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On force-free motion in relativistic quantum mechanics

By

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1. The contents of this publication consist of some results of the application of the operator calculus to those physical systems that can be described by the DIRAC wave equation:

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} + H\psi = 0, \tag{1}$$

with

$$H = c\,\alpha_1\,p_1 + c\,\alpha_2\,p_2 + c\,\alpha_3\,p_3 + \alpha_4\,mc^2; \tag{2}$$

one ordinarily calls such systems DIRAC electrons. We shall restrict ourselves to the force-free case, because we shall concern ourselves with the precisely the fact that the DIRAC system behaves in a relatively-complicated manner even in the absence of external forces. According to BREIT ([1]), the HAMILTONIAN operator (2) is best put into a corresponding parallel with the following expression for the LORENTZ energy of the electron:

$$\frac{mc^2}{\sqrt{1-\beta^2}} = \mathfrak{v}_x\cdot\frac{m\mathfrak{v}_x}{\sqrt{1-\beta^2}} + \mathfrak{v}_y\cdot\frac{m\mathfrak{v}_y}{\sqrt{1-\beta^2}} + \mathfrak{v}_z\cdot\frac{m\mathfrak{v}_z}{\sqrt{1-\beta^2}} + \sqrt{1-\beta^2}\cdot mc^2 .$$

In a certain sense, one has the following correspondence:

$c\,\alpha_1, c\,\alpha_2, c\,\alpha_3$	…	the velocity components $\mathfrak{v}_x, \mathfrak{v}_y, \mathfrak{v}_z$
α_4	…	$\sqrt{1-\beta^2}$
p_1, p_2, p_3	…	the impulse components $m\mathfrak{v}_x/\sqrt{1-\beta^2}$, etc.
m, c, h	…	the well-known constants of nature.

What makes the Ansatz (2) so special is that, with DIRAC, one thinks of *abandoning the trivial connection between velocity and impulse. One then has by no means, say:*

([1]) G. BREIT, Proc. Amer. Acad. **14** (1928), 553.

$$\alpha_1 = \frac{\alpha_4 p_1}{m} .$$

Moreover, it is obvious that the concept of velocity has been emancipated from the concept of impulse. One must understand that the p_1, p_2, p_3 in the wave equation (1) mean the operators $\frac{h}{2\pi i}\frac{\partial}{\partial x_1}$, $\frac{h}{2\pi i}\frac{\partial}{\partial x_2}$, $\frac{h}{2\pi i}\frac{\partial}{\partial x_3}$, while the α_1, α_2, α_3, α_4, however, mean operators that one best expresses by saying that they act upon a fifth variable ζ upon which ψ depends, in addition to x_1, x_2, x_3, t. Nonetheless, that "fifth variable" has only four discrete values – i.e., it is an index – so each value of ψ will exist with four different indices (which cannot be associated with the four world-coordinates, however!). One must then think of the α_k in the wave equation (1) as four four-rowed Hermitian matrices with numbers such as 1, – 1, $\sqrt{-1}$, *et al.*, for their matrix elements that are fixed once and for all. That then means that the action of, for example, α_2 on ψ will be as follows: ψ_1, ψ_2, ψ_3, ψ_4 will be replaced with a certain linear combination of ψ_1, ψ_2, ψ_3, ψ_4 (the arguments x_1, x_2, x_3, t will remain the same). The matrices α_i are chosen in such a way that when each of them is squared, that will give the identity matrix, while they will "anticommute" amongst themselves:

$$\alpha_i\,\alpha_k + \alpha_k\,\alpha_i = 2\,\delta_{ik}\cdot 1 \qquad (i, k = 1, 2, 3, 4). \tag{3}$$

One will be led to that fundamental and momentous requirement by the desire that equation (1) should lead to the *scalar* relativistic wave equation ([1]) for each ψ_i.

