Gamma^{h}_{[ji]} - \chi^{-1} q_{cb}\, x^{a} + \mathcal{A}^{jia}_{cbh}\, \overset{A}{\Omega}{}^{h}_{ji} \\ &\triangleq -q_{cb}\, q^{a}, \end{aligned}$$

and thus, since the equation is invariant:

(72)
$$\overset{R}{S}{}_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = -q_{\mu\lambda}\, q^{\kappa}.$$

Naturally, the Riemannian connection must imply a covariant differential. The differential:

(73)
$$dx^{\mu}\, \overset{R}{\nabla}_{\mu}\, v^{\kappa} = dx^{\sigma} A^{\mu}_{\sigma}\, \overset{R}{\nabla}_{\mu}\, v^{\kappa},$$

is, in fact, covariant, because $x^{\mu}\overset{R}{\nabla}_{\mu}v^{\kappa}$ is annulled and $dx^{\sigma}A_{\sigma}^{\mu}$ transforms like a marked point under the group $\mathfrak{F}$:

(74) $$d'x^{\sigma}A_{\sigma}^{\mu} = \rho\,(dx^{\sigma} + x^{\sigma}\,d\log\rho)\,A_{\sigma}^{\mu} = \rho\,dx^{\sigma}\,A_{\sigma}^{\mu}.$$

The parallel displacement that one obtains by annulling the covariant differential leaves invariant, on the one hand, the quadric, since:

(75) $$\overset{R}{\nabla}_{\mu}\,\mathcal{G}_{\lambda\kappa} = 0,$$

and also, on the other hand, the contact point, due to (69), the hyperplane $\lfloor q_{\lambda}\rfloor$, and the isotropic sphere in $\lfloor q_{\lambda}\rfloor$.

One deduces from (75) by a well-known process [1] that:

(76) $$\overset{R}{\Pi}{}_{\mu\lambda}^{\kappa} = \begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix} + \overset{R}{S}{}_{\mu\lambda}^{\cdot\cdot\cdot\kappa} + \overset{R}{S}{}^{\kappa}_{\cdot\mu\lambda} + \overset{R}{S}{}^{\kappa}_{\cdot\lambda\mu},$$

in which $\begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix}$ is the CHRISTOFFEL relative to $\mathcal{G}_{\lambda\kappa}$. Consequently, one has, from (72):

(77) $$\overset{R}{\Pi}{}_{\mu\lambda}^{\kappa} = \begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix} - q_{\mu\lambda}\,q^{\kappa} - q^{\kappa}_{\cdot\lambda}\,q_{\mu} - q^{\kappa}_{\cdot\mu}\,q_{\lambda}.$$

PROJECTIVE CONNECTIONS DETERMINED BY THE FUNDAMENTAL QUADRIC. – One can demand that the quadric should determine other connections that are more general than the Riemannian connection. Naturally, in that case, one must impose the following condition:

I. – *The quadric is invariant under every displacement that is deduced from the connection, no matter what particular point is chosen along the line of* dx^{κ}.

In order for that condition to be satisfied, it is necessary and sufficient that:

(78) $$r^{\mu}\,\nabla_{\mu}\,\mathcal{G}_{\lambda\kappa} :: \mathcal{G}_{\lambda\kappa} \qquad (:: = \text{proportional to})$$

for each choice of r^{μ}; i.e., that:

(79) $$\nabla_{\mu}\,\mathcal{G}_{\lambda\kappa} = s_{\mu}\,\mathcal{G}_{\lambda\kappa},$$

(1) J. A. SCHOUTEN, *Der Ricci Kalkul* (1924), pp. 73. [See note (1) on page 13.]

in which s_μ is a marked hyperplane that is otherwise undetermined. One easily deduces from that equation that:

$$(80)\qquad \begin{cases} [1] \quad \mathcal{P}_{[\lambda\kappa]} = -\tfrac{1}{2} s \chi \mathcal{G}_{\lambda\kappa}, \\ [2] \quad q_\kappa \mathcal{Q}^{\kappa}_{\cdot\lambda} = \tfrac{1}{2} \chi s_\lambda, \\ s = -s^0 = q^\lambda s_\lambda \qquad \text{(by definition).} \end{cases}$$

Each projective connection ∇ fixes uniquely another connection $\overset{A}{\nabla}$, which is called the *induced affine connection*, and is such that when $\overset{A}{\nabla}$ is applied to an affinor, that will yield the affine part of the result of applying ∇ to the same affinor, and in addition, that $\overset{A}{\nabla}_\mu q^\kappa = 0$, $\overset{A}{\nabla}_\mu q_\lambda = 0$:

$$(81)\qquad \begin{cases} \overset{A}{\nabla}_\mu v^\kappa = A^{\tau\kappa}_{\mu\rho} \nabla_\tau v^\rho, \quad q_\rho v^\rho = 0, \\ \overset{A}{\nabla}_\mu w_\lambda = A^{\tau\sigma}_{\mu\rho} \nabla_\tau w_\sigma, \quad q^\rho v_\rho = 0. \end{cases}$$

Therefore, one deduces that:

$$(82)\qquad A^{\kappa\tau\sigma}_{\rho\mu\lambda} \overset{A}{S}{}_{\tau\rho}^{\cdot\cdot\cdot\rho} = A^{\tau\sigma\kappa}_{\mu\lambda\rho} S_{\tau\sigma}^{\cdot\cdot\cdot\rho},$$

which expresses the idea that the affinorial part of $S_{\mu\lambda}^{\cdot\cdot\cdot\kappa}$ is determined by $\overset{A}{S}{}_{\mu\lambda}^{\cdot\cdot\cdot\kappa}$.

