# On the motion of a point in an $n$-fold manifold. 

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## Introduction

Extending Jacobi's definition, I have (*) defined a dynamical problem to be any problem in mechanics in which one deals with the motion of a system of material points whose number is finite or can also be large without restriction as long as the conditions on the system and the applied forces depend upon only the mutual positions of the points, but not their velocities, and as long as the positions of the points at time $t$ can be established by the values of a finite number of determining data. I referred to the smallest number of determining data that achieve that as the order of the dynamical problem in question. In order to exhibit the differential equations of motion, one must define: first of all, the expression for the virtual work of the system in the time element $(t, \ldots, t+d t)$ :

$$
U^{\prime}=\sum_{\kappa} P_{\kappa} \delta p_{\kappa},
$$

and secondly, the expression for the vis viva of the system at time $t$ :

$$
T=\frac{1}{2} \sum_{\kappa, \lambda} a_{\kappa \lambda} \frac{d p_{\kappa}}{d t} \frac{d p_{\lambda}}{d t} \quad\left(a_{\kappa \lambda}=a_{\lambda \kappa}\right)
$$

The $P_{1}, P_{2}, \ldots, P_{n} ; a_{11}, \ldots, a_{n n}$ in that are functions of the determining data $p_{1}, p_{2}, \ldots, p_{n}$ alone, and the symbols $\kappa, \lambda$ denote the series of numbers $1,2,3, \ldots, n$. If that is the case then that will give the desired differential equations in the second Lagrangian form:

$$
\frac{d}{d t} \frac{\partial T}{\partial \frac{d p_{\mu}}{d t}}-\frac{\partial T}{\partial p_{\mu}}-P_{\mu}=0 \quad(\mu=1,2, \ldots, n)
$$

[^0]One can, however, assign the following meaning to those differential equations. The problem of mechanics can be extended in such a way (*) that one takes the square of the line element $d s$ to be an essentially-positive quadratic form of the differentials of the $n$ independent variables $p_{1}, p_{2}$, $\ldots, p_{n}$. If one then sets:

$$
d s^{2}=\sum_{\kappa, \lambda} a_{\kappa \lambda} d p_{\kappa} d p_{\lambda}
$$

then $T$ can be regarded as the vis viva of a point of mass 1 at time $t$, and the expression $U^{\prime}$ will take on the meaning of the virtual work done on that point during the time element $(t, t+d t)$. Under those assumptions, the differential equations that were just found will prove to be precisely the differential equations of motion of a point of mass 1 in an n-fold manifold whose linear element is $d s$.

In that way, any dynamical problem can be associated with a problem in the motion of one point in an $n$-fold manifold that will require the solution of the same analytical problem. I have then (loc. cit., pp. 325) called two dynamical problems analytically equivalent when they belong to same problem in the motion of a point in an $n$-fold manifold ${ }^{\left({ }^{* *}\right)}$. In the discussion of analytical problems whose solution will yield the solution to infinitely-many associated dynamical problems at a single stroke, it proves to be much more preferable to not proceed in a purely-analytical way, but to always interpret the analytical problem as a problem of the motion of a point in an $n$-fold manifold, and in that way, the intuitiveness and clarity of the presentation will improve immensely. The introduction of that terminology seems all the more justified insofar as it brings one closer to an understanding of increasingly-common considerations that relate to systems of variable quantities by the assistance of higher manifolds, just as that is possible for three variables when one uses geometric measurements.

In what follows, a distinguished class of motions for a point in an $n$-fold manifold will be treated, which will make it possible to arrive at a precise picture of the course of the motion since everything will come down to the examination of an reversal problem that leads to $n$-fold periodic functions of $n$ real variables, as I have shown in a different place ( ${ }^{* * *}$ ).

Whenever one addresses the problem of the motion of a point in an $n$-fold manifold, the opposite question always arises of the problems in mechanics (in the ordinary sense of the term) that are analytically equivalent to that problem. However, in the context of the problem that is to be solved here, I must refrain from going into that question in detail in order to not overextend the scope of this paper and defer the task of communicating my results in regard to it to a later occasion.

[^1]
## I.

## On a distinguished class of motions for a point in an $\boldsymbol{n}$-fold manifold.

Any quadratic differential form:

$$
d s^{2}=\sum_{\kappa, \lambda} a_{\kappa \lambda} d p_{\kappa} d p_{\lambda}
$$

is associated with certain covariants that Beltrami (*) discovered and referred to as differential parameters. In order to define the first-order differential parameter of a function $U\left(p_{1}, p_{2}, \ldots, p_{n}\right)$, one must first exhibit the form that is reciprocal to the form $d s^{2}$ :

$$
\sum_{\kappa, \lambda} A_{\kappa \lambda} d p_{\kappa} d p_{\lambda} \quad\left(A_{\kappa \lambda}=A \lambda \kappa\right)
$$

The first-order differential parameter is then:

$$
\Delta_{1} U=\sum_{\kappa, \lambda} A_{\kappa \lambda} \frac{\partial U}{\partial p_{\kappa}} \frac{\partial U}{\partial p_{\lambda}} .
$$

It has the following meaning: In the $n$-fold manifold whose line element is $d s$, and which will be briefly referred to as a manifold in what follows, the equation $U\left(p_{1}, p_{2}, \ldots, p_{n}\right)=$ const. corresponds to a system of $\infty^{1}(n-1)$-fold manifolds, only one of which belongs to each point $\left(p_{1}, p_{2}, \ldots, p_{n}\right)$, in general. Those ( $n-1$ )-fold manifolds shall be referred to as fields. If one advances from the point $\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ in the manifold in a direction that is normal to the field that goes through that point by a line segment $\delta N$, and in that way, $U$ changes by $\delta U$, then one will have:

$$
\Delta_{1} U=\left(\frac{\delta U}{\delta N}\right)^{2}
$$

Among the functions $U$, one can distinguish the ones that satisfy the partial differential equation:

$$
\Delta_{1} U=f(U),
$$

in which $f(U)$ means a function of only $U$. The existence of that equation is the necessary and sufficient condition for the orthogonal trajectories of the system of fields $U=$ const. to be geodetic lines in the manifold. Now, Beltrami generalized a famous theory of Gauss that the arc-length of such trajectories between any two fields will have the same length by proving that the equation $\Delta_{1} U=f(U)$ defines the geodetic parallelism of the fields $U=$ const.