The manifold of solutions of equation (1) is so exceptionally large that the *method of wave mechanics*, which consists of investigating the behavior of special, representative solutions, *are unimaginably unsuitable here* if we are to arrive at results that have any general interest. The method of q-numbers – or the operator calculus – is splendidly appropriate to that investigation; I have recently pointed out the methodological contradiction ([2]). The gimmick of the operator calculus consists of the fact that one attributes the time variation of the ψ-function to the operators. One therefore calculates, in a certain sense, with only an individual ψ-function; perhaps, to fix ideas, with the one that is true for $t = 0$. At that time-point (to fix ideas), the association of operators with the symbols α_k, p_k that was discussed above might be valid. One now asks: How must I now allow the operators to vary in time in order that when they are coupled with the temporally-unvarying ψ-function, I will come to the same physical statements to which I would come if I had established the association of operators and let ψ vary according to equation (1)? It is highly gratifying that one can now show that the answer to that question is *completely independent* of the initial distribution of the ψ-function, so it will remain entirely arbitrary (except for the normalization condition $\int \psi^* \psi\, dx = 1$), and in addition, the answer *will also read the same way for any arbitrary operator A*; as is known, it will read:

([1]) Cf., the author's *Abhandlungen zur Wellenmechanik*, 2nd ed., pp. 1, 12, 162, 178.
([2]) These Berichte, pp. 300, *et seq.*

$$\frac{h}{2\pi i}\frac{dA}{dt} = HA - AH. \tag{4}$$

I cannot include the brief proof of that here, although it might presently be known to only my closest colleagues. To abbreviate, let us always set:

$$\kappa = \frac{h}{2\pi i}.$$

The *general solution* of equation (1) is:

$$\psi(x_1, x_2, x_3, \zeta, t) = e^{-Ht/\kappa}\,\psi(x_1, x_2, x_3, \zeta, 0). \tag{5}$$

H is an operator, here. (It is obviously time-independent, since we indeed think of equation (5) as being "wave-mechanical.") t is the usual time, so it is not an operator. The operator $e^{-Ht/\kappa}$ is therefore unmistakable. The solution is obviously *single-valued* for a given $\psi(x_1, x_2, x_3, \zeta, 0)$.

The operator equation (4) also has a *single-valued* general solution. One next demands that H *should be constant* in the special case of $A \equiv H$ [also in the sense of the operator calculus. That is certainly not established from the outset, but one does require equation (4)]. Once that has been given, the operator equation (4) can be solved for an arbitrary A in general by:

$$A(t) = e^{Ht/\kappa} A(0)\, e^{-Ht/\kappa}, \tag{6}$$

and the solution will obviously be single-valued.

It must now be shown that $A(t)$ is coupled with $\psi(0)$ in the well-known way by forming the *expectation value* $\bar{A}$, and that will also show how $A(0)$ is coupled with $\psi(t)$. That is all that was required. One must then show that:

$$\int \psi^*(0) \cdot A(t)\psi(0)\,dx = \int \psi^*(t) \cdot A(0)\psi(t)\,dx$$

i.e., from (5) and (6):

$$\int \psi^*(0) \cdot e^{Ht/\kappa} A(0)\, e^{-Ht/\kappa}\psi(0)\,dx = \int e^{H^* t/\kappa}\psi^*(0) \cdot A(0)\, e^{-Ht/\kappa}\psi(0)\,dx.$$

One sees that this is correct when one moves the operator $e^{H^* t/\kappa}$ from the first factor to the second one, while one only switches the "rows and columns": i.e., if H is Hermitian then H will become H^*.

The proof of that requires only the Hermiticity of H that was employed just now, as well as the fact that H does not contain time explicitly [otherwise, (5) would not be the solution to (1)]. Other than that, H can be whatever one wants it to be.

