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On the discontinuities in elastic potentials.

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It is well-known that in the theory of Newtonian potential functions, the discontinuities in these functions and their derivatives play an essential role in both their analytic properties, as well as their physical significance. The same thing is true in the theory of elastic statics. In that theory, the functions that represent the elastic deformation are not given directly by integrals over space and surfaces, but are defined by means of differential equations whose integrals are, however, susceptible to a representation by integrals that are defined over space and surfaces that will endow them with properties that are analogous to those of Newtonian potential functions. It is precisely that analogy that has given rise to almost all of the more recent progress in the theory of elasticity.

In two questions of noteworthy importance, I have especially had occasion to bring to light the necessity of having a complete knowledge of the discontinuities in the functions that are integrals of the equations of elastic statics and that, by an obvious analogy, can be called *elastic potentials*. The first of them is in regard to the deformations that exist in the dielectrics that are interposed between conductions that are laden with electricity, which is a question that is closely connected with the famous views of Maxwell in regard to the way that forces act at a distance. The second question concerns the deformations that are provoked in elastic bodies by slits in well-defined surfaces and successive arbitrary displacements of the two edges of the slit, which are deformations to which Volterra gave the name of *distortions*, and which constitute the more recent chapter in elastic statics.

The determination of those discontinuities, once and for all, and in a general manner, thus seems to be research that is particularly interesting to me. It is all the more so since that research can be carried out with methods that are simple and uniform when one employs functions that are regular and differentiable without limitation and play a role that is analogous to that of the densities in Newtonian potentials, while the rest of it conforms to the physical nature of the problem and to the reasonable criterion that one must begin the study of the problems in their simplest form. Consistent with the preceding viewpoint, it is also possible to summarize the general result that I have reached, and that can be presented as a conclusion of that research by saying that everything reduces, in the final analysis, to the problem of the discontinuities in the second derivatives of the ordinary surface potential functions. That problem has been studied by various analysts and was given a complete solution by *Poincaré* in his *Théorie du Potential Newtonien*. By various methods that were based upon formulas that were established by C. Neumann and E. Beltrami, I have also pointed out a general solution in a note that I recently communicated to this Academy ([1]), which was intended precisely to prepare the analytical elements that would be necessary for the solution of the questions that are treated in this paper.

Finally, it is very important to note that a complete knowledge of that property of discontinuity will permit one to speak of a complete mechanical interpretation of the integral formulas of representation that exhibit a perfect agreement between the various types of integrals that are presented in analysis and the various mechanical processes that can provoke the deformations.

I.

The fundamental formulas

The components u, v, w of the vector that gives the elastic displacement of a deformed elastic medium can be put into a form by which I have been able to reduce the formula, and which I discovered many years ago ([2]), that represents those components by means of definite integrals that are extended over space and the surface of the deformed body:

$$\begin{aligned} u &= \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi_3}{\partial y} - \frac{\partial \Psi_2}{\partial z}, \\ v &= \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_3}{\partial x}, \\ w &= \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y}. \end{aligned} \tag{1}$$

In these formulas, the functions Φ, Ψ_1, Ψ_2, Ψ_3 depend in a simple way upon three other functions A, B, C that are biharmonic potential and four other ones φ, ψ_1, ψ_2, ψ_3 that are harmonic or Newtonian potentials of the surface. One then has:

$$\begin{aligned} \Phi &= \frac{1}{8\pi(\lambda+2\mu)}\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) - \frac{1}{4\pi(\lambda+2\mu)}\varphi, \\ \Psi_1 &= \frac{1}{8\pi\mu}\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) + \frac{1}{4\pi}\psi_1, \\ \Psi_2 &= \frac{1}{8\pi\mu}\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) + \frac{1}{4\pi}\psi_2, \end{aligned} \tag{2}$$

([1]) SOMIGLIANA, "Sulle derivate secondo della funzione potenziale di superficie," Att della R. Acc. delle Sc. di Torino **51** (1916).

([2]) SOMIGLIANA, "Sulle equazioni dell'elasticità," Annali di Matem. (1889).

$$\Psi_3 = \frac{1}{8\pi\mu}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + \frac{1}{4\pi}\psi_3,$$

in which λ, μ are the elastic constants of the medium.

The explicit expressions for these potentials are finally the following ones:

$$(3)\qquad \begin{aligned} A &= \int k\,Xr\,dS + \int L\,r\,ds + 2\mu\int u\frac{\partial r}{\partial n}ds, \\ B &= \int k\,Yr\,dS + \int M\,r\,ds + 2\mu\int v\frac{\partial r}{\partial n}ds, \\ C &= \int k\,Zr\,dS + \int N\,r\,ds + 2\mu\int w\frac{\partial r}{\partial n}ds \end{aligned}$$

$$(4)\qquad \begin{gathered} \varphi = \int\left(u\frac{\partial a}{\partial n} + v\frac{\partial b}{\partial n} + w\frac{\partial c}{\partial n}\right)\frac{ds}{r}, \\ \psi_1 = \int\left(v\frac{\partial c}{\partial n} - w\frac{\partial b}{\partial n}\right)\frac{ds}{r}, \qquad \psi_2 = \int\left(w\frac{\partial a}{\partial n} - u\frac{\partial c}{\partial n}\right)\frac{ds}{r}, \qquad \psi_3 = \int\left(u\frac{\partial b}{\partial n} - v\frac{\partial a}{\partial n}\right)\frac{ds}{r}, \end{gathered}$$

in which:

r	represents the distance from the point (x, y, z) to the moving point of the field of integration (a, b, c).
X, Y, Z	the unitary components of the volume force.
L, M, N	the unitary components of the surface force.
k	the density.
n	the internal normal to the body S that is bounded by the surface s.