It then follows from (53) and (54) that:

$$(83)\qquad q^\mu S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = \tfrac{1}{2} \chi^{-1} (\mathcal{P}^{\kappa}_{\cdot\lambda} - \mathcal{Q}^{\kappa}_{\cdot\lambda}).$$

Since one has:

$$(84)\qquad \partial_\mu q_\lambda - \Pi^{\kappa}_{\mu\lambda} q_\kappa = \nabla_\mu q_\lambda = \chi^{-1} \mathcal{Q}_{\lambda\mu} + q_\lambda s_\mu,$$

due to (54) and (79), one will have:

$$(85)\qquad S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} q_\kappa = q_{\mu\lambda} + \chi^{-1} \mathcal{Q}_{[\mu\lambda]} + q_{[\mu} s_{\lambda]}.$$

Upon combining (82), (83), and (85) suitably, one will deduce that:

$$(86)\qquad \begin{aligned} S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = {} & A^{\kappa\rho\sigma}_{\tau\mu\lambda} \overset{A}{S}{}_{\sigma\rho}^{\cdot\cdot\cdot\tau} - q^\kappa (q_{\mu\lambda} + \chi^{-1} \mathcal{Q}_{[\mu\lambda]} + q_{[\mu} s_{\lambda]}) \\ & - \chi^{-1} q_{[\mu} (\mathcal{P}^{\kappa}_{\cdot\lambda]} - \mathcal{Q}^{\kappa}_{\cdot\lambda]}) + \chi^{-1} q_{[\mu} \mathcal{Q}_{\lambda]\sigma} q^\sigma q^\kappa. \end{aligned}$$

Having done that, we impose the following condition:

II. – *The induced affine connection must be identical to the Riemannian connection of the* $g_{\lambda\kappa}$.

In order to do that, it is necessary that:

(87) $$0 = \overset{R}{\nabla}_\mu\, g_{\lambda\kappa} = A_{\mu\lambda\kappa}^{\tau\sigma\rho}\,\nabla_\tau\,(\mathcal{G}_{\sigma\rho} + q_\sigma\, q_\rho) = A_\mu^\tau\, s_\tau\, g_{\lambda\kappa}\,,$$

so it results that:

(88) $$s_\mu = -\, s\, q_\mu\,.$$

In addition, the first term of (86) is annulled due to the (72), and one will find:

(89) $$S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = -\,\chi^{-1} q_{[\mu}\,(\mathcal{P}_{\cdot\lambda]}^{\kappa} - \mathcal{Q}_{\cdot\lambda]}^{\kappa}) + q^\kappa\,[q_{\mu\lambda} + \chi^{-1}\mathcal{Q}_{[\mu\lambda]} + \chi^{-1} q_{[\mu}\mathcal{Q}_{\lambda]\sigma}\, q^\sigma]$$

and ([1]):

(90) $$\Pi_{\mu\lambda}^{\kappa} = \begin{Bmatrix}\kappa\\ \mu\,\lambda\end{Bmatrix} + S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} + S_{\cdot\mu\lambda}^{\kappa} + S_{\cdot\lambda\mu}^{\kappa} + \tfrac{1}{2}\, s\left(q_\mu \mathcal{A}_\lambda^\kappa + q_\lambda \mathcal{A}_\mu^\kappa - q^\kappa \mathcal{G}_{\lambda\mu}\right),$$

which is an equation that gives $\Pi_{\mu\lambda}^{\kappa}$ as a function of $\mathcal{G}_{\lambda\kappa}$, its derivatives, $\mathcal{P}_{\cdot\lambda}^{\kappa}$, $\mathcal{Q}_{\cdot\sigma}^{\kappa}$, q_λ, and $q_{\lambda\mu}$. One deduces from the two equations (77), (90) that:

(91) $$\Pi_{\mu\lambda}^{\kappa} - \overset{R}{\Pi}{}_{\mu\lambda}^{\kappa} = S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} + S_{\cdot\mu\lambda}^{\kappa} + S_{\cdot\lambda\mu}^{\kappa} + q_{\mu\lambda} q^\kappa + q_{\cdot\lambda}^{\kappa} q_\mu + q_{\cdot\mu}^{\kappa} q_\lambda + \tfrac{1}{2}\, s\left(q_\mu \mathcal{A}_\lambda^\kappa + q_\lambda \mathcal{A}_\mu^\kappa - q^\kappa \mathcal{G}_{\lambda\mu}\right),$$

which is a relation that is valid for any holonomic or anholonomic system.

The two geometric conditions that we have imposed do not suffice to determine the connection uniquely. The projectors $\mathcal{P}_{\cdot\lambda}^{\kappa}$ and $\mathcal{Q}_{\cdot\sigma}^{\kappa}$ must satisfy equations (80.1) and (80.2), but they are not determined completely by those equations.

II. Physical applications.

AUTO-GEODESICS AND TRAJECTORIES. – In this second part, we fix the projective connection by physical conditions, although it is otherwise undetermined. Naturally, the first condition to impose (which is present in spirit) is the following one:

III. – *The auto-geodesics are the trajectories in the space-time of electrically-charged particles. The duration of the radius vector from* $\lfloor x^\kappa \rfloor$ *to* $\lfloor p^\kappa \rfloor$ *depends upon only the quotient e/m and remains constant along the trajectory.*

Those trajectories are given in Riemannian geometry by the well-known equations:

(92) $$\frac{d}{d\tau}\frac{(d\xi)^h}{d\tau} + \Gamma_{ji}^h \frac{(d\xi)^j}{d\tau}\frac{(d\xi)^i}{d\tau} = -\frac{e}{mc}\mathcal{F}_j^{\cdot h}\frac{(d\xi)^j}{d\tau}\,,$$

$$c^2\, d\tau^2 = -\, dx^2 - dy^2 - dz^2 + c^2\, dt^2.$$

In the equation for the auto-geodesics:

([1]) *Der Ricci Kalkul*, pp. 75. [See ([1]) on page 13.]

(93) $$p^\mu \nabla_\mu p^\kappa = 0,$$

the marked point p^κ can be decomposed into a vector and a multiple of q^κ:

(94) $$p^\kappa = p^0 (l\, i^\kappa + q^\kappa), \qquad p^0 = -\, p^\kappa q_\kappa,$$

in which:

(95) $$i^\kappa = \frac{1}{c}\frac{(dx)^\kappa}{d\tau} = \frac{1}{c} A^\kappa_\lambda \frac{(dx)^\kappa}{d\tau} = \frac{1}{c}\mathcal{A}^\kappa_h \frac{(d\xi)^h}{d\tau}, \qquad q_\kappa i^\kappa = 0$$

is the well-known world-velocity vector ([1]), when written in homogeneous coordinates, p^0 is an undetermined coefficient, and l is the duration of the radius vector of p^κ ([2]), which must be constant along the trajectory. From (84) and (93), one will have ([3]):

(96) $$p^\kappa \nabla_\mu (p^0) = -\, p^\kappa p^\mu \nabla_\mu q_\kappa = -\,\chi^{-1} p^\kappa p^\mu \mathcal{Q}_{\lambda\mu} + s\,(p^0)^2.$$

Upon applying the operator $p^\mu \nabla_\mu$ to the equation:

