# On the hypotheses of quantum theory 

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(Presented on 30 March 1916 [cf. supra, pp. 435])
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## I.

§ 1. - The hypotheses of quantum theory have recently been extended by Planck ( ${ }^{1}$ ) and Sommerfeld ( ${ }^{2}$ ) to the case of mechanical systems with several degrees of freedom. The problem consists of giving some principles for the subdivision of phase space into elementary regions. Here, I would like to show that for an important group of mechanical problems, a subdivision of phase space can result in an obvious way when one makes use of canonical variables of a certain type. The subdivision will overlap with the one that Planck and Sommerfeld proposed in many, but not all, cases. Later on, I shall work through two examples that relate to the theory of the electrical splitting of hydrogen lines and to the theory of band spectra. Here, as in all problems that were treated up to now, the proposed subdivision is carried out consistently. In other cases, contradictions can be found only on the boundary of phase space (cf., § 5). For the time being, I have not attempted to bound the region of problems in which our subdivision is consistent from the outset.
§ 2. - I shall restrict myself to mechanical systems whose motion is conditionally periodic ( ${ }^{3}$ ). That is understood to mean the following: If $x_{i}$ mean the coordinates that establish the configuration of the system uniquely, while the $y_{i}$ means the associated impulse components, and $F$ is the energy then the equations of motion for $k$ degrees of freedom will read:

$$
\frac{d x_{i}}{d t}=+\frac{\partial F}{\partial y_{i}}, \quad \frac{d y_{i}}{d t}=-\frac{\partial F}{\partial x_{i}}, \quad i=1,2, \ldots, k .
$$

The conditionally-periodic systems are then characterized by the fact that the solutions will have the form:

[^0]\[

$$
\begin{aligned}
& x_{i}=x_{i}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}, w_{1}, w_{2}, \ldots, w_{k}\right) \\
& y_{i}=y_{i}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}, w_{1}, w_{2}, \ldots, w_{k}\right) .
\end{aligned}
$$
\]

In that way, the $\alpha_{\lambda}(\lambda=1,2, \ldots, k)$ are constants. The $w \lambda$ are angle quantities that are proportional to time:

$$
w_{\lambda}=n_{\lambda} t+\beta_{\lambda}
$$

( $n_{\lambda}$ and $\beta_{\lambda}$ are constants, where $n_{\lambda}$ is the "mean motion" and $\beta_{\lambda}$ is the initial value of the angle $w_{\lambda}$ ). and the $x_{i}, y_{i}$ are periodic of period $2 \pi$ in the angles $w_{\lambda}$.

One can always arrange that the $\alpha_{\lambda}$ are the canonically-conjugate to the $w_{\lambda}$. The equations of motion will then go to:

$$
\frac{d \alpha_{\lambda}}{d t}=-\frac{\partial F}{\partial w_{\lambda}}, \quad \frac{d w_{\lambda}}{d t}=-\frac{\partial F}{\partial \alpha_{\lambda}} .
$$

Since each $\alpha_{\lambda}$ is constant, it will follow that:

$$
\frac{\partial F}{\partial w_{\lambda}}=0, \quad F=F\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}\right)
$$

The energy is a function of only the quantities $\alpha_{\lambda}$. The equations for $d w_{\lambda} / d t$ will imply the "mean motions" of the angle $w_{\lambda}$ as functions of the quantities $\alpha_{\lambda}$ :

$$
\alpha_{\lambda}=+\frac{\partial F}{\partial \alpha_{\lambda}} .
$$

Since $n_{\lambda}$ is a reciprocal time, it will follow from this that $\alpha_{\lambda}$ has the dimension of an action (viz., energy times time). We would then like to refer to the $\alpha_{\lambda}$ as "action variables" (variable with respect to external perturbations of the system), as opposed to the "angle variables" $w \lambda$.

The volume of phase space will be:

$$
\int d \alpha_{1} d w_{1} d \alpha_{2} d w_{2} \cdots d \alpha_{k} d w_{k}
$$

That implies the following natural subdivision of phase space: Each angle variable $w \lambda$ has the limits 0 and $2 \pi$. Each action variable $\alpha \lambda$ has limits such that when one integrates from one limit to the other, one will have:

$$
\int d \alpha_{\lambda} d w_{\lambda}=2 \pi \int d \alpha_{\lambda}=h \quad(h=\text { Planck's quantum of action })
$$

It follows from this that the limits of $\alpha_{\lambda}$ are:

$$
\alpha_{\lambda}=\varepsilon_{\lambda}+m_{\lambda} \frac{h}{2 \pi}, \quad m_{\lambda} \text { are whole numbers, } \varepsilon_{\lambda} \text { are constants. }
$$

In words:

For conditionally-periodic motions, the subdivision of phase space into elementary regions given by saying that the action variables will advance in multiples of $h / 2 \pi$ from one limit to the other, while the angle variables will be enclosed between the values 0 and $2 \pi$.
§ 3. Degenerate cases. - When the representation of the motion of the system is generally possible with the help of less than $k$ angle variables (in other words. when one or more mean motions $n_{\lambda}$ vanishes for arbitrary values of the $\alpha_{\lambda}$ ), the foregoing rule shall be true for only the variable pairs $w \lambda, \alpha_{\lambda}$ "with mean motion," whereas no subdivision of phase space in the remaining pairs of variables $w \lambda, \alpha_{\lambda}$ ("without mean motion") shall result. That extension of the prescription is closely related to the remark that for a vanishing mean motion $n \lambda$, as a result of the equation $n_{\lambda}$ $=\partial F / \partial \alpha_{\lambda}=0$, the energy will be independent of the values of the $\alpha_{\lambda}$, so those variables will have no relationship to the energetic processes in the system.
§ 4. Normalization of the variables. Boundary of phase space. - When one introduces any linear functions of new variables $w_{\lambda}^{\prime}$ with whole-number coefficients in place of the $w \lambda$, the $x_{i}, y_{i}$ will also be periodic of period $2 \pi$ in those new variables $w_{\lambda}^{\prime}$. The new associated canonical variables $\alpha_{\lambda}^{\prime}$ will be linear functions of the $\alpha_{\lambda}$ with whole-number coefficients since it is known that both groups of canonical variables will transform contragrediently under linear substitutions.

If one has found a representation of the motion in a mechanical problem in terms of a number time-proportional angle variables $w \lambda$ then one can subsequently show that one or more commensurabilities of the form:

$$
0=p_{1} n_{1}+p_{2} n_{2}+\ldots+p_{k} n_{k} \quad \text { (the } p_{\lambda} \text { are whole numbers) }
$$

can exist between the mean motions $n_{\lambda}$ and those angles. One can then introduce a number of new variables $w_{i}^{\prime}$ in place of the variables $w_{\lambda}$ by a substitution of the aforementioned type:

$$
q_{i} w_{i}^{\prime}=p_{1} n_{1}+p_{2} n_{2}+\ldots+p_{k} n_{k} \quad \text { (the } q_{i} \text { are whole numbers) }
$$

which have a mean motion of zero. If one chooses the $q_{i}$ suitably (e.g., such that they include the determinant of the $p$ as a factor) then solving them for the $w_{i}$ will give them as whole-number functions of the $w_{i}^{\prime}$. In that way, one can always split the angle variables into two groups in such a way that the variables in the first group will be associated with a mean motion of zero, while no commensurability will exist between the mean motions of the second group. Such a normalization was already tacitly assumed in the discussion of the degenerate cases above.