These expressions for the formulas of representation immediately lend themselves to the introduction of vectorial symbols that I prefer to use over the methods that are more frequently used in mechanics.

The potentials (3), (4) have a well-defined significance in all of space (i.e., including the space that is external to the body S), and I have had occasion to show the utility that one can derive for certain applications precisely by considering them in all of infinite space, as one does with the usual Newtonian potentials. I will then address their singularities when one traverses the surface s independently of whether that surface does or does not bound a finite-dimensional body.

From the preceding formulas, it therefore appears that the left-hand sides of formulas (1) are composed of the second derivatives of the biharmonic potentials A, B, C and the first derivatives of the Newtonian potentials φ, ψ_1, ψ_2, ψ_3 . We wish to determine the discontinuities in not just u, v, w, but also in the components of the deformation and strain, so we will know the formula that gives the discontinuity of the first and second derivative of the Newtonian potentials of the surface, and in addition, the one that gives the dependency of the discontinuity on the first, second, and third derivatives of the spatial potentials and surface biharmonics. The discontinuities in the first and second derivative of the harmonic potentials are known. We can see, first of all, that the other biharmonic potential can be deduced from them, in addition. We can then easily prepare all of the formulas that occur in the calculations that we shall encounter without having to resort to the delicate processes of passing to the limit that complicate the research regarding questions of this genre in the theory of the potential to an extraordinary degree.

We have called the potentials A, B, C in (3) *biharmonic*, since they are composed by the same process as ordinary Newtonian – or harmonic – potentials in space and on surfaces and double layers, except that in place of the elementary potential $1 / r$, one will find the function r. One therefore has relations for the equation:

$$\Delta_2 \, \Delta_2 \, \varphi = 0$$

that are analogous to the ones that the Newtonian potentials have for the Laplace equation:

$$\Delta_2 \varphi = 0.$$

The fundamental formulas (1) are formed from derivatives of biharmonic potentials and Newtonian potentials. Therefore, the expressions (2) contain potentials of two kinds: We generically say *elastic potentials* to mean all expressions that are defined as sums of biharmonic and harmonic potentials (or their derivatives), as in the right-hand sides of (1) and (2), precisely, and that satisfy the indefinite equations of elastic equilibrium.

Let n, n' be the two opposing directions of the normal at a point on the surface s. Let f_n, $f_{n'}$ be the two values of a function f on the two sides of the surface s that correspond to those normals. The jump that the function suffers upon traversing the surface in the direction n will $f_n - f_{n'}$. For brevity, we introduce the notation:

$$f_n - f_{n'} = D\,[f].$$

If U is an ordinary surface potential:

$$U = \int h \frac{ds}{r},$$

in which h is a regular function of the points of s, then one will have:

$$D\,[U] = 0,$$

and the formulas for the discontinuities in the first derivatives can be represented in the following way when one takes the directions of the axes in such a way that z has the direction of the normal n, and the x, y are parallel to the tangent plane at the point considered:

$$D\left[\frac{\partial U}{\partial x}\right] = 0, \quad D\left[\frac{\partial U}{\partial y}\right] = 0, \quad D\left[\frac{\partial U}{\partial z}\right] = -\,4\pi\, h\,.$$

The discontinuities in the second derivatives can also be represented in a simple way if, in addition to the preceding hypothesis with respect to the z-axis, one supposes that the x, y axes are tangents to the lines of curvature of the surface s at the point considered. In such a case, one will have the following relations:

$$(5)\qquad \begin{aligned} &D\left[\frac{\partial^2 U}{\partial x^2}\right] = -\frac{4\pi h}{R_1}, \qquad D\left[\frac{\partial^2 U}{\partial y^2}\right] = -\frac{4\pi h}{R_2}, \qquad D\left[\frac{\partial^2 U}{\partial z^2}\right] = -4\pi h\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \\ &D\left[\frac{\partial^2 U}{\partial x\,\partial z}\right] = -4\pi\,\frac{\partial h}{\partial x}, \qquad D\left[\frac{\partial^2 U}{\partial y\,\partial z}\right] = -4\pi\,\frac{\partial h}{\partial y}, \qquad D\left[\frac{\partial^2 U}{\partial x\,\partial y}\right] = 0, \end{aligned}$$

in which R_1, R_2 represent the radii of curvature of the surface, which correspond to the lines of curvature that are tangent to the x axis and to the y axis, respectively. In addition, the radii of curvature R_1, R_2 must be taken to be positive when their directions (viz., to the center of curvature to the surface) coincide with that of the positive normal n – i.e., in our case, with the positive direction of the z-axis ([1]).

One can establish the corresponding property of the discontinuities in the biharmonic potentials and their derivatives by means of that property of the Newtonian potentials.

II.

