"Sur lest droites qui ont des moments donnés par rapport à des droites fixes," J. f. d. reine u. angewandte Mathematik **97** (1884), 95-110.

On the lines that have given moments with respect to fixed lines.

(By Dr. *Corrado Segre* in Turin.)

Translated by D. H. Delphenich

I.

1. The moment of two arbitrary lines *x*, *y* can be expressed in a manner that will be useful to us by means of the distance from an arbitrary point *X* of one of them *x* to the other one *y*, and the angle that the normal to the plane *Xy* makes with *x*. Let δ be the segment that measures the smallest distance between the lines *x*, *y*, and let φ be the angle between them: Their moment (*) will be $\delta \sin \varphi$. It is clear that upon letting *p* denote the perpendicular to *y* that is based upon the point *X* of *x*, one will get:

$$\delta = p \cos (p\ \delta).$$

Now, if one draws lines through an arbitrary point in space that are parallel to *x*, *y*, δ, *p*, and to the normal *n* of the plane *Xy*, and finally, a line *m* whose direction is normal to *y*, but parallel to the plane of the directions of *x*, *y*, then one will see that the directions of *m*, *n*, *x* form a right triangle at *m* whose hypotenuse will have the angle (*n x*), and whose leg (*m x*) will be the complement of (*x y*) – i.e., of φ – while the other leg (*m n*) will be equal to (δ *p*). One will thus have:

$$\cos (n\ x) = \cos (m\ x) \cos (m\ n) = \sin \varphi \cos (p\ \delta),$$

and as a consequence:

$$\delta \sin \varphi = p \sin \varphi \cos (p\ \delta) = p \cos (n\ x) ;$$

i.e., one will get the desired expression for the moment of *x*, *y*:

$$\text{mom}\ (x, y) = p \cos (n\ x),$$

(*) We consider only the absolute values of these lines, because in order to give a sign to these moments, it will always be necessary to consider a corresponding positive direction for each line, while we propose to study sets of lines independently of the manner by which one fixes the positive directions.

from which, one infers this proposition:

If one multiplies the distance from an arbitrary point of one line to an arbitrary point of another line by the cosine of the angle that former line makes with the normal to the plane that links the other one to that point then one will get a constant product that is equal to the moment of the two lines.

Unless we are mistaken, this known theorem is due to *Drach*, who gave it as a special case of another theorem that he proved by an analytical method (*).

2. In this note, we will study, above all, the set of lines that have the same given moment m with respect to a fixed line r. First of all, we seek those of these lines that lie in an arbitrary given plane π. Let R be the point of intersection of that plane with the fixed line r, and let n be the normal to the plane (which is drawn through R). If we let $(R\ d)$ denote the distance from R to an arbitrary line d of π then we will get:

$$\text{mom}\ (r,\ d) = (R\ d) \cos\ (r\ n)$$

from the preceding theorem. Therefore, in order for d to be such that mom $(r,\ d) = m$, one must have the condition:

$$(R\ d) = \frac{m}{\cos(r\ n)} = \text{const.}$$

We conclude that:

The lines d in space for which mom $(r,\ d) = m$ *define a complex such that those of them that are in an arbitrary plane will envelop a circle that has its center on the line r and has a radius that is expressed by* m / cos $(r\ n)$, *where n represents the normal to that plane.*

These *complex circles* will be equal for all of the planes that are mutually parallel, and more generally, for all of the planes that make the same angle with the fixed line r. The radii of the corresponding circles will be equal to m and form a right cylinder C that has r for its axis and m for its radius for the planes that are perpendicular to r. For all of the other planes, since the angle $(r\ n)$ would no longer be zero, the radii of the complex circles will be larger than m and will tend to increase with that angle. Consequently, for the planes that make a smaller angle with r, the corresponding circles will be very large, and for the planes that are parallel to r, the corresponding circles will become infinite.

3. Consider the intersection that an arbitrary plane π makes with the cylinder of revolution C: One knows that this intersection is an ellipse whose center is the point of intersection R of π with r, and whose focal axis is on the intersection of π with the plane

(*) V. *Drach*, "Zur Theorie der Raumgeraden und der linearen Complexe," Math. Ann. **2**, 128-139 (cf., pp. 132, esp.)

that is perpendicular to π and is drawn through *r*. Upon letting *n* denote the normal to π, one will then see that the focal semi-axis of that ellipse will be given by *m* / cos (*r n*); therefore:

In each plane of space, the complex circle will have a radius that is equal to the focal semi-axis of the ellipse of intersection of that plane with the cylinder C.