In that way, one encounters a special determination of things. I would like to assume that one has found angle variables $w_{\lambda}$ from the outset that give a simple covering of phase space when each angle variable varies from 0 to $2 \pi$ and one varies the $\alpha_{\lambda}$ within certain limits. It shall then be established that with the normalization that was just discussed, the determinant of the old variables with respect to the new ones shall be kept as small as possible, so equal to 1 , if that is feasible.

Without that determination, one would be able to set, e.g., $w_{1}=n w_{1}^{\prime}$ ( $n$ is any whole number). One would then have $\alpha_{1}^{\prime}=n \alpha_{1}$ or $\alpha_{1}=\alpha_{1}^{\prime} / n$, and when one lets $\alpha_{1}^{\prime}$ advance though multiples of $h / 2 \pi$, one would arrive at an arbitrarily-fine subdivision of the variables $\alpha_{1}$, which is obviously physically absurd.

One must further observe the boundary of phase space. Planck has stressed that the elementary regions must fill up all of phase space. The boundary of phase space will generally be determined by some sort of inequalities between the variables $\alpha_{\lambda}$. One seeks the limits of the variables with mean motions by initially dropping all inequalities that include the variables without mean motions at the limits. I would now like to assume that for $l$ variables with mean motions, the limit conditions consist of $l$ inequalities:

$$
\sum_{i} p_{i \lambda} \alpha_{i}>c_{\lambda} \quad \text { (the } c_{\lambda} \text { are constants, the } p_{i \lambda} \text { are whole numbers). }
$$

One can then go to new variables $\alpha_{\lambda}^{\prime}$ and their associated angles $w_{\lambda}^{\prime}$ by the substitution:

$$
\begin{aligned}
& \alpha_{\lambda}^{\prime}=\sum_{i} p_{i \lambda} \alpha_{i}, \\
& w_{\lambda}^{\prime}=\sum_{\lambda} p_{i \lambda} w_{\lambda} .
\end{aligned}
$$

From what was said above, they will exhibit the characteristic properties of our variables, and on the boundary of phase space, they will have the simple form:

$$
\alpha_{\lambda}^{\prime}>c_{\lambda} .
$$

A complete filling-up of phase space by variables with mean motions will result when we choose the initial constants of our subdivision to have only the values:

$$
\varepsilon_{\lambda}=c_{\lambda} .
$$

The variables without mean motions will then be integrated up to the limits that are imposed upon them without subdivision, and in that way all of phase space will be filled.

In many important problems, the foregoing conventions will lead to a consistent subdivision of phase space in a unique manner. An interesting future problem for mathematics would be to find the largest-possible class of problems in which there is consistency and uniqueness from the outset.
§ 5. Thermal and optical properties of the system. - As long as the subdivision of phase space is given, the thermal properties of a body that consists of a large number of such systems in a stationary state can be determined in the way that Planck gave.

As far as the optical properties are concerned, one will get a very simple determination of the frequencies $v$ that are emitted by the system when one accepts the Ansatz of N. Bohr. The frequencies $v$ will then follow from the equation:

$$
h v=F\left(\varepsilon_{1}+\frac{h}{2 \pi} m_{1}, \varepsilon_{2}+\frac{h}{2 \pi} m_{2}, \ldots, \varepsilon_{\lambda}+\frac{h}{2 \pi} m_{\lambda}\right)-F\left(\varepsilon_{1}+\frac{h}{2 \pi} m_{1}^{\prime}, \varepsilon_{2}+\frac{h}{2 \pi} m_{2}^{\prime}, \ldots, \varepsilon_{\lambda}+\frac{h}{2 \pi} m_{\lambda}^{\prime}\right),
$$

in which the $m_{\lambda}$ and $m_{\lambda}^{\prime}$ are arbitrary whole numbers. In that way, just as many whole numbers $m_{\lambda}$ ( $m_{\lambda}^{\prime}$, resp.) will appear as the number of incommensurable mean motions in the system. (That number will be smaller than the degrees of freedom for degenerate systems.)
§ 6. Application of JACOBI's integration method. - One seeks to exhibit the relationship between the original variables $x_{i}, y_{i}$ and the angle and action variables $w \lambda, \alpha \lambda$ by the Jacobi form of the canonical substitution:

$$
y_{i}=\frac{\partial S}{\partial x_{i}}, \quad w_{\lambda}=\frac{\partial S}{\partial \alpha_{\lambda}}, \quad S=S\left(x_{1}, x_{2}, \ldots, x_{k}, \alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}\right) .
$$

$S$ must then be a solution of the partial differential equation:

$$
F\left(x_{i}, y_{i}\right)=F\left(x_{i}, \frac{\partial S}{\partial x_{i}}\right)=\text { const. }
$$

When one has found a solution of that differential equation with $k$ arbitrary constant $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{k}$, which is sufficient for the integration of the problem, those constants will generally still not be the desired constants $\alpha \lambda$, and the values $\partial S / \partial \gamma_{\lambda}$ will not be the desired angles $w_{\lambda}$. However, there is a case that presents itself in the most important problems in which one comes to the desired variables directly from the partial differential equation. One assumes that the variables $x_{i}$ are represented in the form of periodic functions with period $2 \pi$ of just as many auxiliary variables $\eta_{i}$ and that a solution of the partial differential equation in the variables $\eta_{i}$ can be found in the following form:

$$
S=\alpha_{1} \eta_{1}+\alpha_{2} \eta_{2}+\ldots+\alpha_{k} \eta_{k}+T\left(\alpha_{i}, \eta_{i}\right),
$$

in which the $\alpha_{i}$ are integration constants, and $T$ is a function of those constants and the variables $\eta_{\lambda}$ that is again periodic of period $2 \pi$ in the variables $\eta_{\lambda}$.

It will then follow that:

$$
y_{i}=\frac{\partial S}{\partial x_{i}}=\sum_{\lambda} \frac{\partial S}{\partial \eta_{\lambda}} \frac{\partial \eta_{\lambda}}{\partial x_{i}}=\sum_{\lambda} \alpha_{\lambda} \frac{\partial \eta_{\lambda}}{\partial x_{i}}+\frac{\partial T}{\partial \eta_{\lambda}} \frac{\partial \eta_{\lambda}}{\partial x_{i}}, \quad w_{\lambda}=\frac{\partial S}{\partial \alpha_{\lambda}}=\eta_{\lambda}+\frac{\partial T}{\partial \alpha_{\lambda}} .
$$

It follows from the first system of equations that the $y_{i}$ are periodic functions of period $2 \pi$ in the variables $\eta_{\lambda}$, just as we have assumed for the $x_{i}$. It will follow from the second system of equations that $\eta_{\lambda}$ and $w_{\lambda}$ will simultaneously increase by $2 \pi$. If we invert the system of equations by expressing the $\eta_{\lambda}$ as functions of the $w_{\lambda}$ then we will get relations of the form:

$$
\eta_{\lambda}=w_{\lambda}+\text { periodic function }(\text { period } 2 \pi) \text { of the variables } w_{i} .
$$

Therefore, the $\eta_{\lambda}$, and thus, the $x_{i}$ and $y_{i}$, as well, will be periodic functions of the $w_{\lambda}$ of period $2 \pi$. The variables $w_{\lambda}$ are angles of the desired type, and the canonically-associated variables $\alpha_{\lambda}$ are the action variables that we sought, but naturally, except for their normalization and with consideration given to the boundary of phase space.

