"Zur Elektrodynamik. I. Zwei Formen des Prinzips der kleinsten Action in der Elektronentheorie," Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse (1903), 126-131.

## On electrodynamics. I

# Two forms of the principle of least action in the theory of electrons 

By<br>\section*{K. Schwarzschild}<br>Presented at the session on 16 May 1908 by F. Klein<br>Translated by D. H. Delphenich

§ 1. - An amount of electricity $e$ that is found in the spatial element $d \omega$ and moves with a velocity $v_{x}, v_{y}, v_{z}$ in an electric field that is defined by the vectors of the electric and magnetic force, $K$ and $H$, experiences a mechanical force that is determined in Lorentz-Wiechert electrodynamics as follows $\left({ }^{1}\right)$ :

If one denotes the components of the force by $e F_{x}, e F_{y}, e F_{z}$ then one will have:

$$
\begin{equation*}
F_{x}=K_{x}-v_{y} H_{z}+v_{z} H_{y}, \tag{1}
\end{equation*}
$$

along with the values of the other components that emerge by cyclically permuting the indices, although I shall refrain from writing them out here, as in all later analogous cases. (The speed of light is set equal to $V=1$ in that way.)

If one, with Maxwell, introduces the potential $\Phi$ and the vector potential $\Gamma_{x}, \Gamma_{y}, \Gamma_{z}$ instead of the electric and magnetic force with the help of the relations:

$$
\begin{equation*}
K_{x}=-\frac{\partial \Phi}{\partial x}-\frac{\partial \Gamma_{x}}{\partial t}, \quad H_{z}=\frac{\partial \Gamma_{x}}{\partial y}-\frac{\partial \Gamma_{y}}{\partial x} \tag{2}
\end{equation*}
$$

then one will get the following expression for the mechanical force:

$$
\begin{equation*}
F_{x}=-\frac{\partial \Phi}{\partial x}-\frac{\partial \Gamma_{x}}{\partial t}-v_{y}\left(\frac{\partial \Gamma_{x}}{\partial y}-\frac{\partial \Gamma_{y}}{\partial x}\right)+v_{z}\left(\frac{\partial \Gamma_{z}}{\partial x}-\frac{\partial \Gamma_{x}}{\partial z}\right) \tag{3}
\end{equation*}
$$

[^0]or
$$
F_{x}=-\frac{\partial \Phi}{\partial x}+v_{x} \frac{\partial \Gamma_{x}}{\partial x}+v_{y} \frac{\partial \Gamma_{y}}{\partial x}+v_{z} \frac{\partial \Gamma_{z}}{\partial x}-\frac{\partial \Gamma_{x}}{\partial t}-v_{x} \frac{\partial \Gamma_{x}}{\partial x}-v_{y} \frac{\partial \Gamma_{x}}{\partial y}+v_{z} \frac{\partial \Gamma_{x}}{\partial x} .
$$

If one sets:

$$
\begin{equation*}
L=\Phi-v_{x} \Gamma_{x}-v_{y} \Gamma_{y}-v_{z} \Gamma_{z} \tag{4}
\end{equation*}
$$

and employs total derivatives instead of partial ones then can write $F$ in the form:

$$
F_{x}=-\frac{d L}{d x}+\frac{d}{d t} \frac{\partial L}{\partial v_{x}} .
$$

Since that expression has the form of one of Lagrange's variational equations, one sees immediately the validity of the following theorem, which represents a first form of the principle of least action in electrodynamics:

In a given electric field, electricity will move in such a way that the variation of the integral:
(A)

$$
\int d t\left(-T+\sum e L\right)
$$

will vanish when it is taken between fixed initial and final times and positions.
The sum in that extends over all electric charges e, $T$ is the vis viva of the ponderable masses to which the electricity is bound, and L is the quantity $\Phi-v_{x} \Gamma_{x}-v_{y} \Gamma_{y}-v_{z} \Gamma_{z}$, which I would like to refer to as the "electrokinetic potential."

One will obtain a very simple expression for the electrokinetic potential when one introduces the connection between $\Phi$ and $\Gamma$ and the distribution of electricity in space. Namely, if one denotes the spatial density of electricity in the element $d \omega$ at time $t$ by $\chi(t)$ then one will have:

$$
\begin{equation*}
\Phi=\int \frac{d \omega}{r} \chi\left(t^{\prime}-r\right), \quad \Gamma_{x}=\int \frac{d \omega}{r} \chi\left(t^{\prime}-r\right) v_{x}\left(t^{\prime}-r\right) \tag{B}
\end{equation*}
$$

at the point $x^{\prime}, y^{\prime}, z^{\prime}$, and time $t^{\prime}$.
The $r$ in those integrals is the distance from the element $d \omega$ to the reference point $x^{\prime}, y^{\prime}, z^{\prime}$, and the values of the density and velocity to be employed in each spatial element are the ones that were valid at an epoch that preceded it by a light-time. The electrokinetic potential then takes on the following form:
(C)

$$
L=\int \frac{d \omega}{r} \chi\left(t^{\prime}-r\right)\left[1-v^{\prime}\left(t^{\prime}\right) v\left(t^{\prime}-r\right) \cos \left(v^{\prime}, v\right)\right]
$$