[^2]It shall now be assumed that a force function $\Pi\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ for the motion of a point of mass 1 in the manifold in question, so one will have:

$$
P_{\mu}=\frac{\partial U}{\partial p_{\mu}}
$$

and $U^{\prime}=\delta U$. That situation corresponds to the way that one refers to surfaces of constant potential as level surfaces in ordinary mechanics, so the fields of constant force function $\Pi$ ( $p_{1}, p_{2}, \ldots, p_{n}$ ) shall be referred to as level fields. It follows from the definition of the force function that:

$$
\Delta_{1} \Pi=\left(\frac{\delta \Pi}{\delta N}\right)^{2}
$$

gives the square of the force that acts on the moving point in the manifold perpendicular to the level field. Therefore, if one makes the special assumption that the force function satisfies the equation:

$$
\Delta_{1} \Pi=f(\Pi)
$$

then it will follow that the magnitude of that force is constant along a level field, and conversely, if the magnitude of that force is constant along every individual level field then $\Delta_{1} \Pi=f(\Pi)$.

It proves to be convenient for the investigation of the classes of motion that were just characterized to introduce variables $q_{1}, q_{2}, \ldots, q_{n}$ in place of the variables $p_{1}, p_{2}, \ldots, p_{n}$. Indeed, let $q_{1}$ be the arc-length of the orthogonal trajectories, as measured from a particular level field. As Beltrami showed, the variables $q_{2}, q_{3}, \ldots, q_{n}$ can be chosen in such a way that one will get:

$$
d s^{2}=d q_{1}^{2}+\sum_{h, k} b_{h k} d q_{h} d q_{k},
$$

in which $b_{22}, \ldots, b_{n n}$ are functions of $q_{1}, q_{2}, \ldots, q_{n}$, and the indices $h, k$ have to run through the values $2,3, \ldots, n$. With those conventions, $q_{1}=$ const. will be the equation of the level fields, and the line element of those $(n-1)$-fold manifolds $d s$ will be given by the equation:

$$
d \sigma^{2}=\sum_{h, k} b_{h k} d q_{h} d q_{k} .
$$

As a result of the assumption that was made on the force function, one can say something about the dependency of the quantity $d \sigma / d t$ on $t$. Namely, if one assumes that $d \sigma / d t$ vanishes at a location along the path of a moving point then the moving point will go beyond that location along the orthogonal trajectory in the next time element. In the following time element, it would proceed along the geodetic continuation of its path by following only its velocity. However, an accelerating force would act upon it, and indeed in the direction of the orthogonal trajectory of the level field that goes through it. The geodetic continuation will coincide with the trajectory if and only if $\Delta_{1} \Pi$
$=f(\Pi)$, and the moving point will go further along it. Therefore, either it will remain on a trajectory for which $d \sigma / d t$ is continually equal to zero or $d \sigma / d t$ will vanish nowhere. Thus, $\Delta_{1} \Pi=f(\Pi)$ is the necessary and sufficient condition for $d \sigma / d t$ to vanish either everywhere or nowhere.

The quantity $d \sigma / d t$ can be given a mechanical meaning. If one assumes that $d s / d t$ expresses the speed $v$ of the moving point at time $t$ then the equation:

$$
\left(\frac{d s}{d t}\right)^{2}=\left(\frac{d q_{1}}{d t}\right)^{2}+\left(\frac{d \sigma}{d t}\right)^{2}
$$

will show that the speed can be decomposed into two mutually-perpendicular components, the first of which $d q_{1} / d t$ has the direction of the trajectory, while the direction of the second one $d \sigma / d t$ lies on the level field. If $d s / d t$ is known as a function of $t$ then the length $s$ of the path that runs through the point can be obtained by integration. One likewise obtains a quantity $\sigma$ by integrating $d \sigma / d t$ that can be referred to as the lateral deviation of the point of the trajectory along which the point was found at the onset of the motion. A simple connection exists between the speed $v$ of the moving point and the force function $\Pi$. It is:

$$
v^{2}-\bar{v}^{2}=2(\Pi-\bar{\Pi}) .
$$

The initial values of the quantities $v$ and $P$ are characterized by overbars in that. It can and will be assumed that $\bar{v}$ and $\bar{\Pi}$ are finite. It will then be clear that the speed $v$ can increase beyond all limits only when $\Pi$ becomes infinite, that is, when the moving point approaches a level field $\Pi=$ $\pm \infty$ without restriction. As long as $\Pi$ is finite, $d s / d t$ will also be so, and $d \sigma / d t$ will also be finite then. Now it has been proved that either $d \sigma / d t$ always vanishes or it never does. Since the value of $d \sigma / d t$ changes continually, from its mechanical meaning, it will follow that $d \sigma / d t$ will have the same sign as long as $\Pi$ is finite and will therefore continually increase or continually decrease. That can be expressed as follows: The lateral deviation of the moving point from the trajectory that goes through the starting point of the motion will either be always zero, so the point will continually remain on that trajectory, or that deviation will change continually in the same sense as long as the force function $\Pi$ remains finite during the motion.

The level fields on which the force function $\Pi$ becomes infinite must then be examined in particular. The equation:

$$
v^{2}-\bar{v}^{2}=2(\Pi-\bar{\Pi})
$$

shows that the assumption $\Pi=-\infty$ would imply an imaginary value for $v$. It would follow from this that a level field $\Pi=-\infty$ defines a limit for the motion of the point that cannot be exceeded. Therefore, that level field shall not come under consideration in the investigation of $d \sigma / d t$. On the contrary, the field $\Pi=+\infty$ can bring about a change in the sign of $d \sigma / d t$, namely when $d \sigma / d t$ also becomes infinite on it. That is something that can decide in each special case without being able to say anything about in general.

