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Basic electromagnetic equations in bivectorial form

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Following the example of W. R. Hamilton [1]), I understand *bivector* to mean the complex combination of two ordinary vectors R_1, R_2 into:

$$(1) \qquad \rho = R_1 + i\,R_2 , \qquad \text{where } i = \sqrt{-1}.$$

However, other than that, I will concern myself, not with the original quaternion calculus that flows out of Hamilton's treatment of the vectors, but the Heaviside schema for vector algebra and analysis [2]), which is known particularly well-suited to the demands of physicists and which has, moreover, enjoyed a continually increasing acceptance on the continent in recent times.

As far as ordinary vectors are concerned, I will appeal to the Heaviside notation, so the scalar product of two vectors A, B is denoted by simply $A\ B$, while their vector product is denoted by $V\,A\,B$, and the symbols like *curl*, *div*, ∇, and ∇^2 are defined in the usual way.

However, as far as *bivectors* are concerned, only a few remarks of a purely theoretical nature will be necessary here.

In order to distinguish them from ordinary vectors (or even scalars), I will denote bivectors throughout by *Greek* symbols. On the basis of the form (1), I will call R_1 the *first component* of the bivector ρ, while R_2 is the *second component*.

Two bivectors are equal to each other when and only when their components are pair-wise equal to each other; i.e., $\rho = \rho'$ means the same thing as $R_1 = R_1'$, $R_2 = R_2'$, and conversely.

Since any bivector in its basic form is nothing but the sum of ordinary vectors with ordinary *scalar* (if also partially imaginary) coefficients, then with no further assumptions, this illuminates the fact that all of the fundamental operations of vector algebra and analysis can be carried over to bivectors immediately. Therefore, e.g., the sum $\rho + \sigma = R_1 + S_1 + i(R_2 + S_2)$ or difference $\rho - \sigma$ of two bivectors needs no explanation at all; furthermore, the properties $\rho + \sigma = \rho + \sigma$, $\rho + (\sigma + \tau) = (\rho + \sigma) + \tau$, etc., are clear with no further explanation. The scalar product of two bivectors ρ, σ can be developed immediately, in the sense of the remark above, into:

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[1]) Cf., his "Elements of Quaternions."

[2]) O. Heaviside, *Electromagnetic Theory 1*, chap. III.

(2) $$\rho\,\sigma = (R_1 + iR_2)(S_1 + iS_2) = R_1\,S_1 - R_2\,S_2 + i(R_1\,S_2 + R_2\,S_1)\,,$$

and one also has $\sigma\rho = \rho\,\sigma$, as well as $\rho\,(\sigma + \tau) = \rho\,\sigma + \rho\,\tau$, as for ordinary vectors. Similarly, one also has for the product of two bivectors:

(3) $$V\rho\,\sigma = V R_1\,S_1 - V R_2\,S_2 + i\,(V R_1\,S_2 + V R_2\,S_1),$$

and since $V S_1\,R_1 = -\,V R_1\,S_1$, etc., so one also has $V\sigma\rho = -\,V\rho\,\sigma$; this further enlightens us that $V\rho\,(\sigma + \tau) = V\rho\,\sigma + V\rho\,\tau$, as for ordinary vectors. Finally, one remarks that $\rho\,\sigma$ is an (indeed, complex) *scalar*, like say $R_1\,S_1$, while $V\rho\,\sigma$, just like ρ or σ themselves, is a *bivector*. The definition and investigation of $\tau\,V\rho\,\sigma$, $V\tau\,V\rho\,\sigma$, and the like will be left to the reader.

As far as differential operations are concerned, such as, e.g.:

$$\frac{\partial\rho}{\partial t} = \frac{\partial R_1}{\partial t} + i\frac{\partial R_2}{\partial t}\,, \qquad \frac{\partial}{\partial t}(\rho+\sigma) = \frac{\partial\rho}{\partial t} + \frac{\partial\sigma}{\partial t}\,,$$

$$\frac{\partial}{\partial t}(\rho\sigma) = \rho\frac{\partial\sigma}{\partial t} + \sigma\frac{\partial\rho}{\partial t}\,, \qquad \text{etc.}$$

div $(\rho + \sigma)$ = div ρ + div σ, curl $(\rho + \sigma)$ = curl ρ + curl σ, …, I believe they also require no explanation.

