"Diracsches Elektron im Schwerefeld I," Sitzungsber. Preüss. Akad. Wiss., Phys.-Math. Kl. (1932), 105-128.

The Dirac electron in a gravitational field I.

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§ 1. – Introduction.

The union of **Dirac**'s theory of the electron with the general theory of relativity has already been attacked repeatedly by Wigner (¹), Tetrode (²), Fock (³), Weyl (⁴), Zaycoff (⁵). **Poldolsky** (⁶). Most authors introduce an orthogonal axis-cross at each world-point, along with **Dirac** matrices that are numerically-specialized relative to it. With that process, it is a bit hard to see whether **Einstein**'s notion of teleparallelism, which will be referred to in a partially-direct way, actually plays a role or if things are independent of it. Furthermore, it will then be necessary to recast the Riemannian concepts in the lessfamiliar and decidedly more cumbersome form of "bein components." It seems desirable to me to avoid all of that by using only the generalized commutation relations [cf., *infra*, equation (2)], like **Tetrode** $(^{7})$. It shows that one will be led to the important operators Γ_k , whose traces give the four-potential and which **Fock** introduced as "components of the parallel translation of a spinor" in an exceptionally simple and direct way, and likewise directly to the important system of equations [cf., *infra*, (8)] that **Fock** arrived at by way of the detour of bein components. By a restriction to the allowable reference systems (cf., *infra*, § 4) that is completely analogous to the one in the usual special theory of relativity, one will then infer the Hermiticities that are desirable for interpretations, as well as a correspondence between tensor operators and local c-tensors that is likewise completely analogous to the one that **von Neumann** (⁸) presented in the special theory [cf., equation (57) below]. It seems to me that a fundamental advantage of this is that all of the machinery can be constructed almost completely upon pure operator calculus, without referring to the ψ -function. Hopefully, one will not be scared away from the exact *foundation* of that machinery by its scope, for which the author's broad notation is

^{(&}lt;sup>1</sup>) **E. Wigner**, Zeit. Phys. **53** (1928), 592.

^{(&}lt;sup>2</sup>) **H. Tetrode**, Zeit. Phys. **50** (1928), 336.

^{(&}lt;sup>3</sup>) **V. Fock**, Zeit. Phys. **57** (1929), 261.

^{(&}lt;sup>4</sup>) **H. Weyl**, Proc. Nat. Acad. Amer. **15** (1929), 323; Zeit. Phys. **56** (1929), 330.

^{(&}lt;sup>5</sup>) **R. Zaycoff**, Ann. Phys. (Leipzig) **7** (1930), 650.

^{(&}lt;sup>6</sup>) **B. Podolosky**, Phys. Rev. **37** (1931), 1398.

 $^(^{7})$ And also **B. Hoffmann** in a letter to Phys. Rev. **37** (1931), 88.

^{(&}lt;sup>8</sup>) **J. von Neumann**, Zeit. Phys. **48** (1928), 868.

partially responsible. Once those preliminaries have been completed, the *implementation* and the comprehension of the theory might then prove to be simple. – I would like to acknowledge my great debt to the work of my predecessors once and for all, but ask to be allowed to derive everything anew on methodological grounds that have still not been found by anyone else.

§ 2. – Construction of the metric from matrix fields.

We shall call the world-variables:

$$x_0 = ict, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

The first is always pure-imaginary, while the other three are real. **Dirac**'s basic idea was to regard the Euclidian wave operator:

$$\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

as the square of a linear operator:

$$\left(\overset{\circ}{\gamma_0}\frac{\partial}{\partial x_0}+\overset{\circ}{\gamma_1}\frac{\partial}{\partial x_1}+\overset{\circ}{\gamma_2}\frac{\partial}{\partial x_2}+\overset{\circ}{\gamma_3}\frac{\partial}{\partial x_3}\right)^2,$$

in which the γ_k° are 4×4 matrices (¹) that must satisfy:

$$\gamma_i \gamma_k + \gamma_k \gamma_i = 2\delta_{ik}; \qquad (1)$$

i.e., that will be equal to the zero matrix or twice the identity matrix according to whether $i \neq k$ or i = k, resp. One knows that the γ_k° are determined by the requirement (1) precisely, up to a so-called similarity transformation:

$$\gamma_k^{\rm o} = S^{-1} \gamma_k^{\rm o} S ,$$

with an arbitrary, non-singular 4×4 transformation matrix *S*. That freedom in the choice of the γ_k° is obvious, and one knows, as one says, that the freedom is exhausted in that way.

Since one can also start from the square of the line element:

$$dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 ,$$

^{(&}lt;sup>1</sup>) The number of rows will not figure at all in what follows.

in place of the wave operator, that suggests that one can regard the requirements (1) in such a way that the matrices γ_k° , along with the other givens that are involved with the description of the electron also have the purpose of describing the world-metric, which was tentatively assumed to be Euclidian. Should that not be the case, but rather:

 $ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \,,$

then one would have to replace (1) with (**Tetrode**):

$$\gamma_i \gamma_k + \gamma_k \gamma_i = 2 g_{ik} . \tag{2}$$

The γ_k are functions of space and time; i.e., they are 4×4 matrices whose elements are functions of the x_i .

Equations (2) certainly have solutions for the γ_k at every point *P* when one thinks of the g_{ik} as being given *in any way* (but naturally, in such a way that they correspond to a non-singular metric). The freedom in the γ_k that still exists for a given g_{ik} is precisely the same as the freedom in the γ_k° above, namely: Under a transformation by an arbitrary non-singular matrix *S*. One sees the validity of that statement when one argues along the following line:

1. Above all, equations (2) can always be solved by four suitably-chosen linear aggregates of an arbitrary **Dirac** basis system γ_k° – That Ansatz will lead to requirements on the coefficients that can possibly be fulfilled.

2. Conversely: If one has a system of γ_k , about which, one knows only that (2) is fulfilled, then one can give four linear aggregates of those γ_k that fulfill (1), and thus define a **Dirac** basis. If one then has, say, two systems γ_k and γ'_k of solutions of (2) then one can convert them into a **Dirac** basis for each of them by *the same* linear transformation. However, those two **Dirac** bases certainly go to each other by an *S*-transformation. γ_k and γ'_k will also be converted into each other in the same way.

3. The fact that any S-transformation will leave (2) untouched is immediate.

With that, the statements are proved.

A very essential difference between the γ_k° and the γ_k is this: It is known that there is a Hermitian system of γ_k° , but there is, in general, no Hermitian system of γ_k , nor even one in which some of the γ_k are Hermitian and the others are skew-Hermitian. That is connected with the well-known reality properties that are demanded of the g_{ik} , namely, they are pure-imaginary when *one and only one* index 0 is present, and otherwise real. (One must recall that the symmetric product – viz., the anti-commutator – of two Hermitian matrices will always be Hermitian.) We shall go into the question of Hermiticity in detail later on, but for now, we shall only mention that in order to show that for the time being there is not the slightest ground for restricting the transformation *S*, which is arbitrary at each point, to a *unitary* one. Since the γ_k will not be Hermitian without that restriction, one would initially have no reason for imposing the condition of the "conservation of Hermiticity."

We can now derive an important system of differential equations for the γ_k from (2). We think of the γ_k as being given and equations (2) as having been solved at any point *P*, and indeed in such a way that these solutions will be combined into four continuous, differentiable matrix fields, which will obviously be possible.

We now go from a point P to a neighboring point P' and define the complete differential of equation (2) in that sense:

$$\delta \gamma_i \cdot \gamma_k + \gamma_i \cdot \delta \gamma_k + \delta \gamma_i \cdot \gamma_k + \gamma_i \cdot \delta \gamma_k = 2 \frac{\partial g_{ik}}{\partial x_i} \delta x^i.$$
(3)

If we now observe the theorem of **Ricci**, according to which, the covariant derivative of the fundamental tensor g_{ik} vanishes identically:

$$g_{ik;l} \equiv \frac{\partial g_{ik}}{\partial x_l} - \Gamma^{\mu}_{kl} g_{i\mu} - \Gamma^{\mu}_{il} g_{\mu k} \equiv 0, \qquad (4)$$

then the right-hand side of (3) will be equal to:

$$2(\Gamma^{\mu}_{kl}g_{i\mu}+\Gamma^{\mu}_{il}g_{\mu k})\,\delta x^l.$$

One can endow the right-hand side of (3) with that value when one sets:

$$\delta \gamma_i = \Gamma^{\mu}_{il} \,\delta x^l \tag{5}$$

and observes (2). That is, the matrices:

$$\gamma_i + \delta \gamma_i = \gamma_i + \Gamma^{\mu}_{il} \gamma_{\mu} \, \delta x^l \tag{6}$$

will satisfy equation (2) at the point P'when the γ_i satisfy it at the point P.