The examples that follow below will show how simple it is to determine the action variables then.
§ 7. Relationship to PLANCK's subdivision of phase space. - We would like to consider the motion of a point of mass 1 in the plane whose rectangular coordinates are $x_{1}, x_{2}$ under the action of the potential $\frac{1}{2}\left(A_{1}^{2} x_{1}^{2}+A_{2}^{2} x_{2}^{2}\right)$, which corresponds to an anisotropic elastic force. It is known that we will then have:

$$
\begin{gathered}
x_{1}=\gamma_{1} \sin \left(A_{1} t+\beta_{1}\right), \quad x_{2}=\gamma_{2} \sin \left(A_{2} t+\beta_{2}\right), \\
\quad\left(\beta_{1}, \beta_{2}, \gamma_{1}, \gamma_{2} \text { are constants }\right) \\
\dot{x}_{1}=y_{1}=\gamma_{1} A_{1} \cos \left(A_{1} t+\beta_{1}\right), \quad \dot{x}_{2}=y_{2}=\gamma_{2} A_{2} \cos \left(A_{2} t+\beta_{2}\right)
\end{gathered}
$$

We will then have that:

$$
w_{1}=A_{1} t+\beta_{1}, \quad w_{2}=A_{2} t+\beta_{2}
$$

are angles of the desired kind. We must then look for the variables that are canonically conjugate to them. Since the two coordinates obviously move independently of each other, we can consider each coordinate by itself. The energy of the motion of $x_{1}$ is:

$$
F_{1}=\frac{1}{2}\left(x_{1}^{2}+A_{1}^{2} x_{1}^{2}\right)=\frac{1}{2} A_{1}^{2} \gamma_{1}^{2} .
$$

We must have:

$$
\frac{d w_{1}}{d t}=A_{1}=\frac{\partial F}{\partial \alpha_{1}}=A_{1}^{2} \gamma_{1} \frac{\partial \gamma_{1}}{\partial \alpha_{1}}
$$

for the desired canonical variable $\alpha_{1}$. It will then follow that:

$$
1=A_{1} \gamma_{1} \frac{\partial \gamma_{1}}{\partial \alpha_{1}}, \quad \alpha_{1}=\frac{1}{2} A_{1} \gamma_{1}^{2}
$$

and that will correspondingly imply that:

$$
\alpha_{2}=\frac{1}{2} A_{2} \gamma_{2}^{2} .
$$

The boundary conditions on phase space are obviously:

$$
\alpha_{1}>0, \quad \alpha_{2}>0
$$

The boundary of the elementary region will then be determined by the values:

$$
\alpha_{1}=\frac{h}{2 \pi} m_{1}, \quad \alpha_{2}=\frac{h}{2 \pi} m_{2}
$$

The distinguished values of the energy will be:

$$
F=F_{1}+F_{2}=A_{1} \alpha_{1}+A_{2} \alpha_{2}=\frac{h}{2 \pi}\left(A_{1} m_{1}+A_{2} m_{2}\right) .
$$

The volume of the elementary region in phase space will be:

$$
4 \pi^{2} \int d \alpha_{1} d \alpha_{2}=h m_{1} \cdot h m_{2}
$$

We would now like to go to the case of isotropy $A_{1}=A_{2}=A$. The two angles $w_{1}$ and $w_{2}$ will then have the same mean motion, so a degeneracy will occur. If we carry out a linear whole-number substitution with determinant 1 :

$$
w_{1}^{\prime}=w_{1}, \quad w_{2}^{\prime}=w_{2}-w_{1}
$$

then $w_{2}^{\prime}$ will have a mean motion of zero, while $w_{1}^{\prime}$ will have a mean motion of $A$. The associated canonical variables will be:

$$
\alpha_{1}^{\prime}=\alpha_{1}+\alpha_{2}, \quad \alpha_{2}^{\prime}=\alpha_{2} .
$$

According to our rules, only a subdivision in terms of the variables with mean motion $\alpha_{1}^{\prime}$ has to result. Since $\alpha_{1}^{\prime}=0$ is a limit point in phase space, we must set:

$$
\alpha_{1}^{\prime}=\frac{h}{2 \pi} m .
$$

The distinguished energy values will be:

$$
F=A\left(\alpha_{1}+\alpha_{2}\right)=A \alpha_{1}^{\prime}=A \frac{h}{2 \pi} m
$$

The boundary of phase space will be:

$$
\alpha_{1}=\alpha_{1}^{\prime}-\alpha_{2}^{\prime}>0, \quad \alpha_{2}=\alpha_{2}^{\prime}>0,
$$

or

$$
\alpha_{1}^{\prime}>\alpha_{2}^{\prime}>0 .
$$

We then restrict only the domain of the $\alpha_{2}^{\prime}$. The volume of the elementary region will become:

$$
4 \pi^{2} \int d \alpha_{1}^{\prime} d \alpha_{2}^{\prime}
$$

or when $\alpha_{2}^{\prime}$ is integrated between its limits:

$$
4 \pi^{2} \int d \alpha_{1}^{\prime} d \alpha_{2}^{\prime}=4 \pi^{2} \int d\left(\frac{\alpha_{1}^{\prime 2}}{2}\right)
$$

The sum of the first $m$ elementary regions (viz., integration over $\alpha_{1}^{\prime}$ from zero to $\frac{h}{2 \pi} m$ ) will then be: $\frac{1}{2} h^{2} m^{2}$.

As we see, from our prescription, the subdivision of phases will change essentially when we go to isotropy. Whereas for the case of anisotropy, the elementary regions of the $\alpha_{1}, \alpha_{2}$-plane are
 squares, for the case of isotropy, they will become the strips that are separated by the dashed diagonals in the diagram. Since the trajectories in a two-dimensional region overlap in an everywheredense way for anisotropy, but will become closed ellipses for isotropy, it is not unreasonable that a change in the elementary region should also result from that drastic change in character of the motion.

Naturally, one can assert that a certain anisotropy with a physicalobservable effect will always be present, and therefore the quadratic subdivision will also be preserved when there is apparent isotropy.
Such a viewpoint will become even clearer when one regards the case of the isotropic elastic force as a limiting case of a different problem. One combines the elastic force with one that is inversely proportional to the third power of distance, so one takes the expression for the potential to be $A r^{2}+B / r^{2}=\left(r^{2}=x_{1}^{2}+x_{2}^{2}\right)$. For $B=0$, one will then come back to the isotropic elastic force. For $B \neq 0$, the form of the motion that one will get is an ellipse with a rotating major axis, so a precessing perihelion. One will then have two incommensurable periods in the motion here, one of them being the orbital period of the ellipse itself and the other one being the period of the precession. This is a nondegenerate problem then. When one treats it according to the rules above, and then goes to the limit $B=0$ without changing the subdivision, one will get precisely the
subdivision that Planck obtained for this case. Sommerfeld proceeded in the same manner as Planck in some analogous cases. From the rules that were given here, a new subdivision must result for the degenerate case of $B=0$ that is just the one that was given above for isotropy. Planck's subdivision is more plausible than, say, the one that used the variables $\alpha_{1}$ and $\alpha_{2}$ above, insofar as one would prefer to assume a deviation from Hooke's law in the distance function over an anisotropy.

Above all, one sees from the foregoing that one can reconcile Planck's subdivision with the one that was proposed here in this case (and analogously in many others) by replacing the mechanical problem that was given with a less-degenerate neighboring one.