In other words:

The circles that touch at the summits of the focal axis of the ellipses of intersection of planes in space with the right cylinder C will be the circles that are enveloped by the lines of these planes that belong to our complex.

This complex is therefore determined perfectly by the cylinder *C*, and this proposition will provide a remarkable property of the cylinder, in addition. Moreover, we see that all of the circles of the complex will be external to ellipses of *C* that belong to the same plane; i.e.:

The circles of the complex touch all of the cylinder C at two points and present no points in its interior.

One concludes from this that the lines of the complex cannot cut the cylinder; meanwhile, there will be lines that touch it at a point: They will be the ones whose direction is normal to *r*. Moreover, one can also infer these conclusions from the very definition of the complex, and because the product of the smallest distance between *r* and an arbitrary one of its lines *d* by the sine of the angle (*r d*) must be equal to *m*, it is clear that this minimum distance cannot be less than *m* and will be equal to *m* when that angle is equal to $\pi / 2$.

4. We now seek that lines of the complex that pass through an arbitrary point of space. Let *d* be a line that passes through *P*: If *n* is the normal to the plane *Pr* then, from the theorem in no. 1, one will have:

$$\text{mom}\,(r, d) = (P\,r)\cos(n\,d);$$

as a result, if *d* is a line of the complex – i.e., if mom (*r*, *d*) = *m* – then one will have:

$$\cos(n\,d) = \frac{m}{(P\,r)} = \text{const.}$$

Therefore, all of the lines *d* of the complex that pass through the point *P* will define the same angle with the line *n*; i.e.:

The lines of the complex that pass through an arbitrary point P in space define a right cone whose axis is the normal at P to the plane that connects P with r, and whose generators will define an angle with that axis whose cosine is given by m / (*P r*).

These *cones of the complex* are therefore real only for the points *P* whose distance to *r* is not less than *m* – i.e., for the points outside the cylinder *C*: This is in complete agreement with what we saw in the preceding number. One will have cos(*n d*) = 1 for the points *P* that are situated on the cylinder *C*, so the opening of the cone will reduce to zero – i.e., the cone will reduce to just its axis. Therefore, just one real line of the complex will pass through each point of *C*: viz., the normal to the plane that connects that point to *r*. However, the measure of a point *P* extends indefinitely from the cylinder *C* – i.e., the measure of the distance (*P r*) must increase, so cos (*n d*) will diminish indefinitely from 1 to zero: The cone of the complex that corresponds to *P* will thus have an opening that will increase continually, and for a point *P* at infinity that cone will decompose (as we shall soon see more distinctly) into a pair of planes that are parallel to *r*.

5. One can draw two planes that are tangent to a point *P* that is outside the cylinder *C*: The cosine of the angle that each of these planes makes with the normal *n* to the plane *P r* is obviously equal to *m* / (*P r*), and that will also be the cosine of the angle that *n* makes with the lines in these planes that pass through *P* and have directions that are normal to *r*. Therefore, these two lines will belong to the cone of our complex that corresponds to the point *P*, and the two planes that are tangent to *C* in which they are found will be planes that are tangent to that cone along its generators; i.e., one will have the proposition:

All of the cones of the complex are double tangents to the cylinder C. The two lines that pass through an arbitrary point P and are tangent to the curve of intersection of the cylinder C with the plane that is drawn through P normal to r will touch that curve precisely at the two contact points of C with the cone of the complex that belongs to P.

The cylinder *C* thus gives one a very simple construction for the right cone of lines of the complex that pass through the various points of space.

6. We pass on to an examination of how one composes the surfaces of singular points and planes and the congruence of singular lines of our complex. We saw (no. 2) that in each plane *p*, the curve of the complex will be a circle whose center is on the axis *r* and whose radius is *m* / cos (*r n*), or even *m* / sin ($r\ \pi$). This curve can decompose in just two cases:

1. When sin ($r\ \pi$) = 0; i.e., when the circle reduces to the line at infinity (as a double line).
2. When sin ($r\ \pi$) = ∞; i.e., when the circle decomposes into two lines (which will be coincident, as we will confirm), which will touch the absolute (viz., the imaginary circle at infinity) when the radius goes to zero.