That expression agrees with the electrodynamical potential that Clausius exhibited, minus the electrostatic potential, except that the density and velocity of the actual electricity have been replaced with the values that would be true at an epoch that preceded by a light-time. The transition from the theory of action at a distance to the assumption of temporal spreading of the electrical force is completed in the simplest way with that. It implies that all of electrodynamics can be included in the mechanics up to now by the concise statement: In the presence of electric charges, the contribution $\sum e L$, which is summed over all charges $e$, is introduced into the integral in Hamilton's principle.
§ 2. - Whereas in the foregoing, all effects were traced back to the electric charges themselves, in optics, one might, in fact, desire a variational principle that implies not only the ponderomotive forces, but also the field equations for the electric and magnetic force by mere variation. Lorentz placed such a principle at the summit of his basis for the theory of electrons $\left({ }^{1}\right)$, while von Helmholtz defined a different one for Hertzian electrodynamics ( ${ }^{2}$ ), but it also differed from the Lorentzian principle in its entire type of variation. A principle shall be formulated here that yields the equations of the theory of electron, but whose type of variation is analogous to the one in Helmholtz's principle. It is built upon the theorem that was just given and reads:

In order to consider the electromagnetic effects, the quantity:
(D)

$$
\int d t d \omega\left\{\frac{H^{2}-K^{2}}{8 \pi}+\chi L\right\}
$$

must be added to the integral in Hamilton's principle
In that, on the one hand, $K, H$, and $L$ are expressed in terms of $\Phi$ and the $\Gamma$ using (2) and (4), and the last four quantities are varied, while on the other hand, the motion of the ponderable mass is varied, and therefore, at the same time, the electricity that it is endowed with. The integral is to be taken between fixed times, as one always does with Hamilton's principle, and the variations of the position of the mass must vanish for the limiting times, like the variations of the components of the vector potential $\Gamma_{x}, \Gamma_{y}, \Gamma_{z}$.

I would like to suggest how one performs the variation. The variation of the integrand with respect to $\Phi$ yields:

$$
K_{x} \frac{\partial \delta \Phi}{\partial x}+K_{y} \frac{\partial \delta \Phi}{\partial y}+K_{z} \frac{\partial \delta \Phi}{\partial z}+4 \pi \chi \delta \Phi
$$

If one completes the required partial integrations then it will follow that the condition for the vanishing of the variation is:

[^1]$$
4 \pi \chi-\frac{\partial K_{x}}{\partial x}-\frac{\partial K_{y}}{\partial y}-\frac{\partial K_{z}}{\partial z}=0 .
$$

The variation of $\Gamma_{x}$ gives:

$$
H_{z} \frac{\partial \delta \Gamma_{x}}{\partial y}-H_{y} \frac{\partial \delta \Gamma_{x}}{\partial z}+K_{x} \frac{\partial \delta \Gamma_{x}}{\partial t}-4 \pi \chi v_{x} \delta \Gamma_{x}
$$

in the integrand, and therefore the condition equation:

$$
-\frac{\partial H_{z}}{\partial y}+\frac{\partial H_{y}}{\partial z}-\frac{\partial K_{x}}{\partial t}-4 \pi \chi v_{x}=0
$$

That is the first half of Maxwell's system of equations. The second one is already included in the Ansatz (2), in the assertion that the six components of the electric and magnetic forces in that form can be traced back to the four quantities $\Phi, \Gamma$. In fact, one gets:

$$
\frac{\partial K_{x}}{\partial y}-\frac{\partial K_{y}}{\partial x}=-\frac{\partial H_{x}}{\partial t}
$$

from (2) by simple differentiation.
As far as the variation of the motion of the masses and the electric charges that is coupled with it is concerned, one can proceed in two different ways, which correspond to the Lagrangian and the Eulerian pictures in hydrodynamics, respectively. In one case, one sets $d \omega \chi=d e$ and pursues the individual mass particles with their unvarying electric charges $d e$. That is the path that was described in reverse order in the previous paragraphs. In the other case, one considers the changes in the electric density $\chi$ and the velocity that would result at a well-defined location $d \omega$ when the entire motion is varied. I would now like to go down the latter path as a control, although it is intrinsically more complicated.

The coordinates of a mass particle that had the values $x, y, z$ at time $t$ might be equal to $x+\xi$, $y+\eta, z+\zeta$ at the same time under the varied motion. In that way, the density of the electricity at the point $x, y, z$ will go from the initial value of $\chi$ to the value $\chi+\delta \chi$. Corresponding to the hydrodynamical equation of continuity, one will obviously have:

$$
\delta \chi=-\frac{\partial(\chi \xi)}{\partial x}-\frac{\partial(\chi \eta)}{\partial y}-\frac{\partial(\chi \zeta)}{\partial z} .
$$