Yet another type of level field is important for the motion of a point. Namely, it can happen that the level field degenerates to a point through which infinitely-many trajectories go then. Such places in $n$-fold manifolds shall be called poles. In order to examine whether the moving point can attain such a point in the course of motion, I imagine drawing a curve from the starting point of the motion, which might not be a pole, to a pole and goes through no other poles, while its direction varies continually everywhere. If that surface possesses a well-defined tangent at that pole then that tangent will be identical to the tangent to one of the trajectories that goes through the pole. However, in order for that coincidence to occur, it is necessary and sufficient that $d \sigma$ / $d t$ is infinitely-small compared to $d s / d t$ at the location in question. Therefore, $d \sigma / d t$ must vanish for a finite $d s / d t$. However, it will always be equal to zero then, so the moving point must return along the path from the starting point to the pole on a trajectory. If one then excludes the special case in which the force function becomes positively infinite at the pole from consideration then the moving point can attain a pole only when it moves on a trajectory. Those points will therefore likewise define limits for the motion as long as the motion along a trajectory is excluded, so the starting value of $d \sigma / d t$ is assumed to be non-zero.

If one now makes the further assumption that when poles are even present the trajectories will intersect on both sides of the level field of the starting point of the motion at each pole (and this shall be referred to as the regular case) then the course of the motion can be described in the following way: The values of $q_{1}$ that belong to the level fields of the subset of the manifold in question that lie between the two poles might have the upper limit $Q$ and the lower limit $R$, which are limits that are chosen in such a way that only one value $q_{1}$ will belong to every point of the manifold. If no pole exists in one of the two directions then the corresponding value of $Q$ or $R$ is replaced with $\pm \infty$, if level fields with $\Pi=-\infty$ then the limits $Q$ and $R$ must be modified as if those fields were poles. Now, if $d q_{1} / d t$ has, say, a positive value at the start of the motion then the moving point then the moving point can continually preserve a positive $d q_{1} / d t$ only when the limiting value on the direction of positive $q_{1}$ - say $Q$ - is infinite. $d q_{1} / d t$ must vanish once for finite $Q$, while it can vanish for infinite $Q$. Either that vanishing will first occur after an infinitelylong time, and the moving point will approach a well-defined level field asymptotically from one side, or $d q_{1} / d t$ will change sign after a finite time, and the point will then return to a well-defined level field, which can be referred to as the reversal field or also the turning field. After the return, $d q_{1} / d t$ will be negative. If $R$ is infinite then $d q_{1} / d t$ can remain negative for an infinitely-long time, but for finite $R$ it must, and for infinite $R$, the moving point must either asymptotically approach a level field or return to it again. After the reversal, $d q_{1} / d t$ will again be positive, and one can repeat the same argument. Corresponding statements will be true when one considers how the motion of the point must play out in order for it to be able to reach the starting point with the give initial values of $d q_{1} / d t, d q_{2} / d t, \ldots, d q_{n} / d t$.

As a typical form of the motion, one then has the one in which the moving point continually oscillates between two level fields, namely, the reversal fields. As degenerate cases of that type, one has: In one case, the point continually goes further in the same direction, and then the one in which it approaches a certain level field asymptotically. We shall come back to discuss the meaning of the latter phenomenon.

Much can be inferred (*) from merely the assumption that a force function $\Pi$ for the motion exists that satisfies the equation $\Delta_{1} \Pi=f(\Pi)$. However, the investigation can be taken further for one distinguished class of problems. One will arrive at it by the following argument: According to Hamilton and Jacobi, the integration of the differential equation of motion:

$$
\frac{d}{d t} \frac{\partial T}{\partial \frac{d p_{\mu}}{d t}}-\frac{\partial T}{\partial p_{\mu}}-\frac{\partial \Pi}{\partial p_{\mu}}=0
$$

can be converted into the determination of a complete solution to the partial differential equation:

$$
\frac{1}{2} \Delta_{1} W-\left(\Pi+\alpha_{1}\right)=0
$$

in which $\alpha_{1}$ means an arbitrary constant. Namely, if $W$ is a complete solution to that equation, which will then include $n-1$ independent constants $\alpha_{2}, \alpha_{3}, \ldots, \alpha_{n}$, in addition to the constant that is added to it, then:

$$
\frac{\partial W}{\partial \alpha_{1}}=\tau-t, \quad \frac{\partial W}{\partial \alpha_{2}}=\beta_{2}, \ldots, \frac{\partial W}{\partial \alpha_{n}}=\beta_{n}
$$

will be the integral equation for the differential equations of motion. The $\tau, \beta_{2}, \ldots, \beta_{n}$ mean $n$ new constants. In Sections 4 to 6 of my aforementioned Habilitationsschrift, I gave a class of HamiltonJacobi differential equations in $n$ variables for which a complete solution by quadratures existed and discussed the integral equations that would follow from them. In the next section, I will first communicate the results of that treatise and then complete the investigation of the integral equations to the extent that would be necessary in order to apply it to the present problem.

## II.

## On the integration of the Hamilton-Jacobi differential equation by means of separation of variables and a reversal problem that leads to $n$-fold periodic functions of $n$ real variables.

In 1838, Jacobi arrived at the determination of the geodetic lines in a three-axis ellipsoid by introducing elliptic coordinates, and the Hamilton-Jacobi differential equation, to whose integration the problem reduces, can be integrated by separation of variables by applying them. Liouville then recognized in 1846 that this method of integration will allow one to determine the geodetic lines by mere quadratures, and likewise solved certain problems of motion using families of surfaces for which the square of the line element can be put into the form:

[^3]$$
d s^{2}=\left(\kappa\left(q_{1}\right)+\lambda\left(q_{2}\right)\right)\left(d q_{1}^{2}+d q_{2}^{2}\right) .
$$

I have answered the question of how far the power of that method reaches, or in other words, which Hamilton-Jacobi equations:

$$
A_{11}\left(\frac{\partial W}{\partial q_{1}}\right)^{2}+A_{12} \frac{\partial W}{\partial q_{1}} \frac{\partial W}{\partial q_{2}}+A_{22}\left(\frac{\partial W}{\partial q_{2}}\right)^{2}-2\left(\Pi\left(q_{1}, q_{2}\right)+\alpha_{1}\right)=0
$$

admit separation of variables, by saying that the square of the linear element of any surface for which one of the associated Hamilton-Jacobi equations can be integrated by separation of variables can always be brought into the Liouville form by means of a transformation of variables (*):

$$
p_{1}=\Phi\left(q_{1}\right)+\Psi\left(q_{2}\right), \quad p_{2}=\mathrm{X}\left(q_{1}\right)+\Omega\left(q_{2}\right) .
$$

Later, Jacobi showed ( ${ }^{* *}$ ) that the geodetic lines of certain $n$-fold manifold that corresponds to the three-axis ellipsoid can be determined by means of general elliptic coordinates in $n$ variables, and in connection with that, Rosochatius ( ${ }^{* * *}$ ) has examined how the force function must be defined in order for the Hamilton-Jacobi equation to admit separation of variables in general elliptic coordinates. Part of his results are already found in a paper by Liouville $\left(^{\dagger}\right)$.