In the first case, the plane *p* will be parallel to *r*, and in fact, all of the lines of a plane that is parallel to *r* will obviously have the same distance to *r*, so their moment with respect to *r* will no longer depend on their directions, in such a way that all of the lines in a plane that is parallel to one or the other of them will have the given moment *m* with *r*; i.e., they will belong to our complex. These two directions will be imaginary for those planes that are parallel to *r* and cut the cylinder *C*. They will be coincident with the direction of *r* for the ones that touch *C*, and finally, these two directions will be real and distinct for the ones that are outside of *C*. One thus has a first series of singular planes of the complex in the planes that pass through the point at infinity on *r*: The centers of the pencils of lines of the complex that belong to these planes will also be points at infinity. They will likewise be the only real singular planes, because for the other singular planes, one will have $\sin(r\,p) = \infty$, as we have seen. However, if one would like to consider the imaginary elements, as well, one will see that the planes that are tangent to the absolute will form another series of singular planes: In each of these planes, the circle of the complex will obviously reduce, as a locus of points, to the lines that join the point of contact with the absolute to the point of intersection with *r*. However, as an envelope of lines, one sees that this circle must be (as we have seen) doubly tangent to *C*, so it will reduce to two pencils of lines that will have their centers at the two points of intersection of that line with *C*.

As for the singular points, we have seen that the lines of the complex that form a right cone whose axis is the normal that is drawn through *P* to the plane *P r* and are double tangent to *C* will pass through each point *P*. Consequently, each cone of the complex is doubly tangent to either *C* or the absolute. Now, if the point *P* is at infinity then this cone will decompose, by the definition of the complex itself, into two pencils of parallel lines whose planes will be parallel to *r* and equidistant from that axis. If the point *P* is found on the cylinder *C* then we have already remarked that the cone of the complex will have no other real line besides its axis: Since it is doubly tangent to the absolute, moreover, it will decompose into two imaginary planes that are tangent to it and which intersect in that real line.

We further remark that for all of the planes of a pencil of planes that are parallel to each other and to *r*, the curve of the complex will decompose into two points of the axis of that pencil and that the cones of the complex that belong the points of that axis will have it for a double line: Hence, that axis will be a *double line* of the complex. Therefore, in summary, we conclude that:

The double lines of the complex are all lines that pass through the point at infinity of r in the plane at infinity. As a consequence, part of the locus of points of the plane at infinity (counted twice) will be singular, and the point at infinity of r (counted twice) will be part of the envelope of its singular planes: The singular lines that correspond to either these singular points or these singular planes will be precisely the pencil of double lines. In addition, the surface of singular points is composed of the right cylinder C that has r for its axis and m for its radius: The singular points that correspond to the points of C will form a quadratic congruence of lines that touch C and have a direction that is perpendicular to r – i.e., it will cut the line at infinity that is polar to r relative to the absolute. The envelope of singular planes is again comprised of the absolute: The *singular lines that correspond to the tangent planes to the absolute form an (imaginary) quadratic congruence of lines that is based upon r and the absolute.*

II.

1. We shall now study the same complex and find some other properties by the analytical method. Imagine a system of three rectangular axes with the line r as one of the axes. The coordinates of an arbitrary line that joins the points (x, y, z) and (x', y', z') will be:

$$p_{14} = x - x', \qquad p_{24} = y - y', \qquad p_{34} = z - z',$$
$$p_{23} = yz' - zy', \qquad p_{31} = zx' - xz', \qquad p_{12} = xy' - yx'$$

with respect to those axes. The moment of that line with respect to the z-axis – i.e., to r – will be, as one knows (*):

$$\frac{xy' - yx'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}, \qquad \text{or even} \qquad \frac{p_{12}}{\sqrt{p_{14}^2 + p_{24}^2 + p_{34}^2}}.$$