The velocity of a well-defined mass particle in the $x$-direction increases by:

$$
\frac{d \xi}{d t}=\frac{\partial \xi}{\partial t}+v_{x} \frac{\partial \xi}{\partial x}+v_{y} \frac{\partial \xi}{\partial y}+v_{z} \frac{\partial \xi}{\partial z}
$$

under the variation of the motion. In order to get back to the change in velocity at the same location, one must subtract the contribution:

$$
\xi \frac{\partial v_{x}}{\partial x}+\eta \frac{\partial v_{x}}{\partial y}+\zeta \frac{\partial v_{x}}{\partial z}
$$

from that. The total variation of $v_{x}$ will then become:

$$
\delta v_{x}=\frac{\partial \xi}{\partial t}+v_{x} \frac{\partial \xi}{\partial x}+v_{y} \frac{\partial \xi}{\partial y}+v_{z} \frac{\partial \xi}{\partial z}-\xi \frac{\partial v_{x}}{\partial x}-\eta \frac{\partial v_{x}}{\partial y}-\zeta \frac{\partial v_{x}}{\partial z} .
$$

If one substitutes those variations in $\chi L$ then one will get, e.g., the following $\xi$-dependent term:

$$
-\frac{\partial(\chi \xi)}{\partial x}\left(\Phi-v_{x} \Gamma_{x}-v_{y} \Gamma_{y}-v_{z} \Gamma_{z}\right)-\chi \Gamma_{x}\left(\frac{\partial \xi}{\partial t}+v_{x} \frac{\partial \xi}{\partial x}+v_{y} \frac{\partial \xi}{\partial y}+v_{z} \frac{\partial \xi}{\partial z}\right)+\chi \xi\left(\Gamma_{x} \frac{\partial v_{x}}{\partial x}+\Gamma_{y} \frac{\partial v_{x}}{\partial y}+\Gamma_{z} \frac{\partial v_{x}}{\partial z}\right)
$$

and that will yield the contribution to the mechanical force in the $x$-direction:

$$
\begin{gathered}
\chi\left[\frac{\partial \Phi}{\partial x}-\frac{\partial}{\partial x}\left(v_{x} \Gamma_{x}\right)-\frac{\partial}{\partial y}\left(v_{y} \Gamma_{y}\right)-\frac{\partial}{\partial z}\left(v_{z} \Gamma_{z}\right)\right] \\
+\frac{\partial}{\partial t}\left(\chi \Gamma_{x}\right)+\frac{\partial}{\partial x}\left(\chi v_{x} \Gamma_{x}\right)+\frac{\partial}{\partial y}\left(\chi v_{y} \Gamma_{y}\right)+\frac{\partial}{\partial z}\left(\chi v_{z} \Gamma_{z}\right)+\chi\left[\Gamma_{x} \frac{\partial v_{x}}{\partial x}+\Gamma_{y} \frac{\partial v_{y}}{\partial x}+\Gamma_{z} \frac{\partial v_{z}}{\partial x}\right],
\end{gathered}
$$

or when one considers that the equation:

$$
\frac{\partial \chi}{\partial t}+\frac{\partial v_{x} \chi}{\partial x}+\frac{\partial v_{y} \chi}{\partial y}+\frac{\partial v_{z} \chi}{\partial z}=0
$$

is naturally fulfilled during the motion, after an easy conversion, it will yield:

$$
\chi\left[\frac{\partial \Phi}{\partial x}+\frac{\partial \Gamma_{x}}{\partial x}-v_{y}\left(\frac{\partial \Gamma_{y}}{\partial x}-\frac{\partial \Gamma_{x}}{\partial y}\right)+v_{x}\left(\frac{\partial \Gamma_{x}}{\partial z}-\frac{\partial \Gamma_{z}}{\partial x}\right)\right] .
$$

Since that force is understood to mean the force that is exerted upon the system from the outside, one will find that it is in agreement with the Ansatz (1) or (3) of the theory of electrons, even in sign.

With that, the total system of Lorentz-Wiechert electrodynamics has been derived from the variational principle (D).

# On electrodynamics. II 

# The elementary electrodynamical force 

By<br>\section*{K. Schwarzschild}<br>Presented at the session on 16 May 1908 by F. Klein<br>Translated by D. H. Delphenich

§ 1. - Whereas in my previous note the goal was to summarize the theory of electrons in a single formula that would be as succinct as possible, here I shall proceed in the opposite direction and explicitly give the force that an arbitrarily-moving electric point charge exerts upon another such thing with no concern for the complicated form of the expressions that arise.

To that end, one needs only to take a small step beyond Wiechert's investigations "Ueber elektrodynamische Elementargesetze" [Ann. Phys. (Leipzig) (4), Bd. 4].

In the Lorentz-Wiechert electrodynamics, the total force that the electric field $K$ and the magnetic field $H$ exert upon a charge $e$ that moves with a velocity $v$ has the magnitude $e \cdot F$, in which one has:

$$
\begin{equation*}
F_{x}=K_{x}-v_{y} H_{z}+v_{z} H_{y} . \tag{1}
\end{equation*}
$$

(The speed of light is equal to 1 .)
Electric and magnetic forces can be expressed with the help of a scalar potential $\Phi$ and a vector potential $\Gamma$ in the form:

$$
\begin{equation*}
K_{x}=-\frac{\partial \Phi}{\partial x}-\frac{\partial \Gamma_{x}}{\partial t}, \quad H_{z}=\frac{\partial \Gamma_{x}}{\partial y}-\frac{\partial \Gamma_{y}}{\partial x} . \tag{2}
\end{equation*}
$$