We employ the solution (6) merely to see, quickly and without calculation, that commutation relations with constant right-hand sides [such as, e.g., the relations (3)] will remain preserved at all times, even from the operator standpoint. Therefore, from (6), an operator that equals 0 or 1 (or commutes with H, more generally) at time zero (or at any time, more generally) will obviously be *continually* constant. Other than that, we shall now work exclusively with the basis equation (4), to which we shall add merely the explanation for H by (2), the commutation relations (3), and finally, the well-known commutation relations for the p_k with the *operators* x_k :

$$p_k\, x_i - x_i\, p_k = \kappa\, \delta_{ik} \cdot 1. \tag{7}$$

2. If one now takes the time derivative of a coordinate ([1]) using (3) – e.g., x_k – then, from (2), (3), and (7), one will actually get:

$$\frac{dx_k}{dt} = c\, \alpha_i \qquad (k = 1, 2, 3). \tag{8}$$

One might have wondered about that, since, in fact:

$$\left(\frac{dx_k}{dt}\right)^2 = c^2\, \alpha_i^2 = c^2 \cdot 1.$$

(Cf., BREIT, *loc. cit.* and FOCK, *loc. cit.*) The square of each velocity *component* is then capable of taking on only the *measured value* c^2, which must then be likewise the *mean value* (i.e., expectation value) of *many* measurements of the same wave packet. The velocity components themselves admit only the measured values $\pm c$. Their expectation values can and will be small, in general. All the same, one would expect an *order of magnitude c* for them, in general, and might wonder how the center-of-mass of the charge cloud might manage to move that fast and still advance with measurable velocity.

Now, obviously, *it will not move rectilinearly*. In fact, from (4), the α_k are not constant, since they do not commute with H. From (2) and (3) ([2]), one calculates:

$$H\, \alpha_k + \alpha_k\, H = 2\, c\, p_k \qquad (k = 1, 2, 3), \tag{9}$$

and as a result, from (4):

$$\kappa \frac{dx_k}{dt} = H\, \alpha_k - \alpha_k\, H = 2\,(c\, p_k - \alpha_k\, H) = 2\,(H\alpha_k - c\, p_k) \qquad (k = 1, 2, 3). \tag{10}$$

Since the impulse components p_k commute with H, they will be constant in time, so that equation will contain only the variable x_k , and can be integrated. One introduces the variable η_k for x_k :

$$\eta_k = \alpha_k - c\, H^{-1}\, p_k \qquad (k = 1, 2, 3). \tag{11}$$

One then comes to:

([1]) From now on, expressions like "coordinate," "impulse," "energy," etc., will always means the relevant operators. Only t and the constants of nature will be ordinary numbers.

([2]) The α_k commute with the x_i and p_i , since they indeed act upon other variables.

$$\kappa \frac{dx_k}{dt} = -\,2\eta_k\, H = 2\, H\, \eta_k\,. \tag{12}$$

If one integrates that then:

$$\eta_k = \eta_k^0\, e^{-2Ht/\kappa} = e^{2Ht/\kappa} \eta_k^0\,. \tag{13}$$

η_k^0 is the "value" of η_k (i.e., the form of the operator η_k) for $t = 0$. From (12), η_k (and naturally η_k^0, as well) anticommutes ([1]) with H. All operators that anticommute with H depend upon the time parameter in the manner of (13) [that follows from (6) directly] and have constant squares. If one again introduces α_k into (13), from (11), and substitutes the values for α_k that were found in (8) then that will imply:

$$\frac{dx_k}{dt} = c^2\, H^{-1}\, p_k + c\,\eta_k^0\, e^{-2Ht/\kappa}\,. \tag{14}$$

If one integrates this then:

$$x_k = a_k + c^2\, H^{-1}\, p_k\, t - \frac{c\kappa}{2} \eta_k^0\, H^{-1} e^{-2Ht/\kappa}. \tag{15}$$

a_k is the integration constant (operator!) that will not be written as x_0^k, due to the fact that it is not precisely the value of x_k for $t = 0$.