I have proved a theorem that makes it possible to achieve the same thing that Liouville's theorem made possible for two-fold manifolds in the realm of $n$-fold manifolds in my aforementioned Habilitationsschrift. I arrived at that theorem by investigating the cases in which the Hamilton-Jacobi differential equation:

$$
H^{*}=\frac{1}{2} \sum_{\kappa} A_{\kappa}\left(\frac{\partial W}{\partial p_{\kappa}}\right)^{2}-\left(\Pi+\alpha_{1}\right)=0
$$

belongs to the special quadratic differential form with non-vanishing determinant:

$$
d s^{2}=\sum_{\kappa} \frac{d p_{\kappa}^{2}}{A_{\kappa}} .
$$

[^4]It will admit separation of variables when it possesses a complete solution of the form:

$$
W=\sum_{\kappa} \int W_{\kappa}\left(p_{\kappa} ; \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right) d p_{\kappa} .
$$

If one introduces the desired values $W_{\kappa}\left(p_{\kappa} ; \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)$ for $\partial W / \partial p_{\kappa}$ in $H^{*}$ and then differentiates the identity that then arises partially with respect to $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ then one will get the $n$ equations:

$$
\sum_{\kappa} A_{\kappa} \frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{\mu}}=2 \delta_{1 \mu} \quad\left(\delta_{1 \mu}=\left\{\begin{array}{ll}
1 & \mu=1, \\
0 & \mu \neq 0,
\end{array}, \mu=1,2, \ldots, n\right),\right.
$$

from which $A_{1}, A_{2}, \ldots, A_{n}$ can be given in the form:

$$
A_{\kappa}=2 \frac{Q_{\kappa}}{Q}
$$

when one introduces:

$$
\left|\frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{\mu}}\right|=Q=\sum_{\kappa} \frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{1}} Q_{\kappa}
$$

with the notation of $\mathbf{L}$. Kronecker. If one substitutes those values $A_{1}, A_{2}, \ldots, A_{n}$ in $H^{*}$ then that will give:

$$
\Pi=\frac{\sum_{\kappa}\left[W_{\kappa}^{2}-\alpha_{1} \frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{1}}\right] Q_{\kappa}}{\sum_{\kappa} \frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{1}} Q_{\kappa}} \quad(\kappa, \mu=1,2, \ldots, n)
$$

Now, it can be shown that the determinant $Q$ and its sub-determinants $Q_{1}, Q_{2}, \ldots, Q_{n}$ do not vanish identically, and it will then be possible to assign definite values $\bar{\alpha}_{1}, \bar{\alpha}_{2}, \ldots, \bar{\alpha}_{n}$ to the arbitrary constants $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ for which those quantities are not equal to zero. In that way:

$$
\frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{\mu}} \quad \text { might go to } \quad 2 \varphi_{\kappa \mu}\left(p_{\kappa}\right), \quad \text { and } \quad W_{\kappa}^{2}-\alpha_{1} \frac{\partial\left(W_{\kappa}^{2}\right)}{\partial \alpha_{1}} \quad \text { might go to } \quad 2 \varphi_{\kappa 0}\left(p_{\kappa}\right) .
$$

Now let:

$$
\left|\varphi_{\kappa \lambda}\left(p_{\kappa}\right)\right|=\Phi=\sum_{\kappa} \varphi_{\kappa 1}\left(p_{\kappa}\right) \Phi_{\kappa}, \quad(\kappa, \lambda=1,2, \ldots, n)
$$

and

$$
\Phi^{\prime}=\sum_{\kappa} \varphi_{\kappa 0}\left(p_{\kappa}\right) \Phi_{\kappa},
$$

so the result that is obtained can be expressed as:

When the Hamilton-Jacobi equation:

$$
H^{*}=\frac{1}{2} \sum_{\kappa} A_{\kappa}\left(\frac{\partial W}{\partial p_{\kappa}}\right)^{2}-\left(\Pi+\alpha_{1}\right)=0
$$

admits separation of variables, there will necessarily be a system of $n(n+1)$ functions of one variable:

$$
\varphi_{\kappa \nu}\left(p_{\kappa}\right) \quad\binom{\kappa=1,2, \ldots, n}{v=0,1,2, \ldots, n}
$$

with the property that one can set:

$$
A_{1}=\frac{\Phi_{1}}{\Phi}, \quad A_{2}=\frac{\Phi_{2}}{\Phi}, \quad \ldots, \quad A_{n}=\frac{\Phi_{n}}{\Phi} ; \quad \Pi=\frac{\Phi^{\prime}}{\Phi}
$$

Conversely, if the system of functions $\varphi_{\kappa \nu}\left(p_{\kappa}\right)$ is assumed to be completely arbitrary, up to the restriction that none of the determinants $\Phi, \Phi_{1}, \Phi_{2}, \ldots, \Phi_{n}$ vanishes identically, and one defines the quadratic differential form with non-vanishing determinant:

$$
d s^{2}=\sum_{\kappa} \frac{\Phi}{\Phi_{\kappa}} d p_{\kappa}^{2}=\Phi \cdot \sum_{\kappa} \frac{d p_{\kappa}^{2}}{\Phi_{\kappa}}
$$

then that will belong to the Hamilton-Jacobi equation:

$$
\frac{1}{2} \sum_{\kappa} \frac{\Phi_{\kappa}}{\Phi}\left(\frac{\partial W_{\kappa}}{\partial p_{\kappa}}\right)^{2}-\left(\Pi+\alpha_{1}\right)=0 .
$$