Now, Wiechert (loc. cit., pp. 682) derived the following expressions for the potentials $\Phi$ and $\Gamma$ that are created by the point-charge $e$ that moves with a velocity of $v$ at another point $e^{\prime}$ at the time $t^{\prime}$ :

$$
\begin{equation*}
\Phi=e\left\{\frac{1}{r[1-v \cos (v, r)]}\right\}_{t=t^{\prime}-r}, \quad \Gamma_{x}=e\left\{\frac{v_{x}}{r[1-v \cos (v, r)]}\right\}_{t=t^{\prime}-r} \tag{3}
\end{equation*}
$$

The distance $r$ between $e$ and $e^{\prime}$ in those formulas is taken in the sense of $\left(e, e^{\prime}\right)$ (which is the opposite of what Wiechert did). The expressions in curly brackets are defined for an epoch that precedes $t^{\prime}$ by a light-time, as is suggested by the subscript $t=t^{\prime}-r$.

If one would like to calculate the mechanical force itself that a point-charge $e$ exerts upon another one $e^{\prime}$ from $\Phi$ and $\Gamma$ using (1) and (2), resp., then one must do nothing more than differentiate with respect to time $t^{\prime}$ and the coordinates of $e^{\prime}$. However, that is not entirely simple, due to the fact that the light-time, and therefore the position and velocity of $e$, also vary with the position of $e^{\prime}$. One clarifies that relationship as follows:

Let the coordinates of $e^{\prime}$, which are considered to be independent variables during the differentiations, be $x^{\prime}, y^{\prime}, z^{\prime}$, while the coordinates of $e$ at any other time $t$ are $x(t), y(t), z(t)$. One then has:

$$
\begin{equation*}
r^{2}=\left(x^{\prime}-x(t)\right)^{2}+\left(y^{\prime}-y(t)\right)^{2}+\left(z^{\prime}-z(t)\right)^{2}, \tag{4}
\end{equation*}
$$

and only the previous positions of $e$ for which one has:

$$
\begin{equation*}
t=t^{\prime}-r \tag{5}
\end{equation*}
$$

come under consideration in the effect on $e^{\prime}$. If the previous motion of $e$ is known, as is assumed, and prescribed in such a way that $x, y, z$ are given functions of $t$ then that will be a condition on $t$ from which $t$ can be represented as a function of $t^{\prime}, x^{\prime}, y^{\prime}, z^{\prime}$.

I shall now introduce the following convention: All functions of $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}, t$ shall be provided with a prime when they are considered to depend upon those five variables, which are thought of as autonomous. The prime will drop away when one eliminates $t$ in the original representation and replaces it with the function of $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ that emerges from (5). Naturally, when one calculates the forces from (1), (2), the functions are understood in the latter sense, so they will be expressed without primes.

For any arbitrary $f$, one will have:

$$
\frac{\partial f}{\partial x^{\prime}}=\frac{\partial \bar{f}}{\partial x^{\prime}}+\frac{\partial \bar{f}}{\partial t} \frac{\partial t}{\partial x^{\prime}}, \quad \frac{\partial f}{\partial t^{\prime}}=\frac{\partial \bar{f}}{\partial t^{\prime}}+\frac{\partial \bar{f}}{\partial t} \frac{\partial t}{\partial t^{\prime}},
$$

and from (5), one will have:

$$
\begin{equation*}
\frac{\partial t}{\partial t^{\prime}}=\frac{1}{1+\frac{\partial \bar{r}}{\partial t}}, \quad \frac{\partial t}{\partial x^{\prime}}=-\frac{\frac{\partial \bar{r}}{\partial x^{\prime}}}{1+\frac{\partial \bar{r}}{\partial t}} . \tag{6}
\end{equation*}
$$

Furthermore, $\Phi$ and $\Gamma$ can be written in the form:

$$
\begin{equation*}
\Phi=\frac{e}{\bar{r}\left(1+\frac{\partial \bar{r}}{\partial t}\right)}, \quad \Gamma_{x}=\frac{\partial \bar{x}}{\partial t} \cdot \Phi . \tag{7}
\end{equation*}
$$

Let it also be noted in advance that:

$$
\begin{equation*}
\frac{1}{2} \frac{\partial \bar{r}^{2}}{\partial x^{\prime}}=x^{\prime}-\bar{x}(t), \quad \frac{1}{2} \frac{\partial^{2}\left(\bar{r}^{2}\right)}{\partial x^{\prime} \partial t}=-\frac{\partial \bar{x}}{\partial t} . \tag{8}
\end{equation*}
$$

§ 2. Calculating the electric force. - The electric force at the point $x^{\prime}, y^{\prime}, z^{\prime}$ at the time $t^{\prime}$ will be given by:

$$
K_{x}=-\frac{\partial \Phi}{\partial x^{\prime}}-\frac{\partial \Gamma_{x}}{\partial t^{\prime}} .
$$