The Ansatz (5) would generally be contradictory if one wished to apply it to *all* points P' in the vicinity of P. One can convince oneself by a simple calculation that the expression (5) is a complete differential if and only if the *curvature* vanishes at P. However, from what was said above, the γ_i -values at P' (we would like to call them $\gamma_i + \delta' \gamma_i$) can and will still differ from our solution Ansatz (5) [(6), resp.] that was guessed in some way by a similarity transformation, and indeed it will naturally be an infinitely-small one, if continuity is to be valid. That is, there must be an infinitely-small matrix ε such that:

$$\gamma_i + \delta' \gamma_i = (1 - \varepsilon)(\gamma_i + \delta \gamma_i) (1 + \varepsilon) = \gamma_i + \delta \gamma_i + \gamma_i \varepsilon - \varepsilon \gamma_i,$$

or

$$\delta' \gamma_i = \Gamma^{\mu}_{il} \gamma_{\mu} \delta x^l + \gamma_i \, \mathcal{E} - \mathcal{E} \, \gamma_i \,. \tag{7}$$

In itself, ε can have a different, completely arbitrary, value for any neighboring point. However, should γ_i have a correct differential quotient with respect to x_i , then ε would have to be proportional to $\delta \gamma_i$ for an advance in the x_i direction (i.e., for $\delta \gamma_i \neq 0$, while all other components = 0), and so on for each *l*. Therefore, should the change in γ_i under an advance in an arbitrary direction be truly calculable from its differential quotients, then ε would have to be the sum of those four terms. One will then come to the Ansatz:

$$\mathcal{E} = -\Gamma_l \, \delta x^l$$

in which the Γ_l are four matrices that depend upon position and time (naturally, the minus sign is totally arbitrary). When that is substituted in (7), one will get the important system of differential equations that we announced above (¹):

$$\frac{\partial \gamma_i}{\partial x_l} = \Gamma^{\mu}_{il} \gamma_{\mu} + \Gamma_l \gamma_i - \gamma_i \Gamma_l.$$
(8)

We will later express this in the form: The covariant derivative of the fundamental vector γ_k vanishes, in complete analogy to **Ricci**'s theorem, equation (4). On the other hand, the source-free character of the four-current is closely connected with this system of equations. I would like to place special emphasis on the fact that here we have derived it purely from the demands on the *metric*, with no reference being made to the ψ -function, so we *must* take advantage of the transformation degree of freedom in the **Dirac** matrices. The new operations Γ_l will appear in that way – and indeed *unavoidably* – from which, we will see that they are intimately linked with the four-potential (but they do not define a vector!).

We shall examine the necessary conditions for the compatibility of equations (8), namely, that the mixed second differential quotients, when they are calculated in two ways, must agree. When one again expresses the first derivatives that are to be differentiated by (8), one will find that:

$$\Phi_{kl} \gamma_l - \gamma_l \Phi_{kl} = R_{kli}^{\cdots \mu} \gamma_{\mu}.$$
⁽⁹⁾

Here, $R_{kli}^{\dots\mu}$ is the mixed Riemannian curvature tensor in the usual notation (cf., e.g., Levi-Civita, *Der absolute Differentialkalkul*, pp. 91; Springer, Berlin, 1928). Φ_{kl} is an abbreviation that we shall introduce for the six matrices:

^{(&}lt;sup>1</sup>) This agrees with **Fock**, *loc. cit.*, equation (24), in content. The meaning of the sign here is somewhat different from what it was there. If one would like to bring the two into agreement then one should read our Section 5 on Hermiticity!

$$\Phi_{kl} = \frac{\partial \Gamma_l}{\partial x_k} - \frac{\partial \Gamma_k}{\partial x_l} + \Gamma_l \Gamma_k - \Gamma_k \Gamma_l , \qquad (10)$$

which are antisymmetric in the indices k, l, and which, as we will show, have a close relationship to the electromagnetic field. For a given γ_i -field, Γ_l is fixed by (8) and Φ_{kl} is fixed by (9), up to an addend that commutes with all γ_i , so it must be a multiple of the identity matrix. The Φ_{kl} are easy to calculate from (9). Along with the γ_i , one introduces the contravariant ones:

$$\gamma^i = g^{ik} \ \gamma_k \,. \tag{11}$$

Furthermore, one states that:

$$s^{\mu\nu} = \frac{1}{2} (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}).$$
⁽¹²⁾

(For μ , $\nu = 1, 2, 3$, the $s^{\mu\nu}$ correspond in some way to the spin, and for $\mu = 0, \nu = 1, 2, 3$, they correspond to velocity. See below.) We point out that, from (2) and (11):

$$\gamma_i \gamma^k + \gamma^k \gamma_i = 2\delta_i^k \,. \tag{13}$$

Now, one easily finds that:

$$\gamma_i s^{\mu\nu} - s^{\mu\nu} \gamma_i = 2(\delta_i^{\mu} \gamma^{\nu} - \delta_i^{\nu} \gamma^{\mu}).$$
(14)

The $s^{\mu\nu}$ then produce another γ when one commutes it with another γ . That is precisely what one needs in order to solve (9) for Φ_{kl} . Indeed, the right-hand side of (9) can also be written $R_{kl,i\mu} \gamma^{\mu}$, in which $R_{kl,i\mu}$ is the symmetric **Riemann** tensor. With the commutation rules (14), one then confirms that:

$$\Phi_{kl} = -\frac{1}{4} R_{kl,\mu\nu} s^{\mu\nu} + f_{kl} \cdot \mathbf{I}$$
(15)

is the general solutions of (9) (¹). f_{kl} is the remaining free multiplier of unity. The f_{kl} (when multiplied by *i*) will take on the role of the electromagnetic field. One sees that the appearance of those quantities through the construction of the metric from matrices will indeed be very suggestive, but that it is precisely the f_{kl} that are *not* determined by the γ -field, for the time being, but will remain completely free of it.

The $s^{\mu\nu}$ have trace zero as commutators. Hence:

$$\operatorname{Tr} \Phi_{kl} = f_{kl} \cdot \operatorname{Tr} \mathbf{I} = 4 f_{kl} \cdot$$

On the other hand, from (10):

$$\operatorname{Tr} \Phi_{kl} = \frac{\partial}{\partial x_k} (\operatorname{Tr} \Gamma_l) - \frac{\partial}{\partial x_l} (\operatorname{Tr} \Gamma_k)$$

 $^(^{1})$ In content, this essentially coincides with the many-indexed bein equations (46), (48) in **Fock**, *loc. cit.*

because differentiation and taking the trace commute with each other, and the commutation yields no contribution to the trace. If one sets, say:

 $\frac{1}{4}$ Tr $\Gamma_l = \varphi_l$

then:

$$f_{kl} = \frac{\partial \varphi_l}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_l}.$$
 (16)

The traces of the Γ_l are then the four-potential (except for a factor of *i*).

§ 3. – Transformation theory, part one.

From the basic notions of general relativity, a *renaming* of all points:

$$x'_{k} = x'_{k} (x_{0}, x_{1}, x_{2}, x_{3}), \qquad k = 0, 1, 2, 3$$
 (17)

should not change the form of the description of things. Therefore, the function x'_0 should assume only pure-imaginary values, while x'_1 , x'_2 , x'_3 must assume only real ones, so the functional determinant should remain positive. We call that a *point substitution*. The g_{ik} then transform as a second-rank covariant tensor.

