"Ueber die reciproken Figuren der graphischen Statik," Zeit. Phys. **40** (1895), 48-55.

On the reciprocal figures of graphical statics

By

FRIEDRICH SCHUR

in Aachen.

———

With four figures (†).

———

Translated by: D. H. Delphenich

As is known, graphical statics gives rise to a remarkable reciprocity for planar figures, under which any line of the one figure corresponds to a line in the other figure that is parallel to it. Once *Culmann* had sought in vain to exhibit this reciprocity as a projective one, *Maxwell* succeeded in doing that by letting the two figures arise as orthogonal projections of two spatial figures that were polar to each other relative to a paraboloid of rotation, for which one of the two figures must generally be rotated through a right angle. *Cremona* (*) avoided that rotation by replacing the reciprocity relative to a paraboloid of rotation by one that was relative to a so-called "null system." Although the connection between the two planar figures can be expressed most felicitously in that way, the rigorous development of the theory of frameworks derives no advantage from these investigations, in that respect, so can one, like *Cremona*, leave unanswered the question of *whether two given reciprocal figures in graphical statics can always be represented as projections of two reciprocal figures of a null system*. As far as the author knows, this closely-related question did not find an answer anywhere in the later literature (**). The author would therefore like to bring *Cremona's* examination to a conclusion, in the sense that it will be shown that *Cremona* force planes can be constructed with the help of null systems for all frameworks, as long as one can ignore their multiplicity. For the sake of ease of understanding, we link everything to a specific example.

(†) Translator's note: Those figures were not available in the version of the original that was used.

(*) In particular, see *Cremona:* "Les figures réciproques en statique graphique, trad. par Bossut," Paris 1885, where one also finds a more precise bibliography.

(**) Only after this article had gone to press was the author informed of the treatise of *G. Hauck:* "Ueber die reciproken Figuren der graphischen Statik," Journal f. reine u. angew. Math. **100** (1887), pp. 365, fig., in which the solution of the corresponding problem for the so-called *Neumann* type of projection is suggested (pp. 388).

I. First recall some theorems on *null systems* (*). Consistent with our objective, we will then do best to start with their definition in the context of statics. As is known, an arbitrary system of forces in space can be reduced to either a single force or a couple of equal and opposite parallel forces or to two skew forces g and k. We are interested in only the last, so-called general, case. One of the lines of action of the two forces can be chosen entirely arbitrarily in space, with which, both the position and magnitude can be determined. Namely, if g and g' intersect at G, and if K is the point of intersection of k with the plane $[g, g']$ then we will decompose g into two components g' and n along g' and GK, resp., and look for the force k' through K that yields k for its resultant with the force $-n$ that acts along KG. The forces g' and k' will then be obviously equivalent to the two given forces g and k then. Since g' needs only to lie in a plane with g, one can come to any line of action in space by means of it. Our reduction would then be absurd, in general, if g' were also to intersect the line of action k. Such lines are called *null lines* of the force system, since it will produce a rotational moment of zero with respect to any such axis. If we call two lines that can be the lines of action of two skew forces that can replace the forces *conjugate* then null lines will be the lines that simultaneously intersect two conjugate lines. They fill up all of space in such a way that the null lines that run through a point will lie in a plane – viz., the *null plane of the point* – and the null lines in a plane will run through a point – viz., the *null point of the plane*. If one rotates the null plane around a line then the null point will move along the conjugate line, and conversely. Since, on the one had, $g = g' + n$, and on the other hand, $k = k' - n$, in the sense of the calculus of segments, we see that after g and k have been displaced to a common point of attachment they must yield the same resultant as g' and k' after they have been displaced to the same point. If we call this distinguished direction the *axis direction* of the force or null system then it will emerge from its definition that *any two conjugate lines can be projected onto two planes that are parallel to the axis direction.* If we now imagine that our construction of conjugate lines – and thus, the null lines – must produce the same resultant when we leave the positions of g and k unchanged, but they must be increased or decreased by the same ratio, then it will be clear that *a null system is determined completely by a pair of conjugate lines and the axis direction*, in which, the latter must naturally be chosen in such a way that the directions of the two lines will be projected onto two parallel planes. The ratio of the forces that act along g and k will then indeed be known, and so will the line k' that is conjugate to each line g'.