With the notation that was just introduced, it will follow that:

$$
\begin{equation*}
-K_{x}=\frac{\partial t}{\partial t^{\prime}}\left\{\frac{\partial\left(\frac{\partial \bar{x}}{\partial t} \cdot \bar{\Phi}\right)}{\partial t}+\frac{1}{\frac{\partial t}{\partial t^{\prime}}}\left(\frac{\partial \bar{\Phi}}{\partial x^{\prime}}+\frac{\partial \bar{\Phi}}{\partial t} \frac{\partial t}{\partial x^{\prime}}\right)\right\}, \tag{9}
\end{equation*}
$$

and with the use of (6), (7), (8):

$$
\begin{gathered}
K_{x}=-\frac{1}{1+\frac{\partial \bar{r}}{\partial t}} Q_{x}, \\
Q_{x}=-\frac{\partial}{\partial t}\left(\frac{\bar{\Phi}}{2} \cdot \frac{\partial^{2} \bar{r}^{2}}{\partial x^{\prime} \partial t}\right)+\left(1+\frac{\partial \bar{r}}{\partial t}\right) \frac{\partial \bar{\Phi}}{\partial x^{\prime}}-\frac{\partial \bar{r}}{\partial x^{\prime}} \frac{\partial \bar{\Phi}}{\partial t},
\end{gathered}
$$

which can be converted into:

$$
Q_{x}=\frac{\partial}{\partial x^{\prime}}\left\{\bar{\Phi}\left(1+\frac{\partial \bar{r}}{\partial t}\right)\right\}-\frac{\partial}{\partial t}\left\{\bar{\Phi}\left(\frac{\partial \bar{r}}{\partial x^{\prime}}+\frac{1}{2} \cdot \frac{\partial^{2} \bar{r}^{2}}{\partial x^{\prime} \partial t}\right)\right\} .
$$

Since:

$$
\frac{1}{\Phi}=\bar{r}+\frac{1}{2} \frac{\partial \bar{r}^{2}}{\partial t}, \quad-\frac{1}{\Phi^{2}} \frac{\partial \bar{\Phi}}{\partial x^{\prime}}=\frac{\partial \bar{r}}{\partial x^{\prime}}+\frac{1}{2} \frac{\partial \bar{r}^{2}}{\partial x^{\prime} \partial t}
$$

it will easily follow that:
(A)

$$
\begin{gathered}
K_{x}=-\frac{1}{1+\frac{\partial \bar{r}}{\partial t}} \frac{\partial P}{\partial x^{\prime}}, \\
P=\frac{1}{\bar{r}}+\frac{1}{\bar{\Phi}} \frac{\partial \bar{\Phi}}{\partial t}=\frac{1}{\bar{r}} \cdot \frac{1-\frac{1}{2} \frac{\partial^{2} \bar{r}^{2}}{\partial t^{2}}}{1+\frac{\partial \bar{r}}{\partial t}} .
\end{gathered}
$$

The electric force has thus been reduced to a type of potential, namely, $P$. If one expresses the differential quotient of $r$ with respect to $t$ directly in terms of the components of the velocity $v_{x}, v_{y}$, $v_{z}$ and acceleration $w_{x}, w_{y}, w_{z}$ of $e$ at time $t$ then one will get:

$$
\begin{equation*}
P=\frac{1-v_{x}^{2}-v_{y}^{2}-v_{z}^{2}+w_{x}\left(x^{\prime}-x\right)+w_{y}\left(y^{\prime}-y\right)+w_{z}\left(z^{\prime}-z\right)}{r-v_{x}\left(x^{\prime}-x\right)-v_{y}\left(y^{\prime}-y\right)-v_{z}\left(z^{\prime}-z\right)}=\frac{1-v^{2}+r w \cos (r, w)}{r[1-v \cos (r, v)]}, \tag{10}
\end{equation*}
$$

and it will follow upon differentiating with respect to $x^{\prime}$ that:

$$
K_{x}=-\frac{w_{x}}{r[1-v \cos (r, v)]^{2}}+\frac{1-v^{2}+r w \cos (r, w)}{r[1-v \cos (r, v)]}\left(\frac{x^{\prime}-x}{r}-v_{x}\right) .
$$

One can read off the following from that: The electric force that is created by e at the point $x^{\prime}$, $y^{\prime}, z^{\prime}$ is composed of three parts:

A force in the direction of the connecting line:

$$
K_{1}=\frac{e e^{\prime}}{r^{2}} \cdot \frac{1-v^{2}+r w \cos (r, w)}{r[1-v \cos (r, v)]^{3}},
$$

A force in the direction of the velocity $v$ of $e$ :

$$
K_{2}=-v \cdot K_{1},
$$

A force in the direction of the acceleration $w$ of $e$ :

$$
K_{3}=-\frac{e e^{\prime}}{r} \cdot \frac{w}{[1-v \cos (r, v)]^{2}} .
$$

It hardly needs to be repeated that here $v$ and $w$ are understood to mean the velocity and acceleration of $e$ at the moment when the light wave was emitted from $e$.
§ 3. Calculating the magnetic and mechanical forces. - Whereas the explicit representation of the electric force takes a rather complicated form, it proves to be very easy to derive the magnetic and mechanical force from it once the electric force has been calculated.