As long as we make no other demands upon the γ_i besides that they should satisfy equations (2), the question of how they will transform under a point-substitution cannot be answered uniquely, by any means. A similarity transformation with a transformation matrix *S* that varies from point to point will then remain completely free before the point substitution, as well as after it. We *can* generally determine that the γ_i transform as a covariant vector under a pure point substitution, which means that (8) will still be true. The commutator $\Gamma_l \gamma_i - \gamma_i \Gamma_l$ will then transform as a covariant tensor, as will the rest of the equation:

$$\frac{\partial \gamma_i}{\partial x_k} - \Gamma^{\mu}_{il} \gamma_{\mu} , \qquad (18)$$

when γ_i is substituted as a vector. The similarity transformation:

$$\gamma'_k = S^{-1} \gamma_k S \tag{19}$$

is then a thing-in-itself to be considered, so as one easily convinces oneself, the Γ_l will transform as follows:

$$\Gamma_{l}' = S^{-1} \Gamma_{l} S - S^{-1} \frac{\partial S}{\partial x_{l}}, \qquad (20)$$

in order to preserve (8), and thus, *differently* from the γ_k . By contrast, one would find that with that convention, the following aggregate, for which we would like to introduce the symbol ∇_k :

$$\nabla_k = \frac{\partial}{\partial x_k} - \Gamma_k, \qquad (21)$$

will *first of all* (and this is obvious) behave like a covariant vector substitution under a pure point substitution (because that is certainly true for the $\partial / \partial x_k$ by themselves and was established for the Γ_k) and that *secondly*, because of (20), the ∇_k will transform under an *S*-transformation precisely as the γ_k transform as a result of (19), namely:

$$\nabla'_k = S^{-1} \nabla_k S. \tag{22}$$

The meaning of is ∇'_k then that:

$$\nabla'_{k} = \frac{\partial}{\partial x_{k}} - \Gamma'_{k} = \frac{\partial}{\partial x_{k}} - S^{-1} \Gamma_{k} S - S^{-1} \frac{\partial S}{\partial x_{k}}, \qquad (23)$$

and one will then have:

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k} S^{-1} S = \frac{\partial}{\partial x_k} S - \frac{\partial S^{-1}}{\partial x_k} S = S^{-1} \frac{\partial}{\partial x_k} S + S^{-1} \frac{\partial S}{\partial x_k}, \qquad (24)$$

the latter of which is due to the identity:

$$S^{-1} S \equiv \mathbf{I}, \qquad \frac{\partial S^{-1}}{\partial x_k} S + S^{-1} \frac{\partial S}{\partial x_k} \equiv 0.$$

One confirms (22) by substituting (24) in (23).

The Φ_{kl} that are introduced from (10) will *first of all* (and this is obvious) behave like a covariant tensor under point substitutions, and *secondly*, they will behave analogously to (19) under an *S*-transformation:

$$\Phi'_{kl} = S^{-1} \Phi_{kl} S, \qquad (25)$$

where the latter is due to (22), and since, from the definitions (10) and (21), the commutators of the ∇_k are:

$$\Phi_{kl} = \nabla_l \nabla_k - \nabla_k \nabla_l. \tag{26}$$

It should still be added that the traces of the Φ_{kl} – viz., the f_{kl} – will not change under a similarity transformation, due to (25), but the traces of the Γ_l which we should call φ_l , probably will, because no transformation law that is analogous to (19) [(25), resp.] is true for them, but only (20).

We have presented all of this in "would be" form, because the convention that was made is stuck with the arbitrariness that was mentioned to begin with: Since a point substitution generally implies a *change* in the γ_i in any case [indeed, the old γ_i will generally no longer satisfy equations (2)!], for the new choice, an entire *manifold* of γ_i - fields will again be available, whose members will emerge from any one of them by an arbitrary, coordinate-dependent *S*-transformation. Moreover, none of those members is initially distinguished *intrinsically* in any way, nor is the one that was chosen above.

Now, it is strongly suggested (at least, for many purposes) that one might greatly restrict that freedom of choice by employing a (to some, not unavoidable, but still suggestive) desire that Hermiticity should be satisfied, as one likewise cares to do in the special-relativistic **Dirac** theory. In order to see what one can achieve in that regard, one must draw one's attention to the eigenvalues of the γ_k and their double products.

§ 4. – Eigenvalues and Hermitization.

Since:

 $\gamma_k \gamma_k = g_{kk}$ (no summation!),

from (2), γ_k will have the eigenvalues $\pm \sqrt{g_{kk}}$, and indeed each of them twice, because it has a zero trace. One sees the latter when one sets:

$$s_{\mu\nu} = \frac{1}{2} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\mu} \gamma_{\nu}), \qquad (26)$$

analogous to (12). One will then have:

$$\gamma^{i} s_{\mu\nu} - s_{\mu\nu} \gamma^{i} = 2 \left(\delta^{i}_{\mu} \gamma_{\nu} - \delta^{i}_{\nu} \gamma_{\mu} \right), \qquad (27)$$

analogous to (14). Each γ can then be represented as a commutator in many ways, and a commutator will always have trace zero.

Nevertheless, the γ have nothing but real eigenvalues, and as a result, each of them can be made Hermitian by an S-transformation, but that generally does not happen *simultaneously* (for example, for γ_0 and γ_1), because from (2) their symmetric product equals $2g_{01} \cdot I$, so it will be skew-Hermitian (since g_{01} is pure-imaginary).

We shall now further consider the products $\gamma_i \gamma^k$, first for $i \neq k$. The square is [cf., (13)]:

$$(\gamma_i \gamma^k)^2 = \gamma_i \gamma^k \cdot \gamma_i \gamma^k = -\gamma_i \gamma_i \gamma^k \gamma^k = -g_{ii} g^{kk} \text{ (no sum!)}.$$

The eigenvalues are then $\pm \sqrt{g_{ii} g^{kk}}$, and indeed each of them are double, since:

$$\gamma_i \gamma^k = \frac{1}{2} (\gamma_i \gamma^k - \gamma^k \gamma_i),$$

as a commutator, must have trace zero. The eigenvalues of $\gamma^k \gamma_i$ are equal and opposite to the latter, and therefore the same. By contrast, for i = k, one will have:

$$(\gamma_k \gamma^k)^2 = \gamma_k \gamma^k \gamma_k \gamma^k = \gamma_k (2 - \gamma_k \gamma^k) \gamma^k = 2 \gamma_k \gamma^k - g_{kk} g^{kk}$$
(no sum!),
$$(\gamma_k \gamma^k - 1)^2 = 1 - g_{kk} g^{kk}$$
(no sum!).

 $\gamma_k \gamma^k - 1$ will then have the eigenvalues $\pm \sqrt{1 - g_{kk} g^{kk}}$, and since it can be described as a commutator:

$$\gamma_k \gamma^k - 1 = \frac{1}{2} (\gamma_k \gamma^k - \gamma^k \gamma_k) \qquad \text{(no sum)},$$

each of them will be double. $\gamma_k \gamma^k$ will then have the eigenvalues:

$$1\pm\sqrt{1-g_{kk}}g^{kk}$$

and indeed, each of them will double. For k = 0, those values are real, since $g_{00} g^{00} \le 1$.

Of the four matrices:

$$\gamma_0 \gamma^0, \quad \gamma_1 \gamma^1, \quad \gamma_2 \gamma^2, \quad \gamma_3 \gamma^3, \tag{28}$$

only the first one is real then, while the other three have pure-imaginary values. They will then possess (up to a factor *i*) precisely the real behavior that would be reasonable for a physical four-vector $(^1)$. That suggests that one might explore whether those four matrices can be made simultaneously Hermitian (skew-Hermitian, resp.). One can show that this is true as follows, with which, a number of other matrices will be Hermitian at the same time:

When the metric tensor g_{ik} is *real* and *positive-definite*, equations (2) can be satisfied by *Hermitian* γ_k , just as equations (1) can be satisfied by Hermitian γ_k° . I might probably regard that as being known without proof, so one will indeed be dealing with only the projection of a system of γ_k° that is assumed to be Hermitian from a rectangular axis-cross to a skew one, so nothing but real coefficients will appear as direction cosines. Since the g_{ik} are real in that case, the contravariant γ^k will also prove to be Hermitian; that is, one can also satisfy the contravariant equations:

$$\gamma^{i} \gamma^{k} + \gamma^{k} \gamma^{i} = 2 g^{ik}, \qquad (29)$$

$$\mathscr{H} \gamma^{i} \mathscr{H} \gamma^{k} + \mathscr{H} \gamma^{k} \mathscr{H} \gamma^{i} = \mathscr{H} \left(2\delta_{0}^{i} - \mathscr{H} \gamma^{i} \right) \gamma^{k} + \mathscr{H} \left(2\delta_{0}^{k} - \mathscr{H} \gamma^{k} \right) \gamma^{i} = 2(\delta_{0}^{i} \gamma_{0} \gamma^{k} + 2\delta_{0}^{k} \gamma_{0} \gamma^{i}) - 2g_{00} g^{ik}.$$

That is, in fact, *real* when neither of the indices *i*, *k* is zero, while for $i = 0, k \neq 0$, one will have:

$$2 \gamma_0 \gamma^k - 2 g_{00} g^{0k}$$
.

This has, in fact, pure-imaginary eigenvalues, because we know that this is true for $\gamma \gamma^k$, and g^{0k} is pure-imaginary.