As a result, the complex of lines whose moment with respect to r has a given (absolute) value m will have the equation:

$$\frac{p_{12}}{\sqrt{p_{14}^2 + p_{24}^2 + p_{34}^2}} = m,$$

i.e.:

$$p_{14}^2 + p_{24}^2 + p_{34}^2 - \frac{1}{m} p_{12}^2 = 0.$$

Upon adding this equation to:

$$2\lambda\,(p_{14}\,p_{23} + p_{24}\,p_{31} + p_{34}\,p_{12}) = 0,$$

and forming the determinant of the new equation, one will get:

(*) We have given the most general expression for the moment of two lines as a function of their general coordinates in our note "Sulle geometrie metriche dei complessi lineari e delle sfere e sulle loro mutue analogie" (Atti della R. Accademia delle Scienze di Torino, vol. XIX). One can then deduce the equation of our quadratic complex in its most general form from it.

$$\begin{vmatrix} 1 & \lambda & & & & \\ \lambda & 0 & & & & \\ & & 1 & \lambda & & \\ & & \lambda & 0 & & \\ & & & & 1 & \lambda \\ & & & & \lambda & 0 \end{vmatrix}.$$

The value of this determinant is:

$$-\lambda^4\left(\lambda^2+\frac{1}{m^2}\right),$$

and its fifth-order sub-determinants will all have the factor λ^2, while those of fourth-order will not have λ for a common factor. Therefore, the *characteristic* of the complex, in the classification of *Weiler* (*) is:

[(22) 11],

where the symbol (22) corresponds to the quadruple root $\lambda = 0$ of the determinant, while the two 1's correspond to $\lambda^2 + 1 / m^2 = 0$; i.e., to the two simple roots $\lambda = \pm i / m$.

8. The equation of the complex gives us the equation of the cone that belongs to the point $P(x, y, z)$ in variable coordinates x', y', z':

$$(x - x')^2 + (y - y')^2 + (z - z')^2 - \frac{1}{m^2}(xy' - yx')^2 = 0,$$

and that equation obviously represents a right cone whose axis is the normal to the plane:

$$xy' - yx' = 0;$$

i.e., to the plane that joins the point P to the line r. – Since the determinant of that equation, when one renders it homogeneous by means of a factor t, namely:

$$\begin{vmatrix} t^2-\frac{1}{m^2}y^2 & \frac{1}{m^2}xy & 0 & -xt \\ \frac{1}{m^2}xy & t^2-\frac{1}{m^2}y^2 & 0 & -yt \\ 0 & 0 & t^2 & -zt \\ -xt & -yt & -zt & x^2+y^2+z^2 \end{vmatrix}$$

(*) "Ueber die verschiedenen Gattungen der Complexe zweiten Grades," Math. Ann. VII. – The complexes [(22) 11] are studied in no. 23 (pp. 184-186) of that paper.

has all of its primary sub-determinant affected with a common factor with:

$$t^2\left[\frac{1}{m^2}(x^2+y^2)-t^2\right],$$

and it follows from this that the singular surface of our surface will decompose, as a locus of points, into the plane at infinity ($t = 0$), counted twice, and the surface:

$$x^2 + y^2 = m^2;$$

i.e., the right cylinder C that has r for its axis and m for its radius.

9. The equation of the complex likewise gives us the equation for the curve of the complex that belongs to the plane π:

$$\xi x + \eta y + \zeta z + \tau = 0$$

in variable plane coordinates $\xi', \eta', \zeta', \tau'$:

$$(\eta\zeta' - \zeta\eta')^2 + (\zeta\xi' - \xi\zeta')^2 + (\xi\eta' - \eta\xi')^2 - \frac{1}{m^2}(\zeta\tau' - \tau\zeta')^2 = 0.$$

Now, since the equation:

$$(\eta\zeta' - \zeta\eta')^2 + (\zeta\xi' - \xi\zeta')^2 + (\xi\eta' - \eta\xi')^2 = 0$$

is the tangential equation for a cyclic pair of points on the plane π, one sees that the preceding equation is the tangential equation for a circle that has its center at the point:

$$\zeta\tau' - \tau\zeta' = 0;$$

i.e., the point of intersection of the plane π with the axis r. One thus recovers the result that the curves of the complex are circles that have their centers on the fixed line r. Upon once more forming the determinant of the tangential equation of that circle, one sees that its third-order sub-determinants will all have the common factor:

$$\zeta^2(\xi^2 + \eta^2 + \zeta^2),$$

so one concludes that the singular surface will decompose, as an envelope of planes, into the point at infinity ($\zeta = 0$) and the line r, counted twice, and into the curve:

$$\xi^2 + \eta^2 + \zeta^2 = 0;$$

i.e., the absolute.

