# Fulfilling the demand of relativity in classical mechanics 

By E. Schrödinger

Translated by D. H. Delphenich

It is known that E. Mach ${ }^{1}$ ) raised the objection to the classical point mechanics with central forces, whose foundations were worked out clearly by L. Boltzmann ( ${ }^{2}$ ), that it did not satisfy the relativity demand that was imposed from the epistemological standpoint: Its laws are not true for arbitrarily-moving coordinate systems, but only for a group of so-called inertial systems, which move with a uniform translatory motion with respect to each other. Empirically, those would be axis-crosses whose intersections with the fixed stars are at rest or in uniform translatory motion, but the foundations of classical mechanics by no means make the basis for that clear.

General relativity, in its original form $\left(^{3}\right.$ ), could still not satisfy Mach's requirement, as was soon recognized. Once the secular precession of Mercury's perihelion was deduced from its amazing agreement with experiment, one must naively ask: With respect to what does the elliptical orbit perform the rotation in theory that exists with respect to the mean system of fixed stars in experiment? One gets the following answer: The theory requires that this rotation must take place with respect to a coordinate system in which the gravitational potential satisfies certain boundary conditions at infinity. The connection between those boundary conditions and the presence of fixed celestial bodies was by no means clear since the latter were not introduced into the calculations at all.

A way of overcoming that difficulty is currently suggested by the cosmological theories, which demand a spatially-closed universe, and in that way that avoids any boundary conditions at all. Due to the conceptual difficulties that those cosmological theories still exhibit $\left({ }^{4}\right)$, not the least of which are the mathematical difficulties in understanding them, the solution to an important epistemological problem that every structure in natural science immediately poses is pushed into a domain into which few can follow and in which it is not actually easy to maintain a clear distinction between fact and fiction. I have no doubt that when the solution is ultimately achieved in the sense of that theory, it will not only be satisfied to a high degree, but it will also be representable in a form that will afford a real insight into a more advanced viewpoint. However, with the present state of affairs, it is perhaps not pointless to ask whether Mach's relativity demand

[^0]might not be satisfied by some simple modification of classical mechanics that would make the way that inertial systems are determined by the fixed stars understandable in a simple way $\left({ }^{1}\right)$.

The Ansatz for potential energy in point mechanics, and the Newtonian potential in particular, satisfies Mach's postulate with no further assumptions, since it depends upon only the distance between the two mass-points, and not upon their absolute positions in space. For that reason, since it has proved to be preserved, even from the standpoint of that postulate, let it also be regarded only as a first approximation to a law that is, in reality, perhaps more complicated. Things are different for kinetic energy. In classical mechanics, it is determined by the absolute motion in space, while chiefly only relative motions, such as distances and changes of distance between mass-points, are observable. One must then check whether it might not be possible to regard kinetic energy, just like potential energy up to now, not as an energy that is ascribed to the mass-points individually, but as an energy of interaction between two mass-points, and let it depend upon only the distance and the rate of change of the distance between the two points. In order to select one Ansatz from the wealth of possibilities, we will heuristically employ the following demands on the analogy:

1. The kinetic energy, as an energy of interaction, shall depend upon the masses and the distance between the two points in the same way as the Newtonian potential.
2. It shall be proportional to the square of the rate of change of the distance.

One has the following Ansatz for the total interaction energy between two mass-points with the masses $\mu, \mu^{\prime}$ at a distance of $r$ :

$$
\begin{equation*}
W=\gamma \frac{\mu \mu^{\prime} \dot{r}^{2}}{r}-\frac{\mu \mu^{\prime}}{r} . \tag{1}
\end{equation*}
$$

The masses are measured in units that make the gravitational constant equal to 1 here. The temporarily-undetermined constant $\gamma$ has the dimension of the square of a reciprocal velocity here. Since it is supposed to be universal, one would expect that one would be dealing with the speed of light, up to a numerical factor, or that $\gamma$ reduces to a numerical factor when one chooses the unit of time to be the light-second. One would then be permitted to set that numerical factor equal to 3 .