Discontinuities in the biharmonic potential.

a) **Surface potentials.** – Let:

$$V = \int h\, r\, ds$$

be such a potential. We suppose that the function h is regular at all points of the surface s. If one supposes that the point (x, y, z) is outside of that surface then deriving this will give:

$$\frac{\partial V}{\partial x} = \int h\frac{\partial r}{\partial x} ds = -\int h\frac{a}{r} ds + x\int h\frac{ds}{r}$$

when one lets a, b, c indicate the coordinates of the current point on the surface s and when:

$$r = \sqrt{(a-x)^2+(b-y)^2+(c-z)^2}\ .$$

The first derivatives of V are then composed of Newtonian surface potential functions, and there will therefore be no doubt as to their continuity.

Consider the second derivative. One has:

([1]) Cf., the paper that was cited above: "Sulla derivate seconde, etc."

$$\frac{\partial^2 V}{\partial x^2} = \int h\left(\frac{1}{r} + (x-a)\frac{\partial \frac{1}{r}}{\partial x}\right) ds ,$$

namely:

$$\frac{\partial^2 V}{\partial x^2} = \int h\frac{1}{r} ds + x\int h\frac{\partial \frac{1}{r}}{\partial x} ds - \int ha\frac{\partial \frac{1}{r}}{\partial x} ds .$$

The first integral is continuous upon crossing *s*, and the other two are discontinuous, but their discontinuities are equal and opposite in sign. That second derivative is then continuous.

One has, analogously:

$$\frac{\partial^2 V}{\partial x \, \partial y} = \int h\,(x-a)\frac{\partial \frac{1}{r}}{\partial x} ds ,$$

so that derivative is also continuous, and the same thing can be said for all of the remaining second derivatives.

For the third derivatives, one has:

$$\frac{\partial^3 V}{\partial x^3} = 2\int h\frac{\partial \frac{1}{r}}{\partial x} ds + x\int h\frac{\partial^2 \frac{1}{r}}{\partial x^2} ds - \int ha\frac{\partial^2 \frac{1}{r}}{\partial x^2} ds ,$$

$$\frac{\partial^3 V}{\partial x^2 \, \partial y} = \int h\frac{\partial \frac{1}{r}}{\partial y} ds + x\int h\frac{\partial^2 \frac{1}{r}}{\partial x \, \partial y} ds - \int ha\frac{\partial^2 \frac{1}{r}}{\partial x \, \partial y} ds .$$

Suppose that the axes are oriented as we indicated at the end of the preceding section (call this orientation *canonical*), so one sees immediately that these derivatives are continuous on the basis of the formula for the discontinuity in the second derivative of the Newtonian surface potential. The same thing will be true for the other two:

$$\frac{\partial^3 V}{\partial x^3}, \quad \frac{\partial^3 V}{\partial x^2 \, \partial y}$$

by reason of symmetry.

In addition, the formula:

$$\frac{\partial^3 V}{\partial x \, \partial z^2} = \int h\frac{\partial \frac{1}{r}}{\partial y} ds + \int h\,(z-c)\frac{\partial^2 \frac{1}{r}}{\partial z \, \partial x} ds ,$$

$$\frac{\partial^3 V}{\partial y\,\partial z^2} = \int h \frac{\partial \frac{1}{r}}{\partial y} ds + \int h\,(z-c) \frac{\partial^2 \frac{1}{r}}{\partial z\,\partial y} ds\,,$$

$$\frac{\partial^3 V}{\partial x\,\partial y\,\partial z} = \int h\,(x-a) \frac{\partial^2 \frac{1}{r}}{\partial y\,\partial z} ds$$

prove the continuity of that derivative. All that remains to be considered is $\partial^3 V / \partial z^3$, for which one has:

$$\frac{\partial^3 V}{\partial z^3} = \int h \frac{\partial \frac{1}{r}}{\partial z} ds + \int h\,(z-c) \frac{\partial^2 \frac{1}{r}}{\partial z^2} ds\,,$$

from which, it will result immediately that:

(6) $$D\left[\frac{\partial^2 V}{\partial z^3}\right] = -\,8\pi\,h.$$

One can then conclude that:

All of the first and second derivatives of the biharmonic surface potentials are continuous upon traversing the surface in question.

Of the third derivatives, if one supposes that the orientation of the axes is canonical then only $\partial^3 V / \partial z^3$ will be discontinuous, and its corresponding jump will be $-\,8\pi\,h$.

β) **Spatial potentials.** – It is very easy to prove that the biharmonic potential of a space S is:

$$U = \int k\,r\,dS,$$

in which k is the function that is analogous to density, and its first, second, and third derivatives are continuous when one crosses the surface s that bounds the space S.

Indeed, the continuity of U is obvious, and one will have:

$$\frac{\partial U}{\partial x} = -\int k \frac{\partial r}{\partial a} dS = \int k r\,\alpha\, d\sigma + \int \frac{\partial r}{\partial a} r\,dS\,,$$

$$\frac{\partial^2 U}{\partial x^2} = \int_s k\alpha \frac{\partial r}{\partial x} ds + \int \frac{\partial k}{\partial a} \alpha r\,ds + \int \frac{\partial^2 k}{\partial a^2} r\,dS\,,$$

$$\frac{\partial^3 U}{\partial x^3} = \int_s k\alpha \frac{\partial^2 r}{\partial x^2} ds + \int \frac{\partial k}{\partial a} \alpha \frac{\partial r}{\partial x} ds + \int \frac{\partial^2 k}{\partial a^2} \frac{\partial r}{\partial x} dS$$

for its first derivatives, and one will have analogous formulas for the other derivatives.

