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On the reversal of time in natural law

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Introduction. – If the probability of presence:

$$w(x, t_0)\, dx$$

is given for a particle that is diffusing or exhibiting BROWNian motion in the domain of the abscissa $(x, x + dx)$ at time t_0 :

$$w(x, t_0) = w_0(x)$$

then for $t > t_0$ it will be the only solution $w(x, t)$ of the diffusion equation:

$$D\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \tag{1}$$

that is equal to the given function $w_0(x)$ for $t = t_0$. – There is an extensive body of literature on problems of this kind, which have many possible complications and variations that might relate to special experimental arrangements and methods of observation in which the system that is treated does not at all need to be a diffusing particle, but, for example, the needle in an electrometer for K. W. F. KOHLRAUSCH's arrangement for measuring the SCHWEIDLER fluctuations, and a generalization of equation (1) appears in its place, namely, the so-called FOKKER-PLANCK partial differential equation for the system in question that is subject to any sort of random influences ([1]).

Now, such systems give rise to a class of problems in probability that have not been looked at very much, if at all, up to now, and which are already mathematically interesting due to the fact that the answer is not provided by a solution to FOKKER's equations, but, as we will show, by the product of the solutions of two adjoint equations, such that the temporal boundary conditions are not imposed upon the individual solutions, but upon the product. Physically, there exists a close kinship with the

([1]) A. FOKKER, Ann. Phys. (Leipzig) **43** (1914), 812.
M. PLANCK, these Berichte, 10 May 1917.

interesting circle of problems that M. VON SMOLUCHOWSKI ([1]) has unrolled in his beautiful recent work on expectation times and return times of very improbable states in systems of diffusing particles. The conclusions that we will infer in § 6 can actually be read off from SMOLUCHOWSKI's result already, but they can still be surprising in their sharply paradoxical nature. In addition (§ 4), they yield some remarkable analogies with quantum mechanics that seem quite worthy of further thought to me.

§ 1. – The simple example that I would like to treat here is the following one: Let the probability of presence be given, not only for t_0, but also for a second time point $t_1 > t_0$:

$$w\,(x,\,t_0) = w_0\,(x); \qquad w\,(x,\,t_1) = w_1\,(x).$$

How big is it during the *intermediate times*?; i.e., for any t for which:

$$t_0 \leq t \leq t_1\ .$$

Obviously, $w\,(x,\,t)$ is *not* a solution of (1), since any such solution would indeed be determined already by its *initial value* for all later times. w is also probably not a solution of the *adjoint* equation:

$$D\frac{\partial^2 w}{\partial x^2} = -\frac{\partial w}{\partial t}\ , \tag{2}$$

since such a solution would, in turn, be determined by its final value $w_1\,(x)$ for all *previous* times. – Is there a contradiction in the problem statement? Certainly not. One can recognize that immediately in a simple special case that we would like to address. We assume that we have encountered the particle at x_0 at time t_0 and at x_1 at time t_1 . [w_0 and w_1 are then the "leading functions" (*Spitzenfunktionen*) at $x = x_0$ ($x = x_1$, resp.)]. An auxiliary observer observes the position of the particle at time t, but without communicating his result to us. The question that reads: What probabilistic conclusions can we infer about our two observations from the intermediate observation that our assistant made?

The answer is simple. I introduce the notation $g\,(x,\,t)$ for the well-known *basic solution* to (1):

$$g(x,\,t) = \frac{1}{\sqrt{4\pi Dt}}\,e^{-\frac{x^2}{4Dt}}\ . \tag{3}$$

That is the probability density at the location x at time $t > 0$ when the particle started at $x = 0$ at time $t = 0$. – I now let my particle start from $x = x_0$ very many times; say, N times. From these N attempts, I select the ones that take the particle to (x_1; $x_1 + dx_1$) at time t_1 . Their number is:

$$n_1 = N\,g\,(x_1 - x_0\,,\,t_1 - t_0)\,dx_1\ .$$

([1]) M. VON SMOLUCHOWSKI, Bull. Akad. Cracovie A (1913), pp. 418; Göttinger Vorträge (by Teubner, 1914), pp. 89, *et seq.*; Sitz.-Ber. d. Wien Akad. d. Wiss. 2a **123** (1914), 2381; *ibid.*, **124** (1915), pp. 263, 339; Phys. Zeit. **16** (1915), 321; Ann. Phys. (Leipzig) **48** (1915), 1103.