We now imagine that a mass-point $\mu$ is placed in the vicinity of the center of a hollow sphere of radius $R$ that is covered by mass with a density of $\sigma$. We refer all statements to a coordinate system in which the sphere is at rest. Let the mass-point be in motion within it. while its spatial polar coordinates are $\rho, \vartheta, \varphi$, and those of a surface element of the sphere are $R, \vartheta^{\prime}, \varphi^{\prime}$. The distance $r$ between the points of the surface element is given by:

[^1]\[

\left\{$$
\begin{align*}
r^{2} & =R^{2}+\rho^{2}-2 R \rho \cos (R, \rho)  \tag{2}\\
& =R^{2}+\rho^{2}-2 R \rho\left[\cos \vartheta \cos \vartheta^{\prime}+\sin \vartheta \sin \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right)\right]
\end{align*}
$$\right.
\]

The total potential energy is the same in every position and we leave it out of consideration. By differentiation, we will get:

$$
\left\{\begin{align*}
r \dot{r}= & \rho \dot{\rho}-R \dot{\rho}\left[\cos \vartheta \cos \vartheta^{\prime}+\sin \vartheta \sin \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right)\right]  \tag{3}\\
& -R \rho\left[-\sin \vartheta \cos \vartheta^{\prime} \dot{\vartheta}+\cos \vartheta \sin \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right) \dot{\vartheta}-\operatorname{in} \vartheta \sin \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right) \dot{\varphi}\right]
\end{align*}\right.
$$

Since we can orient the coordinate system arbitrarily, it will suffice for us to calculate for $\vartheta=0$. Furthermore, we would only like to calculate the main terms, which will remain when $\rho \ll R$. We can then drop the terms in $\rho$, except when they are multiplied by $\dot{\vartheta}$ or $\dot{\varphi}$. We will also have $r$ $=R$ in that approximation. That will give:

$$
\begin{equation*}
\dot{r}=-\dot{\rho} \cos \vartheta^{\prime}-\rho \dot{\vartheta} \sin \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right) . \tag{4}
\end{equation*}
$$

Hence, from (1):

$$
\begin{align*}
W & =\frac{\gamma \mu \sigma R^{2}}{R} \int_{0}^{2 \pi} d \varphi^{\prime} \int_{0}^{\pi} \sin \vartheta^{\prime} d \vartheta^{\prime}\left[\dot{\rho}^{2} \cos ^{2} \vartheta^{\prime}\right. \\
& \left.+2 \rho \dot{\rho} \dot{\vartheta} \sin \vartheta^{\prime} \cos \vartheta^{\prime} \cos \left(\varphi-\varphi^{\prime}\right)+\rho^{2} \dot{\vartheta}^{2} \sin ^{2} \vartheta^{\prime} \cos ^{2}\left(\varphi-\varphi^{\prime}\right)\right]  \tag{5}\\
& =\frac{4 \pi \gamma \mu \sigma R}{3}\left(\dot{\rho}^{2}+\rho^{2} \dot{\vartheta}^{2}\right)
\end{align*}
$$

That is precisely the value of the kinetic energy that classical mechanics gives, with the mass unit that makes the ordinary mass $m$ of our point (in grams) equal to:

$$
\begin{equation*}
m=\frac{8 \pi \gamma \sigma R}{3} \mu \tag{6}
\end{equation*}
$$

Since, on the other hand, from the Ansatz for potential energy:

$$
\begin{equation*}
m=\frac{\mu}{\sqrt{k}}, \tag{7}
\end{equation*}
$$

in which $k$ is the usual gravitational constant, one must have:

$$
\begin{equation*}
\frac{1}{\sqrt{k}}=\frac{8 \pi \gamma \sigma R}{3} \tag{8}
\end{equation*}
$$

or when one introduces the ordinary surface density $s$ for $\sigma$ :

$$
\begin{equation*}
s=\frac{\sigma}{\sqrt{k}}, \tag{9}
\end{equation*}
$$

one will have:

$$
\begin{equation*}
\frac{4 \pi s R^{2}}{R}=\frac{3}{2 k \gamma}, \tag{10}
\end{equation*}
$$

which is a relation that must be discussed further.
If one expresses the masses in grams then the total interaction energy will be:

$$
\begin{equation*}
W=\frac{\gamma k m m^{\prime}}{r} \dot{r}^{2}-\frac{k m m^{\prime}}{r} . \tag{1'}
\end{equation*}
$$