Here, I will therefore define only two more concepts and give some of their properties, which will be useful to us in the electromagnetic applications:

1. If $\rho = R_1 + iR_2$ and $\rho' = R_1 - iR_2$ then I will call ρ and ρ' two mutually *conjugate* bivectors. I will always denote such objects with the same symbols, *with* and *without* accents.

2. If the components R_1, R_2 of a bivector are *perpendicular* to each other – i.e., if $R_1\,R_2 = 0$ – then I will call ρ an *orthogonal* bivector, and if, in particular, its components are *unit vectors* – i.e., $R_1^2 = R_2^2 = 1$ – then I will call it a *fundamental* bivector.

From these definitions, one immediately finds, due to (2) [(3), resp.] that for any pair of *conjugate bivectors*, one has:

(4) $$\rho\,\rho' = R_1^2 + R_2^2\,,$$

(5) $$V\rho\,\rho' = 2\,i\,V R_2\,R_1,$$

and furthermore, for an arbitrary *orthogonal* bivector $\omega = O_1 + i\,O_2$ $(O_1 O_2 = 0)$:

(6) $$\omega\,\omega \text{ (or } \omega^2) = O_1^2 - O_2^2\,,$$

while for *any* bivector, one has:

$$V \rho \rho = i\, V R_1 R_2 + i\, V R_2 R_1 = 0 \qquad \text{(as for ordinary vectors)},$$

and finally, for a *fundamental* bivector $\varphi = a + i\, b$ ($a\, b = 0$, where a, b are *unit vectors*: $a^2 = b^2 = 1$):

(7) $$\varphi^2 = 0.$$

If a bivector is orthogonal or even fundamental then the bivector that is conjugate to it obviously possesses that same property.

If one then chooses a third unit vector c that is normal to the plane of φ, and indeed, such that $V\, a\, b = c$, so a, b, c define a right-handed system, then one has $V\varphi\, c = -\, b + i\, a$, or:

(8) $$V\varphi\, c = i\, \varphi,$$

which one can easily cloak in words.

After these terse remarks of a general nature, let us go on to the promised electromagnetic applications. I will then denote the electric field, which is an ordinary vector, by E_1 and the magnetic force, which is also such a vector, by E_2, set:

(9) $$E_1 + i\, E_2 = \eta,$$

and call η the *electromagnetic bivector* of the field. On grounds that I will soon clarify, I restrict myself here to the consideration of *empty space*. (One can, moreover, also speak of any isotropic dielectrics with equal dielectric constant and permeability; however, this would only imply a purely formal generalization.)

If, for the sake of brevity in notation, one sets the "critical velocity" = 1, and lets t denote the time then the Maxwell differential equations read, in their usual vectorial form:

(10) $$\begin{cases} \dfrac{\partial E_1}{\partial t} = \operatorname{curl} E_2, & \dfrac{\partial E_2}{\partial t} = -\operatorname{curl} E_1, \\ \operatorname{div} E_1 = 0, & \operatorname{div} E_2 = 0. \end{cases}$$

If one now multiples the second of these equations by i and adds it to the first then one obtains, from (9):

$$\frac{\partial \eta}{\partial t} = \operatorname{curl}\, (E_2 - i\, E_1).$$

However, one has $E_2 - i\, E_1 = -\, i\, (E_1 + i\, E_2)$, so one obtains the remarkable result:

(I) $$\frac{\partial \eta}{\partial t} = -\, i \operatorname{curl}\, \eta;$$

i.e., in place of the two main equations [1]) of the field, in which mixed electric and magnetic vectors appear, one obtains *only one equation with a single unknown variable, namely, the electromagnetic bivector* η, and in fact this differential equation, just like any of the original ones, is of *first order* relative to time.

Likewise, we can proceed with the third and fourth of equations (10), and thus summarize the two solenoidal supplementary conditions into *a single solenoidal supplementary condition* for the electromagnetic bivector:

(II) $$\operatorname{div} \eta = 0.$$

The equations (I), (II) completely replace the four equations (10).