^{(&}lt;sup>1</sup>) In the Euclidian case, they do, in fact, go over to the **Dirac** four-vector (up to a factor *i*). The complication that hinders us from making the γ_k themselves Hermitian, namely, that their symmetric product does not exhibit the required reality properties, will also no longer exist. For $i \neq k$, one has:

which are analogous to (2), by *Hermitian* γ^k when the tensor g^{ik} is real and positivedefinite. Now, that is not generally our tensor g^{ik} , but we can make it so when we distort it and set the "mixed" space-time g^{0k} (k = 1, 2, 3), which were simply omitted initially, equal to zero. Let:

$$a^0, a^1, a^2, a^3$$
 (30)

be a quadruple of Hermitian matrices that satisfy equations (29) with the distorted metric tensor. That is:

$$a^{i} a^{k} + a^{k} a^{i} = 2 g^{ik}$$
(31)

when *none* or *both* indices *i*, *k* are equal to zero, and:

$$a^0 a^k + a^k a^0 = 0 (32)$$

for $k \neq 0$. One now sets:

$$\gamma^{k} = \frac{i}{g^{00}} a^{0} a^{k}, \qquad \text{for } k \neq 0$$
(33)

and

$$\gamma^{0} = \frac{a^{0}}{\sqrt{g_{00} g^{00}}} - \frac{1}{g_{00}} (g_{01} \gamma^{1} + g_{02} \gamma^{2} + g_{03} \gamma^{3}).$$
(34)

One can convince oneself by calculation that these γ^k satisfy the undistorted equations (29).

Since, from (32), a^0 will anticommute with a^k ($k \neq 0$), $a^0 a^k$ will be skew-Hermitian, so from (33), γ^1 , γ^2 , γ^3 will be *Hermitian*. One further calculates from (34) that:

$$\gamma_0 = g_{0k} \ \gamma^k = a^0 \sqrt{\frac{g_{00}}{g^{00}}} =$$
Hermitian. (35)

By our construction, we have then made the *contravariant* γ^1 , γ^2 , γ^3 , as well as the *covariant* γ_0 *Hermitian*. Some further Hermiticities that we establish are: The contravariant pure-space matrices:

$$s^{kl} = \frac{1}{2} (\gamma^k \ \gamma^l - \gamma^l \ \gamma^k)$$
 for $k, l = 1, 2, 3$ (36)

will be skew-Hermitian, since they are the commutators of Hermitian matrices. Furthermore, for $k \neq 0$, the $\gamma \gamma^k$, and likewise the $\gamma^k \gamma_0$, will be skew, because already from (13), γ_0 will anticommute with γ^k ($k \neq 0$). We will then find from (34) and (35) that $\gamma_0 \gamma^0$ and $\gamma^0 \gamma_0$ are Hermitian. It will then follow from this very easily by lowering the index that for $k \neq 0$, both $\gamma_0 \gamma_k$ and $\gamma_k \gamma_0$, and therefore:

$$s_{0k} = \frac{1}{2} (\gamma_0 \gamma_k - \gamma_k \gamma_0),$$

will prove to be skew-Hermitian. However, we expressly observe that *nothing* can be said about the *covariant* s_{kl} for $k, l \neq 0$, and likewise about the *contravariant* s^{0k} ! Similar statements will be true for γ^0 , γ_1 , γ_2 , γ_3 . We shall combine all conventions together. By construction:

$$\begin{array}{ll} \gamma_{0}, \gamma^{\prime}, \gamma^{2}, \gamma^{3}, \gamma_{0} \gamma^{0}, \gamma^{0} \gamma_{0} & \text{are Hermitian,} \\ \gamma_{0} \gamma_{k}, \gamma_{k} \gamma_{0}, \gamma_{0} \gamma^{k}, \gamma^{k} \gamma_{0}, s_{0k}, s^{kl} & \text{are skew-Hermitian} & (k, l \neq 0). \end{array}$$

$$(37)$$

We would now like to free ourselves of the special choice of matrix construction, which served only to prove existence. One can easily see that the requirement that *four* suitably-chosen matrices from the ones that were cited in (37) have the property that was established in it (for example, the requirement that γ_0 , γ^1 , γ^2 , γ^3 should prove to be Hermitian) *is sufficient* to imply that for given g_{ik} , the γ -field is established uniquely, up to a *unitary* transformation. Even *more* freedom will then exist for a given g_{ik} in order for the γ -field to not be that way: namely, transformations by an arbitrary matrix. Should that transformation make the matrices γ_0 , γ^1 , γ^2 , γ^3 Hermitian, from which, *any* matrix, and thus also any *Hermitian* matrix, can be derived by addition and multiplication (¹) then the transformation of *any Hermitian* matrix must be Hermitian; i.e., the transformation must be *unitary*. Q. E. D.

In the future, we would like to admit only those γ fields (one can also say, only those reference systems) for which the matrices γ_0 , γ^1 , γ^2 , γ^3 prove to Hermitian. Everything that was established in (37) will then be true automatically. An "allowable" reference system is determined by the metric up to a *unitary* transformation.

It is very convenient that we have reduced the allowable S-transformations to unitary ones by the new requirement, since they are very easy to work with and trifling. We shall not need to think about them at all, in general, so we can proceed as if the γ -field were determined uniquely by the metric. Naturally, that now poses the problem of how we can determine the transformation law of the γ *more finely* when we start from an allowable γ field and perform a point substitution (17), namely, in such a way that we will again produce an allowable γ -field. The provisional prescription that was given at the beginning of Section 3 – namely, substituting the γ_k as if they formed a covariant vector – does not at all satisfy that requirement, so it does not at all correspond to what happens in

special relativity, where one *does not remotely* substitute the γ_k . When speaking in the language of Section 3, we can say: Any point substitution must be coupled with a completely-determined *S*-transformation (actually, one that is determined up to a unitary factor, but is naturally *not unitary*, in its own right), and that transformation will serve to

$$\gamma_0 = g_{00} \gamma^0 + g_{01} \gamma^1 + g_{02} \gamma^2 + g_{03} \gamma^3,$$

from which, γ^0 can be calculated, since one certainly has $g_{00} \neq 0$.

^{(&}lt;sup>1</sup>) First, it is known of the **Dirac** γ_k that any matrix can be derived from them rationally. One then infers that for the γ_k alone or the γ^k alone. The fact that γ^0 is missing from the quadruple above and γ_k enters in its place does not harm anything, because:

determine the substitution. For that reason, one can aptly speak of an *extended* point substitution. We will deal with that problem in the next section for *infinitely-small* point substitutions.

§ 5. – Transformation theory, part two.

We start from an allowable γ field and go over to the primed variables by the infinitely-small point substitution:

$$x'_{k} = x_{k} + \delta x_{k}$$
 or $x_{k} = x'_{k} - \delta x_{k}$, (38)

which we extend to an infinitely-small S-transformation in the sense that was proposed by way of:

$$S = 1 + \Theta, \qquad S^{-1} = 1 - \Theta. \tag{39}$$

As usual, we shall not explicitly state the replacement of variables in the arguments. The equations between primed and unprimed operators then relate to each other them, not as *equal* values of the arguments, but as *corresponding* ones; i.e., at *the same* point. Now, let:

$$\frac{\partial \delta x_k}{\partial x_l} = a_l^k, \tag{40}$$

to abbreviate.

Those quantities are pure imaginary when one and only one index equals zero, and otherwise real. One will then have:

$$\gamma'_{i} = \gamma_{i} - a_{i}^{l} \gamma_{l} + \gamma_{i} \Theta - \Theta \gamma_{i},$$

$$\gamma'^{k} = \gamma^{k} + a_{l}^{k} \gamma^{l} + \gamma^{k} \Theta - \Theta \gamma^{k}.$$
(41)

If one takes the first equation for i = 0 and multiplies it on the left by the second one then that will give (always precise to only first-order quantities):

$$\gamma_0' \gamma'^k = \gamma_0 \gamma^k - a_0^l \gamma_l \gamma^k + a_l^k \gamma_0 \gamma^l + \gamma_0 \gamma^k \Theta - \Theta \gamma_0 \gamma^k.$$
(42)

We use the right to dispose of Θ in order to remove the second term in the right-hand side of this equation (replace it with another, resp.), since it thwarts the Hermiticity of the result. That can be accomplished by way of:

$$\Theta = -\frac{1}{2g_{00}}a_0^l \gamma_l \gamma_0.$$
⁽⁴³⁾

Namely, we will then have:

$$-2 \Theta \gamma_0 \gamma^k = a_0^l \gamma_l \gamma^k, \qquad (44)$$

and that will give:

$$\gamma'_0 \gamma'^k = \gamma_0 \gamma^k + a_l^k \gamma_0 \gamma^l + \Theta \gamma_0 \gamma^k + \Theta \gamma_0 \gamma^k.$$

We argue on the basis of our conventions (37) that, from (43), Θ is *Hermitian*. Its symmetric product with $\gamma_0 \gamma^k$ is Hermitian or skew according to whether $\gamma_0 \gamma^k$ is. The same thing will be true for the second term on the right-hand side, namely, it will be skew for $k \neq 0$ and Hermitian for k = 0. Thus, the $\gamma'_0 \gamma'^k$ will actually possess the same Hermiticity as the $\gamma_0 \gamma^k$. With that, the status of the γ -field as "allowable" is legitimized.