10. The composition of the singular surface that we thus recovered analytically agrees with what happens in general for quadratic complexes whose characteristic is [(22) 11]. The property of curves of the complex that they are circles that are doubly tangent to the cylinder *C* and the property of cones of the complex that they are cones that are doubly tangent to that same cylinder are nothing but consequences of the general property of conics and cones of an arbitrary quadratic complex that they are quadruply tangent to the singular surface. These properties thus persist for all *homofocal* complexes – i.e., ones that have the same singular surface as our complex.

The equation of the series of complexes that are homofocal to the complex:

$$p_{14}^2 + p_{24}^2 + p_{34}^2 - \frac{1}{m^2} p_{12}^2 = 0$$

is obtained, as one knows (*), in the following manner: Add:

$$2\lambda\,(p_{14}\,p_{23} + p_{24}\,p_{31} + p_{34}\,p_{12}) = 0$$

to the last equation, form the reciprocal equation to the one that one thus obtained, and replace the variables with the complementary line coordinates. One thus immediately obtains:

$$\lambda^2\left[\left(\lambda^2 + \frac{1}{m^2}\right) p_{14}^2 + \left(\lambda^2 + \frac{1}{m^2}\right) p_{24}^2 + \lambda^2 p_{34}^2 - \frac{1}{m^2} p_{12}^2 \right.$$
$$\left. -\; 2\lambda\left(\lambda^2 + \frac{1}{m^2}\right) p_{14}\,p_{23} - 2\lambda\left(\lambda^2 + \frac{1}{m^2}\right) p_{24}\,p_{31} - 2\lambda^2 p_{34}\,p_{12}\right] = 0;$$

i.e., once more dividing by λ^2 and appealing to the general relation between the line coordinates will give:

$$\left(\lambda^2 + \frac{1}{m^2}\right) p_{14}^2 + \left(\lambda^2 + \frac{1}{m^2}\right) p_{24}^2 + \lambda^2 p_{34}^2 - \frac{1}{m^2}\lambda^2 p_{12}^2 + \frac{2}{m^2}\lambda\, p_{12}\,p_{34} = 0.$$

We let λ vary in that equation: We will get all of the complexes that are homofocal to ours. That series is of second degree – i.e., two of these complexes pass through each line in space. Our complex corresponds to $\lambda = \infty$. Two simple roots of the determinant $\lambda = \pm\, i / m$ correspond to the complexes of the series that have the equations:

$$\frac{1}{m^2} p_{34}^2 + \frac{1}{m^2}\frac{1}{m^2} p_{12}^2 \pm \frac{2}{m^2}\frac{i}{m} p_{12}\,p_{34} = 0,$$

or even:

(*) V. *Schur*, "Zur Theorie der Strahlencomplexe zweiten Grades," Math. Ann. XVII, pp. 107-109. Before learning of this note, we found the equation of a homofocal series of quadratic complexes in completely general line coordinates (see our dissertation that was cited above).

$$\left(\frac{1}{m}p_{12} \pm ip_{34}\right)^2 = 0;$$

i.e., the two linear complexes, counted twice:

$$\frac{1}{m}p_{12} + i\,p_{34} = 0, \qquad \frac{1}{m}p_{12} - i\,p_{34} = 0.$$

Since the value $\lambda = \infty$ that corresponds to our quadratic complex is the harmonic conjugate of the root $\lambda = 0$ with respect to the two roots $\lambda = \pm\, i\,/\,m$ of the determinant, we conclude that among the complexes of the homofocal series, ours enjoys some projective peculiarities that distinguish it amongst all of them in that series: The anharmonic ratio, which is the only absolute invariant of quadratic complexes [(22) 11], has the value – 1 for our complex (*). This agrees with the fact that when the cylinder *C* (and, as a consequence, all of the singular surface) is given, our complex itself will also be given completely.