Now, the first of these formulas proves the continuity of the first derivative. The second one proves that of the second derivative (taking into account the continuity that was just proved of the first derivatives of the biharmonic surface potentials). Finally, that of the third derivatives results from the continuity that was just proved of the second derivatives of the biharmonic surface potential.

The spatial biharmonic potentials and their first, second, and third derivatives are continuous upon crossing the surface that bounds the space that is occupied by the mass.

γ) **Double-layer potentials. –** First of all, suppose that the surface on which the double layer is distributed with a moment g is closed. Let its potential be:

$$W = \int g \frac{\partial r}{\partial n} ds,$$

and suppose, in addition, that the function g is continuous inside of it with no interruption in the continuity of its derivatives either. By Green's lemma, we will then have:

$$\int \left(g \frac{\partial r}{\partial \nu} - r \frac{\partial g}{\partial \nu} \right) ds + \int (g \Delta_2 r - r \Delta_2 g)\, dS = 0,$$

if we denote the space that is enclosed by s by S, and this formula will be valid regardless of whether the point (x, y, z) is internal or external to S. Therefore, if it were internal then one could easily see that the only integral to be excluded would have the limit zero.

At this point, it results that the function W has the expression:

$$W = \int \frac{\partial g}{\partial n} r\, ds - 2 \int g \frac{dS}{r} + \int r \Delta_2 g\, dS ,$$

which is valid at every point in space. This formula reduces the search for the discontinuity in W to that of the discontinuity in the surface potentials, the spatial potential that was just studied, and the known Newtonian potentials, and one finds it easily by the prevailing methods that are currently used. It is clear that the limitation on the surface s that it must be closed is not essential, and one can easily remove it by observing that if it is open then it will suffice to prolong the function g in an arbitrarily fashion, continuing arbitrarily until it is closed, but while preserving the continuity.

At this point, one can write:

$$W = W_1 - 2\, W_2 + W_3,$$

in which:

$$W_1 = \int \frac{\partial g}{\partial n} r\, ds, \qquad W_2 = \int g \frac{dS}{r}, \qquad W_3 = \int r \Delta_2 g\, dS .$$

These three functions and their first derivatives are continuous upon crossing s. The same thing will then be true for W and its first derivatives.

As one can see, the second derivatives of W_1 and W_3 are continuous. As for those of W_2, one will easily find that when the axes have the canonical orientation, they will all be continuous, except for $\partial^2 W_2 / \partial z^2$, which has a jump of $-4\pi g$; it will then follow that we will have:

$$D\left[\frac{\partial^2 W}{\partial z^2}\right] = 8\pi g \tag{7}$$

for W, and the remaining second derivatives will be continuous.

We have already seen how the third derivatives of W_1 and W_3 behave; we find the following formulas for those of W_2:

$$D\left[\frac{\partial^3 W_2}{\partial z^3}\right] = 4\pi\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - 4\pi\frac{\partial g}{\partial z},$$

$$D\left[\frac{\partial^3 W_2}{\partial x^2\,\partial z}\right] = -\frac{4\pi}{R_1}g, \qquad D\left[\frac{\partial^3 W_2}{\partial x\,\partial z^2}\right] = -4\pi\frac{\partial g}{\partial x},$$

$$D\left[\frac{\partial^3 W_2}{\partial y^2\,\partial z}\right] = -\frac{4\pi}{R_2}g, \qquad D\left[\frac{\partial^3 W_2}{\partial y\,\partial z^2}\right] = -4\pi\frac{\partial g}{\partial y},$$

while the remaining ones are continuous. One can establish the discontinuities in the third derivatives of W on the basis of this formula and the preceding ones that were found for W_1, W_3. One can conclude:

Of the ten third derivatives of the biharmonic double-layer potentials, five of them are continuous, when the axes are oriented canonically. The discontinuities in the other five are determined by the following formulas:

$$D\left[\frac{\partial^3 W}{\partial z^3}\right] = -8\pi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)g,$$

$$D\left[\frac{\partial^3 W}{\partial x^2\,\partial z}\right] = \frac{8\pi}{R_1}g, \qquad D\left[\frac{\partial^3 W}{\partial x\,\partial z^2}\right] = 8\pi\frac{\partial g}{\partial x}, \tag{8}$$

$$D\left[\frac{\partial^3 W}{\partial y^2\,\partial z}\right] = \frac{8\pi}{R_2}g, \qquad D\left[\frac{\partial^3 W}{\partial y\,\partial z^2}\right] = 8\pi\frac{\partial g}{\partial y}.$$

We note that the values of these discontinuities are expressed in terms of g and the derivatives $\partial g / \partial x$, $\partial g / \partial x$, and that these quantities depend upon only the values of g on the surface s. The intervention of arbitrary auxiliary values of g in all of the space S that we have considered will therefore have no influence on the final result.

III.

Discontinuities in the elastic potentials.

In order to study the discontinuities in the fundamental expressions (1) for the elastic displacements and the corresponding components of the deformations and stresses that are derived from them, we consider separately the expressions in these formulas that depend upon the biharmonic spatial, surface, and double-layer potentials that are contained in A, B, C, and finally, the ones that depend upon the Newtonian potentials ψ_1, ψ_2, ψ_3. The general deformation that results from this will decompose into four types of deformations (and it is easy to verify that each of them satisfy the equations of equilibrium), whose characteristic properties we will study separately. In that way, we can recompose the properties of the complete expression that is represented by formula (1).

Type 1.