Naturally, Θ is not unique, but the value for it that was stated in (43) will have the following meaning: It is uniquely the *Hermitian* component of the infinitely-small matrix that is applied. An arbitrary infinitely-small *skew* component can enter into it. One sees by some reasoning that it would leave all results unchanged; obviously, it indeed also corresponds to only one additional *unitary* transformation!

We shall now connect this with the rigorous definition of a *tensor operator*:

When it is known or has been established for a system of operators:

$$T^{\rho\sigma\cdots}_{\alpha\chi\cdots}$$

that it transforms under any infinitely-small extended point substitution like a tensor with a rank that is suggested by its indices and their positions, but with the introduction of the commutator:

$$T^{\rho\sigma\cdots}_{\alpha\chi\cdots}\Theta-\Theta T^{\rho\sigma\cdots}_{\alpha\chi\cdots},$$

we shall refer to the system of operators as a tensor operator with the rank in question.

The following important theorem is true $(^1)$, which will be obtained from a very easy generalization of the results above:

Let $T_{\alpha\chi\cdots}^{\rho\sigma\cdots}$ be a tensor operator and let it be known that in some reference system, the operators:

$$\gamma_0 T^{\rho\sigma\cdots}_{\alpha\chi\cdots} \tag{46}$$

will be Hermitian or skew according to whether the zero in the indices $\alpha\beta...\rho\sigma$ appears an **even** or **odd** number of times, resp.; that state of affairs will then remain the same in **any** reference system.

One can obviously switch the words "even" and "odd" in that theorem; i.e., one might or might not include the zero in γ_0 . However, what one *cannot* do is to worry about the Hermiticity of $T^{\rho\sigma\cdots}_{\alpha\chi\cdots}$ itself. That is entirely trivial, because it relates to the $\gamma_0 T^{\rho\sigma\cdots}_{\alpha\chi\cdots}$!

One easily confirms that the symbol:

^{(&}lt;sup>1</sup>) The Hermiticity statements have an immediate meaning only when $T_{\alpha\beta\cdots}^{\rho\sigma\cdots}$ does not include the derivative $\partial / \partial x_k$, but is simply a 4×4 matrix with coordinate-dependent elements.

$$\nabla_k = \frac{\partial}{\partial x_k} - \Gamma_k \tag{21}$$

that was introduced in (21) is a vector operator. However, Γ_k by itself is not; it obviously transforms [recall (20)] under an extended point substitution thus:

$$\Gamma'_{k} = \Gamma_{k} - a^{i}_{k} \Gamma_{i} + \Gamma_{k} \Theta - \Theta \Gamma_{k} - \frac{\partial \Theta}{\partial x_{k}}.$$
(47)

The last term in this is superfluous, and conflicts with the vector property. On the other hand, the pure derivative $\partial / \partial x_k$ transforms covariantly in the elementary sense, so it *lacks* the Θ -commutator. If one combines the two then that inconvenience will cancel out, because $\partial \Theta / \partial x_k$ can be regarded as the commutator of $\partial / \partial x_k$ and Θ . When one speaks of "Hermitian" or "skew," naturally, operators like ∇_k that include differentiations will have no immediate sense. Indeed, we have not incorporated this into the definition of a tensor operator.

If one has *two* tensor operators then one easily confirms by multiplying out their transformation formulas [similarly to what was done above in the passage from (41) to (42)] that under "writing them next to each other" (i.e., matrix multiplication), one will get another tensor when the operator that is written *to the left does not include the differential operator. Otherwise, that will not be true*, because it usually does not commute with the substitution coefficients a_i^k . (Indeed, that is no different from the usual tensor calculus, either. Although $\partial / \partial x_k$ would then be a vector, one would still not get a tensor under *ordinary* differentiation of tensor components, but only under *covariant* differentiation.) The tensor character of Φ_{kl} , which is defined by (10) or (26), must be particularly debatable then. However, since we have already seen in § **3** that the Φ_{kl} transform under pure point transformations as a tensor in the elementary sense that one found there, but transform according to (25) under any *S*-substitution, so they obviously define a tensor operator under extended point substitutions in the present refined sense of the term.

We would now like to concern ourselves with the question of what we should understand the term "covariant differentiation" of a tensor operator to mean. We then restrict ourselves to those operators that *do not* include the *differential operator*, so to 4×4 matrices whose elements are coordinate functions. (That does not obstruct the fact that they can have the *form* of differential quotients; for example, Φ_{kl} is acceptable, but not ∇_k .) We will then be dealing with a tensor operator $T^{\rho\sigma\cdots}_{\alpha\beta\cdots}$ under differentiation with respect to x_k and the derivation of supplementary terms that would be suitable additions that transform as a contravariant tensor operator in $\rho\sigma$... and a covariant one in $\alpha\beta$... under extended point substitution.

We shall make use of the fact that an extended point substitution will formally decompose into a pure substitution and a Θ -transformation under which the latter will simply add the commutator Θ . We shall further use the fact that those two infinitely-small transformations obviously commute. If we now consider the covariant differential quotients in the elementary sense:

$$\frac{\partial T^{\rho\sigma\dots}_{\alpha\beta\dots}}{\partial x_{\lambda}} - \Gamma^{\mu}_{\alpha\lambda}T^{\rho\sigma\dots}_{\mu\beta\dots} - \dots + \dots, \qquad (48)$$

and it will be obvious that this substitutes like a tensor of rank $\rho_{\alpha\beta\cdots\lambda}^{\rho\sigma\cdots}$ under a *pure* point substitution. It would still be necessary to show that one would simply add the commutator with Θ under a Θ -transformation, like $T_{\mu\beta\cdots}^{\rho\sigma\cdots}$ itself. Now, in the foregoing expression, that will be true of all terms, with the exception of the first one, for which the term:

$$\frac{\partial (T^{\rho\sigma\cdots}_{\mu\beta\cdots}\Theta - \Theta T^{\rho\sigma\cdots}_{\mu\beta\cdots})}{\partial x_{\lambda}}$$

will get added, instead of:

$$\frac{\partial T^{\rho\sigma\cdots}_{\mu\beta\cdots}}{\partial x_{\lambda}}\Theta-\Theta\frac{\partial T^{\rho\sigma\cdots}_{\mu\beta\cdots}}{\partial x_{\lambda}},$$

under the Θ -transformation. The term:

$$T^{\rho\sigma\dots}_{\mu\beta\dots}\frac{\partial\Theta}{\partial x_{\lambda}} - \frac{\partial\Theta}{\partial x_{\lambda}}T^{\rho\sigma\dots}_{\mu\beta\dots}$$
(49)

will then appear to be *superfluous*. We *discard* it by adding:

$$T^{\rho\sigma\cdots}_{\mu\beta\cdots}\Gamma_{\lambda}-\Gamma_{\lambda}T^{\rho\sigma\cdots}_{\mu\beta\cdots}$$

in (48) as an extension of the commutator, and thus arrive at the final *definition of the covariant differentiation of a tensor operator:*

$$T^{\rho\sigma\cdots}_{\alpha\beta\cdots;\lambda} = \frac{\partial T^{\rho\sigma\cdots}_{\alpha\beta\cdots}}{\partial x_{\lambda}} - \Gamma^{\mu}_{\alpha\lambda}T^{\rho\sigma\cdots}_{\mu\beta\cdots} - \dots + \dots + T^{\rho\sigma\cdots}_{\mu\beta\cdots}\Gamma_{\lambda} - \Gamma_{\lambda}T^{\rho\sigma\cdots}_{\mu\beta\cdots}.$$
 (50)

Proof: According to (47), the added term behaves like this: Under a pure point transformation, it is a tensor of the desired rank, and under a Θ -transformation, one first adds its commutator with Θ and secondly drops the superfluous term (49). The proof that (50) is a tensor is complete with that. One can also write (50) in the form:

$$T^{\rho\sigma\cdots}_{\alpha\beta\cdots;\lambda} = \nabla_{\lambda}T^{\rho\sigma\cdots}_{\mu\beta\cdots} - T^{\rho\sigma\cdots}_{\mu\beta\cdots}\nabla_{\lambda} - \Gamma^{\mu}_{\alpha\lambda}T^{\rho\sigma\cdots}_{\mu\beta\cdots} - \dots + \dots,$$
(51)

which differs from the elementary formula only by the fact that the derivative ∇_{λ} enters in place of the simple $\partial / \partial x_{\lambda}$.

