“Monge e le congruenze generali di rette,” Biblioteca Math. (3) **8** (1907), 321-324. Corrado Segre, *Opera*, a cura della Unione Matematica Italiana, volume IV, Edizione Cremonese, Roma, 1963, pp. 469-473.

Monge and general line congruences

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The paper “Mémoire sur la théorie des déblais et des remblais (Memoir on the theory of cutting and filling)” (1781) by MONGE ([1]) contains the first propositions on *general* line congruences, in addition to ones that relate to *normal* congruences, in particular.

It is divided into two parts that correspond to the problems of transport in a given plane and space, respectively.

Before entering into the subject, the second part establishes, as the author says (pp. 685), some propositions from geometry upon which the following study is founded.

First of all, he states the following proposition in art. XIX (pp. 685-687):

> “If one imagines three lines drawn in space at every point of a plane, and one considers one of these lines then I say that of all of them in its environment that are infinitely close to it, there are generally only two of them that intersect and are consequently in the same plane as it.”

MONGE’s proof coincides with one that one also frequently finds stated nowadays. A line of the given system can be represented by equations in the coordinate variables of the point x, y, z:

$$x - x' + Az = 0, \qquad y - y' + Bz = 0.$$

A and B are functions of x' and y' that are determined by the law that the lines in space obey. In order for the line to meet the infinitely-close line (at the point x, y, z) that corresponds to the values $x' + dx'$, $y' + dy'$ of the parameters, one must have:

$$dx' = z\, dA, \qquad dy' = z\, dB.$$

It will then follow that:

$$dx'\, dB = dy'\, dA.$$

Now, if one substitutes given functions of x' and y' for A and B then one will get a second-degree equation in dy' / dx'; one then deduces the stated theorem.

([1]) *Histoire de l’académie royale des sciences*, Année 1781. Avec les Mémoires de mathématique et de physique, pour la même année, Paris 1784. The main concept of the paper is summarized on pp. 34-38 of *Histoire*. It is then found on pp. 666-704 in *Mémoires*.

Art. XX (pp. 687). "It follows from this that in the system of lines under scrutiny one can always pass any of those lines through another infinitely-close one that is in the same plane as it, and in two different ways. Having said that, let any of those lines in fact pass through one of the lines that intersect it, and then, in the same sense, intersect it with one that cuts the second one, and then intersect it with one that cuts the third in the same sense. It is obvious that if we continue in that way then we will traverse a developable surface. By the same reason, upon continually using the opposite sense, we will traverse another developable surface that will obviously have to cuts the first one along the original line that we considered. Moreover, since there are no other lines upon which one can perform the same operation, it will then follow that all of the lines are nothing but the lines of intersection of two sequences of developable surfaces, such that each surface of the first sequence will cut all of the surfaces in the second sequence along straight lines, and conversely."

After this study of the general systems of lines, one passes on to the normal systems.

Art. XXI (pp. 687-689): "If one imagines all of the possible normals to an arbitrary curved surface then I say that they are all intersections of two sequence of developable surfaces, such that each surface of the first sequence will cut all of the surfaces in the second sequence along straight lines and right angles, and conversely."

In order to prove that, take the system of normals:

$$x - x' + p' (z - z') = 0, \qquad y - y' + q' (z - z') = 0$$

to the surface that is the locus of points (x', y', z'), and one will recover directly the second-degree equations in dy' / dx' for that system that was already adopted in Art. XIX. Then, suppose that the z axis is taken to be parallel to the normal at (x', y', z'), and observe that the resulting equation will have the product of its roots equal to -1. It will then follow that two planes that we now call *focal* are at a right angle.

Art. XXII and the following ones develop a series of considerations and studies that have now become classical and are concerned with the curvature of a surface, the lines of curvature, loci of centers, etc. It is found much later in the treatise *Feuilles d'analyse appliqué à la géométrie* ([2]). However, it appeared for the first time in the paper. We note only that in the brief Art. XXIII (pp. 690), MONGE, while considering the edge of regression of a surface, certainly obtained two surfaces to which all of the lines will be tangent. Now, since it is established in Art. XX that two systems of developables exist for *any* congruence, so MONGE's consideration will also be valid for a general congruence, one will have, in fact, the two *focal* surfaces ([3]).

([2]) 1st ed., Paris, 1795. As one knows, the 3rd ed. (1807) and successive ones were entitled *Application de l'analyse à la géométrie*. It is interesting to observe that the 4th ed. (1809) is preceded by an index of the papers that were published by MONGE in which "*one will find…several questions that were not treated in that work,*" and among which, "sur les déblais et remblais" is listed with the following advertisement: "*In it, one finds the theory of lines of curvature of a surface and the proof of that remarkable proposition in full generality…*" (What then follows is the statement of the fundamental theorem that is contained in Art. XIX that was cited above.)

([3]) Perhaps it will not be pointless to make another observation. On pp. 698 (Art. XXXII) of *Mémoire*, if one has an elementary bush of normals and calculates the area of one of its cross sections at the variable distance u from the point of the given surface then one will find an expression that is proportional to $(u - R)$ $(u - R')$, where R, R' are the radii of curvature. This anticipates the results of MALUS, HAMILTON, and KUMMER in regard to the luminous *intensity* (*clarté*) or *density* of a system of rays.