If we would like to find, e.g., the line k' that is conjugate to a line g' that is parallel to g then it must certainly go through the point of intersection K of k with the plane $[g, g']$. If we then give an arbitrary magnitude and sense to the force that acts along g, with which, the force that acts along k will also be determined, then we will decompose g into two components along g' and the line through K that is parallel to it. Now, the direction of k' is determined by the fact that it yields the resultant $g + k$ with the component g' when it has been brought to K. If g' cuts g at G then k' will be determined more simply as the line of intersection of the plane $[Gk]$ with the plane K, which is parallel to the axis direction and g'.

(*) See *loc. cit.*, introduction by *M. J. Jung*.

II. We now begin with the consideration of an entirely simple framework that possesses the nodes P_1, P_2, P_3, P_4 (Fig. 1) and consists of the two trigons $P_1P_2P_4$ and $P_2P_3P_4$. The forces g_1, g_2, g_3, g_4 (which are denoted by G_0G_1, G_1G_2, G_2G_3, G_3G_0, resp., in the figure), which are found to be in equilibrium, might act at the nodes. If we then think of these forces as having been brought together into the closed force polygon $K_0\,K_1\,K_2\,K_3\,K_0$, and understand C to mean any pole then the associated funicular polygon $S_0\,S_1\,S_2\,S_3\,S_4\,S_5$ must also be closed, so $S_0\,S_1$ must coincide with $S_4\,S_5$.

Moreover, it just so happens that the planar tetragon $g_1\,g_2\,g_3\,g_4$ can be represented as the projection of a spatial tetragon $g_1'\,g_2'\,g_3'\,g_4'$ that is obtained in the direction of the axis of a null system, and the latter spatial tetragon is conjugate in this null system to a tetragon $K_0'\,K_1'\,K_2'\,K_3'$ whose projection is the force polygon $K_0\,K_1\,K_2\,K_3$. In order to arrive at that, we consider the reference plane to be the plane of the base, the direction that is perpendicular to it, to be the axis of the null system, and draw the spatial figures in folded outline (*umgeklappten Aufriss*). In order to fix the null system, we then take the two conjugate lines g_1' and k_1' to be otherwise arbitrary, except that g_1 and $k_1 = K_0\,K_1$ are their base lines, so we choose their outlines g_1' and k_1' to be completely arbitrary, with which, the points K_0' and K_1' will be likewise determined. Now, the next side g_2' of the first spatial tetragon must first of all have g_2 for its outline and secondly, be conjugate to $K_1'\,K_2' = k_2'$ or lie in the null plane $[K_1'\,g_1']$ of K_1', so g_2' will be easy to construct. Conversely, k_2' is now determined as the line conjugate to g_2'. One then finds k_2' as the base line in the null plane $[G_1'\,k_1']$ of G_1', in the event that g_1 and g_2 intersect at G_1 (so g_1' and g_2' will intersect at G_1') or by the method that was given in the previous paragraph (cf., also Fig. 2), in the event that g_1 and g_2 are parallel to g_1' and g_2', resp.. The actual construction is easy to manage in both cases. We can proceed in that way and obtain, in succession, K_2' over K_2 from k_3', then g_3' over g_3, from that, k_3' over $k_3 = K_2K_3$ and K_3', furthermore, g_4' over g_4, and finally, k_4' over $k_4 = K_3K_0$, one must ask only whether k_4' once more goes over K_0'.