Meanwhile, another way that the present process differs from Planck's process should also be pointed out. From the prescription above, for a degenerate problem, the volume of the individual elementary regions in phase space will depend entirely upon the boundary conditions for the variables with no mean motion. By contrast, Planck demanded that the volumes of the elementary regions (except for a power of $h$ ) should be while numbers of a certain type. That demand is already violated in the example above of isotropic elastic force due to the fact that the factor $1 / 2$ appears in the expression $\frac{1}{2} h^{2} m^{2}$ for the sum of the first $m$ elementary regions.

As Planck himself emphasized, only the ultimate result can allow one to decide on the validity of one or the other subdivision.

## II.

§ 8. - As an example, the motion of an electron of charge $-e$ and mass $m$ around a charge $+e$ that is at rest at the origin, in which it is assumed that the entire system is found in a homogeneous external electric field $\mathbf{E}$. This is the problem of the Stark effect for the Bohr model of the hydrogen atom. As I have remarked on a previous occasion $\left(^{1}\right)$, here one is dealing with the motion of a point under Newton's law of attraction between two fixed centers with the specialization that one of the centers has been moved out to infinity. Jacobi himself $\left({ }^{2}\right)$ had already shown that one could integrate that problem very easily with his method by introducing elliptic coordinates.

The external field $\mathbf{E}$ acts on the electron in the direction of the positive $x$-axis. The energy of the system will then read:

$$
F=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)-\frac{e^{2}}{r}-e E x \quad\left(r^{2}=x^{2}+y^{2}+z^{2}\right)
$$

The elliptic coordinates in this special case read:

$$
\lambda=\frac{1}{2}(r+x), \quad \mu=\frac{1}{2}(r-x) .
$$

[^1]The angle of rotation $\phi$ around the $x$-axis will then become the third coordinate. In terms of the variables $\lambda, \mu, \phi$, the energy reads:

$$
F=\frac{1}{2} m(\mu+\lambda)\left(\frac{\dot{\lambda}^{2}}{\lambda}+\frac{\dot{\mu}^{2}}{\mu}\right)+2 m \lambda \mu \dot{\phi}^{2}-\frac{e^{2}}{\lambda+\mu}-e E(\lambda-\mu) .
$$

Upon introducing the impulse components:

$$
\Lambda=\frac{\partial F}{\partial \lambda}=m \frac{(\mu+\lambda)}{\lambda} \dot{\lambda}, \quad \mathrm{M}=m \frac{(\mu+\lambda)}{\mu} \dot{\mu}, \quad \Phi=4 m \lambda \mu \dot{\phi},
$$

the energy will take the form:

$$
F=\frac{1}{2 m(\mu+\lambda)}\left\{\lambda \Lambda^{2}+\mu \mathrm{M}^{2}+\frac{1}{4}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right) \Phi^{2}\right\}-\frac{e^{2}}{\lambda+\mu}-e E(\lambda-\mu) .
$$

One must then treat the partial differential equation:

$$
\frac{1}{2 m(\mu+\lambda)}\left\{\lambda\left(\frac{\partial S}{\partial \lambda}\right)^{2}+\mu\left(\frac{\partial S}{\partial \mu}\right)^{2}+\frac{1}{4}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\left(\frac{\partial S}{\partial \phi}\right)^{2}\right\}-\frac{e^{2}}{\lambda+\mu}-e E(\lambda-\mu)=\gamma_{1}
$$

in which $\gamma_{1}$ means the energy constant.
When one multiplies that by $\mu+\lambda$, one will see immediately that $S$ can be represented as a sum of functions of each of the variables:

$$
S=S_{1}(\lambda)+S_{2}(\mu)+S_{3}(\phi),
$$

and indeed that will imply a solution of the equations:

$$
\begin{aligned}
& \lambda\left(\frac{\partial S_{1}}{\partial \lambda}\right)^{2}-m e^{2}+\gamma_{2}-2 m e E \lambda^{2}+\frac{\gamma_{3}^{2}}{4 \lambda}=+2 m \lambda \gamma_{1} \\
& \mu\left(\frac{\partial S_{2}}{\partial \mu}\right)^{2}-m e^{2}+\gamma_{3}-2 m e E \mu^{2}+\frac{\gamma_{3}^{2}}{4 \mu}=+2 m \mu \gamma_{1} \\
& \frac{\partial S_{3}}{\partial \phi}=\gamma_{3}
\end{aligned}
$$

that includes two arbitrary constants $\gamma_{2}$ and $\gamma_{3}$ in addition to $\gamma_{1}$. One should consider $S_{1}$ more closely. It will follow that:

$$
S_{1}=\int \frac{d \lambda}{\lambda} \sqrt{2 m e E \lambda^{3}+2 m \gamma_{1} \lambda^{2}+\left(m e^{2}-\gamma_{2}\right) \lambda-\frac{1}{4} \gamma_{3}^{2}},
$$

or when one splits the radical into its root factors:

$$
S_{1}=\int \frac{d \lambda}{\lambda} \sqrt{2 m e E\left(\lambda-\lambda_{1}\right)\left(\lambda-\lambda_{2}\right)\left(\lambda-\lambda_{3}\right)} .
$$

One of the three roots will obviously grow to infinity as the external field $\mathbf{E}$ decreases. The other two must have positive real values if a real motion is to be possible at all. The variable $\lambda$ will then remain enclosed between those two real values. If one sets:

$$
\lambda=\frac{1}{2}\left(\lambda_{1}+\lambda_{2}\right)+\frac{1}{2}\left(\lambda_{1}-\lambda_{2}\right) \cos \eta_{1}=a_{1}\left(1-\varepsilon_{1} \cos \eta_{1}\right) \quad\left(\lambda_{2}>\lambda_{1}\right)
$$

then $\lambda$ will be a periodic function of period $2 \pi$ of $\eta_{1}$. The integral of $S_{1}$ will become:

$$
S_{1}=\sqrt{2 m e E} \int \frac{a_{1}^{2} \varepsilon_{1}^{2} \sin ^{2} \eta_{1} d \eta_{1}}{1-\varepsilon_{1} \cos \eta_{1}} \sqrt{\lambda_{3}-a_{1}+a_{1} \varepsilon_{1} \cos \eta_{1}} .
$$

If one sets:

$$
\alpha_{1}=\sqrt{2 m e E} \int_{0}^{2 \pi} \frac{a_{1}^{2} \varepsilon_{1}^{2} \sin ^{2} \eta_{1} d \eta_{1}}{1-\varepsilon_{1} \cos \eta_{1}} \sqrt{\lambda_{3}-a_{1}+a_{1} \varepsilon_{1} \cos \eta_{1}}
$$

then performing the integral will give $S_{1}$ :

$$
S_{1}=\alpha_{1} \eta_{1}+\text { periodic function of } \eta_{1} \text { with period } 2 \pi .
$$