If the force function $\Pi$ can be brought into the form:

$$
\Pi=\frac{\Phi^{\prime}}{\Phi}
$$

then that differential equation will possess the complete solution:

$$
W=\sum_{\kappa} \int \sqrt{2 \varphi_{\kappa 0}\left(p_{\kappa}\right)+\sum_{\lambda} 2 \varphi_{\kappa \lambda}\left(p_{\kappa}\right) \cdot \alpha_{\lambda}} \cdot d p_{\kappa},
$$

and that will give the integral equations of the differential equations of motion in the form:

$$
\begin{aligned}
& \sum_{\kappa} \int \frac{\varphi_{\kappa 1}}{\sqrt{2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}}} d p_{\kappa}=\tau-t, \\
& \sum_{\kappa} \int \frac{\varphi_{\kappa \mu}}{\sqrt{2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}}} d p_{\kappa}=\beta_{\mu} \quad(\mu=2,3, \ldots, n),
\end{aligned}
$$

such that one will be led to the following theorem of dynamics:
If the vis viva for a dynamical problem can be represented by the expression:

$$
T=\frac{1}{2} \sum_{\kappa} \frac{\Phi}{\Phi_{\kappa}}\left(\frac{d p_{\kappa}}{d t}\right)^{2}
$$

while the force function simultaneously has the form:

$$
\Pi=\frac{\Phi^{\prime}}{\Phi}
$$

then the differential equations of motion can be integrated by mere quadratures.
For $n=2$, that theorem will go to precisely the aforementioned theorem of Liouville. Liouville derived the immediate corollary to his theorem that the geodetic lines on certain surfaces, whose line elements are, in fact, given by:

$$
d s^{2}=\left[\varphi_{11}\left(p_{1}\right) \cdot \varphi_{22}\left(p_{2}\right)-\varphi_{12}\left(p_{1}\right) \cdot \varphi_{21}\left(p_{2}\right)\right] \cdot\left(\frac{d p_{1}^{2}}{\varphi_{22}\left(p_{2}\right)}+\frac{d p_{2}^{2}}{\varphi_{11}\left(p_{1}\right)}\right),
$$

can be determined by quadratures. One can convert that form of the line element into the aforementioned Liouville form with no difficulty. Precisely the same thing is implied as a corollary to the theorem that was proved here:

If the square of the line element ds of an n-fold manifold can be represented in the form:

$$
d s^{2}=\Phi \sum_{\kappa} \frac{d p_{\kappa}^{2}}{\Phi_{\kappa}} \quad\left(\Phi=\left|\varphi_{\kappa \lambda}\left(p_{\kappa}\right)\right|=\sum_{\kappa} \varphi_{\kappa 1} \Phi_{\kappa}\right)
$$

then the equations of the geodetic lines of that manifold can be determined by quadratures. In terms of the variables $q_{1}, q_{2}, \ldots, q_{n}$, they read:

$$
\sum_{\kappa} \int \frac{\varphi_{\kappa \mu} d p_{\kappa}}{\sqrt{\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}}}=\beta_{\mu} \quad(\mu=2,3, \ldots, n)
$$

Now, if the integration of the differential equations of motion by quadratures has been accomplished in that way for a large class of dynamical problems, as well, then much less will be learned about the dependency of the determining data $p_{1}, p_{2}, \ldots, p_{n}$ on time $t$, and that will necessitate a new investigation that cannot be carried out with no restricting assumptions on the nature of the functions $\varphi_{\kappa \nu}$.

For $n=1$, one will obtain the equation:

$$
\int \frac{\varphi_{11}\left(p_{1}\right) d p_{1}}{\sqrt{2 \varphi_{10}+2 \varphi_{11} \cdot \alpha_{1}}}=\tau-t
$$

In a posthumous treatise ( ${ }^{*}$ ), Abel considered an equation of the form:

$$
\int \frac{\varphi_{11}\left(p_{1}\right) d p_{1}}{\sqrt{\psi\left(p_{1}\right)}}=t_{1}
$$

and showed that under certain assumptions $p_{1}$ will be determined as a periodic function of $t_{1}$ from that equation. The same thing was treated in detail later by Weierstrass ( ${ }^{* *}$ ), and I have examined it under somewhat-more-general assumptions and by a different method in Section IV of my Inaugural Dissertation.

For $n=2$, when one sets:

$$
2 \varphi_{\kappa 0}+2 \varphi_{\kappa 1} \cdot \alpha_{1}+2 \varphi_{\kappa 2} \cdot \alpha_{2}=\psi_{\kappa}\left(p_{\kappa}\right) \quad(\kappa=1,2),
$$

to abbreviate, one will have:

$$
\begin{aligned}
& \int \frac{\varphi_{11} d p_{1}}{\sqrt{\psi_{1}\left(p_{1}\right)}}+\int \frac{\varphi_{21} d p_{2}}{\sqrt{\psi_{2}\left(p_{2}\right)}}=\tau-t \\
& \int \frac{\varphi_{12} d p_{1}}{\sqrt{\psi_{1}\left(p_{1}\right)}}+\int \frac{\varphi_{22} d p_{2}}{\sqrt{\psi_{2}\left(p_{2}\right)}}=\beta_{2} .
\end{aligned}
$$

The reversal problem can be regarded as a special case of a more general one that Staude (*) had investigated, from which it follows that under certain assumptions, those equations will define $p_{1}$ and $p_{2}$ as single-valued, finite, continuous, bounded periodic functions of time $t$. The important concept of bounded periodic functions that Staude introduced will be discussed below.
(*) "Propriétés remarquables de la function $y=\varphi(x)$, etc.," Euvres completes, nouvelle edition by MM. L. Sylow et S. Lie, t. II, pp. 40.
$\left(^{* *}\right)$ "Ueber eine Gattung reell periodischer Functionen," Monatsberichte der Berliner Akademie (1866), pp. 97.
(*) "Ueber eine Gattung doppelt reell periodischer Functionen zweier Veränderlichen," Math. Ann. 29 (1887), pp. 468, and "Ueber bedingt periodischer Functionen eines beschränkt veränderlichen complexen Argumentes und Anwendungen derselben auf die Mechanik," J. reine angew. Math. 105 (1888), pp. 298.