For these deformations, one can take:

$$A = \int k\,X\,r\,dS, \qquad B = \int k\,Y\,r\,dS, \qquad C = \int k\,Z\,r\,dS,$$

and therefore, in formula (1):

$$\Phi = \frac{1}{8\pi(\lambda+2\mu)}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right),$$

$$\Psi_1 = \frac{1}{8\pi\mu}\left(\frac{\partial B}{\partial z}-\frac{\partial C}{\partial y}\right), \qquad \Psi_2 = \frac{1}{8\pi\mu}\left(\frac{\partial C}{\partial x}-\frac{\partial A}{\partial z}\right), \qquad \Psi_3 = \frac{1}{8\pi\mu}\left(\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}\right).$$

The displacements can also be put into the form:

$$(9^*) \qquad (u, v, w) = -\frac{\lambda+\mu}{\mu}\frac{\partial\Phi}{\partial(x,y,z)}+\frac{1}{8\pi\mu}\Delta_2\,(A, B, C).$$

In this case, A, B, C are biharmonic spatial potentials and they will then be continuous in all of space, along with their first, second, and third derivatives. A, B, C will become infinite at infinity, but their second derivatives will stay finite, and indeed, will be annulled. Therefore, u, v, w are finite and continuous in all of space and are annulled at infinity. The components of the deformations and stresses likewise stay finite and continuous in all of space and go to zero at infinity. These deformations thus present no singularities. They were found by W. Thomson in order to represent the deformations that were produced in an indefinitely-extended space by volume forces that act upon a finite portion of it ([1]).

Type 2.

If we intend that A, B, C should have the values:

$$A = \int L\, r\, ds, \quad B = \int M\, r\, ds, \quad C = \int N\, r\, ds$$

then these deformations can be again put into the form (9).

These are biharmonic surface potentials, and their first and second derivatives are all continuous upon crossing the surface s. The second derivatives are annulled at infinity, so the deformations (9) will have no discontinuities, and they will be annulled at infinity.

On the contrary, since the components of the deformation, are formed linearly from the derivatives of u, v, w, will contain the third derivatives of A, B, C, and we have seen that these are discontinuous, in general. In order to study such a discontinuity at a point of s, we can suppose that the orientation of the axes is canonical with respect to the point that is considered on the surface, and we can then apply the formulas for the discontinuities that were previously established with no further discussion. The passage to arbitrarily-oriented axes can be achieved with the usual transformation formulas.

If we introduce the usual notations for the coefficients of deformation:

$$x_x = \frac{\partial u}{\partial x}, \qquad y_y = \frac{\partial v}{\partial y}, \qquad z_z = \frac{\partial w}{\partial z},$$

$$y_z = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x}, \qquad z_x = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad x_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

then the discontinuities in these expressions can be established immediately on the basis of (6), which gives the only non-null discontinuity in the third derivative of the biharmonic surface potential. One then finds from (1) that:

$$(9) \qquad D\,[x_x] = 0, \qquad D\,[y_y] = 0, \qquad D\,[z_z] = \frac{1}{8\pi}\frac{1}{\lambda+2\mu} D\left[\frac{\partial^2 C}{\partial z^2}\right] = -\frac{1}{\lambda+2\mu} N.$$

One finds, analogously:

$$(9') \qquad D\,[y_z] = -\frac{1}{\mu} M, \qquad D\,[z_x] = -\frac{1}{\mu} L, \qquad D\,[y_z] = 0.$$

([1]) W. THOMSON, "Note on the Integration of the Equations of Equilibrium of an Elastic Solid," Cambridge and Dublin Math. Journ. (1848); Math. and Phys. Papers, vol. 1, art. XXXVII; THOMSON and TAIT, *Treatise on Natural Philosophy*, sect. 731.

When these relations are substituted in the expressions for the components of the stress:

$$\begin{aligned} X_x &= \lambda\,\theta + 2\mu\, x_x, & Y_z &= \mu\, y_z, \\ Y_y &= \lambda\,\theta + 2\mu\, y_y, & Z_x &= \mu\, z_x, \\ Z_z &= \lambda\,\theta + 2\mu\, z_z, & X_y &= \mu\, x_y, \end{aligned}$$

in which, θ denotes the coefficient of cubic dilatation:

$$\theta = x_x + y_y + z_z,$$

one will get discontinuities in the stresses, which are the following:

$$(10)\qquad \begin{aligned} D[X_x] &= -\frac{\lambda}{\lambda+2\mu}N, & D[Y_z] &= -M, \\ D[Y_y] &= -\frac{\lambda}{\lambda+2\mu}N, & D[Z_x] &= -L, \\ D[Z_z] &= -N, & D[X_y] &= 0. \end{aligned}$$

Now, consider a surface element that is normal to the z-axis. The components of the stress that acts upon its positive side will be X_z, Y_z, Z_z, while X_{-z}, Y_{-z}, Z_{-z} will be the components on the negative side, and by the preceding formula, we will have:

$$X_z + X_{-z} + L = 0, \qquad Y_z + Y_{-z} + M = 0, \qquad Z_z + Z_{-z} + N = 0.$$

Now, these are precisely the equations that must be satisfied in order for the element considered to be found in equilibrium under the action of the two elastic stresses that act upon its two sides and an external surface force whose unitary components are L, M, N. More generally, if one abandons the canonical orientation of the axes then the latter relations can be written:

$$X_n + X_{-n} + L = 0, \qquad Y_n + Y_{-n} + M = 0, \qquad Z_n + Z_{-n} + N = 0,$$

which are precisely the equations that must be satisfied on the surface s in order for equilibrium to exist when it is considered to be internal to the body and subject to the actions of the external force L, M, N.