(97) $$\mathcal{Q}_{\lambda\kappa} p^\lambda p^\kappa = (p^0)^2 (l^2 - 1),$$

one will get:

(98) $$0 = (p^0)^2\, l\, p^\mu \nabla_\mu l = p^0 (l^2 - 1)\, [\chi^{-1} \mathcal{Q}_{\mu\lambda} p^\mu p^\lambda - \tfrac{1}{2} s\,(p^0)^2],$$

which is a relation that will remain valid for each choice of l, and thus, the result that:

(99) $$p^\lambda p^\mu \mathcal{Q}_{\mu\lambda} = \tfrac{1}{2}\chi\, s\,(p^0)^2.$$

Upon substituting (94) in (93), because of (96), one will obtain:

(100) $$\begin{aligned} 0 &= p^\mu \nabla_\mu p^0 (l\, i^\kappa + q^\kappa) = p^0\, l\, p^\mu \nabla_\mu i^\kappa + \chi^{-1} p^0 \mathcal{Q}^{\cdot\kappa}_{\cdot\mu}\, p^\mu + p^\mu p^\kappa \nabla_\mu l\, g\,(p^0) \\ &= (p^0)^2\, l^2\, i^\mu \nabla_\mu i^\kappa + (p^0)^2\, l\, \chi^{-1} \mathcal{Q}^{\cdot\kappa}_{\cdot\lambda}\, i^\lambda + (p^0)^2\, l\, \chi^{-1} \mathcal{Q}^{\cdot\kappa}_{\cdot\lambda}\, i^\lambda, \\ &\quad + \chi^{-2} (p^0)^2\, b^\kappa + q^\kappa p^\mu \nabla_\mu (p^0) + \tfrac{1}{2}\, l\, s\,(p^0)^2\, i^\kappa. \end{aligned}$$

in which

(101) $$\begin{cases} b^\kappa = \chi \mathcal{Q}^{\cdot\kappa}_{\cdot\mu}\, q^\mu = \chi \mathcal{Q}^{\cdot\kappa}_{\cdot\mu}\, q^\mu \\ b^\mu q_\mu = \tfrac{1}{2}\chi^2 s \quad \text{[due to (80)].} \end{cases}$$

The transvection of the right-hand side of (100) with q_κ is identically zero due to (96); hence, (100) is equivalent to its vectorial part:

([1]) In G. F. III-VI, we have written $l / c\, i^\kappa$, instead of $l\, i^\kappa$; hence, $1 / c\, i^\kappa$ will be the velocity vector in that case.

([2]) l is a constant with no physical dimensions.

([3]) One must write $\nabla_\mu (p^0)$ for the covariant derivative of p^0, since $\nabla_\mu p^0$ has another significance, namely:

$$\nabla_\mu p^0 = \mathcal{A}^0_\kappa \nabla_\mu p^\kappa.$$

(102) $$0 = i^{\mu} \overset{R}{\nabla}_{\mu} i^{\kappa} + \frac{\chi^{-1}}{l} (\mathcal{P}'^{\kappa}_{\cdot\lambda} + \mathcal{Q}'^{\kappa}_{\cdot\lambda}) i^{\lambda} + \frac{\chi^{-2}}{l^2} b'^{\kappa} + \tfrac{1}{2} \frac{s}{l} i^{\kappa} ,$$

in which $\mathcal{P}'^{\kappa}_{\cdot\lambda}$, $\mathcal{Q}'^{\kappa}_{\cdot\lambda}$, and b'^{κ} are the affinorial parts of $\mathcal{P}^{\kappa}_{\cdot\lambda}$, $\mathcal{Q}^{\kappa}_{\cdot\lambda}$, and b^{κ}, resp.:

(103) $$\mathcal{P}'^{\kappa}_{\cdot\lambda} = A^{\kappa\sigma}_{\rho\lambda} \mathcal{P}^{\rho}_{\cdot\sigma} , \qquad \mathcal{Q}'^{\kappa}_{\cdot\lambda} = A^{\kappa\sigma}_{\rho\lambda} \mathcal{Q}^{\rho}_{\cdot\sigma} , \qquad b'^{\kappa} = A^{\kappa}_{\lambda} b^{\lambda} .$$

Equations (102), when written with respect to the system (*a*):

(104) $$0 = i^{j} \overset{R}{\nabla}_{j} i^{h} + \frac{\chi^{-1}}{l} (\mathcal{P}'^{h}_{\cdot j} + \mathcal{Q}'^{h}_{\cdot j}) i^{j} + \frac{\chi^{-2}}{l^2} b'^{h} + \tfrac{1}{2} \frac{s}{l} i^{h} ,$$

must be compared with the equations of the trajectories (92), which one can write in the form:

(105) $$i^{j} \overset{R}{\nabla}_{j} i^{h} = - \frac{e}{mc^2} \mathcal{F}_{j}^{\cdot h} i^{j} .$$

In order for these two equations to be exactly equivalent, it is necessary that b'^{κ} must be zero. It will then follow, upon taking into account (80) and (101), that:

(106) $$\begin{gathered} b^{\kappa} = -\tfrac{1}{2} \chi^2 s \, q^{\kappa}, \\ \mathcal{Q}_{ab} \, p^a p^b = (p^0)^2 l^2 \, \mathcal{Q}_{\kappa\lambda} \, i^{\kappa} i^{\lambda} + \tfrac{1}{2} (p^0)^2 \chi s, \end{gathered}$$

and consequently, by virtue of (99):

(107) $$\mathcal{Q}'_{(ij)} = 0,$$

so

(108) $$\frac{e}{mc^2} \mathcal{F}_{j}^{\cdot h} i^{j} = \frac{\chi^{-1}}{l} (\mathcal{P}'^{h}_{\cdot j} + \mathcal{Q}'^{h}_{\cdot j}) i^{j} + \tfrac{1}{2} \frac{s}{l} g_{ij} \, i^{i} i^{j} i^{h} ,$$

which is an equation that must be satisfied for any chosen i^h (1). It will then follow that:

(109) $$\frac{k}{c} \mathcal{F}_{ji} = - \chi^{-1} (\mathcal{P}'_{[ji]} + \mathcal{Q}'_{[ji]}) ,$$

which is a relation in which we have introduced the undetermined constant *k* by (2):

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(1) When one takes the induced geodesics, instead of the auto-geodesics, one will find that (G. F. VI, pp. 297):

$$\frac{h}{c} \mathcal{F}_{ji} = - \chi^{-1} \, \mathcal{Q}_{ji} .$$

(2) When one does not fix the sign of χ^2, one will see from (109), and also from the equation of footnote (1) above, that the product $k\chi$ is always real.