I showed that a corresponding theorem is also true for the general case of $n$ variables in my Habilitationsschrift. I started from the more general integral equations:

$$
\sum_{\kappa} \int \frac{\varphi_{\kappa \mu}\left(p_{\kappa}\right) d p_{\kappa}}{\sqrt{\psi_{\kappa}\left(p_{\kappa}\right)}}=t \lambda \quad(\lambda=1,2,3, \ldots, n)
$$

between the real variables $p_{1}, p_{2}, \ldots, p_{n}$ and $t_{1}, t_{2}, \ldots, t_{n}$. In so doing it was assumed that the functions $\psi_{1}, \psi_{2}, \ldots, \psi_{n}$ can be represented in the form:

$$
\psi_{\kappa}\left(p_{\kappa}\right)=\left(p_{\kappa}-a_{\kappa}\right) \cdot\left(b_{\kappa}-p_{\kappa}\right) \cdot \chi_{\kappa}\left(p_{\kappa}\right),
$$

in which $a_{\kappa}$ and $b_{\kappa}$ are real quantities, and $a_{\kappa}$ is smaller than $b_{\kappa}$, and in which $\chi_{1}, \chi_{2}, \ldots, \chi_{n}$ have finite positive values in the region:

$$
a_{\kappa} \leq p_{\kappa} \leq b_{\kappa} \quad(\kappa=1,2, \ldots, n),
$$

which might be denoted by $\mathfrak{B}$ in what follows. It will then be further assumed that the determinant:

$$
\left|\frac{\varphi_{\kappa \lambda}\left(p_{\kappa}\right)}{\sqrt{\psi_{\kappa}\left(p_{\kappa}\right)}}\right|=\frac{\Phi}{\sqrt{\psi_{1} \cdot \psi_{2} \cdots \psi_{n}}}
$$

does not vanish identically in the region $\mathfrak{B}$, so one will have the equations:

$$
\sum_{\kappa} \int_{a_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa \lambda}\left(p_{\kappa}\right) d p_{\kappa}}{\sqrt{\left(p_{\kappa}-a_{\kappa}\right)\left(b_{\kappa}-p_{\kappa}\right)} \cdot \sqrt{\chi_{\kappa}\left(p_{\kappa}\right)}}=t \lambda
$$

in which the symbol $\sqrt{\chi_{\kappa}\left(p_{\kappa}\right)}$ is assigned a positive value, while it can still be demanded of the sign of $\sqrt{\left(p_{\kappa}-a_{\kappa}\right)\left(b_{\kappa}-p_{\kappa}\right)}$ that $p_{1}, p_{2}, \ldots, p_{n}$ are single-valued, finite, continuous functions of $t_{1}$, $t_{2}, \ldots, t_{n}$ in the region $\mathfrak{B}$, which will make the system of values $p_{1}=a_{1}, p_{2}=a_{2}, \ldots, p_{n}=a_{n}$ belong to the system of values $t_{1}=0, t_{2}=0, \ldots, t_{n}=0$. If new variables $w_{1}, w_{2}, \ldots, w_{n}$ are introduced in place of the variables $p_{1}, p_{2}, \ldots, p_{n}$ by way of:

$$
p_{\kappa}=\frac{a_{\kappa}+b_{\kappa}}{2}+\frac{a_{\kappa}-b_{\kappa}}{2} \cos w_{\kappa},
$$

and if is it established that the system of values $p_{1}=a_{1}, p_{2}=a_{2}, \ldots, p_{n}=a_{n}$ should correspond to the system of values $w_{1}=0, w_{2}=0, \ldots, w_{n}=0$ then the integral equations will go to:

$$
\sum_{\kappa} \int_{0}^{w_{\kappa}} h_{\kappa \lambda}\left(w_{\kappa}\right) d w_{\kappa}=t_{\lambda}
$$

in which the functions $h_{\kappa \lambda}\left(w_{\kappa}\right)$ are single-valued, finite, even functions of period $2 \pi$ in their arguments $w_{\kappa}$ that can be regarded as real variables that vary without bound. Now, if the one last assumption is added that the functions $\varphi_{\kappa \lambda}\left(p_{\kappa}\right)$ do not change sign in the region $\mathfrak{B}$, and therefore the functions $h_{\kappa \lambda}\left(w_{\kappa}\right)$ will not change sign for arbitrary values of the $w_{\kappa}$ either, then the following theorem will be true:

The equations:

$$
\sum_{\kappa} \int_{a_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa \lambda}\left(p_{\kappa}\right) d p_{\kappa}}{\sqrt{\left(p_{\kappa}-a_{\kappa}\right)\left(b_{\kappa}-p_{\kappa}\right)} \cdot \sqrt{\chi_{\kappa}\left(p_{\kappa}\right)}}=t_{\lambda} \quad(\lambda=1,2, \ldots, n)
$$

define $p_{1}, p_{2}, \ldots, p_{n}$ in the region:

$$
a_{\kappa} \leq p_{\kappa} \leq b_{\kappa} \quad(\kappa=1,2, \ldots, n)
$$

to be single-valued, finite, continuous, even functions of $t_{1}, t_{2}, \ldots, t_{n}$ that are $n$-fold periodic with the system of periods:

$$
2 \omega_{\mu 1}, 2 \omega_{\mu 2}, \ldots, 2 \omega_{\mu n} \quad(\mu=1,2,3, \ldots, n)
$$

In that way, the periods are given by:

$$
\omega_{\mu \lambda}=\int_{a_{\kappa}}^{b_{\kappa}} \frac{\varphi_{\kappa \lambda}\left(p_{\kappa}\right) d p_{\kappa}}{\sqrt{\left(p_{\kappa}-a_{\kappa}\right)\left(b_{\kappa}-p_{\kappa}\right)} \cdot \sqrt{\chi_{\kappa}\left(p_{\kappa}\right)}},
$$

in which the quantity $\sqrt{\left(p_{\kappa}-a_{\kappa}\right)\left(b_{\kappa}-p_{\kappa}\right)}$ is assigned the positive sign. All of the systems of values $t_{1}, t_{2}, \ldots, t_{n}$ that belong to a system of values $p_{1}, p_{2}, \ldots, p_{n}$ are represented by:

$$
\pm t_{\lambda}^{0}+\sum_{\kappa} 2 m_{\kappa} \omega_{\kappa \lambda},
$$

in which $m_{1}, m_{2}, \ldots, m_{n}$ mean whole numbers. The system of values $t_{1}^{0}, t_{2}^{0}, \ldots, t_{n}^{0}$ belongs to the region:

$$
t_{\lambda}=\sum_{\kappa} \tau_{\kappa} \omega_{\kappa \lambda} \quad\left(0 \leq \tau_{\lambda} \leq 1, \lambda=1,2,3, \ldots, n\right)
$$

and it is the only one in that region that includes the system of values $p_{1}, p_{2}, \ldots, p_{n}$.
One easily convinces oneself that this theorem coincides with the one that Staude found for $n$ $=2$. In that case, Staude represented the bounded periodic functions $p_{1}$ and $p_{2}$ of $t_{1}$ and $t_{2}$ by
doubly-infinite trigonometric series. A corresponding representation in terms of $n$-fold infinitely trigonometric series can be found in the general case that is considered here for $p_{1}, p_{2}, \ldots, p_{n}$ as functions of $t_{1}, t_{2}, \ldots, t_{n}$, but we shall not go further into the details of that here.