If a mass-point $m$ (e.g., a planet) moves in the neighborhood of a large mass $m^{\prime}$ (e.g., the Sun) then, in addition to the kinetic energy (5) with respect to the "mass horizon," its potential and kinetic energy ( $1^{\prime}$ ) with respect to $m^{\prime}$ must be brought under consideration. One will then get the total energy of the "one-body problem":

$$
\begin{equation*}
W=\left(\frac{m}{2}+\frac{\gamma k m m^{\prime}}{r}\right) \dot{r}^{2}+\frac{m}{2} r^{2} \dot{\varphi}^{2}-\frac{k m m^{\prime}}{r} . \tag{11}
\end{equation*}
$$

In addition to its gravitation attraction, the presence of the Sun then has the effect that the planet takes on an inertial mass that is somewhat larger "radially" than it is "tangentially." By applying the law of areas, which suffers no alteration:

$$
\begin{equation*}
r^{2} \dot{\varphi}=f \tag{12}
\end{equation*}
$$

and the substitution:

$$
\begin{equation*}
r^{-1}=\xi, \tag{13}
\end{equation*}
$$

after eliminating time from (11), one will get:

$$
\begin{equation*}
\left(1+2 \gamma k m^{\prime} \xi\right)\left(\frac{d \xi}{d \varphi}\right)^{2}+\xi^{2}-\frac{2 k m^{\prime}}{f^{2}} \xi-\frac{2 W}{m f^{2}}=0 \tag{14}
\end{equation*}
$$

in the usual way. With:

$$
\begin{equation*}
\xi=\eta+\frac{k m^{\prime}}{f^{2}}, \quad C=\frac{2 W}{m f^{2}}+\frac{k^{2} m^{\prime 2}}{f^{4}}, \tag{15}
\end{equation*}
$$

one will get:

$$
\begin{equation*}
d \varphi=\frac{d \eta \sqrt{1+\frac{2 \gamma k^{2} m^{\prime 2}}{f^{2}}+2 \gamma k m^{\prime} \eta}}{\sqrt{C-\eta^{2}}} \tag{16}
\end{equation*}
$$

which differs from the usual form by the square root factor in the numerator. One easily convinces oneself that it will define only a minor correction when it is applied to the planetary orbit if $\gamma$ has the order of magnitude of the reciprocal square of the speed of light. We can then be satisfied with the approximation:

$$
\begin{equation*}
\varphi=\left(1+\frac{\gamma k^{2} m^{\prime 2}}{f^{2}}\right) \arcsin \eta-\gamma k m^{\prime} \sqrt{C-\eta^{2}}+\text { const. } \tag{17}
\end{equation*}
$$

Whereas the second term on the right-hand side implies only an exceptionally small periodic perturbation, the first one implies a secular precession of the perihelion that contributes:

$$
\begin{equation*}
\Delta=\frac{2 \pi \gamma k^{2} m^{\prime 2}}{f^{2}} \tag{18}
\end{equation*}
$$

per orbit in the sense of the orbit (viz., $\varphi$ runs from the angle $2 \pi+\Delta$ up to $\eta$, and in so doing, $r$ will return to the same value and the same phase of motion). Now, from known formulas:

$$
\begin{equation*}
k m^{\prime}=\frac{4 \pi^{2} a^{2}}{\tau^{2}}, \quad f=\frac{2 \pi a b}{\tau}, \tag{19}
\end{equation*}
$$

so

$$
\frac{k^{2} m^{\prime 2}}{f^{2}}=\frac{4 \pi^{2} a^{4}}{b^{2} \tau^{2}}=\frac{4 \pi^{2} a^{4}}{\tau^{2}\left(1-s^{2}\right)}
$$

( $\tau, a, b, c, \varepsilon$ are the orbital period, semi-major and semi-minor axes, and the numerical eccentricity of the ellipse, resp.). That gives:

$$
\begin{equation*}
\Delta=\frac{8 \pi^{3} \gamma a^{2}}{\tau^{2}\left(1-\varepsilon^{2}\right)} . \tag{20}
\end{equation*}
$$

One will get agreement with the precession of perihelion that is derived in the general theory of relativity $\left({ }^{1}\right)$, so with experiments in the case of Mercury, when one sets:

$$
\begin{equation*}
\gamma=\frac{3}{c^{2}} . \tag{21}
\end{equation*}
$$

( ${ }^{1}$ ) A. Einstein, loc. cit., last page.