One first finds that the magnetic force is:

$$
\begin{aligned}
H_{z} & =\frac{\partial \Gamma_{x}}{\partial y^{\prime}}-\frac{\partial \Gamma_{y}}{\partial x^{\prime}}=\frac{\partial}{\partial y^{\prime}}\left(\frac{\partial x}{\partial t} \Phi\right)-\frac{\partial}{\partial x^{\prime}}\left(\frac{\partial y}{\partial t} \Phi\right) \\
& =\frac{\partial \bar{x}}{\partial t} \frac{\partial \bar{\Phi}}{\partial y^{\prime}}-\frac{\partial \bar{y}}{\partial t} \frac{\partial \bar{\Phi}}{\partial x^{\prime}}+\frac{\partial t}{\partial y^{\prime}} \frac{\partial}{\partial t}\left(\frac{\partial \bar{x}}{\partial t} \bar{\Phi}\right)-\frac{\partial t}{\partial x^{\prime}} \frac{\partial}{\partial t}\left(\frac{\partial \bar{y}}{\partial t} \bar{\Phi}\right) .
\end{aligned}
$$

If one considers the formulas that were derived above:

$$
\begin{gathered}
-\frac{1}{\bar{\Phi}^{2}} \frac{\partial \bar{\Phi}}{\partial x^{\prime}}=\frac{\partial \bar{r}}{\partial x^{\prime}}+\frac{1}{2} \frac{\partial^{2} \bar{r}^{2}}{\partial x^{\prime} \partial t}=\frac{\partial \bar{r}}{\partial x^{\prime}}-\frac{\partial \bar{x}}{\partial t}, \\
\frac{\partial t}{\partial x^{\prime}}=-\frac{\frac{\partial \bar{r}}{\partial x^{\prime}}}{1+\frac{\partial \bar{r}}{\partial t}}=-\frac{\partial \bar{r}}{\partial x^{\prime}} \frac{\partial t}{\partial t^{\prime}},
\end{gathered}
$$

and the corresponding ones that are valid for the other coordinates, then one will get:

$$
H_{z}=\frac{\partial \bar{r}}{\partial x^{\prime}}\left\{\frac{\partial \bar{\Phi}}{\partial y^{\prime}}+\frac{\partial t}{\partial t^{\prime}} \frac{\partial}{\partial t}\left(\frac{\partial \bar{y}}{\partial t} \bar{\Phi}\right)\right\}-\frac{\partial \bar{r}}{\partial y^{\prime}}\left\{\frac{\partial \bar{\Phi}}{\partial x^{\prime}}+\frac{\partial t}{\partial t^{\prime}} \frac{\partial}{\partial t}\left(\frac{\partial \bar{x}}{\partial t} \bar{\Phi}\right)\right\},
$$

or when one recalls the expression (9) for the electric force:

$$
H_{z}=\frac{\partial \bar{r}}{\partial x^{\prime}}\left\{-K_{y}-\frac{\partial \bar{\Phi}}{\partial t} \frac{\partial \bar{r}}{\partial y^{\prime}} \frac{\partial t}{\partial t^{\prime}}\right\}-\frac{\partial \bar{r}}{\partial y^{\prime}}\left\{-K_{x}-\frac{\partial \bar{\Phi}}{\partial t} \frac{\partial \bar{r}}{\partial x^{\prime}} \frac{\partial t}{\partial t^{\prime}}\right\},
$$

so ultimately:
(C)

$$
H_{z}=\frac{\partial \bar{r}}{\partial y^{\prime}} K_{x}-\frac{\partial \bar{r}}{\partial x^{\prime}} K_{y} .
$$

Since the differential quotients $\partial \bar{r} / \partial x^{\prime}$, etc., simply mean the direction cosines of $r$, that result can be expressed in words as:

The magnetic force is perpendicular to the radius vector and the electric force and has the magnitude:

$$
H=K \sin (K, r) .
$$

The mechanical force that the amount of electricity $e^{\prime}$ experiences when it moves with velocity $v^{\prime}$ is composed of the electric force and an additional force that is perpendicular to the magnetic force and $v^{\prime}$, and has a magnitude of $H v^{\prime} \sin \left(H, v^{\prime}\right)$. A simple geometric consideration will show that as a result of the connection between $K$ and $H$ that was just found, that additional force must fall in the plane ( $K, r$ ) and have the magnitude:

$$
K v^{\prime} \sin (K, r) \cos \left(v^{\prime},[K, r]\right) .
$$

That can be put into words as:

The mechanical force that a charge e exerts upon another charge $e^{\prime}$ that moves with a velocity $v^{\prime}$ is composed of the electric force $K$ that e generates at the location of $e^{\prime}$ and an additional force. The additional force lies in the plane that is determined by the direction of the electric force and the connecting line r from e to $e^{\prime}$ perpendicular to $v^{\prime}$ and has the magnitude:

$$
K u^{\prime} \sin (K, r),
$$

where $u^{\prime}$ means the projection of $v^{\prime}$ onto that plane.
Of the two possible directions for the additional force, one then chooses the one that points away from $u^{\prime}$ in the sense of rotation of $K, r$.