I only hinted at how one can apply the results of the investigation of the more general reversal problem that was just considered to the discussion of the previously-found integral equations of the dynamical problems in my Habilitationsschrift. However, since it is precisely that viewpoint that has an essentially significance in the present work, I must treat it more thoroughly here.

A moving point might be found at a location $\left(\bar{p}_{1}, \bar{p}_{2}, \ldots, \bar{p}_{n}\right)$ at the beginning of its motion at time $t=\bar{t}$. The direction in which it begins its motion will be determined by the initial values of $\frac{d p_{1}}{d t}, \frac{d p_{2}}{d t}, \ldots, \frac{d p_{n}}{d t}$, which shall be denoted by $\bar{p}_{1}^{\prime}, \bar{p}_{2}^{\prime}, \ldots, \bar{p}_{n}^{\prime}$, resp. The values of the functions $\varphi_{\kappa \lambda}$ at the starting point of the motion shall be indicated by overbars. Upon fixing those initial conditions for the motion, the integral equations will take the form:

$$
\begin{aligned}
& \sum_{\kappa} \int_{\bar{p}_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa 1} d p_{\kappa}}{\sqrt{2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}}}=\bar{t}-t, \\
& \sum_{\kappa} \int_{\bar{p}_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa \mu} d p_{\kappa}}{\sqrt{2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}}}=0,
\end{aligned}
$$

in which the constants $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ appear. However, in order to determine those constants from the initial conditions, one will get:

$$
\sum_{\kappa} \frac{\bar{\varphi}_{\kappa \mu} \bar{p}_{\kappa}^{\prime}}{\sqrt{2 \bar{\varphi}_{\kappa 0}+\sum_{\lambda} 2 \bar{\varphi}_{\kappa \lambda} \cdot \alpha_{\lambda}}}=-\delta_{1, \mu}
$$

by differentiation. One easily convinces oneself that $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ are determined uniquely from the initial conditions by means of those equations.

One might set:

$$
2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda}=\psi_{\kappa}\left(p_{\kappa}\right),
$$

to abbreviate. It shall be assumed that the functions $\varphi_{\kappa \mu}$ are arranged such that the application of the previously-developed theorem would be possible, and the system of values $\bar{p}_{1}, \bar{p}_{2}, \ldots, \bar{p}_{n}$ belongs to the region $\mathfrak{B}$. If one then writes the integral equations in the form:

$$
\begin{aligned}
& \sum_{\kappa} \int_{a_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa 1} d p_{\kappa}}{\sqrt{\psi_{\kappa}}}=\sum_{\kappa} \int_{a_{\kappa}}^{\bar{p}_{\kappa}} \frac{\varphi_{\kappa 1} d p_{\kappa}}{\sqrt{\psi_{\kappa}}}+\bar{t}-t \\
& \sum_{\kappa} \int_{a_{\kappa}}^{p_{\kappa}} \frac{\varphi_{\kappa \mu} d p_{\kappa}}{\sqrt{\psi_{\kappa}}}=\sum_{\kappa} \int_{a_{\kappa}}^{\bar{p}_{\kappa}} \frac{\varphi_{\kappa \mu} d p_{\kappa}}{\sqrt{\psi_{\kappa}}} \quad(\mu=2,3, \ldots, n)
\end{aligned}
$$

then that will give $p_{1}, p_{2}, \ldots, p_{n}$ as functions of time when one sets:

$$
\begin{aligned}
& t_{1}=\sum_{\kappa} \int_{a_{\kappa}}^{\bar{p}_{\kappa}} \frac{\varphi_{\kappa 1} d p_{\kappa}}{\sqrt{\psi_{\kappa}}}+\bar{t}-t \\
& t_{\mu}=\sum_{\kappa} \int_{a_{\kappa}}^{\bar{p}_{\kappa}} \frac{\varphi_{\kappa \mu} d p_{\kappa}}{\sqrt{\psi_{\kappa}}}
\end{aligned}
$$

respectively, in the single-valued, finite, continuous, even $n$-fold periodic functions $p_{1}, p_{2}, \ldots, p_{n}$ of $t_{1}, t_{2}, \ldots, t_{n}$ that were obtained before. It is clear that those functions of time satisfy the differential equations of motion and that the initial conditions are simultaneously fulfilled.

The functions $p_{1}, p_{2}, \ldots, p_{n}$ of $t_{1}, t_{2}, \ldots, t_{n}$ will remain unchanged when their arguments are increased by a system of periods:

$$
\sum_{\kappa} 2 m_{\kappa} \omega_{\kappa \lambda} \quad(\lambda=1,2, \ldots, n)
$$

Now, should $t_{1}, t_{2}, \ldots, t_{n}$ have the fixed values that were given above, then such an addition would be permissible only when one has precisely:

$$
\sum_{\kappa} 2 m_{\kappa} \omega_{\kappa \lambda}=0 \quad(\lambda=2,3, \ldots, n)
$$

for certain values of the numbers $m_{1}, m_{2}, \ldots, m_{n}$. It is only when $m_{1}, m_{2}, \ldots, m_{n}$ are whole numbers that $p_{1}, p_{2}, \ldots, p_{n}$ are periodic functions of time, and indeed with the period:

$$
2 \Omega=\sum_{\kappa} 2 m_{\kappa} \omega_{\kappa 1} .
$$

Staude referred to such functions of time as bounded periodic functions (*). For him, they appeared for the case $n=2$, in which one condition equation must exist between the quantities $\omega_{\kappa \mu}$, and

[^5]Staude then spoke of simple bounded periodic functions in that case. The functions that appear here are characterized as $(n-1)$-fold bounded periodic by an analogous relation.