If we now denote the point of k_4' that lies over K_0 by K_0'' and understand C' to mean any point over the pole C then the five rays from C' to K_0', K_1', K_2', K_3', K_0'' will be conjugate to five rays of a plane that must intersect in four points S_1', S_2', S_3', S_4' of g_1', g_2', g_3', g_4', respectively; the same thing will then be true for their projections relative to g_1, g_2, g_3, g_4 . These projections must then define a funicular polygon $S_0\,S_1\,S_2\,S_3\,S_4\,S_5$, since they are parallel to the rays from C to K_0, K_1, K_2, K_2, K_0 . However, by assumption, the first and last side of it must coincide. The same thing will also be true then for rays whose projections they are, since they will lie in the same plane, so it is ultimately true for $C'K_0'$ and $C'K_0''$, as well, such that K_0' and K_0'' then coincide, and as a result, g_4', along with g_1', must lie in the same plane that contains K_0'. The first part of our problem is then solved, and one likewise sees that the same process can also be applied to arbitrarily many forces g_1, g_2, …, g_n .

The rest is very easy to obtain in our case. The points P_1', P_2', P_3', P_4', whose projections are the nodes P_1, P_2, P_3, P_4 of the framework, resp., now lie on the sides g_1', g_2', g_3', g_4', respectively, of the first spatial tetragon. The generally open hexahedron that consists of the six triangles $g_1' g_2'$, $g_2' g_3'$, $g_3' g_4'$, $g_4' g_1'$, $P_1' P_2' P_4'$, and $P_2' P_3' P_4'$ is now the spatial figure whose edges have the rods of the framework and the forces that act upon them for their projections. It corresponds in the null system to the hexangle that consists of the points K_1', K_2', K_3', K_0', H_1', and H_2', whose edges have, on the one hand, the sides of the force polygon, and on the other hand, the stresses in the rods of the framework, as their projections. The latter are, in each case, the connecting lines of those two points that are projections of the null points of the two faces of the hexahedron in which the edges that belong to the rods in question lie. If one would like to find these stresses quickly from the figure then one would do well to assign the symbols K_1, K_2, K_3, K_4, H_1, and H_2 to the projections of those faces. In practice, this notation will yield a much simpler overview than the one that is usually employed, which denotes the rods and their associated stresses with the same numbers, since the latter segments frequently partially overlap or go through each other.

III. If one is dealing with a more complicated framework then the determination of the spatial n-gon $g_1' g_2' \ldots g_n'$ whose projections are the lines of action of the given forces will come about in entirely the same way. One likewise determines the spatial points P_1', P_2', …, P_n' whose projections are the points of application of each force. Since we do not intend to discuss all possible kinds of frameworks here, we will make the assumption that is made in practice that the points of application of the given forces lie on the boundary of the framework, so the rods that connect the nodes will always belong to a field of the framework. We must then make the further assumption that at most three of the nodes P_1, P_2, …, P_n, in general, will belong to the field of the framework. The space point that belongs to a node of the field must then lie in a plane, which will be true for only three such cases with no further conditions. Now, should more nodes lie on the boundary, at which no forces act, then we would think of them as each being distributed between two of the previously-treated nodes P_i and P_{i+1}, between which they will lie during a single traversal of the entire boundary. Now, we have to think of a zero force as having been brought to these nodes, which will be represented by the point K_i in the force polygon, such that the space points that correspond to it are all to be found in the plane $[g_i', g_{i+1}']$. If j is the number of points that lie between P_i' and P_{i+1}' then they and the points (g_i', g_{i+1}') will define a j+3-edged face that belongs to the first spatial figure.

As far as the space points that correspond to the internal nodes are concerned, no general rules can be given for their determination. We will first have to observe whether the framework decomposes into fields in such a way that every internal rod belongs to two and only two fields or the introduction of ideal nodes (*) would achieve that. We understand that term to mean the point of intersection of segments that represent rods that, in reality, only pass over each other. Four segments that correspond to rods in the

(*) See *loc. cit.*, Appendice by Saviotti, pp. 63.

force plane that coincide in such an ideal node will, in fact, define a parallelogram, such that one will obtain the same stress for each rod in the various parts of it. However, the case in which an ideal point has more than two rods passing over it requires special attention. Ordinarily, an indeterminacy will then emerge that will be lifted by the condition that the stresses that result in every rod that is thought of as being broken by the ideal point are equal and opposite.