Furthermore, let η' be the *conjugate electromagnetic bivector* (to η); i.e.:

(11) $$\eta' = E_1 - i\, E_2 \,.$$

One then, in turn, obtains a single bivectorial differential equation in place of the two main Maxwell equations in a manner that is completely similar to the one above, and indeed:

(I′) $$\frac{\partial \eta'}{\partial t} = i \operatorname{curl} \eta',$$

as well as a single supplementary condition:

(II′) $$\operatorname{div} \eta' = 0.$$

Naturally, equations (I′), (II′) say exactly the same thing as (I), (II). However, when taken *together*, *both* pairs are not without interest. (One remarks, moreover, that $\frac{1}{2}(\eta + \eta') = E_1$, $\frac{1}{2i}(\eta - \eta') = E_2$.) However, in any case, a *single* bivector (which might be η or η') suffices for the complete treatment of the phenomena in question, so in the entire course of a calculation that relates to it, one does not need to perhaps split the electric and the magnetic vector. This situation seems to me to also define the most suitable formal expression for the physical state of affairs: One can treat any time-varying field with the electric and magnetic forces as unseparated from each other.

The next thing to address is the question of the *electromagnetic energy* of the field. If one denotes its spatial *density* by e; i.e., if one sets:

(12) $$e = \tfrac{1}{2}(E_1^2 + E_2^2)$$ [2])

[1]) Which is what I call the *first* two equations (10), while I refer to the last two as *solenoidal supplementary conditions.*

[2]) By the choice of the so-called Heaviside "rational units," the usual, but bothersome, factor of $1/4\pi$ drops out; the same remark is also true for the expression of the energy flux that follows shortly.

then one can express e in the simplest way by the conjugate electromagnetic pair η, η'. From the model of formula (4) for such bivectors, one, in fact, obtains immediately:

(III) $$e = \tfrac{1}{2}\,\eta\eta',$$

or, in words: *The density of the field energy is equal to one-half the scalar product of the mutually conjugate electromagnetic bivectors.*

In order to obtain the electromagnetic *energy flux F*, which is defined (as an ordinary vector) by the requirement that:

(13) $$\frac{\partial e}{\partial t} = -\operatorname{div} F,$$

we scalar multiply equation (I) by η', and likewise equation (I′) by η, and add the results; in this way, it follows that:

$$\eta'\frac{\partial \eta}{\partial t} + \eta\frac{\partial \eta'}{\partial t} = \frac{\partial}{\partial t}\eta\eta' = i\,(\eta \operatorname{curl} \eta' - \eta' \operatorname{curl} \eta).$$

One now gets the known formulas for the ordinary vectors:

$$\operatorname{div} V A B = B \operatorname{curl} A - A \operatorname{curl} B,$$

and this may be extended to bivectors with no further assumptions; i.e., one also has:

$$\operatorname{div} V \rho\, \sigma = \sigma \operatorname{curl} \rho - \rho \operatorname{curl} \sigma,$$

which one can, moreover, immediately verify from (3). When applied to our case, this gives the relation:

$$\frac{\partial}{\partial t}(\eta\eta') = -\, i \operatorname{div} V\, \eta\, \eta',$$

so, from (III) and (13), except for a purely additive solenoidal vector:

(IV) $$F = \frac{i}{2} V\, \eta\, \eta',$$

or, in words: *The energy flux – or "Poynting vector" – is equal to* $\tfrac{1}{2}$ *times the vector product of the mutually conjugate electromagnetic bivectors.*

One can also immediately convince oneself from equation (5) that (IV) is identical with the usual expression for the energy flux, namely $F = -\,V E_2\, E_1 = +\,V E_1\, E_2$.

I will leave the comparison of (IV) with (III) and the phrasing of the their closely-related combination in words to the reader.