(110) $$l = k\frac{mc}{e}.$$

One easily deduces from (80), (101), and (107) that:

(111) $$\begin{cases} \mathcal{P}_{\lambda\kappa} = \mathcal{P}'_{[\lambda\kappa]} - \tfrac{1}{2}s\chi g_{\lambda\kappa} + \tfrac{1}{2}s\chi q_\lambda q_\kappa, \\ \mathcal{Q}_{\lambda\kappa} = \mathcal{Q}'_{[\lambda\kappa]} + \tfrac{1}{2}s\chi q_\lambda q_\kappa. \end{cases}$$

THE CONSERVATION OF ENERGY AND QUANTITY OF MOTION. – In the usual theory of relativity, the vector mci^h represents the kinetic energy and impulse; the potential energy and impulse are represented by the vector $(e/c)\varphi^h$, in which φ^h is the potential vector, which is determined up to an additive gradient vector. In the projective theory, a gradient vector is a very particular object. In general, a gradient is a marked hyperplane. One can then represent the total energy by a marked point, which is the sum of the vector $mci^\kappa + (e/c)\ \varphi^\kappa$ and a marked point that is an arbitrary gradient. It is possible to choose that gradient point in such a manner that the marked point of total energy coincides with the marked point p^κ, so the equation of the trajectories will be, at the same time, the equation of the conservation of total energy and impulse. We impose the condition that this identification must be possible, and we formulate it in the following manner:

IV. – *The marked point:*

(112) $$p^\kappa = p^0\,(li^\kappa + q^\kappa)$$

differs from the total energy vector:

(113) $$mci^\kappa + \frac{e}{c}\varphi^\kappa$$

only by a gradient.

It then follows, first of all, that:

(114) $$\begin{cases} p^0 = \dfrac{mc}{l}, \\ p^\kappa = mci^\kappa + \dfrac{e}{k}q^\kappa. \end{cases}$$

Hence, p^0 is constant, so, by virtue of (96) and (99):

(115) $$s = 0,$$

and by virtue of (111):

(116) $$\begin{cases} \mathcal{P}_{\lambda\kappa} = \mathcal{P}'_{[\lambda\kappa]}, \\ \mathcal{Q}_{\lambda\kappa} = \mathcal{Q}'_{[\lambda\kappa]}; \end{cases}$$

in other words, $\mathcal{P}_{\lambda\kappa}$ and $\mathcal{Q}_{\lambda\kappa}$ *are bivectors.*

Secondly, it is necessary that:

$$\mathcal{F}_{\lambda\kappa} = 2\,\partial_{[\mu}\,\varphi_{\lambda]} = 2\frac{m}{e}\frac{c^2}{l}\,\partial_{[\mu}\,q_{\lambda]} = 2\frac{c}{k}\,q_{\mu\lambda}\,, \tag{117}$$

and that condition will be fulfilled when:

$$\mathcal{P}_{\lambda\kappa} + \mathcal{Q}_{\lambda\kappa} = -\,2\chi\,q_{\mu\lambda} = -\chi\frac{k}{c}\,\mathcal{F}_{\lambda\kappa}\,. \tag{118}$$

Under those conditions, there is no longer anything mysterious about the marked points p^κ in the local spaces of the points of a trajectory. Those points are nothing but the geometric representation of total energy. The reason for the indeterminacy that the potential φ_λ is affected with appears clearly: It comes down to the fact that the true – i.e., well-defined – potential $(c/k)\ q_\lambda$ is not a vector, but a marked hyperplane, and that the affine theory, in which it is impossible to represent that hyperplane, can do no better than to resort to an undetermined vector φ_λ that differs from only by a gradient $(c/k)\ q_\lambda$.

THE TWO BIVECTORS $\mathcal{P}_{\kappa\lambda}$ AND $\mathcal{Q}_{\kappa\lambda}$. – We have seen that the two quantities $\mathcal{P}_{\lambda\kappa}$ and $\mathcal{Q}_{\lambda\kappa}$ are bivectors and that their sum is the electromagnetic bivector, up to a constant factor. Now, in physics, there is only one bivector, namely, the electromagnetic bivector $\mathcal{F}_{\kappa\lambda}$. One must then impose the following supplementary condition:

V. – *The bivectors $\mathcal{P}_{\kappa\lambda}$ and $\mathcal{Q}_{\kappa\lambda}$ differs from the electromagnetic bivector $\mathcal{F}_{\kappa\lambda}$ by only some constant factors:*

$$\begin{cases} \mathcal{P}_{\mu\lambda} = -\chi\,p\,q_{\mu\lambda} = -\tfrac{1}{2}\,p\,\dfrac{k}{c}\,\mathcal{F}_{\mu\lambda}, \\ \mathcal{Q}_{\mu\lambda} = -\chi q\,q_{\mu\lambda} = -\tfrac{1}{2}\,q\,\dfrac{k}{c}\,\mathcal{F}_{\mu\lambda}. \end{cases} \tag{119}$$

Upon substituting those values in (89) and (91), one will find that:

$$S_{\mu\lambda}^{\cdot\cdot\cdot\kappa} = (q-1)\,q_{\mu\lambda}\,q^\kappa + (q-p)\;q_{[\mu}\,q_{\lambda]}^{\cdot\kappa} \tag{120}$$

and

$$\Pi_{\mu\lambda}^{\kappa} = \left\{ \begin{matrix} \kappa \\ \mu\,\lambda \end{matrix} \right\} + (q-1)\,q_{\mu\lambda}\,q^\kappa + (1-p)\,q_\mu\,q_\lambda^{\cdot\kappa} + (1-q)\,q_\lambda\,q_\mu^{\cdot\kappa}\,. \tag{121}$$

Hence, the condition *V* reduces the choice of the geometry that one must adopt, up to the determination of the constants *p*, *q*, and *k*.