Finally (Art. XXXIV, pp. 699, *et seq.*), he returns to the problems of *déblais et remblais:*

"Suppose one is given two equal volumes in space and each of which is bounded by one or more given curved surfaces. Find the point in the second volume to which each molecule of the first one must be transported in order for the sum of the products of the molecules, each multiplied by the space traversed, to be a *minimum.*"

MONGE supposed essentially that in order to make that transport, one must travel along *rectilinear* paths. It then results that all of the lines that are conjugate to the corresponding elements in the two volumes must form a *congruence* (and not a complex). Consequently, from Art. XIX, XX, those lines will be the intersections of two systems of developables, such that any surface of the first system will cut the surfaces of the second system along straight lines. However, in order to get the aforementioned *minimum*, one sees that the developables must cut at a right angle. Hence, by applying Art. XXI, MONGE concluded that the desired paths must follow the normal lines to one of those surfaces ([4]).

I believe that it is worth the pain to exhibit the geometric content of the paper of MONGE for those who have not consulted it, since it does not seem to be sufficiently well-known ([5]).

In fact, the majority of writers on line geometry believe that MONGE considered only *normal* congruence of lines and the *more general* consideration was found for the first time in the well-known work of MALUS ([6]), who was an old disciple of MONGE. Consequently, the decomposition of that congruence into two systems of developables, and therefore the existence of two focal surfaces, is attributed to MALUS.

HAMILTON already did that ([7]). Among the moderns, one has MANNHEIM ([8]), DARBOUX ([9]), LIE ([10]), ZINDLER ([11]), and others.

([4]) As you know, after MONGE, DUPIN and other addressed the question of *déblais et remblais,* while introducing hypotheses that were more conformable to the practical cases.

([5]) The *Mémoire* of MONGE is mentioned in the *Vorlesungen über Geschichte der Mathematik herausgeben von* M. CANTOR, v. 4, Leipzig, 1908, in the article by V. KOMMERELL, pp. 451 *et seq.* Perhaps that is because the author assumed that its geometric substance was all found in the *Feuilles d'analyse*, in which one found that only minutely (pp. 559, *et seq.*). However, MONGE said nothing about general line congruences there. (Cf., the note on pp. 322.)

([6]) *Optique*, Journ. de l 'éc. polyt. **7** (= 14th letter) (1808). In addition to a host of other noteworthy original ideas, one will find the results of MONGE that were cited above recalled with no citations.

([7]) "Theory of systems of rays," Trans. Irish Acad. **15** (1828). "First supplement to an essay on the theory…" *ibid.* **16** (1830). Cf., especially, the bottom of pp. 22, in the second work.

KUMMER, "Allgemeine Theorie der geradlinigen Strahlensysteme," Journal für Math. **57** (1859). In the introduction, he cites MONGE only in regard to normal congruences.

([8]) "Mémoires sur les pinceaux de droites et les normalies," Journal de math. (2) **17** (1872). (Cf., the bottom of pp. 121).

([9]) *Leçons sur la théorie générale des surfaces*, 2nd Partie, Paris 1889; cf., the note on pp. 280.

([10]) *Geometrie der Berührunstransformationen, dargestelellt von* LIE *und* SCHEFFERS, Leipzig 1896. Cf., the historical notes on pp. 268 *et seq.* The third note at the bottom of pp. 271, which relates to normal

From what I have said, the opinion above must be corrected ([12]).

congruences, is characteristic: "*As was mentioned above, this special case was already considered by* MONGE. *Some other authors have probably attributed the general considerations to* MONGE *incorrectly, which were presented by* MALUS." (The citation "Cf., also the Mémoires de l'Académie 1781, pp. 684," which is found on pp. 268, seems to have been removed from another memoir of MONGE that is also cited there, and which already contains the term *incorrectly*, pp. 684.)

([11]) "Die Entwicklung und der gegenwärtige Stand der differentiellen Liniengeometrie," Jahresber. der Deutschen Math.-Verein **15** (1906). Cf., the contrast between MALUS and MONGE that he makes in the note ([2]) on pp. 186. It is found in the early part of the brief history on pp. 128 of the treatise by the same author: *Liniengeometrie mit Anwendungen*, II Bd., Leipzig 1906.

([12]) Permit me to add another small historical observation on line geometry:

The law by which the planes that are tangent to a non-developable ruling vary at the points of one of its rectilinear generators is commonly attributed to CHASLES ["Mémoire sur les surfaces engendrées par une ligne droite," Correspondance mathém. et physique, (3) **3** (1838)]. However, HAMILTON already studied that law on pp. 108-109 of the "Theory of systems of rays" that was cited above and established the equation in the form $\delta \tan P = u$, which is now well-known. The constant u, which is now called the *parameter*, was given the more expressive term "coefficient of undevelopability" by HAMILTON. One should also compare the expression $\Delta = \sqrt{u^2+\delta^2} \cdot d\Theta$, which gives the distance from a moving point on a generator to the successive generator (which makes the angle $d\Theta$ with it). In particular, one infers that it is equal to the minimum distance between the two lines, divided by the angle between them.