In a completely-analogous way, for $\mu$, one must replace the form:

$$
-2 m e E \mu^{3}+2 m \gamma_{1} \mu^{2}+\left(m e^{2}+\gamma_{2}\right) \mu-\frac{1}{4} \gamma_{3}^{2}
$$

with its decomposition into root-factors:

$$
-2 m e E\left(\mu-\mu_{1}\right)\left(\mu-\mu_{2}\right)\left(\mu-\mu_{3}\right),
$$

and introduce a variable $\eta_{2}$ in place of $\mu$ by the relation:

$$
\mu=\frac{1}{2}\left(\mu_{1}+\mu_{2}\right)+\frac{1}{2}\left(\mu_{1}-\mu_{2}\right) \cos \eta_{2}=a_{2}\left(1-\varepsilon_{2} \cos \eta_{2}\right),
$$

and one will then get:

$$
S_{2}=\alpha_{2} \eta_{2}+\text { periodic function of } \eta_{2} \text { with period } 2 \pi,
$$

in which $\alpha_{2}$ is the integral:

$$
\alpha_{2}=\sqrt{2 m e E} \int_{0}^{2 \pi} \frac{a_{2}^{2} \varepsilon_{2}^{2} \sin ^{2} \eta_{2} d \eta_{2}}{1-\varepsilon_{2} \cos \eta_{2}} \sqrt{a_{2}-\mu_{3}+a_{2} \varepsilon_{2} \cos \eta_{2}}
$$

Ultimately, that will give the differential equation for $S_{3}$ :

$$
S_{3}=\gamma_{3} \phi .
$$

The total result for $S$ will then be:

$$
S=S_{1}+S_{2}+S_{3}=\alpha_{1} \eta_{1}+\alpha_{2} \eta_{2}+\gamma_{3} \phi+\text { periodic function of period } 2 \pi \text { of } \eta_{1} \text { and } \eta_{2} .
$$

Since the position of the moving point, as expressed by $\eta_{1}, \eta_{2}$, and $\phi$, is periodic in each of the variables with a period of $2 \pi$, that is precisely the form for $S$ that was treated in § 7, from which it emerges that $\alpha_{1}, \alpha_{2}$, and $\alpha_{3}=\gamma_{3}$ will be the desired action variables, up to constants and a possible whole-number linear substitution.

In the calculation of $\alpha_{1}$ and $\alpha_{2}$, we would like to restrict ourselves to the case of a weak external field $\mathbf{E}$ and large values of $\lambda_{3}$ and $\mu_{3}$. We will then have $\alpha_{1}$ in the first approximation:

$$
\alpha_{1}=\sqrt{2 m e E \lambda_{3}} \int_{0}^{2 \pi} \frac{a_{1}^{2} \varepsilon_{1}^{2} \sin ^{2} \eta_{1} d \eta_{1}}{1-\varepsilon_{1} \cos \eta_{1}}\left[1-\frac{a_{1}\left(1-\varepsilon_{1} \cos \eta_{1}\right)}{2 \lambda_{3}}\right] .
$$

The integral can be performed by means of known simple formulas, and that will give:

$$
\alpha_{1}=\sqrt{2 m e E \lambda_{3}} a_{1}\left[1-\sqrt{1-\varepsilon_{1}^{2}}-\frac{a_{1}}{4 \lambda_{3}} \varepsilon_{1}^{2}\right],
$$

or by means of the relations $a_{1}=\frac{1}{2}\left(\lambda_{1}+\lambda_{2}\right), a_{1} \varepsilon_{1}=\frac{1}{2}\left(\lambda_{1}-\lambda_{2}\right)$, it can be expressed in terms of the roots $\lambda_{1}, \lambda_{2}, \lambda_{3}$ :

$$
\alpha_{1}=\sqrt{2 m e E \lambda_{3}}\left[\frac{1}{2}\left(\lambda_{1}+\lambda_{2}\right)-\sqrt{\lambda_{1} \lambda_{2}}-\frac{1}{16 \lambda_{3}}\left(\lambda_{1}-\lambda_{2}\right)^{2}\right] .
$$

The roots $\lambda$ of the third-degree form are easy to determine for small $\mathbf{E}$. We will get, with sufficient precision:

$$
\begin{aligned}
\lambda_{1}+\lambda_{2} & =-\frac{m e^{2}-\gamma_{2}}{2 m \gamma_{1}}-\frac{e E}{\gamma_{1}}\left[\left(\frac{m e^{2}-\gamma_{2}}{2 m \gamma_{1}}\right)^{2}+\frac{\gamma_{3}^{2}}{8 m \gamma_{1}}\right], \\
\lambda_{1} \lambda_{2} & =-\frac{\gamma_{3}^{2}}{8 m \gamma_{1}}-\frac{e E}{\gamma_{1}} \frac{m e^{2}-\gamma_{2}}{2 m \gamma_{1}} \cdot \frac{\gamma_{3}^{2}}{8 m \gamma_{1}},
\end{aligned}
$$

$$
\lambda_{3}=-\frac{\gamma_{1}}{e E}+\frac{m e^{2}-\gamma_{2}}{2 m \gamma_{1}} .
$$

If one substitutes that in $\alpha_{1}$ then one will get:

$$
\alpha_{1}=-\frac{\gamma_{3}}{2}+\frac{m e^{2}-\gamma_{2}}{2 \sqrt{-2 m \gamma_{1}}}-\frac{m e E}{\sqrt{-2 m \gamma_{1}}}\left[\frac{3}{8}\left(\frac{m e^{2}-\gamma_{2}}{2 m \gamma_{1}}\right)^{2}+\frac{\gamma_{3}^{2}}{16 m \gamma_{1}}\right],
$$

which is correct up to higher powers of $E$. An entirely-analogous calculation will yield:

$$
\alpha_{2}=-\frac{\gamma_{3}}{2}+\frac{m e^{2}+\gamma_{2}}{2 \sqrt{-2 m \gamma_{1}}}-\frac{m e E}{\sqrt{-2 m \gamma_{1}}}\left[\frac{3}{8}\left(\frac{m e^{2}+\gamma_{2}}{2 m \gamma_{1}}\right)^{2}+\frac{\gamma_{3}^{2}}{16 m \gamma_{1}}\right] .
$$

We would like to look for the expression for the energy $\left(F=\gamma_{1}\right)$ in terms of the quantities $\alpha_{1}$, $\alpha_{2}$, and $\alpha_{3}\left(=\gamma_{3}\right)$. We initially find that:

$$
\begin{aligned}
\alpha_{1}+\alpha_{2}+\alpha_{3} & =\frac{m e^{2}}{\sqrt{-2 m \gamma_{1}}}-\frac{3}{2} \frac{m^{2} e^{3} E \gamma_{2}}{\left(-2 m \gamma_{1}\right)^{5 / 2}} \\
\alpha_{2}-\alpha_{1} & =\frac{\gamma_{2}}{\sqrt{-2 m \gamma_{1}}}+\text { terms of order } E .
\end{aligned}
$$

It will follow from this that:

$$
\frac{m e^{2}}{\sqrt{-2 m \gamma_{1}}}=\alpha_{1}+\alpha_{2}+\alpha_{3}+\frac{3}{8} \frac{e^{3} E}{\gamma_{1}^{2}}\left(\alpha_{2}-\alpha_{1}\right)
$$

up to quantities of order $E^{2}$. One sees from this that the energy $\gamma_{1}$ depends upon only the two combinations of the variables $\alpha_{1}+\frac{1}{2} \alpha_{3}$ and $\alpha_{2}+\frac{1}{2} \alpha_{3}$. One will then have:

$$
\frac{\partial F}{\partial \alpha_{3}}=\frac{1}{2}\left(\frac{\partial F}{\partial \alpha_{1}}+\frac{\partial F}{\partial \alpha_{2}}\right)
$$

or the commensurability of the mean motions:

$$
2 n_{3}=n_{1}+n_{2} .
$$

A degeneracy will also enter into this problem. From our general prescription, in such a case, we have to go from that motion to one with mean motion zero by means of a whole-number substitution with a determinant that is as small as possible. Here, we find the substitution with a
determinant 1: $w_{1}^{\prime}=w_{1}, w_{2}^{\prime}=w_{3}-w_{1}, w_{3}^{\prime}=-2 w_{3}+w_{1}+w_{2}$, which will make the mean motion of $w_{3}^{\prime}$ vanish. The associated substitution of the action variables reads:

$$
\alpha_{1}^{\prime}=\alpha_{1}+\alpha_{2}+\alpha_{3}, \quad \alpha_{2}^{\prime}=2 \alpha_{2}+\alpha_{3}, \quad \alpha_{3}^{\prime}=\alpha_{3}
$$