THE VARIATIONAL EQUATIONS OF THE FIELD. – In a well-constructed theory, it is necessary that the field equations can be derived from a variational principle that is based upon a universal function that is an invariant, and preferably, the simplest invariant. Now, the simplest invariant of our geometry is the scalar curvature N, which is defined by the equations:

$$(122)\qquad \begin{cases} N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = -2\partial_{[\nu}\Pi_{\mu]\lambda}^{\kappa} - 2\Pi_{\mu|\tau|}^{\kappa}\Pi_{\mu]\lambda}^{\tau}, \\ N = N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}\mathcal{G}^{\mu\lambda}. \end{cases}$$

We are then led to impose the condition that:

VI. – *The equations that are induced from:*

$$(123)\qquad \delta\int \mathfrak{R}\, dx^0 \dots dx^4 = 0, \qquad \mathfrak{R} = N\sqrt{\mathfrak{G}}$$

(in which x^κ and χ remain constant, and in which one makes the $\mathcal{G}_{\kappa\lambda}$) *give both the equations of gravitation and the ones from electromagnetism that consist of* MAXWELL's *second equation* [1] *in a vacuum.*

Upon performing the variation, one will be led to the equations [2]:

$$(124)\qquad K_{ij} - \tfrac{1}{2}K\,g_{ij} - 2\frac{q^2 - 2pq + 2pk^2}{q^2}(\mathcal{F}_i^{\cdot h}\mathcal{F}_{jh} - \tfrac{1}{2}\mathcal{F}_{hl}\mathcal{F}^{hl}g_{ij}) = 0,$$

$$(125)\qquad 2\,(q^2 - 2pq + 2p)\,\frac{k}{qc}\overset{R}{\nabla}_j\,\mathcal{F}_i^{\cdot j} = 0,$$

in which K_{ji} and K are the well-known curvature tensor and scalar of Riemannian geometry. (125) is the second MAXWELL equation in a vacuum, if:

$$(126)\qquad \begin{cases} q^2 - 2pq + 2p \neq 0, \\ q \neq 0. \end{cases}$$

Compare (124) with the equation for energy and impulse:

$$(127)\qquad K_{ij} - \tfrac{1}{2}K\,g_{ij} - 2\frac{\kappa}{c^2}(\mathcal{F}_i^{\cdot h}\mathcal{F}_{jh} - \tfrac{1}{2}\mathcal{F}_{hl}\mathcal{F}^{hl}g_{ij}) = 0,$$

in which κ is the gravitational constant. One sees that it is necessary and sufficient that:

$$(128)\qquad 2\,(q^2 - 2pq + 2p)\,k^2 = \kappa q^2,$$

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[1] The first MAXWELL equation is already a consequence of (117).
[2] G. F. VI, pp. 308.

so it will follow that (1):

(129) $$q^2 - 2pq + 2p > 0$$

and

(130) $$k = \frac{q}{\sqrt{q^2 - 2pq + 2p}}\sqrt{\frac{\kappa}{2}}\,.$$

THE ORDINARY DIRAC EQUATION IN THE PROJECTIVE THEORY. – It is well-known that in the usual theory of relativity, one can obtain the current term in the second MAXWELL equation by adding another universal function M to N that is deduced from the DIRAC equations for the matter wave. We shall first present that method by using the language of projective geometry.

The equation:

(131) $$\alpha^{(\kappa}\,\alpha^{\lambda)} = \mathcal{G}^{\kappa\lambda}$$

defines an associative hypercomplex number system of sixteen numbers that are generated by the five numbers α^0, α^1, …, α^4. These five hypercomplex numbers can be represented by co-contravariant affinors of valence two in a four-dimensional local auxiliary space, namely, the *spin space* (2):

(132) $$\alpha^{0A}_{\cdots B},\ \alpha^{1A}_{\cdots B},\ \alpha^{2A}_{\cdots B},\ \alpha^{3A}_{\cdots B},\ \alpha^{4A}_{\cdots B}\,.$$

In that case, one then has two transformations to take into consideration: On the one hand, the affine transformations of the system of frames (a) in the projective local spaces, and on the other, the homogeneous affine transformations of the system of frames (A) in the local spin spaces. Those transformations are absolutely independent of each other. Obviously, we impose the condition that all of our equations must be invariant with respect to those two types of transformations. One will define the vectors and affinors of that spin space in the usual manner with respect to the group of transformations of (A), and one calls them *spin vectors* and *spinors.* The magnitude:

(133) $$\alpha^{\kappa A}_{\cdots B}$$

is a *projector spinor*, which belongs to the local projective space by way of its index κ and to the local spin space, by way of its indices A and B. In the spin space, the allowable transformations are not just the real coordinate transformations, but also all of the transformations with complex coefficients of the homogeneous affine group. That fact will lead us to the following consequence: Since the usual transformation of spin vectors is given by the equations (3):

(1) When one takes $\chi^2 < 0$, it is necessary that $k^2 < 0$ and $q^2 - 2pq + 2p < 0$.

(2) G. F. V.

(3) α^A_B is the *identity spinor.*

(134) $$\psi^{A'} = \alpha_A^{A'}\psi^A, \qquad \eta_{B'} = \alpha_{B'}^{B}\eta_B,$$

one can define another type of spin vector that is called a *spin vector of the second kind*, which transform, like the preceding ones, except that the coefficients of the transformations $\bar{\alpha}_{\bar{A}}^{\bar{A}'}$, $\bar{\alpha}_{\bar{B}'}^{\bar{B}}$ are the complex conjugates of the $\alpha_A^{A'}$, $\alpha_{B'}^{B}$ ([1]):

(135) $$\zeta^{\bar{A}'} = \bar{\alpha}_{\bar{A}}^{\bar{A}'}\zeta^{\bar{A}}, \qquad \theta_{\bar{B}} = \bar{\alpha}_{\bar{B}'}^{\bar{B}}\theta_{\bar{B}}.$$

It is obvious that the complex conjugates of the components of an ordinary spin vector are the components of a spin vector of the second kind, and *vice versa.* In addition, there also exist spinors (and spinor densities) that carry both overbarred indices and simple indices, for example:

(136) $$\sigma^{\bar{A}'}_{\cdot B'\bar{C}'} = \bar{\alpha}_{\bar{B}'}^{\bar{B}}\bar{\alpha}_{\bar{A}\bar{C}'}^{\bar{A}'\bar{C}'}\sigma^{\bar{A}}_{\cdot B\bar{C}'}.$$

Those spinors are called *Hermitian spinors.* The complex conjugates of the components of a HERMITIAN spinor are the components of another HERMITIAN spinor, which are called the conjugate spinor to the first one. The principal letters of the two conjugate spinors are always the same, but one of them is overbarred and the other one is not. The most important HERMITIAN spinors are the covariant or contravariant spinors that are symmetric or alternating. That property (which must not, in fact, be confused with the proper of symmetry or alternation of ordinary spinors) is defined by the equations:

(137) $$\rho_{A\bar{B}} = \pm\bar{\rho}_{\bar{B}A}, \quad \tau^{A\bar{B}} = \pm\,\bar{\tau}^{\bar{B}A}.$$

When one multiplies a symmetric HERMITIAN spinor by $\sqrt{-1}$, an alternating spinor will result, and *vice-versa.*

W. PAULI proved that there exists an invariant symmetric HERMITIAN spinor of rank 4 in spin space, namely, $\omega_{\bar{A}B}$, that we call the covariant *fundamental HERMITIAN tensor* in spin space ([2]). One deduces the *fundamental contravariant tensor* $\varpi'^{A\bar{B}}$ from $\omega_{\bar{A}B}$:

(138) $$\omega_{\bar{B}C}\,\varpi'^{C\bar{A}} = \bar{\alpha}_{\bar{B}}^{\bar{A}}, \qquad \varpi'^{A\bar{C}}\omega_{\bar{C}B} = \alpha_B^A.$$

One can deduce $\omega_{\bar{A}B}$ from the α^κ by the equation ([3]):

([1]) $\bar{\alpha}_{\bar{B}}^{\bar{A}}$ is the *identity spinor of the second kind.*

([2]) *Loc. cit.*, pp. 347. The Hermitian form that corresponds to that spinor is found already in the work of E. CARTAN, "Les groupes réels, simples et continus," Ann. de l'école Norm. Sup. **31** (1914), 265-355. The spinor was rediscovered independently by V. BARGMANN, "Bemerkungen zur allgemeinrelativistischen Fassung der Quantentheorie," Berl. Sitzungsber. (1932), 347-354.

([3]) W. PAULI, *loc. cit.*, pp. 3, see also G. F. V, pp. 413-417. See also J. A. SCHOUTEN and J. HAANTJES, "Konforme Feldtheorie II; R_6 and Spinraum," Ann. d. Pisa (1934).

(139) $$\bar{\alpha}^{\kappa\bar{A}}_{\cdots\bar{B}} = \omega_{\bar{B}C}\,\alpha^{\kappa C}_{\cdots D}\,\bar{\varpi}'^{D\bar{A}}.$$

One can prove that the spin-affinors:

(140) $$\begin{cases} \omega\alpha_\lambda = \omega_{\bar{A}C}\,\alpha_{\lambda\cdot B}^{\cdot C}, & \alpha_\lambda\omega^{-1} = \alpha_{\lambda\cdot C}^{\cdot A}\,\bar{\varpi}'^{D\bar{A}}, \\ \omega\alpha_{[\kappa\lambda\mu\nu]} = \omega_{\bar{A}C}\,\alpha_{[\kappa\cdot|D|}^{\cdot C}\alpha_{\lambda\cdot|E|}^{\cdot D}\alpha_{\mu\cdot|F|}^{\cdot E}\alpha_{\nu]\cdot B}^{\cdot F}; & \alpha_{[\kappa\lambda\mu\nu]}\,\omega^{-1} \end{cases}$$

are symmetric and that the spin-affinors:

(141) $$\begin{cases} \omega\alpha_{[\lambda\mu]}, & \alpha_{[\lambda\mu]}\omega^{-1}, \\ \omega\alpha_{[\lambda\mu\nu]}, & \alpha_{[\lambda\mu\nu]}\omega^{-1} \end{cases}$$

are alternating ([1]).

In Euclidian space-time, the DIRAC equation can be written:

(142) $$\alpha^\mu\left(\frac{h}{i}\partial_\mu - \frac{e}{c}\varphi_\mu + mc\,q_\mu\right)\psi^A = 0$$

in homogeneous coordinates, in which ψ^A is the spin vector that represents the matter wave. In the presence of a gravitational field, one must replace ∂_μ with a symbol ∇_μ for the covariant differentiation of spin vectors. That covariant differentiation is deduced from the condition that the derivative of $\alpha^{\kappa C}_{\cdots A}$ must be zero.

One confirms the remarkable fact that there exists just one possible differentiation for contravariant densities of weight 1/4 of spin vectors and for covariant densities of weight – 1/4 of spin vectors, while the covariant differentiation of spin vectors themselves remains undetermined. Meanwhile, that indeterminacy does not pose any difficulty: One must simply always take a spin vector density ψ^A of weight + 1/4 to represent a matter wave, and consider $\alpha^{\kappa C}_{\cdots A}$ to be a spinor of weight + 1/4 in C and of weight – 1/4 in A. The parameters $A^A_{B\mu}$ of the differentiation of a density of spin vector ψ^A of weight 1/4 satisfy the equation:

(143) $$A^A_{A\mu} = 0.$$

With the aid of that equation and:

(144) $$0 = \nabla_\mu\alpha^{\kappa A}_{\cdots B} = \partial_\mu\alpha^{\kappa A}_{\cdots B} + \Pi^\kappa_{\mu\lambda}\alpha^{\lambda A}_{\cdots B} + A^A_{C\mu}\,\alpha^{\kappa A}_{\cdots B} - A^C_{B\mu}\,\alpha^{\kappa A}_{\cdots C},$$

one easily deduces that:

(145) $$A^A_{B\mu} = -\tfrac{1}{4}\Pi^{\ \kappa}_{\mu\lambda}\alpha^{\lambda\cdot A}_{\cdots\kappa\cdot B} + \tfrac{1}{4}\alpha^{\kappa A}_{\cdots C}\,\partial\alpha^{\cdot C}_{\kappa\cdot B}.$$

([1]) G. F. V, pp. 413. Equation (140) and (141) will no be longer valid when one takes the signature $+---+$.

When one replaces $\Pi_{\mu\lambda}^{\kappa}$ with $\overset{R}{\Pi}{}_{\mu\lambda}^{\kappa}$, one will find the parameters:

$$\overset{R}{A}{}_{B\mu}^{A} = -\tfrac{1}{4}\overset{R}{\Pi}{}_{\mu\lambda}^{\kappa}\alpha_{\cdots\kappa\cdot B}^{\lambda\cdot A} + \tfrac{1}{4}\alpha_{\cdots C}^{\kappa A}\,\partial\alpha_{\kappa\cdot B}^{\cdot C}, \tag{146}$$

which defines a Riemannian connection.