11. Among these projective peculiarities, there is one that concerns singular lines. For an arbitrary value of λ, the corresponding complex of the homofocal series has its singular lines given by the equations:

(1) $$\left(\lambda^2 + \frac{1}{m^2}\right)p_{14}^2 + \left(\lambda^2 + \frac{1}{m^2}\right)p_{24}^2 + \lambda^2 p_{34}^2 - \frac{1}{m^2}\lambda^2 p_{12}^2 + \frac{2}{m^2}\lambda p_{12}p_{34} = 0,$$

$$\left(\lambda^2 p_{34} + \frac{1}{m^2}\lambda p_{12}\right)\left(-\frac{1}{m^2}\lambda^2 p_{12} + \frac{1}{m^2}\lambda p_{34}\right) = 0\,;$$

i.e., they form two quadratic congruences that belong to the linear complexes:

(2) $$p_{12} + \lambda\, m^2\, p_{34} = 0,$$
(3) $$\lambda\, p_{12} - p_{34} = 0,$$

respectively (which are not generally in involution, contrary to what *Weiler* said). For the first of these congruences, upon substituting equation (2) in (1), one will get the following two other equations:

$$p_{14}^2 + p_{24}^2 - \frac{1}{m^2}p_{12}^2 = 0, \qquad p_{14}^2 + p_{24}^2 - m^2\lambda^2 p_{34}^2 = 0,$$

which shows that this congruence is composed of lines tangent to the fixed cylinder *C* that cut the plane at infinity at points of the conic that has the equation:

(*) See the theorem on the absolute invariants of quadratic complexes that we gave in no. 141 of our paper "Sulla geometria della retta e delle sue serie quadriche" (Memorie della R. Accademia delle scienze di Torino, serie II, tomo XXXVI).

$$x^2 + y^2 - m^2 \lambda^2 z^2 = 0$$

in that plane. In other words, for an arbitrary λ of the series, that congruence will be composed of lines that are tangent to the cylinder C and make the same angle with its axis r, whose cosine will be given by $1 / \sqrt{1+m^2\lambda^2}$. Now, for our complex, we will have $\lambda = \infty$, and that angle, in turn, will reduce to a right angle: The conic of the plane at infinity will reduce to the (double) line $z = 0$, which will also be the axis of the special complex $p_{34} = 0$, to which the linear complex (2) will reduce in this case.

For the congruence of singular lines that belong to the linear complex (3), one will get, upon combining that equation with (1):

$$p_{14}^2 + p_{24}^2 + p_{34}^2 = 0, \qquad p_{14}^2 + p_{24}^2 + \lambda^2 p_{12}^2 = 0,$$

so one sees that this congruence is composed of lines that cut the absolute and touch an (imaginary) right cylinder that has r for its axis and a radius equal to i / λ. For our quadratic complex, this right cylinder will reduce to its axis; i.e., the linear complex (3) that this congruence of singular lines belongs to will become the special complex $p_{12} = 0$ that will have r for its axis when $\lambda = \infty$.

12. One will not get the most general complex of the class [(22) 11] upon projectively transforming our quadratic, but in fact one of the ones that belong to the category of complexes of lines that cut two second-order surfaces harmonically, which is a category that we studied with our friend *Loria* in a recent paper (*). This will result immediately from some characteristics that we found for the complex [(22) 11] of that category in that paper. Moreover, one can verify, with no difficulty, that the lines of our complex will cut the two quadrics (which have r for their axis of rotation)

$$x^2 + y^2 + \frac{m^2}{k^2} z^2 = m^2 - k^2, \qquad x^2 + y^2 - \frac{m^2}{k^2} z^2 = m^2 + k^2,$$

harmonically, where k^2 represents an arbitrary parameter. In particular, if one sets $k^2 = m^2$ then one will see that our complex can be considered to be composed of chords of the surface:

$$x^2 + y^2 - z^2 = 2m^2,$$

which are seen as a right angle from its center.

