On the conformally-invariant form of Maxwell’s equations and the electromagnetic impulse-energy equations

By J. A. SCHOUTEN and J. HAANTJES
Translated by D. H. Delphenich

Abstract. It will be shown that Maxwell’s equations and the electromagnetic impulse-energy equations can be written in a conformally-invariant form. The theory of electromagnetism can then be also constructed upon the basis of conformal geometry, so it requires either a metric or a parallelism. Metric and parallelism then originate in material phenomena, not electromagnetic ones.

It is known that Maxwell’s equations are invariant under not only the Lorentz group, but also under conformal transformations (\(^1\)). Weyl (\(^2\)) has shown that the equations remain invariant when one replaces \(g_{hi}\) with \(\lambda g_{hi}\). It must then be possible to present Maxwell’s equations in a conformal geometry – i.e., in a metric geometry in which \(g_{hi}\) is given only up to an arbitrary (non-constant) numerical factor. Such a geometry is characterized by a tensor density (\(^3\)) \(\Phi_{hi}\) of weight \(-1/2\) that one obtains from \(g_{hi}\) (which is given, up to a numerical factor) as follows:

\[
\Phi_{hi} = (-g)^{-1/2} g_{hi}, \quad g = \text{Det}(g_{hi}). \tag{1}
\]

The determinant of \(\Phi_{hi}\) is equal to \(-1\). The raising and lowering of indices shall henceforth come about by way of \(\Phi_{hi}\) and its inverse \(\Phi^{hi}\) (of weight \(+1/2\)). The weight is not invariant under that process then. Let the signature of \(\Phi_{hi}\) be \(-+\).

The tensor density \(\Phi_{hi}\) does not fix any translation. Indeed, one can set:

\[
0 = \nabla_j \Phi_{hi} = \partial_j \Phi_{hi} - \Gamma^k_{jh} \Phi_{ki} - \Gamma^k_{ji} \Phi_{hk} + \frac{1}{2} \Gamma^l_{ji} \Phi_{hl}, \quad \Gamma^h_{[ji]} = 0, \tag{2}
\]

but these equations will imply only that:

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\(^3\) This idea goes back to T. Y. Thomas, Proc. Nat. Acad. of Sci. 11 (1925), 722-725.
\[ \Gamma^h_{\mu} = \left\{ \begin{array}{ll} \frac{1}{4} \mathfrak{g}_{ij} \eta^h + \frac{1}{2} A^h_i \eta_i + \frac{1}{2} A^h_j \eta_j, & A^h_i = \delta^h_i = \left\{ \begin{array}{ll} 0 & h \neq i \\ 1 & h = i, \end{array} \right. \end{array} \right. \] (3)

in which \( \{ h_{ij} \} \) is the Christoffel symbol of \( \mathfrak{g}_{hi} \), and \( \eta \) is a geometric object that can be chosen freely and has the transformation law:

\[ \eta' = A^i_j \eta - \partial_i \ln \text{Det} \left( A^h_i \right). \] (4)

Since \( \{ l_{ij} \} = 0 \), it will follow from (3) that:

\[ \Gamma^i_{\mu} = \eta_i. \] (5)

Although the operator \( \nabla \) is then endowed with indeterminacy, that indeterminacy will be lifted automatically in all physically-important cases, as we will now show.

We begin with the covariant electromagnetic bivector \( F_{hi} \), which satisfies the equation:

\[ F_{hi} = 2 \partial_{[h} \varphi_{i]}, \quad \partial_h = \frac{\partial}{\partial \xi^h}, \] (6)

and as a result, the equation:

\[ \nabla_{[j} F_{hi]} = 0, \] (7)

as well, since it is, in fact, known that \( \nabla_j \) can be replaced with \( \partial_h \) here. We construct the following bivector density of weight + 1 from \( F_{hi} \):

\[ \tilde{s}^{hi} = \mathfrak{g}^{hi} \mathfrak{g}^{ij} F_{ij}. \] (8)

Now, it is, in turn, known that \( \nabla_j \tilde{s}^{hi} \) is independent of the basic translation, and therefore:

\[ s^h = - \nabla_j \tilde{s}^{hi} = - \partial_j \tilde{s}^{hi}, \] (9)

will be a well-defined contravariant vector density of weight + 1.

Naturally, no line element, in the usual sense, is established by \( \mathfrak{g}_{hi} \), but every displacement \( d\xi^h \) is associated with a scalar density \( ds \) of weight \(-1/4\) by means of the equation:

\[ (ds)^2 = \mathfrak{g}_{hi} d\xi^h d\xi^i. \] (10)

Now, if \( de \) is the electric charge of the four-dimensional volume \( d\omega \) (a density of weight \(-1\)) then the equation (1):

\[ \text{(1')} \quad \text{Cf., H. Weyl, loc. cit., pp. 201.} \]
will establish a "charge density" of weight + 3/4, and the following relation must exist:

\[ s^h = \rho \frac{d\xi^h}{d\xi} \]

in which the weights actually agree (1 = \( \frac{3}{4} + \frac{1}{4} \)). Since \( s^h \) has weight + 1, it is known that \( \nabla_j s^j \) will once more be independent of the choice of translation, and (9) will imply the continuity equation:

\[ \nabla_j s^j = \partial_j s^j = 0. \]

When calculated from \( F_{hi} \) and \( s^h \), the force density:

\[ f_i = - F_{hi} s^h \]

will have weight + 1. From \( F_{ij} \) and \( \tilde{s}^{hi} \) one can calculate the affinor density of weight + 1 of impulse and energy

\[ \mathbb{G}^i = - \tilde{s}^{hi} F_{ij} + \frac{1}{4} F_{ij} \tilde{s}^{lj} A^l. \]

Although \( \nabla_j \mathbb{P}^j \) now depends upon the choice of displacement for an arbitrary choice of affinor density \( \mathbb{P}^j \), remarkably, \( \nabla_j \mathbb{G}^j \) is entirely independent of that choice, since (one observes that \( \nabla_j \mathbb{G}_{hi} = 0 \)):

\[
\nabla_j \mathbb{G}^j = - F_{il} \nabla_j \tilde{s}^{lj} - \tilde{s}^{lj} \nabla_j F_{il} + \frac{1}{2} \tilde{s}^{lj} \nabla_i F_{lj} \\
= - F_{il} \nabla_j \tilde{s}^{lj} + \tilde{s}^{lj} \nabla_i F_{lj} + \frac{1}{2} \tilde{s}^{lj} \nabla_i F_{lj} \\
= - F_{il} s^l - f_i.
\]

With that, we have, however, also found the conformally-invariant form for the law of conservation of energy and impulse. The entire theory of electromagnetism can then be constructed upon the basis of a conformal geometry, so one will need either a metric or a parallelism. A metric and a parallelism originate in material phenomena, not electromagnetic ones. Naturally, that last remark is true only as long as the electromagnetic phenomena can be represented purely by Maxwell's equations, and therefore it would no longer be true when, e.g., any correction terms would prove to be necessary that would not be consistent with conformal invariance.