Ultimately, the ordinary DIRAC equation is written in homogeneous coordinates while suppressing the indices of the spin space:

$$\alpha^{\mu}\left(\frac{h}{i}\overset{R}{\nabla}_{\mu} - \frac{e}{c}\varphi_{\mu} + mc\,q_{\mu}\right)\psi = \alpha^{\mu}\left[\frac{h}{i}(\partial_{\mu} + \overset{R}{\Lambda}_{\mu}) - \frac{e}{c}\varphi_{\mu} + mc\,q_{\mu}\right]\psi = 0. \tag{147}$$

Upon introducing the system of frames (a), it is easy to write down that equation in the well-known four-dimensional form.

THE GENERALIZED DIRAC EQUATION. – The operator $\overset{R}{\nabla}_{\mu}$ is an operator of Riemannian geometry, and it is probable that in our projective theory, one must replace it with ∇_{μ}; one will then obtain the wave equation in the form:

$$\alpha^{\mu}\left(\frac{h}{i}\nabla_{\mu} - \frac{e}{c}\varphi_{\mu} + mc\,q_{\mu}\right)\psi = \alpha^{\mu}\left[\frac{h}{i}(\partial_{\mu} + \Lambda_{\mu}) - \frac{e}{c}\varphi_{\mu} + mc\,q_{\mu}\right]\psi = 0. \tag{148}$$

When one takes (91), (115), (120), (145), and (146) into account, one will find that the difference of the left-hand sides of equations (147) and (148) is:

$$\begin{aligned}\frac{h}{i}\alpha^{\mu}\left(\nabla_{\mu} - \overset{R}{\nabla}_{\mu}\right) = \frac{h}{i}\alpha^{\mu}\left(\Lambda_{\mu} - \overset{R}{\Lambda}_{\mu}\right) &= \frac{1}{4}\frac{h}{i}(p-2q)\,\alpha^{\mu\lambda\kappa}q_{\lambda}\,q_{\mu\kappa} \\ &= \frac{1}{4}\frac{h}{i}(p-2q)\,\alpha^{[\mu\lambda\kappa]}q_{\lambda}\,q_{\mu\kappa},\end{aligned} \tag{149}$$

from which, one deduces that the passage to a projective theory will modify the known results that relate to quanta only in the case where $p - 2q \neq 0$.

THE VARIATIONAL EQUATION OF THE FIELD AND THE MATTER WAVE. – On the usual theory, one obtains the universal invariant by multiplying the left-hand side of equation (147) by $2\kappa / c\,\bar{\psi}\omega$:

$$\overset{R}{M} = \frac{2\kappa}{c}\bar{\psi}\,\omega\alpha^{\mu}\left(\frac{h}{i}\overset{R}{\nabla}_{\mu} - \frac{e}{c}\varphi_{\mu} + mc\,q_{\mu}\right)\psi. \tag{150}$$

That invariant is not real, but the solution to the variational equation is real, because $\overset{R}{M}$ is "practically real"; in other words, because the imaginary part of $\overset{R}{M}$ is a divergence ([1]). Indeed, since $\omega\,\alpha^\mu$ is symmetric, one will have:

$$\overset{R}{M} - \overline{\overset{R}{M}} = \frac{2\kappa}{c}\frac{h}{i}\left(\bar\psi\,\omega\alpha^\mu\,\overset{R}{\nabla}_\mu\,\psi + \psi\,\bar\omega\bar\alpha^\mu\,\overset{R}{\nabla}_\mu\,\bar\psi\right) = \frac{h}{i}\frac{2\kappa}{c}\overset{R}{\nabla}_\mu\left(\bar\psi\,\omega\alpha^\mu\psi\right). \tag{151}$$

The fundamental invariant of the projective theory must obviously be:

$$M = \frac{2\kappa}{c}\bar\psi\,\omega\alpha^\mu\left(\frac{h}{i}\overset{R}{\nabla}_\mu - \frac{e}{c}\varphi_\mu + mc\,q_\mu\right)\psi\,. \tag{152}$$

We shall show that M is also "practically real." One has:

$$\begin{cases} M - \overset{R}{M} = +\dfrac{1}{4}\dfrac{2\kappa}{c}\dfrac{h}{i}\bar\psi\omega(p-2q)\alpha^{[\mu\lambda\kappa]}q_\lambda q_{\mu\kappa}\psi \\ \qquad\qquad = \dfrac{1}{4}\dfrac{2\kappa}{c}\dfrac{k}{2c}(p-2q)\dfrac{h}{i}\bar\psi\omega\,\alpha^{[\mu\lambda\kappa]}\psi\chi^{-1}x_\lambda\mathcal{F}_{\mu\kappa}. \end{cases} \tag{153}$$

Since $\omega\,\alpha^{[\mu\lambda\nu]}$ is alternating, the expression will be pure imaginary, and $M - \overset{R}{M}$ will be real ([2]).

The variational equation:

$$\delta\int(\mathfrak{M}+\mathfrak{N})\;dx^0\,dx^1\,dx^2\,dx^3\,dx^4 = 0, \qquad \mathfrak{M} = M\sqrt{\mathfrak{G}} \tag{154}$$

leads to the equations ([3]):

$$\begin{cases} K_{ij} - \tfrac{1}{2}Kg_{ij} - \dfrac{\kappa}{c^2}\left(\mathcal{F}_j^{\cdot h}\mathcal{F}_{ih} - \tfrac{1}{4}\mathcal{F}_{hl}\mathcal{F}^{kl}\,g_{ji}\right) \\ + \dfrac{\kappa}{c^2}\left[\mathfrak{Re}\dfrac{h}{i}e\bar\psi\omega\alpha'_{(j}\left(\overset{R}{\nabla}_{i)} - \dfrac{ie}{hc}\varphi_{i)}\right)\psi - \dfrac{p-2q}{2q}\dfrac{hk}{i}\mathcal{F}_{l(i}\,\bar\psi\alpha'^{\,l}_{\cdot j)}\alpha^0\psi\right] = 0, \end{cases} \tag{155}$$

$$\overset{R}{\nabla}_j\,\mathcal{F}_i^{\cdot j} - e\bar\psi\,\omega\alpha'_i\psi + \frac{p-2q}{2q}\frac{hk}{i}\overset{R}{\nabla}{}^l\,\bar\psi\,\omega\alpha'_{[l}\,\alpha'_{i]}\,\alpha^0\psi = 0 \qquad (\alpha'_\lambda = A^\kappa_\lambda\alpha_\kappa)\,. \tag{156}$$

([1]) H. WEYL, *Gruppentheorie und Quantenmechanik*, 2nd ed., pp. 188.