I once more select from them the ones that:

1. Take the particle to $(x; x + dx)$ at time t and then
2. Take it to $(x_1; x_1 + dx_1)$ at time t_1 .

Their number is:

$$n = N\, g\,(x - x_0\,,\, t - t_0)\; dx\; g\,(x_1 - x,\, t_1 - t)\; dx_1\,.$$

The probability in question is obviously the quotient n / n_1; i.e.:

$$w\,(x, t) = \frac{g(x - x_0, t - t_0)\, g(x_1 - x, t_1 - t)}{g(x_1 - x_0, t_1 - t_0)}\,. \tag{4}$$

That is the solution for the special case in which one is *certain* about the position of the particle at times t_0 and t_1 .

§ 2. – We now consider the general case. The plan of attack is the following: We let a very large number N of particles start at time t_0, and in fact:

$$N\, w_0\,(x_0)\; dx_0 \tag{5}$$

of them from the domain $(x_0\,;\, x_0 + dx_0)$. We observe that at time t_1 we will encounter:

$$N\, w_1\,(x_1)\; dx_1 \tag{6}$$

of them in the domain $(x_1\,;\, x_1 + dx_1)$. (Incidental remark: This observation should be more or less *surprising*, since it will brand our series of attempts as being more or less *exceptional*. One would then *expect* to have:

$$N\, dx_1 \int_{-\infty}^{+\infty} w_0(x_0)\, g(x_1 - x_0, t_1 - t_0)\, dx_0\,, \tag{6′}$$

instead of (6). However, that is not true here. We assume that the distributions (5) and (6) are *true* and must draw our conclusions from that *fact.*)

The solution to this more general problem is apparently much more difficult than the special case that was treated before. If we wished to know how many of the particles (5) are found among the ones at (6) then we would have to multiply that number by (4) and integrate over x_0 and x_1 from $-\infty$ to $+\infty$. The determination of the aforementioned number is the main problem.

We divide the x-axis into cells of equal size whose lengths we define to be the unit length, for the sake of simplicity. We let a_k denote the number (5) that start in the k^{th} cell at t_0, and let b_1 denote the number (6) that are encountered in the l^{th} cell at t_1 . Let g_{kl} be the *a priori* probability for a particle that starts in the k^{th} cell to enter the l^{th} one; i.e., g_{kl} is a notation for $g\,(x_1 - x_0,\, t_1 - t_0)$ that is suitable for the present purposes, and one has $g_{lk} =$

g_{kl} . Finally, let c_{kl} be the number of particles that arrive in the l^{th} cell from k^{th} one. One then has the following equations:

$$\left.\begin{aligned} \sum_l c_{kl} &= a_k \quad \text{for any } k \\ \sum_k c_{kl} &= b_l \quad \text{for any } l. \end{aligned}\right\} \tag{7}$$

There exists one and only one identity between equations (7), which arises from:

$$\sum_k a_k = \sum_l b_l = N. \tag{8}$$

Naturally, the system of numbers c_{kl} is *not* given. The actually-observed shift of the particles can come about for any of the systems of c_{kl} that are compatible with (7). However, it will be correct in the limit $N = \infty$ (which is indeed what is always intended), assuming that it endows the system of c_{kl} that has the highest probability with complete certainty.