For the determination of the space points that correspond to the internal, real as well as ideal, nodes, one must start from the fact that they are associated with one and the same field of the framework and lie in the same plane. One then looks for points that should lie in a plane with three already-known space points, and then seeks to gradually come to all space points of the first figure. In that way, one will often not be able to go to work directly, but one must first choose one or more of the unknown points arbitrarily, in order to arrive at one's objective by the study of their motion. We will explain this process with some examples and content ourselves here with the general remark that the determination of the internal space point will become impossible, so the framework will become statically-indeterminate when all that is present are triangles that consist of internal, real or ideal, nodes.

IV. As a first example, we choose the framework that consists of the four external nodes P_1, …, P_4 (Fig. 2) and the internal node P_5 ; it decomposes into the two triangular fields $P_1P_4P_5$and $P_3P_4P_5$ and the tetrangular field $P_1P_2P_3P_5$. The three forces g_1, g_2, g_3 might act at P_1, P_2, P_3, resp., whose force polygon is $K_0K_1K_2K_0$. One then determines the trigon $g_1' \, g_2' \, g_3'$ (the sketch is folded on one side in the figure) using the method that was given in **II**, and then P_1', P_2', P_3'. Furthermore, P_4 is determined by the fact that it must lie in the plane $[g_3', g_1']$, and finally, P_5', in such a way that it must line in the plane $P_1' P_2' P_3'$. The first spatial figure is thus determined completely.

The so-called *French roof truss carrier* (Fig. 3) yields a similar example in which the advantage of our notation will, at the same time, become clear. The equal and opposite forces g_2, g_3, …, g_8 might act at the upper nodes P_2, P_3, …, P_8, resp., while the support relations $g_1 = g_6 = -\frac{1}{2}(g_2 + \ldots + g_8)$ are verified at P_1 and P_9 ; the associated force polygon is $K_0K_1\ldots K_8K_0$. A choice of g_1' and k_1' first yields the lines g_2', …, g_9' again, and with them, the points P_1', $P_2', \ldots, P_9'$, and the points P_{10}', P_{11}', P_{12}', P_{13}' are then determined in such a way that they must lie in the plane $[g_1', g_9']$, and finally, P_{14}' and P_{15}' are determined in such a way that must lie in the plane $P_5' P_{11}' P_{12}'$. The null system then immediately yields the force plane that belongs to the framework as the projection of the spatial twenty-two-hedron that corresponds to the twenty-two-gon. As is known, one does not arrive at it by the ordinary methods of decomposition into components, so one requires some gimmicks. (In the figure, one thinks of H_4 as being mobile along the parallel to $P_3 P_{13}$ through H_3, whereby H_6 will move along a line that goes through S; H_6 can then be found from any position of the moving point H_6'.)

As a last example, we treat the *framework with six nodes* P_1, P_2, …, P_6 (Fig. 4), *and the sides and diagonals of the hexangle that they define as rods.* The latter will cross over each other at three points A, B, C, which we must then introduce as ideal nodes. The six forces g_1, g_2, …, g_6 might then act at the six nodes, and we let $K_0K_1\ldots K_5K_0$ be the associated force polygon. By a choice of g_1' and k_1', we will again find g_2', …, g_6', and from that, P_1', P_2', …, P_6'. We cannot give the points A', B', C' that correspond to the ideal nodes directly at this point. However, if we choose one of these points arbitrarily over A then B' and C' will also be determined immediately as lying in the planes $P_1'P_1'A'$ (?) and $P_5'P_6'A'$, and one then asks only whether B' and C' also lie in a plane with P_3' and P_4'. Naturally, that will not be the case for an arbitrary choice of A'. However, when A' moves along the vertical over A, $P_3'P_4'B'$ and $P_3'P_4'C'$ will describe two projective pencils of planes that have the corresponding vertical plane through $P_3'P_4'$ in common, such that one will generally obtain one and only one position that solves the problem. The further pursuit of these considerations would also show us if more than one solution or only an imaginary solution exists. Meanwhile, since some the easy understanding of this would entail some practice in the projective geometry of space, and the case in which the three diagonals run through a point must also be treated specially, we refrain from using a process that also links the actual drawing of the force plane to the train of thought that one must follow.