([2]) When one takes the signature + – – – +, $M - \overset{R}{M}$ will no longer be real, and one must take $p - 2q = 0$ in order to make M practically real.

([3]) $\mathfrak{Re}$ signifies "real part of."

The second equation is identical with the complete second MAXWELL equation, up to the term that contains that factor $(p - 2q)$. Therefore:

VII. – *In order for the variational equation* (154) *to lead to the second Maxwell equation with no additional term*, it is necessary and sufficient that:

$$p - 2q = 0. \tag{157}$$

By virtue of that equation and (118), one will have:

$$p = \frac{4}{3}, \qquad q = \frac{2}{3}, \tag{158}$$

and consequently (120):

$$S_{\mu\lambda\kappa} = -\tfrac{1}{3}\, q_{\mu\lambda}\, q_\kappa - \tfrac{1}{3}\, q_{\lambda\kappa}\, q_\mu - \tfrac{1}{3}\, q_\lambda\, q_{\lambda\mu} = -\, q_{[\mu\lambda}\, q_{\kappa]} = S_{[\mu\lambda\kappa]}\,. \tag{159}$$

Hence, $S_{\mu\lambda\kappa}$ is *alternating.*

A connection for which $S_{\mu\lambda\kappa}$ is alternating has a very remarkable property. The symmetric part $\Pi_{(\mu\lambda)}^{\kappa}$ of $\Pi_{\mu\lambda}^{\kappa}$ (for holonomic systems) constitutes a new connection, and one easily proves that in order for the derivative of $G_{\lambda\mu}$ to be annulled for that connection, it is necessary and sufficient that $S_{\mu\lambda\kappa}$ must be alternating. In that case, and only in that case, the auto-geodesics are identical with the auto-geodesics of symmetric geometry.

We summarize the results of the preceding calculations in a figure. In the plane p, q, the cases that correspond to the points of the hyperbola and the line $q = 0$ must be discarded by condition VI, and the cross-hatched parts, by the condition that M should be "practically" real. Condition III leads to the line $p + q = 2$ when one takes the auto-geodesics, and to the line $q = 2$ when one takes the induced geodesics. Condition VII leads to the line $p - 2q = 0$. The theory of A. EINSTEIN and W. MAYER (when made projective) is represented by the line $p = 0$, the theory of O. VEBLEN and B. HOFFMANN is represented by the point $Q : p = 1$, $q = 1$ (symmetric connection), and that point also represents the theory of W. PAULI (*loc. cit.*, 1933). The point P is the only point that fulfills all of our conditions I-VII. One sees that the auto-geodesics are preferable, since the induced geodesics lead to a line that penetrates into the forbidden part of the plane and cuts the line $p - 2q = 0$ at a point S (permissible) that belongs to the implausible signature $+ - - - +$. In addition, the point Q, which symbolizes the symmetric connection, is found on the line $p + q = 2$. There is no longer any difference between the two types of geodesics at the point R (because $\mathcal{P}_{\cdot\lambda}^{\kappa} = 0$), but that point is not as good because it gives neither symmetry nor the MAXWELL equations without the additional term. On the contrary, the point Q does allow symmetry, and it is possible that nature has such a preference for symmetric geometries that it will even accept additional terms into the MAXWELL equation in order to preserve the symmetry. That is the PAULI viewpoint. Meanwhile, one must recall that the Riemannian connection itself is not symmetric when one expresses it in projective language, which is a warning that one must not exaggerate the value of the symmetry without bound.

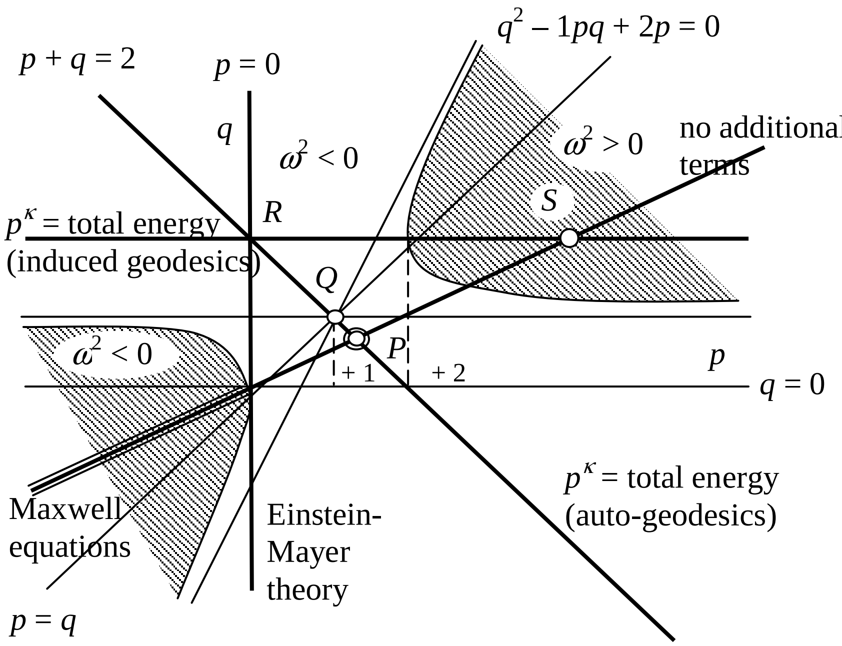

Figure 1 ([1]).

The additional terms in equation (156) are extremely small. Indeed, they contain the factor:

(160)
$$k = \frac{q}{\sqrt{q^2 - 2pq + 2p}}\sqrt{\frac{\kappa}{2}},$$

which is very small for all points that are not situated in the immediate neighborhood of the hyperbola. That makes it quite improbable that experiments can permit us to decide whether one must add those terms to the equation. Meanwhile, it is not excluded that upon developing the relativistic theory of quanta, one will arrive at some very strong arguments in order to be able to choose between P and Q definitively.

The projective theory realizes the unification of the theories if gravitation and electromagnetism in a very satisfying manner, but it is incapable of relating to the theory of material particles, because the two functions M and N do not follow from one and the same principle. That is a grave defect. The present theory then represents only a first approximation, and it is clear that one must introduce a new principle of one wishes to arrive at a complete unification of all of the physical theories.

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([1]) In this figure, one reads χ, instead of ω.