"Über die konforminvariante Gestalt der Maxwellschen Gleichungen und der elektromagnetischen Impuls-Energiegleichungen," Physica **1** (1934), 869-872.

On the conformally-invariant form of Maxwell's equations and the electromagnetic impulse-energy equations

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Abstract. It will be shown that Maxwell's equations and the electromagnetic impulse-energy equations can be written in a conformally-invariant form. The theory of electromagnetism can then be also constructed upon the basis of conformal geometry, so it requires either a metric or a parallelism. Metric and parallelism then originate in material phenomena, not electromagnetic ones.

It is known that **Maxwell's** equations are invariant under not only the **Lorentz** group, but also under conformal transformations ([1]). **Weyl** ([2]) has shown that the equations remain invariant when one replaces g_{hi} with $\lambda\, g_{hi}$. It must then be possible to present **Maxwell**'s equations in a conformal geometry – i.e., in a metric geometry in which g_{hi} is given only up to an arbitrary (non-constant) numerical factor. Such a geometry is characterized by a tensor density ([3]) $\mathfrak{G}_{hi}$ of weight – 1/2 that one obtains from g_{hi} (which is given, up to a numerical factor) as follows:

$$\mathfrak{G}_{hi} = (-\,\mathfrak{g})^{-1/2}\, g_{hi}\,, \qquad \mathfrak{g} = \mathrm{Det}\,(g_{hi}). \tag{1}$$

The determinant of $\mathfrak{G}_{hi}$ is equal to – 1. The raising and lowering of indices shall henceforth come about by way of $\mathfrak{G}_{hi}$ and its inverse $\mathfrak{G}^{hi}$ (of weight + 1/2). The weight is not invariant under that process then. Let the signature of $\mathfrak{G}_{hi}$ be $- - - +$.

The tensor density $\mathfrak{G}_{hi}$ does not fix any translation. Indeed, one can set:

$$0 = \nabla_j\, \mathfrak{G}_{hi} = \partial_j\, \mathfrak{G}_{hi} - \Gamma^k_{jh}\, \mathfrak{G}_{ki} - \Gamma^k_{ji}\, \mathfrak{G}_{hk} + \tfrac{1}{2}\Gamma^l_{jl}\, \mathfrak{G}_{hi}\,, \qquad \Gamma^h_{[ji]} = 0, \tag{2}$$

but these equations will imply only that:

([1]) **E. Cunningham**, Proc. Lond. Math. Soc. **8** (1910), 77-88; **H. Bateman**, *ibidem*, pp. 223-264 and 469-488.

([2]) **H. Weyl**, *Raum, Zeit, Materie*, 4th ed., pp. 260; cf., also pp. 201 and pp. 208.

([3]) This idea goes back to **T. Y. Thomas**, Proc. Nat. Acad. of Sci. **11** (1925), 722-725.

$$\Gamma^h_{ji} = \left\{ {h \atop j\,i} \right\} - \tfrac{1}{4}\mathfrak{G}_{ji}\,\eta^h + \tfrac{1}{4}A^h_j\eta_i + \tfrac{1}{4}A^h_i\eta_j\,, \quad A^h_i \overset{*}{=} \delta^h_i = \begin{cases} 0 & h \neq i \\ 1 & h = i, \end{cases} \tag{3}$$

in which $\left\{ {h \atop j\,i} \right\}$ is the **Christoffel** symbol of $\mathfrak{G}_{hi}$, and η_i is a geometric object that can be chosen freely and has the transformation law:

$$\eta_{i'} = A^i_{i'}\,\eta_i - \partial_{i'}\ln \operatorname{Det}\,(A^{h'}_h)\,. \tag{4}$$

Since $\left\{ {l \atop i\,l} \right\} = 0$, it will follow from (3) that:

$$\Gamma^l_{jl} = \eta_j\,. \tag{5}$$

Although the operator ∇ is then endowed with indeterminacy, that indeterminacy will be lifted automatically in all physically-important cases, as we will now show.