The Ansatz (1) then takes on the more precise determination:

$$
\begin{equation*}
W=\frac{3 \mu \mu^{\prime} \dot{r}^{2}}{r}-\frac{\mu \mu^{\prime}}{r}, \tag{1"}
\end{equation*}
$$

when the units of time and mass are chosen in such a way that the speed of light and the gravitational constant are both equal to 1 . (10) will become:

$$
\frac{4 \pi s R^{2}}{R}=\frac{c^{2}}{2 k}=6.7 \times 10^{27} \mathrm{c} . \mathrm{g} . \mathrm{s} .
$$

If one imagines that the "mass horizon" consists of individual mass-points and assigns them irregularly-distributed velocities that do not, however, have a higher order of magnitude relative to a suitably-chosen coordinate system than the ones that are measured experimentally at the center then nothing in the result (5) will change for a sufficiently-large $R$ than the facts that, first of all, that result will be true relative to the stated coordinate system, for which the center of mass of the mass horizon is at rest, and secondly, that an additional constant term will enter in that originates in the radial velocity of the mass horizon, but which has no influence on the motion.

Moreover, it is clear that one can also replace the surface distribution of the mass horizon with a spatial distribution that is arranged spherically-symmetrically about the point of observation in in the large-scale average, as long as the situation is such that the innermost shell of that spatial distribution, for which $R$ is still not sufficiently large to justify the omissions that were made above, yields only a vanishing contribution to the total inertial effect. Let $d$ be the spatial density of that distribution in $\mathrm{g} / \mathrm{cm}^{3}$ and let $R$ be its outer radius, so obviously in place of ( $10^{\prime}$ ), we will now have:

$$
\begin{equation*}
\int_{0}^{R} \frac{4 \pi \rho^{2} d}{\rho} d \rho=2 \pi R^{2} d=\frac{c^{2}}{2 k}=6.7 \times 10^{27} \text { c.g.s. } \tag{10"}
\end{equation*}
$$

in which we have performed the integration for a $d$ that is constant inside of $R$. That remarkable relation says that the (negative) potential of all masses at the point of observation, when calculated with the gravitational constant that is valid at the point of observation, is equal to one-half the square of the speed of light.

A rough estimate of the integral in $\left(10^{\prime \prime}\right)$ for the luminous masses of our galaxy gives a value of $10^{16} \mathrm{c}$. g. s. for it. In that way, one assumes that a ball of radius $R=200$ parsec ( 1 parsec $=3.09$ $\times 10^{18} \mathrm{~cm}$ ) is filled uniformly with stars with the mass of the Sun in such a way that 30 such stars will be contained in a ball of radius 5 parsec. Only an entirely-vanishing fraction of inertial effects that are observed on the Earth and in the solar system can originate in their interaction with the masses in our Milky Way. That is a very gratifying result in regard to the admissibility of the picture that was developed here because if the order of magnitude of the situation were only slightly different then it would be possible to explain the lack of any anisotropy in the terrestrial and planetary inertia only in a very contrived way. A mass distribution like the one that is
established by the luminous stars must have as a consequence that a body will oppose an acceleration within the galactic plane with a greater inertial resistance than it does perpendicular to it. The fact that we are probably not found at precisely the center of that mass distribution must have similar consequences. The order of magnitude that was established above seems to me to place the inertial anisotropy that originates in the asymmetric placement of the masses in our Milky Way just below the limit of astronomical observability, as one can roughly estimate by comparing with the anisotropy of the mass of Mercury, which is just barely detectible.

By contrast, it generally seems to me that the question will pop up anew of why our inertial system is indeed free of rotation with respect to our own galaxy (or the latter with respect to the former) if it is not "anchored" in our galaxy, but in stellar masses that are much further away. As seen from our entirely naïve elementary standpoint, the reason for that (or even better, the fact of the matter) is that empirically, nothing but relatively minor relative stellar velocities appear, namely, only ones that are noticeably smaller than the speed of light. Our Ansatz (1") suggests no basis for that state of affairs at all.