13. The homofocal series that we have considered is, as we have already remarked, composed of ∞^1 quadratic complexes whose conics are all circles and whose cones are all

(*) See the note "Sur les différentes espèces de complexes du 2e degree des droites qui coupent hamoniquement deux surfaces du second ordre," Math. Ann. XXIII, pp. 213-234. Our complex here is considered from the projective viewpoint on page 230.

right cones (which are indeed the most general quadratic complexes that enjoy that property). In addition, these circles and these cones are all doubly tangent to the cylinder *C*. In an arbitrary plane π, these circles thus form one of two series of circles that are doubly tangent to the ellipse of intersection of π with *C*; we have already recognized that one of these circles (viz., the one that belongs to our complex) touches that ellipse at the summit of its focal axis: Therefore, that series of circles that are doubly tangent to the ellipse is the one whose chords of contact are parallel to that axis; i.e., whose centers have the other axis of the ellipse for their locus. Among these circles, there are two of radius zero whose centers are the imaginary foci of the ellipse: These circles reduce, as envelopes, to their centers, doubly counted, and they thus correspond to the two complexes of the homofocal series that reduce to two fundamental linear complexes, doubly counted. We thus obtain a remarkable property of the right cylinder:

An arbitrary right cylinder is cut by each plane in space along an ellipse whose two imaginary foci are the points that correspond to that plane with respect to two linear complexes (which are conjugate imaginary and in involution with each other).

This property does not persist for real foci or for quadric surfaces that are not right cylinders, in general.

14. One can give a geometric definition of an arbitrary quadratic complex of the series that is homofocal to our complex that is analogous to the one that was given for it. The equation (no. 10) of that arbitrary complex (λ) can be written in the following manner:

$$\left(\lambda^2 + \frac{1}{m^2}\right)(p_{14}^2 + p_{24}^2 + p_{34}^2) - \frac{1}{m^2}(p_{34}^2 + \lambda^2 p_{12}^2 - 2\lambda p_{12} p_{34}) = 0;$$

i.e.:

$$\frac{\lambda p_{12} - p_{34}}{\sqrt{p_{14}^2 + p_{24}^2 + p_{34}^2}} = \pm m \sqrt{\lambda^2 + \frac{1}{m^2}}\,.$$

Set $\lambda = -1 / k$; this equation will then become:

$$\frac{p_{12}}{\sqrt{p_{14}^2 + p_{24}^2 + p_{34}^2}} + k\frac{p_{34}}{\sqrt{p_{14}^2 + p_{24}^2 + p_{34}^2}} = \mp\sqrt{m^2 + k^2}\,,$$

and it can also be written, by virtue of known formulas, and upon letting *d* denote the line coordinate p_{ik} :

$$\text{mom}(r, d) + k \cos (r\, d) = \mu,$$

where μ is a constant that is given by:

$$\mu^2 - k^2 = m^2.$$

We thus conclude that any complex of the homofocal series is composed of lines d for which the quantity mom(r, d) + k cos (r d) has a constant absolute value μ, where the two constants k and μ vary with the complex (since $k = -1 / \lambda$) in such a fashion that $\mu^2 - k^2 = m^2$. Now, *Klein* (*) has called the quantity mom (r, r') + ($k + k'$) cos (r r') the *moment* of two linear complexes whose axes are r and r' and whose parameters are k and k': In particular, the *moment* of a line d and the linear complex that has r for its axis and k for its parameter (i.e., which has $p_{12} + k\, p_{34} = 0$ for its equation) will be the quantity mom(r, d) + k cos (r d). We can then say that any of the complexes of the series is *the complex of lines that has a given moment with respect to a given linear complex* whose equation is (3), and we can state the following proposition:

Each quadratic complex of the class [(22) 11] *is a projective transformation of a complex of lines that has a given moment with respect to a fixed linear complex. The homofocal series of the latter quadratic complex is also composed of complexes of lines that have the same given moment m with respect to a linear complex whose axis r is always the same, while the parameter k varies with the complex of the series in such a fashion that* $\mu^2 - k^2$ *has a constant value. Upon calling this value* m^2, *the singular surface of the entire homofocal series will be composed of the absolute and the right cylinder that has r for its axis and m for its radius. The homofocal series of quadratic complexes thus corresponds uniquely to a pencil of linear complexes that have r for its axis; the corresponding congruence of two complexes is composed of lines that cut the absolute and form one of the two quadratic congruences of singular lines of the quadratic complex. Among these linear complexes, the one whose parameter k is zero corresponds to a quadratic complex of the homofocal series that will be the locus of lines that have the same moment m with respect to the line r.*

III.

