“Sur une classe de problèmes de Dynamique,” C. R. Acad. Sci. Paris **116** (1893), 485-487.

On a class of problems in dynamics

Note by **P. STAECKEL**, presented by Darboux

Translated by D. H. Delphenich

One knows that the surfaces whose linear elements are reducible to the Liouville form constitute a class for which the problem of geodesic lines admits an integral that is homogeneous of degree two with respect to the velocities.

With the goal of generalizing that theorem, I imagine some problems in dynamics in which the force function is constant. Let $q_1, q_2, \ldots, q_n$ be the independent variables upon which the position of the moving system depends. Let $q_1', q_2', \ldots, q_n'$ denote their derivatives with respect to time, and furthermore let $2T$ be the *vis viva*, which is defined by the formula:

$$2T = \sum_{k,\lambda} a_{k,\lambda} \, q_k' \, q_\lambda' \qquad (k, \lambda = 1, 2, \ldots, n),$$

in which the coefficients are given functions of $q_1, q_2, \ldots, q_n$. Moreover, let:

$$\varphi_{k\lambda} (q_k) \qquad (k, \lambda = 1, 2, \ldots, n)$$

be n^2 functions that depend upon only the indicated argument and whose determinant we denote by:

$$\Phi = \underset{(k,\lambda=1,2,\ldots,n)}{|\varphi_{k\lambda}|} = \sum_{k=1}^{n} \varphi_{k\lambda} \, \Phi_{k\lambda} \qquad (\lambda = 1, 2, \ldots, n).$$

Now suppose that the quadratic form of the differentials $dq_1, dq_2, \ldots, dq_n$:

$$\sum_{k,\lambda} a_{k,\lambda} \, dq_k \, dq_\lambda$$

is reducible to the form:

$$\sum_{k=1}^{n} \frac{\Phi}{\Phi_{k1}} dq_k^2 \, .$$

I therefore say that there exists not only the *vis viva* integral:

$$\sum_{k=1}^{n} \frac{\Phi}{\Phi_{k1}} q_k'^{\,2} = \alpha_1 ,$$

but also n – 1 *other integrals of the differential equations of motion that are homogeneous of degree two with respect to the velocities*, namely:

$$\sum_{k=1}^{n} \frac{\Phi \cdot \Phi_{k\lambda}}{\Phi_{k1}^2} q_k'^{\,2} = \alpha_\lambda \qquad (\lambda = 2, 3, \ldots, n),$$

in which the quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Having said that, one easily sees that the problem is soluble by quadratures, and one finds the integrable equations:

$$\sum_{k=1}^{n} \int \frac{\varphi_{k1}\, dq_k}{\sqrt{\sum_{\lambda=1}^{n} \alpha_\lambda\, \varphi_{k\lambda}}} = \tau - t ,$$

$$\sum_{k=1}^{n} \int \frac{\varphi_{k\mu}\, dq_k}{\sqrt{\sum_{\lambda=1}^{n} \alpha_\lambda\, \varphi_{k\lambda}}} = \beta_\mu \qquad (\mu = 2, 3, \ldots, n),$$

in which the quantities $t, \beta_2, \beta_3, \ldots, \beta_\mu$ are arbitrary constants.

For n = 2, one recovers the equations that Liouville gave ([1]).

([1]) One can also consult the celebrated paper by Liouville: "Sur les équations différentielles du movement d'un nombre quelconque de points matériels," J. de Math. (4), t. **14**, in which one will find a special case of the remarkable theorem that was discovered by Staeckel that is already given for *arbitrary n*.

(Note by DARBOUX)