This property immediately gives the mechanical significance of the deformations of the type considered. It represents the deformation of an indefinite medium when surface forces whose unitary components are L, M, N act upon a surface s that is situated at a finite distance, and the medium can be assumed to be homogeneous and immobile at infinity.

These deformations can be considered to be the limiting deformations to which the deformations of the first type will reduce when the space in which the volume force acts becomes a surface.

One can make an interesting observation on the basis of formula (10). Consider a surface element that passes through the z-axis – i.e., it is normal to the element ds that was considered first and has the x-axis for its normal. The components of the stress that correspond to it are X_x, Y_x, Z_x , and the values that these quantities take on when the surface element is considered to belong to the region into which the positive z axis penetrates are different from the ones that they have when the element belongs to the region of negative x. As a result of (10), one will have:

$$D\,[X_x] = -\frac{\lambda}{\lambda+2\mu}N, \qquad D\,[Y_x] = 0, \qquad D\,[Z_x] = -L,$$

while those differences must all be zero in order for those stresses to be equal.

If the element considered has the y-axis for its normal then one will have:

$$D\,[X_y] = 0, \qquad D\,[Y_y] = -\frac{\lambda}{\lambda+2\mu}, \qquad D\,[Z_y] = -M,$$

instead.

An analogous fact will be true for an arbitrary surface element that passes through the z-axis, namely, for the normal to the surface s at the point considered.

That singular fact will not impede continuous elastic equilibrium from existing for the elements considered, since the elastic stresses will be continuous when one crosses those elements in the sense of their normal, and therefore they will always be subject to equal and opposite stresses in the region of the body to which they belong.

Type 3.

If we set:

$$A = \int u\frac{\partial r}{\partial n}ds, \qquad B = \int v\frac{\partial r}{\partial n}ds, \qquad C = \int w\frac{\partial r}{\partial n}ds$$

then the deformations of this type can be written in the form (9*):

$$4\pi\,u = \alpha\,\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right) + \Delta_2 A,$$

$$4\pi\,v = \alpha\,\frac{\partial}{\partial y}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right) + \Delta_2 B, \qquad \alpha = -\frac{\lambda+\mu}{\lambda+2\mu},$$

$$4\pi\,w = \alpha\,\frac{\partial}{\partial z}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right) + \Delta_2 C.$$

Suppose that the axes are oriented canonically, so only the second derivatives with respect to z of the three biharmonic double-layer potentials A, B, C will be discontinuous, and those discontinuities will be determined from formula (7). Namely, one then finds immediately that:

(11) $$D[u] = 2u, \qquad D[v] = 2v, \qquad D[w] = 2(\alpha + 1)\,w,$$

in which relations the values of u, v, w in the right-hand sides will be the same as the ones that appear in the integrals A, B, C. In order to calculate the discontinuities in the components of the deformations, recall the continuity of the third derivatives of the double-layer potentials that correspond to the derivation symbols:

$$D_x^3, \qquad D_y^3, \qquad D_x^2 D_y, \qquad D_x D_y^2, \qquad D_x\, D_y\, D_z\,.$$

We will then have:

$$4\pi D[x_x] = D\left[\alpha\frac{\partial^2 C}{\partial x^2\,\partial z} + \frac{\partial^2 A}{\partial x\,\partial z^2}\right],$$

$$4\pi D[y_y] = D\left[\alpha\frac{\partial^2 C}{\partial y^2\,\partial z} + \frac{\partial^2 B}{\partial y\,\partial z^2}\right], \qquad \text{etc.}$$

The results that one obtains when one applies the formulas that relate to these discontinuities (8) are the following ones:

(12)
$$\begin{aligned}
D[x_x] &= 2\alpha\,\frac{w}{R_1} + 2\,\frac{\partial u}{\partial x}, \\
D[y_y] &= 2\alpha\,\frac{w}{R_2} + 2\,\frac{\partial v}{\partial y}, \\
D[z_z] &= 2\alpha\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w, \\
D[y_z] &= 4\alpha\,\frac{v}{R_2} + 2\,(2\alpha + 1)\frac{\partial w}{\partial y}, \\
D[z_x] &= 4\alpha\,\frac{u}{R_2} + 2\,(2\alpha + 1)\frac{\partial w}{\partial x}, \\
D[x_y] &= 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned}$$

in which:

$$D[\theta] = 2\,(\alpha + 1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Those formulas lead to the following discontinuities for the components of the stress:

$$
(13)\qquad
\begin{aligned}
D\,[X_x] &= 2\lambda\,(\alpha+1)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+4\mu\left(\frac{\partial u}{\partial x}+\alpha\frac{w}{R_1}\right),\\
D\,[Y_y] &= 2\lambda\,(\alpha+1)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+4\mu\left(\frac{\partial v}{\partial y}+\alpha\frac{w}{R_2}\right),\\
D\,[Z_z] &= -\,2\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-4\mu\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w.
\end{aligned}
$$

The other three components of the stress have discontinuities that are proportional to the right-hand sides of the last three equations (12).

The mechanical significance of this third type of deformation will result from composing it with the fourth type, which we shall now study.

Type 4.