We will then have the following relation for the energy $\gamma_{1}$ :

$$
\frac{m e^{2}}{\sqrt{-2 m \gamma_{1}}}=\alpha_{1}^{\prime}+\frac{3}{8} \frac{e^{3} E}{\gamma_{1}^{2}}\left(\alpha_{2}^{\prime}-\alpha_{1}^{\prime}\right),
$$

or when solved for $\gamma_{1}$ :

$$
-\gamma_{1}=\frac{m e^{4}}{2 \alpha_{1}^{\prime 2}}+\frac{3}{2} \frac{E \alpha_{1}^{\prime}}{m e}\left(\alpha_{1}^{\prime}-\alpha_{2}^{\prime}\right)
$$

Finally, the boundary of phase space must be determined. In phase space, the roots of the thirddegree form that appears above must be real, so the boundary of phase space must be given by the condition that two of the roots coincide.

If $\lambda_{1}$ coincides with $\lambda_{2}$ then $\varepsilon_{1}=0$, and it will follow from this that: $\alpha_{1}=0$. On the other hand, if $\lambda_{1}$ coincides with $\lambda_{3}$ then one will have:

$$
\sqrt{\lambda_{3}-a_{1}+a_{1} \varepsilon_{1} \cos \eta_{1}}=\sqrt{\frac{1}{2}\left(\lambda_{2}-\lambda_{1}\right) \cdot 2\left(1+\cos \eta_{1}\right)},
$$

and it will follow that:

$$
\alpha_{1}=\sqrt{2 m e E} \int_{0}^{2 \pi} \frac{2 a_{1}^{2} \varepsilon_{1}^{2} \sqrt{\varepsilon_{1}} \sin ^{2} \eta_{1} \cos \eta_{1} / 2 d \eta_{1}}{1-\varepsilon_{1} \cos \eta_{1}}
$$

One likewise sees that this value also vanishes. One will then get one limit of phase space from the reality conditions on the roots $\lambda$ :

$$
\alpha_{1}=0 .
$$

We will likewise get another limit on phase space from the reality condition on the roots $\mu$ :

$$
\alpha_{2}=0 .
$$

Now, a root $\lambda$ and a root $\mu$ can also not vanish simultaneously since a collision of the electron with the nuclear would result at the origin in the course of the motion. That condition would lead to the limits:

$$
\gamma_{3}=\alpha_{3}=0 .
$$

With a suitable definition of the angles $\phi, \eta_{1}, \eta_{2}$, one can always arrange that $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are positive quantities for real motions. Phase space is then bounded completely by the inequalities:

$$
\alpha_{1}>0, \quad \alpha_{2}>0, \quad \alpha_{3}>0 .
$$

When those inequalities are adapted to the normalized variables $\alpha_{1}^{\prime}, \alpha_{2}^{\prime}, \alpha_{3}^{\prime}$ they will real:

$$
\alpha_{1}^{\prime}-\alpha_{2}^{\prime}+\alpha_{3}^{\prime}>0, \quad \alpha_{2}^{\prime}-2 \alpha_{3}^{\prime}>0, \alpha_{3}^{\prime}>0
$$

The following restriction on $\alpha_{1}^{\prime}$ and $\alpha_{2}^{\prime}$ will emerge from that:

$$
2 \alpha_{1}^{\prime}-\alpha_{2}^{\prime}>0, \alpha_{2}^{\prime}>0
$$

In order to consider those boundary conditions, we will perform one last transformation:

$$
\alpha_{1}^{\prime \prime}=2 \alpha_{1}^{\prime}-\alpha_{2}^{\prime}, \quad \alpha_{2}^{\prime \prime}=\alpha_{2}^{\prime}
$$

The boundary conditions then read: $\alpha_{1}^{\prime \prime}>0, \alpha_{2}^{\prime \prime}>0$. Those will then be the variables that advance by multiples of $h / 2 \pi$. The initial values are zero. The distinguished values are then:

$$
\alpha_{1}^{\prime \prime}=\frac{h}{2 \pi} m_{1}, \quad \alpha_{2}^{\prime \prime}=\frac{h}{2 \pi} m_{2}, \quad\left(m_{1}>0, m_{2}>0\right)
$$

If one expresses energy in terms of $\alpha_{1}^{\prime \prime}$ and $\alpha_{2}^{\prime \prime}$ and introduces those distinguished values then one will get the result that:

$$
-F=2 \pi^{2} \frac{m e^{4}}{h^{2}} \cdot \frac{4}{\left(m_{1}+m_{2}\right)^{2}}+\frac{3}{8} \frac{h^{2}}{4 \pi^{2}} \frac{E}{m e}\left(m_{1}^{2}-m_{2}^{2}\right)
$$

The Bohr Ansatz will then imply the frequencies that will be radiated by our system:

$$
v=N\left[\frac{4}{\left(m_{1}+m_{2}\right)^{2}}-\frac{4}{\left(m_{1}^{\prime}+m_{2}^{\prime}\right)^{2}}\right]+\frac{3}{8} \frac{h}{4 \pi^{2}} \frac{E}{m e}\left[\left(m_{1}^{2}-m_{2}^{2}\right)-\left(m_{1}^{\prime 2}-m_{2}^{\prime 2}\right)\right]
$$

in which $N=2 \pi^{2} \frac{m e^{4}}{h^{2}}$ is the Ritz number.
If one sets: $m_{1}+m_{2}=4, m_{1}^{\prime}+m_{2}^{\prime}=6,8,10, \ldots$ in this then one will get the Balmer series for hydrogen in a vanishing external field $(\mathbf{E}=0)$. The change in the frequencies that is caused by the external field will be:

$$
\begin{aligned}
\Delta \nu & =\frac{3}{8} \frac{h}{4 \pi^{2}} \frac{E}{m e}\left[\left(m_{1}^{2}-m_{2}^{2}\right)-\left(m_{1}^{\prime 2}-m_{2}^{\prime 2}\right)\right] \\
& =\frac{3}{2} \frac{h}{4 \pi^{2}} \frac{E}{m e}\left[2\left(2-m_{1}\right)-M\left(M-m_{1}^{\prime}\right)\right], \quad M=3,4,5, \ldots
\end{aligned}
$$

For odd values of $m_{1}+m_{2}$ and $m_{1}^{\prime}+m_{2}^{\prime}$, one will get frequencies that lie very close to the lines in the helium spectrum. In that way, one can come to the realization that perturbing a hydrogen atom by an external electric field can make it radiate helium frequencies.
§ 9. - In regard to the historical basis for that result for $\Delta v$, the following should be remarked: Warburg ( ${ }^{1}$ ) was the first to show that when quantum theory was applied to the Bohr hydrogen atom, it would yield the splitting that Stark observed, or at least its order of magnitude. Bohr $\left({ }^{2}\right)$ has already derived a closely-related formula by considering the motions that travel along the $x$ axis. Sommerfeld ( ${ }^{3}$ ) has shown that by extending quantum theory to orbits of arbitrary eccentricity (in the absence of and external field), the multiplicity of the hydrogen lines should make one expect a splitting into perhaps just as many components as Stark had observed. Above all, I infer from Sommerfeld's beautiful presentation that one would achieve that goal when one couples his arguments with the familiar conditionally-periodic nature of the motion under the attraction to two fixed centers that I have discussed.