However, it proves to be entirely uncontrived when we combine our knowledge of the mechanics of our solar system (which is all that we have employed up to now) as a purelyempirical foundation with the observations of the appreciable increase in inertia when we approach the speed of light (e.g., deflection experiments with electrons). Those experiments show that the Ansatz ( $1^{\prime \prime}$ ) can be regarded as only an approximation for small velocities and needs to be corrected for large $\dot{r}$, i.e., ones that are comparable to unity. If we regard the "relativistic" energy formula:

$$
\begin{equation*}
\text { Kin. en. }=m c^{2}\left(\frac{1}{\sqrt{1-\beta^{2}}}-1\right) \tag{22}
\end{equation*}
$$

as the expression for the observations then it will be easy to give a modification of ( $1^{\prime \prime}$ ) that will lead to precisely (22) for arbitrary velocities. One sets:

$$
W=\frac{\mu \mu^{\prime}}{r}\left(\frac{2}{\left(1-\dot{r}^{2}\right)^{3 / 2}}-3\right)
$$

If we replace $\dot{r}$ in this according to (4) and carry out the calculations that are analogous to (5) [while dropping the second term in parentheses in ( $1^{\prime \prime \prime}$ ), which yields only a constant] then:

$$
W=\frac{2 \mu \sigma R^{2}}{R} \int_{0}^{2 \pi} d \varphi^{\prime} \int_{0}^{\pi} \frac{\sin \vartheta^{\prime} d \vartheta^{\prime}}{\left\{1-\left[\dot{\rho} \cos \vartheta^{\prime}+\rho \dot{\vartheta} \sin \vartheta^{\prime} \cos \left(\varphi^{\prime}-\varphi\right)\right]^{2}\right\}^{3 / 2}} .
$$

If we next set:

$$
x=\cos \vartheta^{\prime}, \quad y=\sin \vartheta^{\prime} \cos \left(\varphi^{\prime}-\varphi\right)
$$

then $x$ and $y$ will traverse the surface of the unit circle twice when $\vartheta^{\prime}, \varphi^{\prime}$ sweep out their entire domain. One finds that:

$$
W=4 \mu \sigma R \int^{x^{2}+y^{2} \leq 1} \frac{d x d y}{\left\{1-[\dot{\rho} x+\rho \dot{\vartheta} y]^{2}\right\}^{3 / 2} \sqrt{1-x^{2}-y^{2}}} .
$$

We now introduce "plane polar coordinates" $r, \psi$ for $x$ and $y$ such that we likewise prefer to choose:

$$
\sqrt{1-r^{2}}=z
$$

instead of $r$ as the variable. That will give:

$$
\begin{aligned}
W & =4 \mu \sigma R \int_{0}^{2 \pi} d \psi \int_{0}^{1} \frac{d z}{\left(1-a^{2}+a^{2} z^{2}\right)^{3 / 2}}=4 \mu \sigma R \int_{0}^{2 \pi} \frac{d \psi}{1-a^{2}} \\
& =4 \mu \sigma R \int_{0}^{2 \pi} \frac{d \psi^{\prime}}{1-v^{2} \cos ^{2} \psi^{\prime}}
\end{aligned}
$$

with the abbreviations:

$$
\begin{aligned}
& a=\dot{\rho} \cos \psi+\rho \dot{\vartheta} \sin \psi, \\
& v=\sqrt{\dot{\rho}^{2}+\rho^{2} \dot{\vartheta}^{2}} .
\end{aligned}
$$

One now sees most simply by a series development of the last integral (or by direct calculation or complex integration) that ultimately:

$$
\begin{equation*}
W=\frac{8 \pi \mu \sigma R}{\sqrt{1-v^{2}}}=\frac{8 \pi \mu \sigma R}{\sqrt{1-\dot{\rho}^{2}-\rho^{2} \dot{\vartheta}^{2}}} \tag{23}
\end{equation*}
$$

which agrees with the variable part of (22) according to (6) and (21) since we have indeed taken the speed of light to be unity from the outset in the present calculation.