The particle shift that is actually-observed can even come about in many different ways for a fixed c_{kl} . That possible realization endows the observed fact with the probability:

$$\prod_k \prod_l (g_{kl})^{c_{kl}} \ . \tag{9}$$

However, as we said, there are many such equally-probable possible realizations, namely:

$$\prod_k \frac{a_k\,!}{\prod_l c_{kl}\,!} \tag{10}$$

of them. The product of (9) and (10) yields the total probability that the observed fact is associated with a well-defined system of numbers c_{kl} :

$$\prod_k a_k\,! \cdot \prod_k \prod_l \frac{(g_{kl})^{c_{kl}}}{c_{kl}\,!} \ . \tag{11}$$

We now look for the system of c_{kl} that makes (11) a maximum under the auxiliary condition (7) in the usual way. We easily find that:

$$c_{kl} = g_{kl}\ \psi_k\ \phi_l \ . \tag{12}$$

The ψ_k and ϕ_l are LAGRANGE multipliers. They are determined from the auxiliary conditions:

$$\left.\begin{aligned} \psi_k \sum_l g_{kl}\,\phi_l &= a_k \quad \text{for any } k \\ \phi_k \sum_l g_{kl}\,\psi_l &= b_l \quad \text{for any } l. \end{aligned}\right\} \tag{13}$$

We now have to translate (12) and (13) back into the language of continua. a_k, b_l are given by (5) and (6). ψ_k, ϕ_l are functions of x, and indeed we would like to set:

$$\psi_k = \sqrt{N}\ \psi(x_0)\, dx_0\,, \qquad \phi_l = \sqrt{N}\ \psi(x_1)\, dx_1\,.$$

Furthermore, one has $g_{kl} = g\,(x_1 - x_0\,,\, t_1 - t_0)$. Therefore:

$$\left.\begin{aligned} \psi(x_0)\int_{-\infty}^{+\infty} g(x_1 - x_0, t_1 - t_0)\,\phi(x_1)\,dx_1 &= w_0(x_0), \\ \phi(x_1)\int_{-\infty}^{+\infty} g(x_1 - x_0, t_1 - t_0)\,\psi(x_1)\,dx_1 &= w_1(x_1), \end{aligned}\right\} \tag{13′}$$

and:

$$c\,(x_0, x_1)\,dx_0\,dx_1 = N\,g\,(x_1 - x_0\,,\, t_1 - t_0)\ \psi(x_0)\ \phi(x_1)\,dx_0\,dx_1 \tag{12′}$$

will be desired number of particles that diffuse from $(x_0\,,\, x_0 + dx_0)$ to $(x_1\,,\, x_1 + dx_1)$. If we multiply (12′) by (4) and integrate over x_0 and x_1 then (after dividing by N) we will get the probability density at the location x at time t:

$$w\,(x, t) = \int_{-\infty}^{+\infty} g(x - x_0, t - t_0)\,\psi(x_0)\,dx_0 \cdot \int_{-\infty}^{+\infty} g(x_1 - x, t_1 - t)\,\phi(x_1)\,dx_1\,. \tag{14}$$

This is the solution to the problem, expressed in terms of solutions of the pair of integral equations (13′).

§ 3. – A discussion of this pair of equations would certainly be interesting, but presumably not entirely easy, since it is nonlinear. I regard the existence and uniqueness of the solution (except for perhaps especially treacherous givens w_0, w_1) as being agreed upon, due to the reasonableness of the problem statement that led to these equations in an entirely unique and precise manner. For the moment, we are less interested in how one *actually* constructs the ψ and ϕ from well-defined givens w_0 and w_1 than we are in the general form of $w\,(x,\,t)$. It is extremely transparent, namely: *It is the product of any solution of* (1) *with any solution of* (2). The first factor in (14) is then nothing but any solution of (1), but characterized by $\psi\,(x_0)$, viz., its distribution of values at time t_0 ; an analogous statement will be true for the second factor in (14), which relates to equation (2). Furthermore, it easily follows from (1) and (2) that a product of two solutions will have a time-independent $\int_{-\infty}^{+\infty} dx \ldots$, and will therefore remain normalized to 1 if it were normalized to 1 at any arbitrary time. (Naturally, the following restriction must be made: Only those two solutions can be employed whose product has a *finite* $\int_{-\infty}^{+\infty} dx \ldots$, such that one can normalize it to 1.) Hence, one can then choose arbitrarily *any two* time-points within those time-spans for which the product of the solutions remains regular to be the time points t_0, t_1 at which the probability densities are observed. (Naturally, they are observed in such a way that the values of the product are even.) The product then yields the probability density for *intermediate times.*