From (15), the coordinate x_k consists of two summands, for which we introduce special sign:

$$\left.\begin{aligned} x_k &= \tilde{x}_k + \xi_k, \\ \tilde{x}_k &= a_k + c^2 H^{-1} p_k t, \\ \xi_k &= -\frac{c\kappa}{2}\eta_0^k H^{-1} e^{-2Ht/\kappa} = -\frac{c\kappa}{2}\eta_k H^{-1} = \frac{c\kappa}{2} H^{-1}\eta_k. \end{aligned}\right\} \tag{16}$$

The first summand $\tilde{x}_k$ grows linearly with time, and in fact, with a *velocity* that corresponds to the *impulse* p_k, and which has nothing to do with the α_k and does not at all need to have an order of magnitude of c. In fact, if, say, the ψ-function – i.e., the "wave packet" – consists only a superposition of energy-impulse eigenfunctions on a small domain – i.e., plane wave of small wave lengths on a narrow domain of the wave normal – then the operators $c^2\, H^{-1}\, p_k$ will come about as follows: Multiply:

([1]) One does not adhere to the demand that η_k^0, as the "initial value" of η_k, *per definitionem*, must be constant, and therefore it does not have to commute with H. That is analogous to the way that in ordinary mechanics, the initial value x_0 of x is also certainly constant, but it will in no way follow that $\left(\frac{dx}{dt}\right)_0$ must vanish.

$$c^2 \cdot \frac{\sqrt{1-\beta^2}}{mc^2} \cdot \frac{m\mathfrak{v}_k}{\sqrt{1-\beta^2}} = \mathfrak{v}_k ,$$

in which $\mathfrak{v}_k$ and β are connected with the DE BROGLIE wave length in the usual way that has been confirmed by experiment (not with the α_k , as was discussed in the corresponding parallel manner to begin with).

However, a second summand ξ_k must be added to that, which has an obvious periodic character (since $\kappa = h / 2\pi i$ is pure imaginary), and in fact, a truly-complicated "almost-periodic" character, in general. $c\alpha_k$ is the velocity of that high-frequency, small-amplitude, rapid *zitterbewegung* (cf., what follows), which is overlaid with the uniform, rectilinear motion. One can also say that $c\alpha_k$ is the *instantaneous velocity* of the center-of-mass of the charge cloud. In fact, for very short time intervals, x_k will be given by *an entirely different* linear function. If one develops that *e*-function in (15) and combines it with the linear term that is already present there then one will obviously get $c\alpha_k^0 t$, since otherwise, (15) would indeed not be a correct solution to (8). Hence, $c^2 H^{-1} p_k$ is, so-to-speak, the macroscopic velocity of the electron, while $c\alpha_k$ is the microscopic velocity, or when one expresses that more intuitively: The expectation values of those operators will give the macroscopic (microscopic, resp.) velocity of the center-of-mass of the charge cloud.

The *amplitude* of the zitterbewegung is especially interesting. From (16), it can be estimated by:

$$\xi_k = \frac{c\kappa}{2} H^{-1} \eta_k , \tag{16′}$$

with no further assumptions. η_k , just like α_k , "has order of magnitude 1," and for slow motion of the electron, H will have order of magnitude mc^2 ($\kappa = h / 2\pi i$). The expectation value of x_k then has the order of magnitude:

$$\bar{\xi}_k \approx \frac{h}{4\pi mc} \approx 10^{-11} \text{ cm.} \tag{17}$$

That is the well-known critical length dimension (viz., the Compton wave length) that one cannot reduce a wave packet to without having to accept an oscillation in the impulse of the enormous magnitude of *mc*, according to the uncertainty relation. For an electron with a more-or-less *well-defined* macroscopic velocity, the deviations of the center-of-mass from the rectilinear path will be much smaller than the size of the charge cloud. One can get an even finer estimate by squaring (16′), since x_k, just like η_k, anticommutes with H, and thus will have a *constant* square:

$$\xi_k^2 = -\frac{c^2\kappa^2}{4} H^{-1} \eta_k^0 .$$

Now, from (11), one has:

$$H\,\eta_k = H\,\alpha_k - c\,p_k , \qquad \eta_k\,H = \alpha_k\,H - cp_k ,$$

$$H^2\eta_k^0 = H^2 - c\, p_k\,(H\,\alpha_k + \alpha_k\,H) + c^2\,p_k^0 .$$

Hence, from (9):

$$H^2\eta_k^0 = H^2 - c^2\,p_k^0 .$$

Thus:

$$\xi_k^2 = \frac{h^2c^2}{16\pi^2} H^{-2}\,(1 - H^{-2}\,c^2\,p_k^0). \tag{18}$$

When that is applied to a wave structure of more-or-less unitary energy-impulse, an argument that is similar to the one above will imply that:

$$\overline{\xi_k^2} = \frac{h^2}{16\pi^2 m^2 c^2}(1-\beta_k^2)(1-\beta^2)\,, \tag{19}$$

in which β and β_k are the total velocity and its k-components, which correspond to the *energy and impulse*, and have nothing to do with the α_k . Apparently, the amplitude of the zitterbewegung will then decrease as one approaches the speed of light.