Let it be mentioned in passing that the Ansatz ( 1 "' $)$ includes the Lagrangian function:

$$
\begin{equation*}
L=\frac{\mu \mu^{\prime}}{r}\left(\frac{2}{\sqrt{1-\dot{r}^{2}}}-4 \sqrt{1-\dot{r}^{2}}+3\right) \tag{24}
\end{equation*}
$$

which satisfies the equation:

$$
\begin{equation*}
\dot{r} \frac{d L}{d \dot{r}}-L=W=\frac{\mu \mu^{\prime}}{r}\left(\frac{2}{\left(1-\dot{r}^{2}\right)^{3 / 2}}-3\right) . \tag{25}
\end{equation*}
$$

If one integrates $L$ using (24), similarly to what was done before with $W$ for the interaction of our mass-point with the hollow sphere, then one will get (except for a constant) the well-known relativistic Lagrangian function of a mass-point:

$$
\begin{equation*}
L=-m c^{2} \sqrt{1-\beta^{2}}, \tag{26}
\end{equation*}
$$

where $\beta$ once more denotes the ratio of the speed of the mass-point to the speed of light.

The most serious objection that can be raised against the possibility of conceptualizing matters in the way that was described in this note is that it seems to revert to the principle of action-at-adistance in a manner that is unheard of nowadays. Obviously, no one in this day and age (not even the author) is moved to actually interpret the Ansätze (1), (1), etc., in that way. However, just as we have suggested that a star that is at a distance of many light-years can exert a minimal (and apparently instantaneous) influence on each oscillation of a terrestrial second pendulum by its gravitational field, even though, in reality, gravitation propagates only with the speed of light, I believe that we can likewise calculate with the terms in our Ansätze that depend upon $\dot{r}$ without sinning against the basic law that all effects propagate with a finite speed as long as the situation is such that, on average, we do not need to know whether we are calculating with the instantaneous state of motion of the distant celestial body or the retarded state of motion.

In other cases, one would generally next encounter certain difficulties if one wished to get serious about considering the latency time. It would then be largely impossible to give $\dot{r}$. One could define it in a purely-empirical way by the observed Doppler effect, but that would not be the same for two observers at two different mass-points that sent light signals to each other "at the same moment." The kinetic energy of the interaction (which would then contract to unity, for the time being) would necessarily split into two terms again. Moreover, the reason for the difference in the Doppler effect when the two celestial bodies have, say, equal masses could be perceived only in the existence of all remaining celestial bodies, which would then have to define an inertial system for light, just as it does for point motion.

I regard it as likely that upon pursuing that line of reasoning, after many alterations, one would ultimately arrive at the general theory of relativity, because it probably represents a framework that no future theory will demolish completely, even if is still not filled with concrete and vivid ideas to this day, for the most part. I consider the picture that was employed here that a change in the relative, not the absolute, state of motion of a body would demand an expenditure of work to be, at the very least, a legitimate and useful intermediate step that might allow one to understand a simple experimental state of affairs by means of conceptual pictures that everyone is familiar with in a simple, but still not basically incorrect, sort of way.

Zurich, Physikalisches Institut der Universität.


[^0]:    $\left.{ }^{( }{ }^{1}\right)$ E. Mach, Die Mechanik in ihrer Entwicklung, Leipzig, F. A. Brockhaus, $3^{\text {rd }}$ ed., 1897. Cf., esp. Chap. II, 6.
    $\left(^{2}\right)$ L. Boltzmann, Vorlesungen über die Principe der Mechanik, Leipzig, J. A. Berth, 1897.
    $\left(^{3}\right)$ A. Einstein, Ann. Phys. (Leipzig) 49 (1916), pp. 769.
    $\left({ }^{4}\right)$ H. Weyl, Raum, Zeit, Materie, $5^{\text {th }}$ ed., Berlin, J. Springer, 1923, § 39. Cf., also the article "Massenträgheit und Kosmos," by the same author in volume 12 (1924) of Naturwissenschaften.

[^1]:    ( ${ }^{1}$ ) The solution to that problem is actually already present in the presentation of the law of inertia that Mach gave. If has probably found so little approval mainly due to the fact that Mach believed that one would have to assume a reciprocal inertial effect that is independent of distance (loc. cit., pp. 228, et seq.)