If the electromagnetic bivector is given in all of space for $t = 0$ – say, $\eta = \eta_0$ – then the entire spectrum of electromagnetic phenomena is given on the basis of the differential equation (I), at least, inside of a domain of continuity. The details of carrying out the integration process in given special cases are not consistent with the basic theme of this treatise. I would thus only like to remark here that one can write down the symbolic solution of (I) with no further assumptions. Namely, when one denotes a combined operator by {} and lets *e* denote the base of natural logarithms, one understands that:

(V) $$\eta_t = \{e^{-i\,t\,\cdot\,\mathrm{curl}}\}\,\eta_0 = \{\cos(t\cdot\mathrm{curl}) - i\sin(t\cdot\mathrm{curl})\}\,\eta_0\,,$$

which coincides precisely with the "symbolic integrals" that were obtained from the two ordinary vectorial equations in a previous treatise [1]) by a completely circumstantial method. (The condition (II) does not especially need to be considered, since due to the fact that div $\mathrm{curl}^n = 0$ ($n = 1, 2, 3, \ldots$), it follows from (V) that div η_t = div η_0 . Thus, if only the initial field is prescribed according to (II) then div $\eta_t = 0$ remains true for all time; moreover, this likewise follows simply from equation (I) itself.) Furthermore, I would not like to place the emphasis on equation (V), from now on, but on equations (I) to (IV).

Pure electromagnetic waves. – Waves that would deserve this term can be characterized by the fact that the electric force is everywhere *perpendicular* to the magnetic one and that one-half of the energy in any region of space is electric, while the other half is magnetic; i.e., the relations $E_1^2 = E_2^2$ and $E_1\,E_2 = 0$ exist (for the vacuum). From the definition that was given in the introduction, and from equation (7), one can comprehend this concisely when one says:

For *pure* waves, one distinguishes η from a *fundamental* bivector only by a real scalar factor *s*:

(14) $$\eta = s\,\varphi,$$

such that $\eta^2 = 0$.

From this, it also follows that $\eta(\partial\eta / \partial t) = 0$, or from (I), η curl $\eta = 0$.

For the conjugate electromagnetic bivector, one likewise has in this case that $\eta' = s\varphi'$, where φ' is conjugate to φ. From (8) and (14), one has:

(15) $$V\,\eta\,c = i\,\eta = i\,s\cdot\varphi,$$

while for η':

(15′) $$V\,\eta'\,c = -\,i\,\eta' = i\,s\cdot\varphi',$$

since the two components of $\varphi'(a, -b)$ define a *left*-handed system with *c*.

In conclusion, I would only like to treat the special case of *plane* waves. In this case, the fundamental bivector φ, which likewise yields the plane wave, is constant in time and space, and only the scalar factor *s* varies. One thus has $\partial\eta / \partial t = \varphi\cdot\partial s / \partial t$, so from (I), one has:

$$\frac{\partial\eta}{\partial t} = \varphi\,\frac{\partial s}{\partial t} = -\,i\ \mathrm{curl}\,(s\,\varphi).$$

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[1]) L. Silberstein, Ann. d. Phys. **6**, pp. 373 *et seq.*, 1901.

If one lets z denote the scalar length measured in the direction of the wave normal c then, since $\partial / \partial x = 0$, $\partial / \partial x = 0$, the Hamiltonian operator ∇ equals $c(\partial / \partial z)$; now, since it is completely general that curl = $V\nabla$, one has:

$$-\text{ curl } (s\ \varphi) = -\ Vc\ \frac{\partial(s\varphi)}{\partial t} = \frac{\partial s}{\partial z} V\varphi\, c = \frac{\partial}{\partial z} V\eta\, c = i\ \frac{\partial \eta}{\partial z},$$

[from (15)], so:

$$\frac{\partial \eta}{\partial t} = -\ \frac{\partial \eta}{\partial z}, \tag{16}$$

from which it follows that the most general integral is:

$$\eta = f(z - t), \tag{17}$$

where f means an arbitrary function of the argument $z - t$; the wave, or as one can also say, the electromagnetic bivector η thus propagates in the c direction with velocity 1; i.e., with the critical velocity. As one can easily convince oneself by a cursory comparison of (15′) with (15) and (I′) with (I), the *conjugate* bivector η would propagate in the *exact opposite* direction, as is known to be the case.

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