15. The lines that have given moments m, m_1 with respect to two fixed lines r, r_1 belong to two quadratic complexes that are defined by these two conditions, respectively, and thus form a fourth-degree congruence. However, one also knows that if one gives the absolute value of the ratio of the moments of a line with respect to two fixed lines r, r_1 then that line will belong to one or the other of two certain linear complexes in involution, relative to which r, r_1 will be conjugate lines. Therefore, our fourth-degree congruence will decompose into two quadratic complexes that belong to these two linear complexes. – In an arbitrary plane, the lines of the two quadratic complexes will, as we have seen, envelop two circles that have their centers on r, r_1, respectively. The line that joins these two centers will consequently belong to these two stated linear complexes. The four common tangents to these circles will cut pair-wise on that line at their two centers of similitude: One deduces from this that they will be the two points that correspond to our plane with respect to these two linear complexes. Therefore, in each plane, the two lines of a quadratic congruence will be the two tangents that cut at a center of similitude, and those of the other congruence will be the ones that cut at the other

(*) "Die allgemeine lineare Transformation der Liniencoordinaten," Math. Ann. II, pp. 368.

center of similitude. – Among the four lines that have the moments m, m_1 with respect to r, r_1, resp., and pass through an arbitrary point, one can likewise easily distinguish the ones that belong to the one quadratic congruence from the ones that belong to the other one, since the planes of the two pairs of lines will intersect in the line that passes through that point and cuts the two lines r, r_1 (since that line will belong to two linear complexes that contain these quadratic congruences).

Any of these two quadratic congruences can be considered to be the intersection of a quadratic complex of lines that have the moment m with respect to r with an arbitrary linear complex: The line r_1 conjugate to r with respect to it will enjoy the property that all of the lines of that intersection will have a fixed moment m_1 with it (which is equal to the product of m with the *modulus* of the linear complex with respect to two conjugate lines r_1, r). The line that joins the points at infinity of r and r_1 will be a double line of the quadratic congruence, because it will belong to our linear complex and will be a double line of the quadratic complex. It will therefore also be a double line of the focal surface of that congruence. In addition, since just one pencil (whose plane passes through r) of lines of the quadratic complex passes through each point of the absolute, that point will be a focus for the lines of the congruence belongs to that pencil. Therefore:

The focal surface of our quadratic congruence has a double line at infinity and cuts the plane at infinity again along the absolute.

This focal surface is therefore a "complex surface," and it is easy to see that it presents no other singularities from the projective-geometric point of view.

16. The lines that have given moments m, m_1, m_2 with respect to three given lines r, r_1, r_2, resp., belong to the intersection of three quadratic complexes and consequently form a ruled surface of degree 16. However, one sees in the same manner that for the congruences that were considered above that ruled surface will decompose into four ruled surfaces of degree four that belong to four linear congruences, respectively, that will be common to each of three linear complexes for which the pairs of lines rr_1, $r_1 r_2$, $r_2 r$, resp., will be conjugate lines. Each of these four linear congruences will thus contain lines that cut rr_1r_2 and consequently will have directrices that are two generators (of the same mode of generation) of the quadratic surface that is determined by the generators rr_1r_2 . These two directrices will be double directions for the corresponding fourth-degree ruled surface, which will present no other projective peculiarity.

17. The lines that have given moments m, m_1, m_2, m_3 with respect to four given lines r, r_1, r_2, r_3, resp., are 32 in number, and they divide into eight groups of four, where each group belongs to a system of generators of a quadric surface that contains the two lines that simultaneously cut r, r_1, r_2, r_3 in the same system. This again results from the consideration of linear complexes of lines whose moments with respect to rr_1, rr_2, rr_3 have the given ratios $m : m_1$, $m : m_2$, $m : m_3$, resp. Each of these conditions will determine two linear complexes: One can thus take three of these complexes – one for each pair – in eight different ways, and the desired lines must belong to one of the eight quadratic ruled surfaces of the intersections of these complexes.

Turin, 6 January 1884.