One now recognizes that the important system of differential equations (8), upon which we based our arguments to begin with, say nothing more than that the vanishing of the covariant derivatives of the metric vector γ_k . That is completely analogous to **Ricci**'s

theorem, which says the same thing about the metric tensor g_{ik} . Moreover, entirely the same thing is true for the tensor that is derived from the γ_k by multiplication and addition with *constant* coefficients; e.g., γ^k , $s_{\mu\nu}$, $s^{\mu\nu}$, etc. All of these have zero covariant derivatives. That is an immediate consequence of equations (8).

§ 6. – Interpretation in terms of the Ψ -spinor.

The restriction of the γ -field to the ones that we called "allowable" will seem particularly convenient when one uses a four-component ψ -function – i.e., a so-called *spinor* – upon which the operators act as the basis for the interpretation of those operators. Should a system of equations:

$$T^{\rho\sigma\cdots}_{\mu\beta\cdots}\psi = 0 \tag{52}$$

be true in *any* reference system when it is true in *one of them*, then one would have to determine that ψ transforms as an invariant under a pure point substitution, but as follows under an S-transformation:

$$\psi' = S^{-1} \psi. \tag{53}$$

The first statement is obvious, and under an *S*-transformation, it will actually follow from (52) upon left-multiplying by S^{-1} that:

$$S^{-1}T^{\rho\sigma\cdots}_{\mu\beta\cdots}S S^{-1}\psi = T^{\prime\rho\sigma\cdots}_{\mu\beta\cdots}\psi^{\prime} = 0.$$

Under an extended infinitely-small point substitution, one will then have to set:

$$\psi' = \psi - \Theta \psi, \tag{54}$$

in which Θ is the Hermitian matrix (43). Now, since ∇_k is a vector operator, it will follow that, among other things: When all four numbers:

$$\nabla_k \,\psi = \frac{\partial \psi}{\partial x_k} - \Gamma_k \,\psi \tag{55}$$

vanish in *some* reference system, they will vanish in all of them. It would then be appropriate to refer to them as the *covariant derivative* of the spinor ψ .

One will get ordinary numbers from the operators (q-numbers) that will be interpreted physically as having the flavor and notations of occupation probabilities, densities of electricity, current densities, transition probabilities, *et al.*, in the following way: One applies the operator A in question to a spinor $\psi(\text{viz.}, A\psi)$ and thus defined the so-called Hermitian inner product of the two spinors ψ and $A\psi$; i.e., one multiplies the first component of the complex conjugate ψ^* by the first component of $A\psi$, the second component of ψ^* with the second of $A\psi$, etc., and adds those four products. We would like to write:

$$\psi^* A \psi$$
 (56)

for that, for brevity (¹). If A does *not* include the differential operator $\partial / \partial x_k$, but is simply a 4×4 matrix with coordinate-dependent elements, then we can also say: We enter the components of ψ^* and ψ as arguments in the bilinear form that is defined by that matrix.

Now, when the matrix is Hermitian (skew, resp.), the *c*-number (56) will always prove to be real (pure imaginary, resp.), which would be necessary for the components of *c*-tensors if one is to interpret them physically. Now, we saw in § 5 that: When $T^{\rho\sigma...}_{\alpha\beta...}$ is a tensor operator, the Hermiticity of *its* components will not be preserved at all under an allowable transformation (i.e., under an *extended* point substitution), but only those of $\gamma_0 T^{\rho\sigma...}_{\alpha\beta...}$. Hence, it will not be, say, the *c*-numbers $\psi^* T^{\rho\sigma...}_{\alpha\beta...}\psi$, but the *c*-numbers:

$$\mathsf{T}^{\rho\sigma\cdots}_{\alpha\beta\cdots} = \psi^* \gamma_0 T^{\rho\sigma\cdots}_{\alpha\beta\cdots} \psi, \tag{57}$$

that exhibit the required reality behavior for a physical tensor of rank $\rho_{\alpha\beta\cdots}^{\rho\sigma\cdots}$. We would now like to show that it is also *those c-numbers* that actually transform like a *c*-tensor of rank $\rho_{\alpha\beta\cdots}^{\rho\sigma\cdots}$, and for that reason, can be used for the physical interpretation of the tensor operators. Namely, when one performs the extended point substitution (38), (40), one will first obtain the following:

$$\mathsf{T}^{\rho\sigma\cdots'}_{\alpha\beta\cdots} = (\psi^* - \Theta^* \psi^*)(\gamma_0 - a_0^l \gamma_l + \gamma_0 \Theta - \Theta \gamma_0) (T^{\rho\sigma\cdots}_{\alpha\beta\cdots} - a_\alpha^l T^{\rho\sigma\cdots}_{l\beta\cdots} - +\cdots)(\psi - \Theta \psi)$$

$$= \mathsf{T}^{\rho\sigma\cdots}_{\alpha\beta\cdots} - \Theta^* \psi^* \gamma_0 T^{\rho\sigma\cdots}_{\alpha\beta\cdots} \psi - \psi^* \Theta \gamma_0 T^{\rho\sigma\cdots}_{\alpha\beta\cdots} \psi - a_0^l \psi^* \gamma_l T^{\rho\sigma\cdots}_{\alpha\beta\cdots} \psi - a_0^l \mathsf{T}^{\rho\sigma\cdots}_{\alpha\beta\cdots} - + \dots$$
(58)

(Two terms in Θ , namely, the ones that arise from $-\Theta\psi$ and $\gamma_0 \Theta$, will cancel each other. Obviously, terms of second order in Θ and a_k^l have been suppressed.) The second, third, and fourth term on the right-hand side drop out, so: The second and third ones are equal to each other, since Θ^* will go over to the other factor under "transposition" and therefore, since it is Hermitian, to Θ . From (43):

$$-2\psi^*\Theta\gamma_0T^{\rho\sigma\cdots}_{\alpha\beta\cdots}\psi=\frac{1}{g_{00}}a_0^l\psi^*\gamma_l\gamma_0\gamma_0T^{\rho\sigma\cdots}_{\alpha\beta\cdots}\psi=a_0^l\psi^*\gamma_lT^{\rho\sigma\cdots}_{\alpha\beta\cdots}\psi$$

will then cancel the fourth term, as asserted. One will then get the usual substitution formula for the c-tensor (57):

^{(&}lt;sup>1</sup>) The notation that was chosen here will *not* factor in what follows. $A\varphi B\chi$ will mean *the same thing* as $B\chi A\varphi$, namely, it will always mean: First component of $A\varphi$ times the first one of $B\chi$ plus the second of $A\varphi$ times the second of $B\chi$ plus, etc.

$$\mathsf{T}^{\rho\sigma\cdots'}_{\alpha\beta\cdots} = \mathsf{T}^{\rho\sigma\cdots}_{\alpha\beta\cdots} - \boldsymbol{\alpha}^{l}_{\alpha}\mathsf{T}^{\rho\sigma\cdots}_{l\beta\cdots} - +, \tag{59}$$

etc. One specifically observes that in this proof, the operator $T_{\alpha\beta\cdots}^{\rho\sigma\cdots}$ itself will have to be either transposed with an a_k^l or switched with it. The proof will then also be true then; i.e., $T_{\alpha\beta\cdots}^{\rho\sigma\cdots}$ will also transform as a *c*-tensor then when $T_{\alpha\beta\cdots}^{\rho\sigma\cdots}$ includes the differential operator $\partial / \partial x_k$. It is merely the statements about Hermiticity that will then have no immediate sense for the local tensor components.

It will be convenient for what follows to extend formula (55) to the case in which one is not dealing with a spinor, but its complex-conjugate. The complex conjugate of (55) would be:

$$\frac{\partial \psi^*}{\partial x_k^*} - \Gamma_k^* \psi^*,$$

but that would *not* go to the ordinary derivative for k = 0 ($x_0 = ict$!) in the Euclidian case, but to its negative, which would be very inconvenient. Unfortunately, it will, in turn, be necessary to change the sign for k = 0 and to define the covariant derivative of ψ^* to be:

$$\nabla_{\kappa} \psi^* = \frac{\partial \psi^*}{\partial x_k} \mp \Gamma_k^* \psi^* \tag{60}$$

(upper sign for k = 1, 2, 3; lower for k = 0).