With:

$$
h=6.55 \times 10^{-27}, \quad \frac{e}{m}=5.31 \times 10^{17}, \quad e=4.7 \times 10^{-10},
$$

one will get:

$$
\Delta v=1.990 \times 10^{6} \mathrm{E}(\text { volt }) \cdot Z,
$$

in which $Z$ is the whole number: $Z=2\left(2-m_{1}\right)-M\left(M-m_{1}^{\prime}\right)$.
For the individual hydrogen lines and the field strength of 104,000 volt/cm that Stark used $\left(^{4}\right)$, one will then get, in Angström units:

| $H_{\alpha} \quad \Delta \lambda=$ | $2.972 Z, \quad M=3$ |  |
| :--- | :--- | :--- |
| $H_{\beta}$ | $1.632 Z$, | 4 |
| $H_{\gamma}$ | $1.301 Z$, | 5 |
| $H_{\delta}$ | $1.161 Z$, | 6. |

If one calculates $Z$ from this for the splitting in the lines that Stark observed by starting with the observed values of $\Delta \lambda$ then one will find that:

| $H_{\alpha}$ | 0.0 | 0.9 | 2.1 | 3.0 | 3.9 |  |  |  |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $H_{\beta}$ | 0.0 | 2.1 | 4.1 | 6.1 | 8.1 | 10.1 | 11.9 |  |  |  |  |  |  |  |
| $H_{\gamma}$ | 0.0 | 2.1 | 3.0 | 5.1 | 7.5 | 8.1 | 10.2 | 12.2 | 13.3 | 15.3 | 17.5 | 18.4 | 20.2 | $(22.6)$ |
| $H_{\delta}$ | 0.0 | 2.1 | 4.5 | 6.4 | 8.3 | 10.3 | 12.4 | 14.8 | 16.9 | 18.3 | 20.8 | 22.2 | 24.6 | 26.2 |

[^2](Numbers in parentheses correspond to questionable lines, according to Stark.)
The whole-number character of the values of $Z$ that are derived from the observations is so distinct in $H_{\alpha}$ and $H_{\beta}$ that it is only in that way that a real interpretation of the theory will already be guaranteed. As far as the theoretical values of the whole numbers $Z$, one has the following:

When one lets the numbers $m_{1}, m_{2}, m_{1}^{\prime}, m_{2}^{\prime}$ run through all of the positive values that are consistent with the conditions $m_{1}+m_{2}=4, m_{1}^{\prime}+m_{2}^{\prime}=6,8, \ldots$, the greatest values of $Z$ that occur for the four lines will be: $9+4=13,16+4=20,25+4=29,36+4=40$. Those numbers are known to exceed the observed greatest numbers $Z$ considerably. Sommerfeld (loc. cit., pp. 448) proposed that only transitions to smaller values of the action variables (with our terminology) should be regarded as permissible, so one must demand that $m_{1}^{\prime} \geq m_{1}, m_{2}^{\prime} \geq m_{2}$. With that demand, one will get the following whole numbers $Z$ :

| $H_{\alpha}$ | 0 | 1 | 2 | 3 | 4 | 5 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| ---: | :--- | :--- | :--- | :--- | :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $H_{\beta}$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |  |  |  |  |  |  |  |  |  |  |  |  |  |
| $H_{\gamma}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | - | 15 | 16 | - | 18 | - |
| $H_{\delta}$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | - | 32 |  |  |  |

The theory demands one line more than the observation gives for $H_{\alpha}$, although Stark has still not observed the line number completely in that case. Complete agreement exists between theory and observation for $H_{\beta}$, and much agreement exists for $H_{\gamma}$ and $H_{\delta}$ without seeing that clearly in detail.

Later on, one must also consider the observed polarizations and intensities of the lines. If the theory can therefore not be seen to be closed then it is nonetheless remarkable how exceptionally close one can already come to the observed behavior by means of this initial rigorous implementation of quantum theory with the use of Bohr's Ansatz for the oscillation frequencies $\left.{ }^{1}\right)$.

## III.

§ 10. - As a second example, the force-free motion of a rotating body about its center of mass will be treated.

The energy in this case reads:

$$
F=\frac{1}{2}\left(A p^{2}+C r^{2}\right),
$$

in which $C$ is the moment of inertia around the rotational axis, $A$ is the moment of inertia about an equatorial axis, and $p$ and $r$ are the associated angular velocities. If one lets $\vartheta$ and $\psi$ denote the

[^3]spherical coordinates of the figure's axis in space, and $\phi$ denotes the angle of rotation of the body around the figure's axis from the nodal line then one will have:
\[

$$
\begin{aligned}
& p^{2}=\dot{\vartheta}^{2}+\dot{\psi}^{2} \sin ^{2} \vartheta, \quad r=\dot{\phi}+(\cos \vartheta) \dot{\psi} \\
& F=\frac{1}{2} A\left(\dot{\vartheta}^{2}+\dot{\psi}^{2} \sin ^{2} \vartheta\right)+\frac{1}{2} C[\dot{\phi}+(\cos \vartheta) \dot{\psi}]^{2}
\end{aligned}
$$
\]

The impulse components will be:
$\Theta=\frac{\partial F}{\partial \dot{\vartheta}}=A \dot{\vartheta}, \quad \Psi=\frac{\partial F}{\partial \dot{\psi}}=A \sin ^{2} \vartheta \dot{\psi}+C \cos \vartheta[\dot{\phi}+(\cos \vartheta) \dot{\psi}], \quad \Phi=\frac{\partial F}{\partial \dot{\phi}}=C[\dot{\phi}+(\cos \vartheta) \dot{\psi}]$.

In terms of the impulse components, the energy reads:

$$
F=\frac{\Theta^{2}}{2 A}+\frac{\Phi^{2}}{2 C}+\frac{(\Psi-\Phi \cos \vartheta)^{2}}{2 A \sin ^{2} \vartheta}
$$

Here, one rapidly arrives at one's goal when one recalls that the general motion of the free, symmetric top is regular precession. If one chooses the (fixed) impulse direction to be the axis $\vartheta$ $=0$ then one will have:

$$
\vartheta=\text { const. }, \quad \Theta=0, \quad \psi=n_{1} t+\text { const. }, \quad \phi=n_{2} t+\text { const. }
$$

for the regular precession. $n_{1}$ is the angular velocity of precession, and $n_{2}$ is the angular velocity of the rotation. In so doing, the following condition must be fulfilled:

$$
\frac{d \Theta}{d t}=\frac{\partial F}{\partial \dot{\vartheta}}=0 .
$$

Calculation will then give:

$$
\Phi=\Psi \cos \vartheta
$$

Here, it is obvious that $\psi$ and $\phi$ are angle variables of the desired type, and the associated impulse components $\Psi$ and $\Phi$ are the desired action variables (up to a possible whole-number substitution).