We begin with the covariant electromagnetic bivector F_{hi} , which satisfies the equation:

$$F_{hi} = 2\partial_{[h}\,\varphi_{i]}\,, \qquad \partial_h = \frac{\partial}{\partial \xi^h}\,, \tag{6}$$

and as a result, the equation:

$$\nabla_{[j}F_{hi]} = 0, \tag{7}$$

as well, since it is, in fact, known that ∇_j can be replaced with ∂_h here. We construct the following bivector density of weight + 1 from F_{hi} :

$$\mathfrak{F}^{hi} = \mathfrak{G}^{hl}\,\mathfrak{G}^{ij}\,F_{lj}\,. \tag{8}$$

Now, it is, in turn, known that $\nabla_j\,\mathfrak{F}^{hi}$ is independent of the basic translation, and therefore:

$$\mathfrak{s}^h = -\nabla_j\,\mathfrak{F}^{hi} = -\partial_j\,\mathfrak{F}^{hi}, \tag{9}$$

will be a well-defined contravariant vector density of weight + 1.

Naturally, no line element, in the usual sense, is established by $\mathfrak{G}_{hl}$, but every displacement $d\xi^h$ is associated with a scalar density $d\mathfrak{s}$ of weight – 1/4 by means of the equation:

$$(d\mathfrak{s})^2 = \mathfrak{G}_{hi}\,d\xi^h\,d\xi^i\,. \tag{10}$$

Now, if de is the electric charge of the four-dimensional volume $d\omega$ (a density of weight – 1) then the equation ([1]):

([1]) Cf., **H. Weyl**, *loc. cit.*, pp. 201.

$$de\, d\mathfrak{s} = \rho\, d\omega \tag{11}$$

will establish a "charge density" of weight + 3/4, and the following relation must exist:

$$\mathfrak{s}^h = \rho \frac{d\xi^h}{d\mathfrak{s}}, \tag{12}$$

in which the weights actually agree ($1 = \frac{3}{4} + \frac{1}{4}$). Since $\mathfrak{s}^h$ has weight + 1, it is known that $\nabla_j\, \mathfrak{s}^j$ will once more be independent of the choice of translation, and (9) will imply the continuity equation:

$$\nabla_j\, \mathfrak{s}^j = \partial_j\, \mathfrak{s}^j = 0. \tag{13}$$

When calculated from F_{hi} and $\mathfrak{s}^h$, the force density:

$$\mathfrak{f}_i = -F_{hi}\, \mathfrak{s}^h \tag{14}$$

will have weight + 1. From F_{ij} and $\mathfrak{F}^{hi}$ one can calculate the affinor density of weight + 1 of impulse and energy

$$\mathfrak{S}^{h}_{\cdot i} = -\mathfrak{F}^{hj} F_{ij} + \tfrac{1}{4} F_{lj}\, \mathfrak{F}^{lj} A^h_i\,. \tag{15}$$

Although $\nabla_j \mathfrak{P}^{j}_{\cdot i}$ now depends upon the choice of displacement for an arbitrary choice of affinor density $\mathfrak{P}^{j}_{\cdot i}$, remarkably, $\nabla_j \mathfrak{S}^{j}_{\cdot i}$ is entirely independent of that choice, since (one observes that $\nabla_j\, \mathfrak{G}_{hi} = 0$):

$$\begin{aligned}\nabla_j \mathfrak{S}^{j}_{\cdot i} &= -F_{il}\, \nabla_j\, \mathfrak{F}^{jl} - \mathfrak{F}^{jl}\, \nabla_j\, F_{il} + \tfrac{1}{2} \mathfrak{F}^{hj}\, \nabla_i\, F_{hj} \\ &= -F_{il}\, \nabla_j\, \mathfrak{F}^{jl} + \tfrac{3}{2} \mathfrak{F}^{hj}\, \nabla_{[i}\, F_{hj]} \\ &= +F_{il}\, \mathfrak{s}^l = -\mathfrak{f}_i\,.\end{aligned} \tag{16}$$

With that, we have, however, also found the conformally-invariant form for the law of conservation of energy and impulse. The entire theory of electromagnetism can then be constructed upon the basis of a conformal geometry, so one will need either a metric or a parallelism. A metric and a parallelism originate in material phenomena, not electromagnetic ones. Naturally, that last remark is true only as long as the electromagnetic phenomena can be represented purely by **Maxwell**'s equations, and therefore it would no longer be true when, e.g., any correction terms would prove to be necessary that would not be consistent with conformal invariance.

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