It is a very interesting result (at least in the context of ordinary mechanics) that the elementary force between two point-charges proves to depend upon only the first and second derivatives of the coordinates with respect to time. In general, one is then dealing with the values of those quantities for two different times, namely, the velocity and acceleration of the source electron at the moment that the force-wave is emitted and the velocity of the charge in question at the moment when it is overtaken by the wave. The equations of motion for a system of point-charges are then, in turn, differential equations of order only two, while the complexity lies in the fact that they are, at the same time, functional equations.
§ 4. A different representation of the mechanical force. - If one substitutes the value (C) for the magnetic force in the expression (1) for the mechanical force (in which the components of $v$ are provided with a prime, since one is dealing with the effect on $e^{\prime}$ ) and employs the representation (A), (B) for the electric force that was found above then one will get the following value for the mechanical force:

$$
F=-\frac{1-v^{\prime} \cos (r, v)}{1+\frac{\partial \bar{r}}{\partial t}} \cdot \frac{\partial P}{\partial x^{\prime}}+\frac{\frac{\partial \bar{r}}{\partial x^{\prime}}}{1+\frac{\partial \bar{r}}{\partial t}}\left(v_{x}^{\prime} \frac{\partial P}{\partial x_{1}}+v_{y}^{\prime} \frac{\partial P}{\partial y_{1}}+v_{z}^{\prime} \frac{\partial P}{\partial z_{1}}\right) .
$$

One introduces new variables that take the form of the coordinates of the point at which $e^{\prime}$ is found when the wave is emitted from $e$, so when it already has the velocity $v^{\prime}$ over the entire time interval. They will be given by:

$$
\begin{aligned}
& \mathfrak{x}=x^{\prime}-v_{x}^{\prime} \bar{r}, \\
& \mathfrak{y}=y^{\prime}-v_{y}^{\prime} \bar{r}, \quad \bar{r}=\sqrt{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}, \\
& \mathfrak{z}=z^{\prime}-v_{z}^{\prime} .
\end{aligned}
$$

It then follows that for any function $f$ of $x^{\prime}, y^{\prime}, z^{\prime}$ :

$$
\begin{gathered}
\frac{\partial f}{\partial x^{\prime}}=\frac{\partial f}{\partial \mathfrak{x}}-\frac{\partial \bar{r}}{\partial x_{1}}\left(v_{x}^{\prime} \frac{\partial f}{\partial \mathfrak{x}}+v_{y}^{\prime} \frac{\partial f}{\partial \mathfrak{y}}+v_{z}^{\prime} \frac{\partial f}{\partial \mathfrak{z}}\right) \\
v_{x}^{\prime} \frac{\partial f}{\partial x^{\prime}}+v_{y}^{\prime} \frac{\partial f}{\partial y^{\prime}}+v_{z}^{\prime} \frac{\partial f}{\partial z^{\prime}}=\left(1-v^{\prime} \cos \left(r, v^{\prime}\right)\right)\left(v_{x}^{\prime} \frac{\partial f}{\partial \mathfrak{x}}+v_{y}^{\prime} \frac{\partial f}{\partial \mathfrak{y}}+v_{z}^{\prime} \frac{\partial f}{\partial \mathfrak{z}}\right) .
\end{gathered}
$$

When applied to $F_{x}$, that will give:
(D)

$$
F_{x}=-\frac{1-v^{\prime} \cos \left(r, v^{\prime}\right)}{1-v \cos (r, v)} \cdot \frac{\partial P}{\partial \mathfrak{x}},
$$

which will also imply that the total mechanical force, after removing a suitable factor, can be obtained by differentiating the potential $P$ above with respect to the new variables $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$.
§ 5. First special case: Two charges moving with uniform, equal, and equally-directed velocity. - Here, it would be best to continue with the last way of representing the mechanical force (D). If $v=v^{\prime}$ then it will follow that:

$$
F_{x}=-\frac{\partial P}{\partial \mathfrak{x}} .
$$

That will immediately imply that the mechanical force has a potential in that case. If one lays the $x$-axis in the common direction of motion, to simplify, then the value of the potential that is given by (10) will be:

$$
P=\frac{1-v^{2}}{r-v \cos \left(x^{\prime}-x\right)} .
$$

The introduction of the variables $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ in place of $x^{\prime}, y^{\prime}, z^{\prime}$ will yield:

$$
P=\frac{1-v^{2}}{\sqrt{(\mathfrak{x}-x)^{2}+\left[(\mathfrak{h}-y)^{2}+(\mathfrak{z}-z)^{2}\right]\left(1-v^{2}\right)}}
$$

by a simple calculation. On the basis of known laws of forces, one will then find the forces that electric charges exert upon each other under a common uniform translation.
§ 6. Second special case: Mechanical force on a charge that moves arbitrarily that is due to a uniformly-moving one. - When the acceleration $w$ of the source charge $e$ vanishes, the potential $P$ will become:

$$
P=\frac{1-v^{2}}{r-v_{x}\left(x^{\prime}-x\right)-v_{y}\left(y^{\prime}-y\right)-v_{z}\left(z^{\prime}-z\right)},
$$

and it will follow from § $\mathbf{2}$ that the electric force is:

$$
K_{x}=\frac{1-v^{2}}{r^{3}[1-v \cos (r, v)]^{3}}\left[x^{\prime}-x-v_{x} r\right] .
$$