On the basis of the formulas (1), (2), the deformation of type 4 can be written:

$$
\begin{aligned}
4\pi u &= -\frac{\lambda}{\lambda+2\mu}\frac{\partial\varphi}{\partial x}+\frac{\partial\psi_3}{\partial y}-\frac{\partial\psi_2}{\partial z},\\
4\pi v &= -\frac{\lambda}{\lambda+2\mu}\frac{\partial\varphi}{\partial y}+\frac{\partial\psi_1}{\partial z}-\frac{\partial\psi_3}{\partial x},\\
4\pi w &= -\frac{\lambda}{\lambda+2\mu}\frac{\partial\varphi}{\partial z}+\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_1}{\partial y},
\end{aligned}
$$

in which the functions φ, ψ_1, ψ_2, ψ_3 are defined by formula (4).

These displacements are discontinuous when crossing the surface s, as are the first derivatives of the Newtonian surface potentials. Taking into account the formula that gives those discontinuities and the canonical orientation of the axes, which one again assumes, one finds immediately that:

$$
D\left[\frac{\partial\varphi}{\partial x}\right]=0,\qquad D\left[\frac{\partial\varphi}{\partial y}\right]=0,\qquad D\left[\frac{\partial\varphi}{\partial z}\right]=-\,4\pi w,
$$

$$
D\left[\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial z}\right]=-\,4\pi u,\qquad D\left[\frac{\partial\psi_1}{\partial z}-\frac{\partial\psi_3}{\partial x}\right]=-\,4\pi v,\qquad D\left[\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_1}{\partial y}\right]=0.
$$

from which, it follows that:

$$D[u] = -u, \quad D[v] = -v, \quad D[w] = -\frac{\lambda}{\lambda+2\mu}w, \tag{14}$$

in which, as in the preceding case, the values of u, v, w in the right-hand side are the same as the ones that appeared in the integrals (4).

In order to establish the discontinuity in the components of the deformation, recall that we denoted the direction cosines for the normal to the surface s by α, β, γ, and if we take the origin to be the point on it at which we wish to study the discontinuity and suppose that the axes have the canonical orientation then, taking into account only the terms of second order in x and y, we will have:

$$z = -\frac{1}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right)$$

for the equation of the surface, if the signs of the radii of curvature preserve the previous convention that we used.

We will then have:

$$\alpha = -\frac{\partial z}{\partial x}\gamma, \qquad \beta = -\frac{\partial z}{\partial y}\gamma, \qquad \gamma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2},$$

and at the origin of the coordinates:

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 1,$$

$$\frac{\partial\alpha}{\partial x} = \frac{1}{R_1}, \qquad \frac{\partial\beta}{\partial x} = 0, \qquad \frac{\partial\gamma}{\partial x} = 0,$$

$$\frac{\partial\alpha}{\partial y} = 0, \qquad \frac{\partial\beta}{\partial x} = \frac{1}{R_2}, \qquad \frac{\partial\gamma}{\partial x} = 0.$$

If one makes these assignments and applies the formula (5) then one will find that:

$$D\left[\frac{\partial^2\varphi}{\partial x^2}\right] = -4\pi\frac{w}{R_1}, \qquad D\left[\frac{\partial^2\psi_2}{\partial x\,\partial y}\right] = 0,$$

$$D\left[\frac{\partial^2\psi_2}{\partial x\,\partial z}\right] = -4\pi\left[\frac{\partial}{\partial x}(w\alpha - u\gamma)\right]_0 = -4\pi\left(\frac{w}{R_1} - \frac{\partial u}{\partial x}\right),$$

and so on. The results that one derives for the discontinuities in the components of the deformation are the following ones:

$$D\,[x_x] = -\,2\alpha\,\frac{w}{R_1} - \frac{\partial u}{\partial x}, \quad D\,[y_y] = -\,2\alpha\,\frac{w}{R_2} - \frac{\partial v}{\partial v}, \quad D\,[z_z] = 2\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

(15)

$$D\,[y_z] = -\,4\alpha\,\frac{v}{R_2} - 2(2\alpha+1)\frac{\partial w}{\partial y},$$

$$D\,[z_x] = -\,4\alpha\,\frac{u}{R_1} - 2(2\alpha+1)\frac{\partial w}{\partial x},$$

$$D\,[x_y] = -\,4\alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

from which, it will follow that:

$$D\,[x_x + y_y + z_z] = 0.$$

One will then have:

$$D\,[X_x] = 2\mu\,D\,[x_x], \text{ etc.}$$

(16)

$$D\,[Y_z] = \mu\,D\,[y_z], \text{ etc.}$$

for the components of the stress.

The search for the discontinuities that relate to the four types of deformation that appear in the fundamental formulas (1) is now complete.