By the way, one must beware of the following closely-related error: From (16), one probably has:

$$\bar{\xi}_k = \bar{x}_k - \bar{\tilde{x}}_k\,,$$

or, when written more clearly:

$$\bar{\xi}_k = \overline{x_k - \tilde{x}_k}\,;$$

i.e., $\bar{\xi}_k$ is probably the mean distance from the charge cloud to a plane that is perpendicular to the k-direction and goes through the point whose coordinates (c-numbers!) have the expectation values $\tilde{x}_k$ and moves in a uniform, rectilinear way. However, *by no means* is – say – $\overline{\xi_k^2}$ the mean square of that distance for the points of the charge cloud. Otherwise, one would indeed read off from (18) that the DIRAC equations admit only charge clouds with the linear dimension $\dfrac{h}{4\pi mc}$, no more and no less, and that would obviously be absurd. Moreover, one must imagine things as follows: The *true* statistics of position (= charge cloud) that are described by the operators x_k will be described by the *detour* from the statistics of position of a *fictitious* point $\tilde{x}_k$ that deviates from it only slightly and is less simple than if it corresponded, in the mean, to a uniform, rectilinear motion and contained no operators that act upon the index coordinate ζ. ξ_k will then describe the statistics of the true position relative to the fictitious one. Each point of the fictitious charge cloud will be, to a certain extent, atomized into a small charge cloud, and indeed in the same way. That small charge cloud will be the one that has linear dimensions of order $\frac{h}{4\pi mc}$, no more and no less, and constant (= time-independent) quadratic moment relative to the fictitious point ([1]).

3. One is inclined to regard the statistics of position that are described by x_k as the actual model of the internal structure of the electron "after removing the translation," and one might attempt to connect it with the phenomenon of spin. One arrives at the concept of spin by the remark that the impulse moment operator is not constant under force-free motion either, and the vector sum of it and another operator is constant, moreover ([2]). We will now appeal to the three-dimensional vector symbolism. The symbols p, α, x, $\tilde{x}$, ξ, η, etc. *with no index* shall mean three-vectors with the components p_1, p_2, p_3; α_1, α_2, α_3, etc., and furthermore, $(\alpha\, p) = \alpha_1\, p_1 + \alpha_2\, p_2 + \alpha_3\, p_3$ shall mean the inner product whose components are $\alpha_2\, p_3 - \alpha_3\, p_2$, etc. $[\alpha\, p]$ shall mean the vector product, in which one must remember that the vector product of a vector with itself does *not* vanish, in general, and especially that one must observe the sequence of factors upon differentiating; for example:

$$\frac{d}{dt}[\alpha, \alpha] = \left[\frac{d\alpha}{dt}, \alpha\right] + \left[\alpha, \frac{d\alpha}{dt}\right]. \tag{20}$$

We write down equations (10) vectorially as:

$$\begin{aligned} \frac{\kappa}{2}\frac{d\alpha}{dt} &= H\alpha - cp, \\ \frac{\kappa}{2}\frac{d\alpha}{dt} &= cp - \alpha H, \end{aligned} \tag{10′}$$

and then vectorially multiply the first one by α on the right and the second one on the left, and add them together. Since the α commute with the p, only the sign will change with the order of factors in their vector product. Due to the general equation (4), one will get:

$$\frac{\kappa}{2}\frac{d}{dt}[\alpha, \alpha] = \kappa\frac{d}{dt}[\alpha, \alpha] + 2c\,[\alpha, p]$$

or

([1]) Not only is ξ_k^2 constant, but also $\xi_k\, \xi_i$ for $k \neq l$. That follows from its anticommutation with H:

$$k\,\frac{d}{dt}(\xi_k\, \xi_i) = H\, \xi_k\, \xi_i - \xi_k\, \xi_i\, H = -\, \xi_k\, H\, \xi_i - \xi_k\, \xi_i\, H = -\, \xi_k\,(H\, \xi_i - \xi_i\, H) = 0.$$