If one sets:

$$
\begin{array}{ll}
\xi=x^{\prime}-x-v_{x} r, & r^{2}=\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}, \\
\eta=y^{\prime}-y-v_{y} r, & \\
\zeta=z^{\prime}-z-v_{z} r, & \rho^{2}=\xi^{2}+\eta^{2}+\zeta^{2}
\end{array}
$$

then $\xi, \eta, \zeta$ will be the relative coordinates of $e^{\prime}$ relative to the simultaneous location of $e, \rho$ is the actual distance between both charges, and both are valid at the moment when the wave overtakes $e^{\prime}$. If one eliminates $x^{\prime}, y^{\prime}, z^{\prime}$ with the help of $\xi, \eta, \zeta$ then one will get:

$$
K_{x}=\frac{1-v^{2}}{\left[1-v^{2} \sin ^{2}(v, \rho)\right]^{3 / 2}} \cdot \frac{\xi}{\rho^{3}} .
$$

It then follows from this that:

The electric force points away from the simultaneous position of the source charge and has a magnitude of:

$$
K=\frac{1}{\rho^{2}} \cdot \frac{1-v^{2}}{\left[1-v^{2} \sin ^{2}(v, \rho)\right]^{3 / 2}} .
$$

In order to find the additional mechanical force, one observes that the triangle that is defined by the positions of $e$ and $e^{\prime}$ when the wave reaches the latter and the position of $e$ when the wave is emitted has sides $r, \rho$, and $r \cdot v$, resp., and the following equation will then be true:

$$
r \sin (r, \rho)=r \sin (r, K)=r \cdot v \sin (v, \rho)
$$

or

$$
\sin (r, K)=v \sin (v, \rho) .
$$

It then follows from § $\mathbf{3}$ that:
The additional force lies in the plane that includes the connecting line from $e$ to $e^{\prime}$ and the velocity $v$ of $e$ and is perpendicular to the velocity $v^{\prime}$ of $e^{\prime}$, and it has the magnitude:

$$
\frac{1-v^{2}}{\left[1-v^{2} \sin ^{2}(v, \rho)\right]^{3 / 2}} \cdot \frac{v \cdot v^{\prime}}{\rho^{2}} \sin (v, \rho) \cos \left(v^{\prime},[\rho, v]\right) .
$$

That immediately reminds one of Grassmann's elementary law of the older electrodynamics, and one can, in fact, express our result exactly as:

One will get the mechanical force that a uniformly-moving charge exerts upon another one that moves arbitrarily when one adds the usual Coulomb electrostatic force and Grassmann's elementary force and multiplies the result by the correction factor:

$$
\frac{1-v^{2}}{\left[1-v^{2} \sin ^{2}(v, \rho)\right]^{3 / 2}},
$$

in which $v$ is the velocity of the source charge, and $\rho$ is the connecting line between the two charges ${ }^{1}$ ).

It must be pointed out that this rule probably applies to the study of the motion of individual electrons, but that for a closed electric current, the acceleration cannot be neglected in comparison to the velocity, due to the curvature of the current path. Hence, one would perhaps not like to infer from this that the correction factor above should be added to the older law of induction for closed currents.
§ 7. Third special case: Field at a great distance from the source charge. - From (10), the potential from which the electric force was derived in $\S 2$ has the magnitude:

$$
P=\frac{1-v^{2}}{r(1-v \cos (r, v))}+\frac{w \cos (r, w)}{1-v \cos (r, v)} .
$$

[^2]At sufficiently-large distances from the source charge, the first part of $P$ will vanish in comparison to the second one, when $w$ is not precisely zero, and all that will remain is the part that depends upon the acceleration $w$ :

$$
P=\frac{w \cos (r, w)}{1-v \cos (r, v)} .
$$

Since that still depends upon the direction of $r$, but no longer depends upon its magnitude, it will follow that:

$$
\frac{\partial P}{\partial r}=0
$$

i.e., at a great distance from the source charge, the electric force is perpendicular to the radius vector. One can infer immediately from § $\mathbf{3}$ that at a great distance from the source charge, the magnetic force will be equal to the electric force. At the same time, it will be perpendicular to the electric force and the radius vector.

Therefore, the relationships that one is used to for light waves are always in effect (even for vanishing acceleration) at a great distance from the source charge.

If the velocity $v$ is small compared to the speed of light then that will yield an approximate value for the absolute value of the electric and magnetic forces:

$$
K=H=\frac{w \sin (w, r)}{r} .
$$

One sees from this, as was suggested before in a different way, that we will not observe the square of the amplitude or velocity of an oscillating electron from the intensity of the light that it creates, but its acceleration.


[^0]:    ${ }^{(1)}$ In particular, one should confer: E. Wiechert, "Electrodynamische Elementargestze," Drude’s Annalen, Bd. 4, pps. 676, 677.

[^1]:    ( ${ }^{1}$ ) Lorentz, La Théorie de Maxwell.
    $\left(^{2}\right)$ Helmholtz, Gesammelte Abhandl., Bd. III, pp. 476.

[^2]:    $\left.{ }^{( }{ }^{1}\right)$ In the reference on the elementary laws of electrodynamics by Reif and Sommerfeld that is found in print (Enzyklopädie der mathematischen Wissenschaften, Bd. V, 12), Sommerfeld emphasized the close kinship between the theory of electrons and Clausius's older law of potential, as well as Grassman's elementary law. His viewpoint was confirmed entirely and, at the same time, made more precise in my previous note, along with the current one in §§ 3 and 6.