§ 4. – Today, what is probably the most interesting thing about our result is probably its striking formal analogy with quantum mechanics. A certain relationship between the basic equation of wave mechanics and FOKKER's equation has probably been imposed upon each of them, as well as the statistical conceptual picture that is linked to both of them that is sufficiently familiar in both circles of ideas. Just the same, upon closer consideration, they exhibit two very serious discrepancies. One of them is that in the classical theory of random systems, the probability densities themselves are subject to the linear differential equation, while in wave mechanics, it is the so-called probability amplitudes, from which all probabilities will be constructed bilinearly, that are subject to such an equation. The second discrepancy lies in the fact that in both cases, the differential equation indeed has first order in time, but the appearance of a factor of $\sqrt{-1}$ in the wave equations gives it a hyperbolic (or, physically speaking, reversible) character, in contrast to the parabolic-irreversible character of the FOKKER equation.

In those two regards, the example that was treated above exhibits many close analogies with wave mechanics, even though it related to a classical, truly-reversible system. The probability density is not, as in wave mechanics, the solution of *one* FOKKER equation, but the product of the solutions of two equations that differ only by the sign of time. Therefore, the answer distinguishes no time direction. If one switches $w_0(x)$ and $w_1(x)$ then one will get precisely the opposite evolution of $w(x, t)$ between t_0 and t_1 . (In a certain sense, that is generally true for the simpler problem statement with only one temporal boundary-value function: If merely the probability density is given at time t_0, and nothing else, then it will have precisely the same value at time $t_0 + t$ that it had at time $t_0 - t$.)

I would still rather not predict whether or not the analogy will prove useful in clarifying quantum-mechanical concepts. In spite of it all, it is obvious that the aforementioned $\sqrt{-1}$ represents a very deep-rooted difference. – I cannot help myself from mentioning some words of A. S. EDDINGTON on the interpretation of the space of wave mechanics (even if they are quite gloomy) that appeared on pp. 216, *et seq.* of his Gifford lectures ("The nature of the physical world," Cambridge 1928):

"The entire interpretation is very vague, but it seems to depend upon whether one is dealing with a probability by which one knows what is happening or a probability for the sake of prediction. $\psi\psi^*$ will be preserved when one introduces two symmetric systems of ψ-waves that move in opposite directions; presumably, one of them has something to do with a probabilistic conclusion about the known (or assumed to be known) state of the system at a later time-point."

§ 5. – We would now like to write (14) in the form:

$$w\,(x,\,t) = \Psi\,(x,\,t)\,\Phi\,(x,\,t), \tag{15}$$

in which it is assumed that Ψ is a solution of (1), Φ is a solution of (2), and the product $\Phi\Psi$ is normalized to 1:

$$D\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial\Psi}{\partial t}, \qquad D\frac{\partial^2\Phi}{\partial x^2} = -\frac{\partial\Phi}{\partial t}, \qquad \int_{-\infty}^{+\infty}\Phi\Psi\,dx = 1. \tag{16}$$

If one multiplies the first equation by $x\Phi$, the second one by $-\,x\Psi$, and adds them, then that will give:

$$\frac{\partial}{\partial t}(x\Phi\Psi) = Dx\frac{\partial}{\partial x}\left(\Phi\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Phi}{\partial x}\right).$$

One forms $\int_{-\infty}^{+\infty} dx \ldots$ and integrates by parts:

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{+\infty} xw\,dx &= -\,D\int_{-\infty}^{+\infty}\left(\Phi\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Phi}{\partial x}\right)dx \\ &= 2\,D\int_{-\infty}^{+\infty}\Psi\frac{\partial\Phi}{\partial x}\,dx\,.\end{aligned}$$