([2]) The fact that one must still distinguish between orbital moment and spin moment in the absence of an external field was recently proved by E. FUES and H. HELLMANN in an interesting paper [Phys. Zeit. **31** (1930), 465], with which, the present one has many intrinsic points of contact, but an entirely different methodology.

$$\frac{\kappa}{2}\frac{d}{dt}[\alpha,\alpha] = -c\,[\alpha, p] = -\frac{d}{dt}[x, p]\,. \tag{21}$$

The last equation follows from (8) and the constancy of p. Hence, one will have:

$$[x, p] + \frac{\kappa}{4}[\alpha,\alpha] = \text{const.} \tag{22}$$

$[x, p]$ – if one adopts the classical definition – the impulse moment (viz., "orbital impulse"). We will straightaway see that it is not constant by itself, even for the force-free motion that we consider here, but only the sum above will be constant. That alone already justifies (without referring to the extra term in H that first appears in a field) referring to the vector whose three components read, e.g.:

$$\frac{h}{8\pi i}(\alpha_1\,\alpha_2 - \alpha_2\,\alpha_1) = \frac{h}{4\pi i}\,\alpha_1\,\alpha_2 = \frac{h}{4\pi}s_3$$

as an "extra impulse moment" (i.e., "spin moment"). We then introduce the new vector s by:

$$\alpha_1\,\alpha_2 = i\,s_3\,,\ \text{etc.} \qquad \text{or} \qquad [\alpha, \alpha] = 2is. \tag{23}$$

Its components are Hermitian, since those of $[\alpha, \alpha]$, as commutators, will anticommute.) The following relations are easily verified:

$$[s, s] = -2is, \qquad [\alpha, s] = +[s, \alpha] = 2i\,\alpha. \tag{24}$$
$$s_1^2 = s_2^2 = s_3^2 = 1.$$

The s_k, like the α_k, have only the eigenvalues ± 1 then. Now, equation (21) then reads:

$$\frac{h}{4\pi}\frac{ds}{dt} = -c\,[\alpha, p]. \tag{22′}$$

The components of $[\alpha, p]$ belong to the quantities that anticommute with H; in fact:

$$H\,[\alpha, p] + [\alpha, p]\,H = [H\alpha + \alpha H, p] = 2c\,[p, p] = 0.$$

[That employs (9), as well as the fact that the p_k commute with each other and with H.] As a result, it will depend upon time in the manner of (13):

$$[\alpha, p] = [\alpha, p]_0\, e^{-2Ht/\kappa} = e^{2Ht/\kappa}\,[\alpha, p]_0\,. \tag{25}$$

When that is substituted into (22′) and integrated, that will give:

$$\frac{h}{4\pi}s = \frac{h}{4\pi}\tilde{s} + \frac{c\kappa}{2}[\alpha, p]_0 H^{-1} e^{-2Ht/\kappa}. \tag{26}$$

$\tilde{s}$ is an integration constant (as an operator and a vector) that is connected with the "initial values" of s, α, p in a way that is easy to give ([1]). With that, the variable part of the spin moment is exhibited explicitly. If the macroscopic velocity $c^2 H^{-1} p$ is small in comparison to the speed of light then the amplitude of the variable part of s will be small in comparison to the constants, and indeed of first order in β. Namely, from (25), one can also write:

$$\frac{h}{4\pi}s = \frac{h}{4\pi}\tilde{s} + \frac{h}{4\pi i}[\alpha, cH^{-1}p] \tag{26'}$$

for (26). s and α are of order 1 (with eigenvalues ± 1), $c H^{-1}p$ has order β. If one scalar-multiplies (26′) by p then one will find upon applying the elementary commutation rules of vector algebra to the three-product:

$$(sp) = (\tilde{s}p) = \text{const.} \tag{27}$$

The component of the spin in the direction of the (linear) impulse) is constant. Indeed, that also follows already from (22′), since the change in the velocity of s "is perpendicular to p." Admittedly, one must be cautious with such conclusions; i.e., observe non-commutation. For example, that scalar product with the other factor $(s\alpha)$ is in no way constant, as one can easily convince oneself.