We would now like to investigate the covariant derivative of the *c*-tensor (57), which probably must be connected with the covariant derivative of the *tensor operator* that was defined in (50) in some way. We first obtain:

$$\mathbf{T}_{\alpha\beta\cdots;\lambda}^{\rho\sigma\cdots} = \frac{\partial \mathbf{T}_{\alpha\beta\cdots}^{\rho\sigma\cdots}}{\partial x_{\lambda}} - \Gamma_{\lambda\alpha}^{\mu} \mathbf{T}_{\mu\beta\cdots}^{\rho\sigma\cdots} - \dots + \dots$$

$$= \frac{\partial \boldsymbol{\psi}^{*}}{\partial x_{\lambda}} \gamma_{0} T_{\alpha\beta\cdots}^{\rho\sigma\cdots} \boldsymbol{\psi} + \boldsymbol{\psi}^{*} \frac{\partial \gamma_{0}}{\partial x_{\lambda}} T_{\alpha\beta\cdots}^{\rho\sigma\cdots} \boldsymbol{\psi} + \boldsymbol{\psi}^{*} \gamma_{0} \frac{\partial T_{\alpha\beta\cdots}^{\rho\sigma\cdots}}{\partial x_{\lambda}} \boldsymbol{\psi} + \boldsymbol{\psi}^{*} \gamma_{0} T_{\alpha\beta\cdots}^{\rho\sigma\cdots} \frac{\partial \boldsymbol{\psi}}{\partial x_{\lambda}} - \Gamma_{\lambda\alpha}^{\mu} \boldsymbol{\psi}^{*} \gamma_{0} T_{\mu\beta\cdots}^{\rho\sigma\cdots} \boldsymbol{\psi} - + \dots$$

We can extend the four derivatives that enter into this to *covariant* derivatives using (60), (8), (50), (55), in which the derivatives of γ_0 will vanish. We will then get:

$$\mathsf{T}^{\rho\sigma\dots}_{\alpha\beta\dots;\lambda} = \nabla_{\lambda}\psi^{*}\gamma_{0}T^{\rho\sigma\dots}_{\alpha\beta\dots}\psi + \psi^{*}\nabla_{\lambda}\gamma_{0}T^{\rho\sigma\dots}_{\alpha\beta\dots}\psi + \psi^{*}\gamma_{0}T^{\rho\sigma\dots}_{\alpha\beta\dots;\lambda}\psi + \psi^{*}\gamma_{0}T^{\rho\sigma\dots}_{\alpha\beta\dots}\nabla_{\lambda}\psi , \qquad (61)$$

plus a *remainder*, which we will now show must vanish. That remainder is:

remainder = $\pm \Gamma^{\mu}_{\lambda} \psi^* \gamma_0 T^{\rho \sigma \dots}_{\alpha \beta \dots} \psi$

$$= \psi^* [\Gamma^{\mu}_{0\alpha} \gamma_{\mu} T^{\rho\sigma\dots}_{\alpha\beta\dots} + (\Gamma_{\lambda} \gamma_0 - \underline{\Gamma_{\lambda} \gamma_0}) T^{\rho\sigma\dots}_{\alpha\beta\dots} - \underline{\gamma_0} T^{\rho\sigma\dots}_{\alpha\beta\dots} \Gamma_{\lambda} + \gamma_0 \Gamma_{\lambda} T^{\rho\sigma\dots}_{\alpha\beta\dots}] \psi + \underline{\psi^* \gamma_0} T^{\rho\sigma\dots}_{\alpha\beta\dots} \Gamma_{\lambda} \psi .$$

The underlined terms cancel each other. $\pm \Gamma_{\lambda}^{*}$, like $\pm \Gamma_{\lambda}^{\dagger}$, will go to the other factor (¹). What will remain is:

remainder =
$$\psi^* A T^{\rho\sigma\cdots}_{\alpha\beta\cdots} \psi$$
 with

$$A = \Gamma^{\mu}_{0\lambda} \gamma_{\mu} + (\Gamma_{\lambda} \pm \Gamma^{\dagger}_{\lambda}) \gamma_0.$$

The proof will be complete when we can show that:

$$\frac{1}{2g_{00}}A \ \gamma_0 \equiv \frac{1}{2}(\Gamma_\lambda \pm \Gamma_\lambda^\dagger) + \frac{1}{2g_{00}}\Gamma_{0\lambda}^\mu \gamma_\mu \gamma_0 \tag{62}$$

vanishes. [It will then follow from this that $A \equiv 0$, since γ_0 has the non-vanishing eigenvalue $\pm \sqrt{g_{00}}$. If A = 0 then the "remainder" will vanish, and equation (61) will be proved.]

Now, in the case of the upper sign, which will be true for $\lambda = 1, 2, 3$, the operator (62) will be the *Hermitian* component of:

$$\Gamma_{\lambda} + \frac{1}{2g_{00}} \Gamma^{\mu}_{0\lambda} \gamma_{\mu} \gamma_{0}, \qquad (63)$$

while in the case of the lower sign, it will be the *skew-Hermitian* component. One can see with no great effort that when one commutes this operator with the *Hermitian* matrices γ_0 , γ^1 , γ^2 , γ^3 according to (37), in the cases $\lambda = 1, 2, 3$, it will be *Hermitian* without exception, while in the case of $\lambda = 0$, it will be *skew-Hermitian without exception*. Therefore, its Hermitian (skew-Hermitian, resp., when $\lambda = 0$) component *commutes* with γ_0 , γ^1 , γ^2 , γ^3 , and will thus be a multiple of the identity. In other words, the component whose vanishing is at issue will reduce to:

real part of Trace
$$(\Gamma_{\lambda} + \frac{1}{2g_{00}}\Gamma_{0\lambda}^{\mu}\gamma_{\mu}\gamma_{0})$$
 for $\lambda = 1, 2, 3$
Imaginary part of Trace $(\Gamma_{0} + \frac{1}{2g_{00}}\Gamma_{00}^{\mu}\gamma_{\mu}\gamma_{0})$.

Now, Trace $\gamma_{\mu} \gamma_0 = 4 g_{00}$ and:

$$g_{\mu 0} \Gamma^{\mu}_{0\lambda} = \Gamma_{0,0\lambda} = \frac{1}{2} \frac{\partial g_{00}}{\partial x_{\lambda}};$$
 for $\lambda = 0, 1, 2, 3.$

The question then comes down to whether one actually has:

 $^(^{1})$ The dagger \dagger refers to the transposed complex-conjugate matrix, which is almost always the convention (but sadly, one can only say *almost* always).

real part of Trace
$$\Gamma_{\lambda} = -\frac{\partial \ln g_{00}}{\partial x_{\lambda}}$$
, for $\lambda = 1, 2, 3$ (64)
Imaginary part of Trace $\Gamma_{\lambda} = -\frac{\partial \ln g_{00}}{\partial x_{0}}$.

We will now show that we have promised too much. Namely, we *cannot establish* the foregoing equations, and indeed that is due to the fact that the Γ_{λ} were originally introduced and employed up to now exclusively in such a way that *only their commutators* with other matrices played any role, which implies precisely that their *traces* will vanish. The first time that they played a role was in the covariant derivatives of *spinors*, equations (55) and (60), which we first made use of in precisely equation (61), which was to be proved. What we can prove is only that we are free to *define* the trace part in question by (64), and that is actually the case. First of all, it is certainly true in *some* reference system, because the right-hand side of (64) possesses the required reality. Thanks to (47) and (43), we can then show that the convention that was made at the time is invariant under allowable transformations. I shall suppress the proof of that.

The covariant derivative of the spinor will be made precise by that convention. However, the convention is actually desirable in yet another regard. Namely, if the trace parts that were spoken of were *not* represented as the derivatives of one and the same function (- ln g_{00}) then the traces of the Φ_{kl} would then produce *pure-imaginary* electromagnetic field strengths. One could avoid that in the following way: The *real part* of Trace Γ_0 and the *imaginary part* of Trace Γ_{λ} ($\lambda = 1, 2, 3$), from which the *real* field strengths are derived, remain *free*, as before.