On the left-hand side is the *velocity* by which the *center of mass* of the probability density displaces. However, the integral on the right-hand side is constant, since:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Psi\frac{\partial\Phi}{\partial x}\,dx = \int_{-\infty}^{+\infty}\left(\frac{\partial\Psi}{\partial t}\frac{\partial\Phi}{\partial x} + \Psi\frac{\partial^2\Phi}{\partial x\,\partial t}\right)dx = \int_{-\infty}^{+\infty}\left(\frac{\partial^2\Psi}{\partial t^2}\frac{\partial\Phi}{\partial x} - \Psi\frac{\partial^3\Phi}{\partial x^3}\right)dx$$

$$= \int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial x}\frac{\partial\Phi}{\partial x} - \Psi\frac{\partial^2\Phi}{\partial x^2}\right)dx = 0.$$

The center of mass then moves with constant velocity from the initial position to the final position.

In the special case in which the initial and final positions of the particles are known precisely, equation (4) allows one to establish that the probability *maximum* also displaces uniformly from the initial to the final position. If (4) is a GAUSSian distribution for every time point then the center of mass and the maximum will agree at every time point.

§ 6. – In one special case, the solution to the pair of integral equations (13′) can be given immediately. Namely, it is the one when the density distribution w_1 that is prescribed at the end of the interval is precisely the one that one would have developed from the initial distribution w_0 by the free reign of the diffusion equation (1); i.e., when:

$$w_1(x_1) = \int_{-\infty}^{+\infty} g(x_1 - x_0, t_1 - t_0) w_0(x_0) \, dx_0 .$$

Namely, one must obviously then set:

$$\phi \equiv 1; \qquad \psi \equiv w_0 .$$

$w(x, t)$ will then satisfy equation (1) in the whole time interval. If one represents it by a diffusion process involving many particles then it will be a thermodynamically completely *normal* diffusion process.

However, the solution of (13′) would be just as simple even when, conversely, the *initial distribution* w_0 is precisely the one that would evolve from the final distribution w_1 during the time $t_1 - t_0$ by a free reign of the *normal* (!) diffusion equation (1), or in other words: when the final distribution w_1 is given in such a way that it would emerge from the initial distribution by the *inverted* diffusion equation (2) in time $t_1 - t_0$. The assumption is then written:

$$w_0(x_0) = \int_{-\infty}^{+\infty} g(x_1 - x_0, t_1 - t_0) \, dx_1 ,$$

and the solutions of (13′) are:

$$\phi \equiv w_1; \qquad \psi \equiv 1.$$

$w(x, t)$ then satisfies the "inverted" equation (2) in the entire time interval, so the diffusion process that is being represented will be as thermodynamically abnormal as one can imagine.

Naturally, that originates in the comical choice of boundary conditions, but nonetheless admits an entirely interesting application in reality, namely, to the manner by which one can construct especially improbable exceptional states for a system in thermodynamic equilibrium, which are to be expected now and then, even if they are also rarely extraordinary.

Assume, in fact, that we have established the normal uniform spatial distribution for a system of diffusing particles at time t_0 that goes to a quite significantly deviated one at time t_1, but not so significant that it would not go back to a markedly uniform distribution in a time $t_1 - t_0$ according to the diffusion law. Furthermore, it should be known with certainty that at the intermediate times the system would be left in unperturbed thermodynamic equilibrium, or in other words, that the abnormal distribution that we observe is actually a spontaneous thermodynamic fluctuation phenomenon. In our opinion, if we were to then ask which *past history* that the observed strongly-abnormal distribution of probable ways might have had then we would have to reply that its first manifestation would probably lie as far back as necessary for its last traces to vanish once more. That is, such that from that first manifestation onward, an unimaginable swelling of the abnormality would take place due to the diffusion current, which almost always points almost exactly in the direction of concentration *gradient* (not its decline!), but corresponds precisely to the material constant D, except for that difference in sign. In brief: such that the abnormality would probably arise from a regular diffusion process by precisely a time reversal. Of course, this statement of opinion on the probable past history would only be a judgment based upon probability, but I believe that it would take on precisely the high degree of "almost certainty" as the corresponding statement of opinion about the probable later history; i.e., about the normal diffusion process that one would expect for $t > t_1$.