However, in order find the mechanical significance of the deformations that depend upon the surface values u, v, w in (1), we must consider the deformation that results from the composition of the last two types that we considered. The property of the discontinuity that corresponds to this resultant deformation can be deduced immediately by summing the right-hand sides of (11), (14), and (12), (15), and (13), (16), respectively. In that way, one will find the following characteristic property of the discontinuity in the indicated deformation:

(17) $$D\,[u] = u, \qquad D\,[v] = v, \qquad D[w] = w;$$

i.e., the jumps that the components of the displacement suffer when one passes across the surface s are precisely equal to the values of the functions u, v, w that appear in the integrals φ, ψ. In addition, one has:

(18)
$$\begin{aligned}
D\,[x_x] &= \frac{\partial u}{\partial x}, & D\,[x_x] &= 0,\\
D\,[y_y] &= \frac{\partial v}{\partial y}, & D\,[z_x] &= 0,\\
D\,[z_z] &= -\frac{\lambda}{\lambda+2\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), & D\,[x_y] &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},
\end{aligned}$$

and for the components of the stress, one will have:

(19)
$$D\,[X_x] = \frac{2\lambda\mu}{\lambda+2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x},$$
$$D\,[Y_y] = \frac{2\lambda\mu}{\lambda+2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y},$$
$$D\,[Z_z] = 0,$$
$$D\,[Y_z] = 0, \qquad D\,[Z_x] = 0, \qquad D\,[X_y] = \mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

These are then the properties of the discontinuities in the components of the deformation and stress for the deformation considered. However, it is easy to see, as was proved recently ([1]), that if one associates the following three of the four preceding conditions:

$$D\,[X_z] = 0, \qquad D\,[Y_z] = 0, \qquad D\,[Z_z] = 0$$

with the conditions (17), which can be written:

(20)
$$D\,[X_n] = 0, \qquad D\,[Y_n] = 0, \qquad D\,[Z_n] = 0$$

when one abandons the canonical orientation, and one must assume that the resultant deformation that corresponds to indefinitely-extended space is determined completely and uniquely verified on all of the surface s, then this deformation, for the above, is nothing but the result of composing types 3 and 4 that are being considered.

From the mechanical standpoint, the significance of such a deformation is the following:

It is the deformation that is produced in an indefinitely-extended elastic body when one makes a slit in a finite surface s and displaces the edges with respect to each other in such a way that any point experiences a displacement that is represented by the vector (u, v, w) *when no force – either volume or surface – acts upon the body.*

It is clear that the analytical conditions that the components u, v, w must satisfy in that case (other than continuity, in general, and vanishing at infinity) will be precisely the relations (17) for the surface s. However, the stresses must satisfy the condition that the deformation must maintain equilibrium on the two edges of the slit, and that condition is the one that is represented by the relations (20) precisely. Therefore, mechanical intuition agrees with the analytical result that the problem and its conditions determine uniquely.

One can then conclude that the deformation represents a *distortion* of the indefinitely-extended elastic space, which is assumed to be fixed at infinity, with the significance that has been given to that term in all of the recent research that was cited above.

([1]) Rend. della R. Acc. dei Lincei (1914-15).

Furthermore, that deformation is precisely the one that has served me for a general study of the problem of elastic distortions.

It should be mentioned here that credit is due to Weingarten, in particular, for having drawn attention of mechanicians to that class of deformations in a note that was inserted in the "Rendiconti della R. Accademia dei Lincei" in 1901, and that in 1905, Volterra subsequently published an interesting series of studies in those "Rendiconti" in which the pecularities of those deformations were studied in the cases that their physical realization can present. However, the conditions that Weingarten established for the two edges of the slit are more restrictive than the ones that result from (20), in which one assumes that all six components of the stress must be continuous upon crossing the slit.

The conditions that must be satisfied in order for this to occur will result immediately from our formulas (18), (19). The vectors (u, v, w) along the slit will no longer be arbitrary in this case, but they must satisfy the conditions:

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

for all points of the surface, in which it is intended that the variables x, y are referred to the canonical orientation.

The Volterra conditions are even more restrictive, since, in addition to the Weingarten conditions, one supposes that *the first and second derivatives* of the components of the stress are continuous upon crossing the slit, as well, which amounts to conditions that are even more limiting on the vector (u, v, w). It turns out that the two edges of the slit cannot be subjected to displacements that are relatively rigid.

The opportunity of giving a definition that is broader in scope than the one that I proposed for the deformations that are produced in elastic bodies by slits and successive relative displacements of the two edges results from the fact that is imposed upon us by physical reality, in which we will proceed in a different way and exclude phenomena from the theory that actually exist in nature, as has been pointed out on various occasions.

However, the proposed definition presents itself spontaneously from a standpoint that is very important analytically, as well. Therefore, if one accepts that definition on the basis of the results that were obtained in this note, and from its most general conclusion, one can state the following proposition:

An arbitrary deformation of a bounded isotropic body is the superposition of the three deformations that are provoked in a homogeneous, elastic space that is indefinitely extended and assumed to be fixed at infinity, namely:

1. *A system of volume forces that act in the space that the body occupies.*
2. *A surface force that acts upon its surface.*
3. *A distortion that is due to a slit in the surface of the body and a relative displacement of the various points of the slit that is equal to the surface values of the displacement* ([1]).

([1]) Cf., GEBBIA, "Le deformazione tipiche dei corpi solidi elastici," Annali di Matematica, v. VII, pp. 3; MAGGI, "Sull'interpretazione del nuovo teorema di Volterra sulla teoria dell'elasticità," Rend. Acc. Lincei (1905), 2nd sem.; "Sugli spostamenti elastici discontinui," Rend. Acc. Lincei," (1908), 1st sem.

The analogy between this theorem and the property of Newtonian potentials that they can always be considered to the superposition of a spatial potential, a surface potential, and a double-layer potential is obvious. This characterizes the elastic potentials analytically, in a way, establishes the mechanical significance of the integrals that appear in the general formulas, and can be considered to the most general foundation for the application of the methods of the theory of potentials to the problems of elastic equilibrium.

Secretary of the Academy

CORRADO SEGRE