We must now cast our gaze upon the pure *unitary* transformations, which are in and of themselves also allowable, along with the extended point substitutions. The only thing that remains to be said is that such a unitary transformation, which one would like to perform, must obviously also be performed on ψ , from the prescription (53). Thus, such a unitary transformation will be completely harmless and irrelevant. In particular, the components of the *c*-tensors (57) will be *completely* insensitive to it, and likewise for the trace part, which was defined in (64).

The essential results of this section are:

1. The determination of the transformation law (54) and the covariant derivative (55) for the spinor.

2. The association of *c*-tensor components with the tensor operator by (57) and the proof that they actually transform like ordinary tensor components *of the same rank*.

3. The presentation of a relatively-simple formula (61) for the calculation of the covariant derivative of a c-tensor, which is a formula that is of interest mainly because its validity requires the in itself welcome (... text missing from original)

4. The *normalization* of those trace components of Γ_{λ} that would give rise to the appearance of pure-imaginary electromagnetic field strengths if they *were not* normalized.

§ 7. – The Dirac equation.

The operator $\gamma^k \nabla_k$ is an invariant that one can aptly refer to as the "magnitude of the gradient." The generalized **Dirac** equation demands that (¹):

$$\gamma^k \, \nabla_k \, \psi = \mu \, \psi, \tag{65}$$

in which μ is a universal constant:

$$\mu = \frac{2\pi mc}{h}.$$

The *c*-vector that belongs to γ^k under the assignment (57) is *i* S^k , so:

$$i S^{k} = \psi^{*} \gamma_{0} \gamma^{k} \psi.$$
 (69)

Since the covariant derivative of the *operator* γ^k vanishes, from (61), that will reduce the covariant derivative of S^k to:

$$i S^{k}_{; \lambda} = \nabla_{\lambda} \psi^{*} \gamma_{0} \gamma^{k} \psi + \psi^{*} \gamma_{0} \gamma^{k} \nabla_{\lambda} \psi.$$

If one forms the covariant divergence of this by contraction:

(¹) One can generally try to "symmetrize" this and take:

$$\frac{1}{2} \left(\gamma^k \, \nabla_k + \nabla_k \, \gamma^k \right) \tag{66}$$

to be the left-hand side of (65). However, this expression can be converted. The vanishing of the covariant derivative of γ^k says that:

$$\nabla_l \gamma^k - \gamma^k \nabla_l = -\Gamma_{l\mu}^k \gamma^{\mu}.$$

Contraction will yield:

$$\nabla_{k} \gamma^{k} - \gamma^{k} \nabla_{k} = -\Gamma_{k\mu}^{k} \gamma^{k} = -\frac{\partial \ln \sqrt{g}}{\partial x_{\mu}} \gamma^{\mu} .$$
(67)

As a result:

$$\frac{1}{2} \left(\gamma^k \, \nabla_k + \nabla_k \, \gamma^k \right) = \gamma^k \, \nabla_k - \frac{\partial \ln \sqrt{g}}{\partial x_\mu} \, \gamma^\mu = g^{1/4} \, g^k \, \nabla_k \, g^{-1/4}. \tag{68}$$

That is *not* an invariant operator, which should not surprise us. Namely, $\nabla_k \gamma^k$ is not one either, nor does it have any obligation to be such a thing. We have already emphasized above that the product of two tensor operators will be certain to have the tensor property only when the *left-hand* factor does not include the derivative. Furthermore, the use of the Ansatz (66) would still proceed in the same way again, except that one must merely put $g^{-1/4} \psi$ in place of ψ ; that is, $g^{-1/4} \psi$ must transform as a spinor. For that reason, we shall keep the Ansatz (65).

$$i S^{\lambda}_{; \lambda} = \nabla_{\lambda} \psi^{*} \gamma_{0} \gamma^{\lambda} \psi + \psi^{*} \gamma_{0} \gamma^{\lambda} \nabla_{\lambda} \psi,$$

then the first summand will be negative of the complex conjugate of the second one $(^{1})$, but from (65), that will be:

which will then be real, since γ_0 is Hermitian. Hence:

$$S^{\lambda}_{;\lambda} = 0. \tag{70}$$

The source-free character of the *four-current*, which, from our assignment (57), belongs to the *contravariant metric vector* as a *c*-vector, will then follow from the **Dirac** equation and the fundamental equations (8) (cf., **Fock**, *loc. cit.*, pp. 267)

We would now like to square the **Dirac** equation in order to compare the result with the ones that are familiar from the special theory (ψ has been omitted, for brevity):

$$\gamma^{\lambda} \nabla_{\lambda} \gamma^{l} \nabla_{l} = \mu^{2}. \tag{71}$$

One switches the first two factors by means of equation (67) (in the remark) and employs the fact that from (2) and (12), one will have:

$$\gamma^{\lambda} \gamma^{l} = g^{kl} + s^{kl}. \tag{72}$$

That will give:

$$\nabla_k (g^{kl} + s^{kl}) \nabla_l + \frac{\partial \ln \sqrt{g}}{\partial x_{\mu}} \gamma^{\lambda} \gamma^l \nabla_l = \mu^2.$$

It follows from the vanishing of the covariant derivative of s^{kl} that:

$$abla_k s^{kl} - s^{kl} \nabla_k = - \frac{\partial \ln \sqrt{g}}{\partial x_\mu} s^{\mu l}.$$

That gives [with another use of (72)]:

$$abla_k g^{kl} \nabla_l + s^{kl} \nabla_k \nabla_l + \frac{\partial \ln \sqrt{g}}{\partial x_{\mu}} g^{\mu l} \nabla_l = \mu^2.$$

From (26), and due to the antisymmetry in s^{kl} , the second term will be equal to $-\frac{1}{2}s^{kl}\Phi_{kl}$. The first and third ones (in which one replaces μ with k) combine into the generalized **Laplace** operator; one therefore ultimately gets:

^{(&}lt;sup>1</sup>) The *Hermitian* operator $\gamma_0 \gamma^0$ goes over to $(\gamma_0 \gamma^0)^*$ in the first factor, while the skew operator $\gamma_0 \gamma^{\lambda} (\lambda \neq 0)$ will go to $-(\gamma_0 \gamma^{\lambda})^*$. However, in exchange, ∇_0 will include a sign change, but not $\nabla_{\lambda} (\lambda \neq 0)$. Compare the above to equation (60) in the text, as well as noting the remarks that were made in regard to equation (56).

$$\frac{1}{\sqrt{g}}\nabla_k \sqrt{g} g^{kl} \nabla_l - \frac{1}{2} s^{kl} \Phi_{kl} = \mu^2.$$
(73)

It is interesting to substitute the expression for Φ_{kl} in the expression (15) that was found before. In that way, the invariant:

$$\frac{1}{8} R_{kl,\mu\nu} s^{kl} s^{\mu\nu}$$

will appear. Due to the symmetry of the covariant **Riemann**ian curvature tensor in the first and second index-pair, that will equal:

$$\frac{1}{16}R_{kl,\mu\nu}(s^{kl}s^{\mu\nu}+s^{\mu\nu}s^{kl}).$$

If one actually calculates the symmetric product of the s^{kl} now (which I would not like to do here *in extenso*) then one will use the known cyclic symmetry:

$$R_{kl,\mu\nu}+R_{l\mu,k\nu}+R_{\mu k,l\nu}=0.$$

Having made use of that, one will finally get:

$$\frac{1}{8}R_{kl,\mu\nu}s^{kl}s^{\mu\nu} = -\frac{1}{4}s^{k\mu}s^{l\nu}R_{kl,\mu\nu} = -\frac{R}{4},$$

in which *R* is the *invariant curvature*. Hence, the substitution of Φ_{kl} from (15) in (73) will yield the following:

$$\frac{1}{\sqrt{g}} \nabla_k \sqrt{g} g^{kl} \nabla_l - \frac{R}{4} - \frac{1}{2} f_{kl} s^{kl} = \mu^2.$$
(74)

In the third term on the left-hand side, one recognizes the well-established effect of the field strengths on the spin tensor, and indeed the pure trace part of Φ_{kl} has already been solved for in f_{kl} , which should probably be referred to as field strengths in the real sense, and as we have often mentioned, they will be completely free of the metric.

The second term seems to me to have considerable theoretical interest. Of course, it is many, many powers of ten too small to be capable of *replacing*, say, the term on the right-hand side. μ will then be the reciprocal **Compton** wave length, which is approximately 10¹¹ cm⁻¹. At any rate, it seems meaningful that in the generalized theory, any term that is equivalent to the enigmatic mass term would come into play at all (¹).

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^{(&}lt;sup>1</sup>) See above, **Veblen** and **B. Hoffman**, Phys. Rev. **36** (1930), 821.