Naturally, one should not be led by that finding to the mistaken belief that perhaps a diffusion flux *in the direction of* the gradient and with a magnitude that corresponds precisely to the material constant D should be *in and of itself* much less improbable than any magnitude that is more arbitrarily inclined. Our judgment of probability is not based upon only the mechanism of the diffusion process, but also very essentially upon the strongly-abnormal final state that is assumed to be actually observed. It turns out that it can be attained by a precise reversal of the law of diffusion infinitely more easily and with a much more extraordinary probability than it can in any other less radical way.

One can probably transfer the statement with no further thought to arbitrary thermodynamic fluctuation phenomena, as long as they exceed the domain of normal fluctuation processes. The so-called irreversible laws of nature actually distinguish no time direction when one interprets them statistically. What they say in a special case then depends upon only the temporal boundary conditions at two "cross-sections" (t_0 and t_1), and is completely symmetric with respect to those two cross-sections, without arriving at their temporal sequence in any way. That will be disguised only by the fact that we generally regard *just one* of the two cross-sections as being actually observed, while the other one will obey the trustworthy rule that when that cross-section is moved to a sufficient distance in time, a state of great disorder or maximal entropy must prevail there. The fact that this rule is the correct one is actually quite remarkable, and I believe that it is not logically deducible. However, in any event, it also does not distinguish a time direction, since it is true just the same for either of the two time directions that one also shifts the second cross-section, as long as it is merely sufficiently separated in time from the first one.

Moreover, all of that was probably an opinion that BOLTZMANN expressed before. One should understand nothing else when he states the following, for example, at the end of his treatise "Über die sogenannte *H*-Kurve" [Math. Ann. **50** (1898), pp. 325; Ges. Abh. III, no. 128] ([1]):

"There is no doubt that a world would be just as conceivable in which all natural processes evolved in the opposite sequence. However, a man that lived in that reversed world would by no means perceive matters differently from the way that we do. He would just refer to what we call the future as the past, and conversely."

([1]) See also, *Gastheorie*, Part II, § 90; furthermore, Nature **51** (1895), 413; Wied. Ann. **60** (1897), 392; one can then confer the aforementioned papers of SMOLUCHOWSKI; of the more recent authors, G. N. LEWIS especially went into the principle of the "symmetry of time" [e.g., Phys. Rev. **35** (1930), 1533, and elsewhere].

Whoever then thinks that the thorough founding of that older thesis on the diffusion process that SMOLUCHOWKSI studied so thoroughly in connection with this is trivial and superfluous must forgive me if I prefer to agree with him. In discussions of these things, I occasionally meet up with noteworthy contradictions that make me uncertain. The opinion has been expressed that laws for the *creation* of strongly-abnormal state from a normal one by fluctuations are not nearly as compelling as the ones for its *vanishing*, and that, moreover, a well-defined abnormal state will be attained *relatively* often by a completely disordered evolution that is *not* the mirror image in time of a normal evolution when one piles up the rare cases in which it arises by sufficiently long observation times.

§ 7. – The arguments of the first three paragraphs are applicable to many complicated cases with little alteration: e.g., several spatial coordinates, variable diffusion coefficient, external forces that are any sort of function of position. For the probability density, one always gets the product of the solutions to two adjoint equations that generally differ by not only the sign of time, but also in their other terms. For the basic solutions [cf., *supra*, equation (3)] to the adjoint equations, one finds the simple (and certainly not new) theorem that they go to each other by permuting the coordinates of the starting point and the singular point and changing the sign of time. However, I would not like to go further into these matters until it has been established whether one can actually employ them to obtain a better understanding of quantum mechanics.