If one sets the energy $\Theta=0$ and introduces $\cos \vartheta=\Phi / \Psi$ then one will get the representation of the energy in terms of action variables:

$$
F=\frac{\Psi^{2}}{2 A}+\frac{1}{2}\left(\frac{1}{C}-\frac{1}{A}\right) \Phi^{2}
$$

All that remains now is to consider the limits of phase space.

One can always distinguish an axis direction in a top and the axis direction $\vartheta=0$ such that $\Psi$ and $\Phi$ will be positive, so:

$$
\Psi>0, \quad \Phi>0
$$

It will then follow from the relation $\Phi=\Psi \cos \vartheta$ that:

$$
\Psi>\Phi .
$$

No other limiting conditions exist.
If one now sets:

$$
\Psi-\Phi=\alpha_{1}, \quad \Phi=\alpha_{2} \quad(\text { substitution of determinant } 1)
$$

then those conditions will read:

$$
\alpha_{1}>0, \quad \alpha_{2}>0
$$

Therefore, the desired action variables are found in the form of $\alpha_{1}$ and $\alpha_{2}$. The distinguished values are:

$$
\alpha_{1}=\Psi-\Phi=\frac{h}{2 \pi} m_{1}, \quad \alpha_{2}=\Phi=\frac{h}{2 \pi} m_{2}
$$

The distinguished values of energy read:

$$
F=\frac{h^{2}}{8 \pi^{2}}\left[\frac{\left(m_{1}+m_{2}\right)^{2}}{A}+\left(\frac{1}{C}-\frac{1}{A}\right) m_{2}^{2}\right] .
$$

When all three moments of inertia are equal $(A=C=J)$, one will come back to the case of the spherical top that has already been treated many times. From a well-known formula, the distinguished energy values will be:

$$
F=\frac{h^{2}}{8 \pi^{2} J} \cdot m^{2}, \quad m=0,1,2, \ldots
$$

§ 11. - One can apply the foregoing calculations to the theory of band spectra. One imagines that a system of electrons orbits around a molecule that is itself rotating. The charges that are present in the molecule might be distributed in such a way that a reciprocal influence of the rotation of the molecule and the motions of the electrons does not exist. If $F_{0}$ is the energy of the motion of the electrons for a distinguished state (so it will depend upon some sort of whole numbers) and one assumes that the moments of inertia of the molecule are all equal then the total energy of the system for the distinguished states will be:

$$
F_{0}+\frac{h^{2}}{8 \pi^{2} J} \cdot m^{2}
$$

For a different distinguished state, it will be: $F_{0}^{\prime}+\frac{h^{2}}{8 \pi^{2} J} \cdot m^{\prime 2}$. According to Bohr, one will get the frequencies that are emitted by the system from:

$$
v=\frac{F_{0}-F_{0}^{\prime}}{h}+\frac{h^{2}}{8 \pi^{2} J}\left(m^{2}-m^{\prime 2}\right) .
$$

If one fixes all of the whole numbers that occur up to $m$ then one will get a series of lines that goes up to violet

$$
v=a+b m^{2}, \quad b=\frac{h^{2}}{8 \pi^{2} J}
$$

which corresponds to the Deslandres formula that is valid approximately for many bands. Varying $m^{\prime}$ will give a band that goes to red.

One sees that one can calculate the moments of inertia for the molecule in question directly from the constant $b$ in the band formula/

I shall give a few examples:

| Material | Wavelength of the <br> beginning of the band | $J$ | Remark |
| :---: | :---: | :---: | :---: |
| O | $6277 \mathrm{~A} . \mathrm{U}$. | $8 \times 10^{-39}$ | 1 |
| $\prime \prime$ | 6868 | $10 \times 10^{-39}$ |  |
| $"$ | 7594 | $14 \times 10^{-39}$ |  |
| Hg | 4215 | $1.8 \times 10^{-39}$ | 2 |
| N | 3883 | $40 \times 10^{-39}$ | 3 |
| $"$ | 3577 | $14 \times 10^{-39}$ | 4 |
| $\mathrm{H}_{2} \mathrm{O}$ | 3200 | $0.3 \times 10^{-39}$ | 5 |

1. The three known, very-regular bands for oxygen in the Solar spectrum $\alpha, B, C$ according to Higgs, Astronomy and Astrophysics (1893), pp. 547.
2. A value for many similarly-formed bands according to E. Liess, Zeitschrift für wissenschaftliche Photographie (1912), pp. 349.
3. The so-called cyan band, according to Kayser and Runge, Phys. Abh. d. Berl. Akad. d. Wiss. (1889).
4. A value for many similarly-formed bands according to Deslandres, C. R. Acad. Sci. Paris, t. 138, pp. 317.
5. The bands for $\mathrm{H}_{2} \mathrm{O}$ follow Deslandres's law only very roughly. We have taken the value of $b$ for the first line at the head of the band that Meyerheim [Zeitschrift für wissenschaftliche Photographie (1904), pp. 139] denoted by $C_{1}$ and $C_{3}$ since the remaining bands are not pursued all the way up to that head.

The moments of inertia have the expected order of magnitude. In particular, for steam, one will get $J=0.22 \times 10^{-39},\left(J=0.10 \times 10^{-39}\right.$, resp. $)$ from the equidistant absorption bands that were observed by Rubens and Hettner in the infrared range ( ${ }^{1}$ ) (for the Bjerrum or Eucken series, resp.), which is not too far from the value above. That would suggest that one might imagine that the same rotations of the molecules whose frequencies express themselves immediately in the infrared as equidistant absorption strips will generate effective bands in the ultraviolet according to the quadratic law of Deslandres when one uses the Bohr Ansatz.

One can represent a while series of bands with the same form with the help of the formula for molecules with two different moments of inertia. Since molecules will mostly possess three different moments of inertia in reality, and also an interaction between the rotation of the molecule and the motion of the orbiting electrons, in general, it is understandable that the detailed structure of most band spectra is so complicated.

[^4]
[^0]:    $\left.{ }^{( }{ }^{1}\right)$ Verhandlungen der Deutschen Physikalischen Gesellschaft (1915), pps. 407 and 438.
    $\left.{ }^{(2}\right)$ Sitzungsber. der Kgl. Bayer. Akad. d. Wiss. (1915), pp. 425.
    $\left(^{3}\right)$ Cf., e.g., C. V. L. Charlier, Mechanik des Himmels, Bd. I, Leipzig, 1902.

[^1]:    $\left({ }^{1}\right)$ Verhandl. der Deutschen Physikalischen Gesellschaft (1914), pp. 20.
    $\left(^{2}\right) \quad$ Vorlesungen über Dynamik, pp. 221.

[^2]:    ${ }^{(1)}$ ) Verhandlungen der Deutschen Physik. Gesellschaft (1913), pp. 1259.
    $\left({ }^{2}\right)$ Phil. Mag. 27 (1914), pp. 516.
    $\left(^{3}\right)$ Loc. cit., pp. 444.
    $\left({ }^{4}\right)$ Sitzungsberichte der Ges. d. Wiss. zu Göttingen, Math.-phys. Klasse (1914), pp. 427.

[^3]:    $\left({ }^{1}\right)$ (Remark added in proof.) In the meantime, P. Epstein communicated in a letter that he had treated the problem independently in a somewhat-different way, to which one would be led using our rules when one used a less-degenerate neighboring mechanical system as a basis. He achieved an almost-complete agreement with experiments and also found a rule for the polarizations (Phys. Zeit., to appear).

[^4]:    ( ${ }^{1}$ ) These Situzungsberichte (1916), pp. 167.