Obviously, we can think of every rod of the framework as being extended when we replace it with two equal and opposite points that are applied at its endpoints and correspond to the stress that acts in it. If we then think of the framework as being fixed by the addition of an ideal rod then we can also give our problem the form: Determine the magnitudes and sense of the forces that are applied to the endpoints of the rod to be extended in such a way that they will provoke zero stress in the ideal rod with the given force system (*); one can once more neglect the ideal rod then. However, we can break this problem into two parts: First, one determines that stress in the ideal rod that comes from the given force system, and then the stress in the ideal rod that comes from any two equal and opposite forces that act on the extended rod. Corresponding to the demands of our problem, the determination of the force that acts on the extended rod then requires only the search for a fourth proportional. The stresses that arise from the two force systems are then summed when they act simultaneously, and the stresses that are produced from the two equal and opposite forces will change that proportionally. The problem will obviously yield only an indeterminate solution then or only infinite solutions when the stress in the ideal rod is zero for any magnitude of force in the extended rod.

In our case, we extend the rod P_1P_4 and add the ideal rod P_2P_6, such that only the ideal node B will remain. The spatial figure that belongs to the given force system is indeed known immediately; B' must then lie in the plane $P_3'P_4'P_5'$ here. The force polygon $T_?T_1T_0$ might belong to the second force systems s and $-s$, which acts in P_1P_4; we can then choose s' and t' over s and T_0T_1, with which, T_0' and T_1' will also be determined. Moreover, P_1', P_2', P_3', and P_4' lie in the null plane $[T_0's']$ of T_0', P_4', P_5', P_6', P_1' lie in the null plane $[T_1's']$ of T_1', and B' again lies in the plane $P_3'P_4'P_5'$. Obviously,

(*) See *Henneberg: Statik der starren systems*, Darmstadt, 1886, pp. 228, figure.

zero stress will result in the ideal rod P_2P_6 if and only if the two planes $P_6'P_1'P_2'$ and $P_2'B'P_6'$ coincide – that is, B' lies along the line of intersection of the two planes $P_6'P_1'P_2'$ and $P_3'P_4'P_5'$ or along the connecting lie of the two points $\alpha' = (P_1'P_2', P_3'P_4')$ and $\gamma' = (P_4'P_5', P_6'P_1')$. However, the three points $\alpha = (P_1P_2, P_4P_5)$, $B = (P_2P_5, P_3P_6)$, and $\gamma = (P_5P_4, P_6P_1)$ will also lie along a line, so the hexangle $P_1P_2P_5P_4P_3P_6$ will then be a *Pascal* hexangle, and *the six given nodes must lie on a conic section* (*). Conversely, if the six nodes lie on a conic section, so α, B, γ lie along a line, then B' will lie along the line $\alpha'\gamma'$ or in the plane $P_6'P_1'P_2'$, and the stress that results in the ideal rod P_2P_6, which might also have the magnitude *s*, will always be zero. Therefore, if the given force system is not to be arranged such that it likewise produces zero stress in P_2P_6 then it must produce infinite stress in the original framework. If the six nodes lie on a conic section then our framework will generally be impracticable.

This example will suffice to show how *Cremona's* ideas can be employed in order to examine the static determinacy or feasibility of a given framework, and the use of the null system in the construction of force plane then takes on the character of a general method.

———————

(*) Cf., e.g., *Müller-Breslau: Die graphische Statik der Bauconstructionen*, 2nd ed., Leipzig, 1887; Bd. I, pp. 208, fig.