Now, in order to come to the connection between spin and our "internal position statistics" ξ_k, we assume (16):

$$\xi = \frac{c\kappa}{2}H^{-1}\eta = \frac{ch}{4\pi i}H^{-1}\eta, \tag{16'}$$

and multiply it vectorially on the right by p:

$$[\xi p] = \frac{ch}{4\pi i}H^{-1}[\eta p].$$

Due to (11):

$$\eta = \alpha - c H^{-1} p, \tag{11}$$

one will get:

$$[\xi p] = \frac{ch}{4\pi i}H^{-1}[\alpha p] = -[\alpha, c H^{-1} p]. \tag{28}$$

That is equal and opposite to the variable part of the spin impulse [cf., (26′)]; i.e., $[\xi p]$ is the variable part of the orbital impulse, which is illuminated immediately by (16) ($[\tilde{x}\, p]$ is constant).

([1]) Namely, one obviously has: $\tilde{s} = s_0 + ic\,[\alpha, p]_0\, H^{-1}$.

One asks whether one can also present the constant part of the spin impulse by combining the "lever arm ξ" with a suitably-chosen linear impulse. For that, one appeals to the impulse quantities that belong to the *actual* ("microscopic") velocity $c\alpha$, from which, the linear impulse will arise upon multiplying by H / c^2. It is somewhat more convenient in calculations to first work with $c\eta$, instead of $c\alpha$; i.e., the microvelocity that has been corrected by the macrovelocity. We then choose the "micro-impulse" to be, perhaps:

$$\frac{\eta H}{c}$$

and calculate, from (16′) and (12′):

$$\left[\xi, \frac{\eta H}{c}\right] = \frac{h}{4\pi i}[\eta, \eta] = \{[\alpha, \alpha] - [\alpha, c\, H^{-1} p] - [c\, H^{-1} p, \alpha]\}$$

$$= \frac{h}{4\pi i}\{[\alpha, \alpha] - 2[\alpha, c\, H^{-1} p]\}. \tag{29}$$

[Two changes of sign compensate here. α "anticommutes with H, up to a commutator const. $\cdot\, p$"; cf., (9).] Hence, from (23):

$$\left[\xi, \frac{\eta H}{c}\right] = \frac{h}{2\pi}s - \frac{h}{2\pi i}[\alpha, c\, H^{-1} p] = 2 \cdot \frac{h}{4\pi}\tilde{s}\,. \tag{30}$$

[Cf., (26′)] In regard to Hermiticity, things are such that ηH is indeed skew-Hermitian, since η and H skew commute. The vector product is, however, again Hermitian, since it is essentially a commutator, as (29) shows. One then gets *twice* the constant part of the spin impulse moment in that way. As far as I can see, this not-entirely-suspected result cannot be avoided by any rearrangement of the factors or by employing α instead of η, but only when one combines the level arm ξ with one-half of the micro-impulse that was employed above. For small macrovelocities, the $[\alpha,\alpha]$ would always take on a factor of 2 then. In particular, the total spin would also seem to be the vector product of ξ with an impulse that is perhaps only one-half as large as the one that was introduced above (always thinking: for *small* macrovelocity). One easily finds from (28) and (30) that:

$$\frac{h}{4\pi}s = \left[\xi, \frac{\eta H}{c} - p\right],$$

which can be rewritten in various ways, from (9) and (11), without however being required to express s in terms of x and η or in terms of x and α alone, but not p.

The strangely-bizarre relationships that come about for the DIRAC equation, even for a force-free mass point, seem to me to be worth presenting, regardless of whether I can point out any conclusive results of that investigation at the moment.

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