It emerges from the foregoing that it is possible to represent the $n$ determining data $p_{1}, p_{2}, \ldots$, $p_{n}$ as functions of time in the dynamical problem under consideration as long the initial values $\bar{p}_{1}$, $\bar{p}_{2}, \ldots, \bar{p}_{n} ; \bar{p}_{1}^{\prime}, \bar{p}_{2}^{\prime}, \ldots, \bar{p}_{n}^{\prime}$ are given. The investigation of the problem cannot be considered to be complete with that alone. Rather, the question now arises of how the various motions are connected with the various corresponding initial conditions, and one must then determine what modifications must be made to the course of motion when the initial conditions are varied. That way of thinking leads to the important result that the solution to the reversal problem, as it was explained in the foregoing, is still not sufficient and needs to be completed in an essential way. How to manage that would probably be best explained by an example. In Sections III-VI of my Inaugural Dissertation, I examined the motion of a material point on a surface of rotation under the assumption that a force function exists that is constant on the parallel circles (*). In that way, one will find that under very general assumptions on the nature of the surface and the force function, the way that the motion proceeds is that the moving point oscillates between two fixed parallel circles that one can refer to as reversal circles, or also turning circles. If one now lets the material point begin its motion from a certain starting point but varies the direction and magnitude of the initial velocity then that will show that for a continuous variation of the initial velocity, the position of the reversal circles will likewise vary continuously, in general. However, if the direction and magnitude of the initial velocity assume certain values that can be given at the outset then the motion will degenerate into a parallel circle as an asymptotic approximation, and if the variation is continued further, as well, the position of one of the reversal circles will change in a discontinuous way, while it will next experience continuous changes again under further variations of the initial velocity. Staude $\left(^{* *}\right)$ has given an explanation for this and similar, even stranger, phenomena. Namely, just as two branches of a curve can merge together when a constant in its equation varies, whereby a transition to a singularity will be created at that moment, one must also regard a motion as branched in some situations, two branches of the motion can merge together under variation of the initial conditions. "A singular form of motion will then be squeezed between the divided and undivided form of the motion," e.g., an asymptotic approach to a parallel circle. "In that way, one will get around the possible branching of the trajectory at the critical parallel circle, because according to Kirchhoff (Vorlesungen über mathematische Physik, Vorl. I, § 2), the coordinates $x, y, z$ of the moving point are single-valued functions of time for the duration of the motion."

I have already emphasized the fact that the given solution of the general reversal problem produced only one branch of the structure ( $p_{1}, p_{2}, \ldots, p_{n}$ ) in my Habilitationsschrift. However, as soon as one knows the zeroes of the function $\psi_{\kappa}\left(p_{\kappa}\right)$, and as soon as the previous assumptions are

[^6]fulfilled in the region $\mathfrak{B}$, which is defined by such zeroes in the way that was given above, it will also be possible to obtain the other branches of the structure by the same process. One then goes over to the dynamical problem in question, so one has:
$$
\psi_{\kappa}\left(p_{\kappa}\right)=2 \varphi_{\kappa 0}+\sum_{\lambda} 2 \varphi_{\kappa \lambda} \cdot \alpha_{\lambda},
$$
in which the arbitrary constants $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ depend upon the initial values of the quantities $p_{1}$, $p_{2}, \ldots, p_{n} ; \frac{d p_{1}}{d t}, \frac{d p_{2}}{d t}, \ldots, \frac{d p_{n}}{d t}$ in the previously-given way. If one then investigates the changes to the course of motion when one changes the initial velocity for a fixed starting point, that is, when the quantities $\bar{p}_{1}, \bar{p}_{2}, \ldots, \bar{p}_{n}$ are regarded as constant and the quantities $\bar{p}_{1}^{\prime}, \bar{p}_{2}^{\prime}, \ldots, \bar{p}_{n}^{\prime}$ as variable, then everything will come down to how the roots of the equations $\psi_{\kappa}\left(p_{\kappa}\right)=0$ change. The quantities $a_{\kappa}$ and $b_{\kappa}$ will both change continuously under continuous changes to $\bar{p}_{1}^{\prime}, \ldots, \bar{p}_{n}^{\prime}$, in general, and the motion will preserve its original character until one of those roots merges with one of the other roots, if such things are present. At that moment, the equation in question $\psi_{\kappa}\left(p_{\kappa}\right)$ $=0$ will take on a double root, and that will generally come about in such a way that the system of values considered $p_{1}, p_{2}, \ldots, p_{n}$ will belong to an infinitely-large value of $t$, that is, the moving point will approach that location asymptotically.

Those considerations will suffice for us to investigate the special problem that shall be treated in the next section, which is that of exhibiting the line of reasoning that must be pursued, and that will, at the same time, yield a verification of the general theorems.

## III.

## On a class of motions of a point in an $n$-fold manifold that corresponds to the Jacobi motion on a surface of revolution.

One now addresses the problem of treating the cases in which the differential equations of the motion of the dynamical problem that defined the subject of the first section can be integrated. In order to apply the results of the foregoing section to it, one must recall that one had:

$$
d s^{2}=d q_{1}^{2}+\sum_{h, k} b_{h k} d q_{h} d q_{k},
$$

at the time, and the force function $\Pi$ was a function of only $q_{1}$. One must then next assume that the sum in $d s^{2}$ can be put into the simpler form:

$$
\sum_{h} b_{h} d q_{h}^{2}
$$

and then investigate when it would be possible to find a system of $n(n+1)$ functions:

$$
\varphi_{\kappa \mu}\left(p_{\kappa}\right) \quad\binom{\kappa=1,2, \ldots, n}{\mu=0,1,2, \ldots, n}
$$

such that when one sets:

$$
\begin{array}{rlr}
\left|\Phi_{\kappa \lambda}\right|=\Phi & =\sum_{\kappa} \varphi_{\kappa 1}\left(q_{\kappa}\right) \cdot \Phi_{\kappa}, & \\
\Phi^{\prime} & =\sum_{\kappa} \varphi_{\kappa 0}\left(q_{\kappa}\right) \cdot \Phi_{\kappa}, & (\kappa, \lambda=1,2, \ldots, n)
\end{array}
$$

one will have the equations:

$$
\Pi\left(q_{1}\right)=\frac{\Phi^{\prime}}{\Phi}, \quad 1=\frac{\Phi}{\Phi_{1}}, \quad b_{2}=\frac{\Phi}{\Phi_{2}}, \quad \ldots, \quad b_{n}=\frac{\Phi}{\Phi_{n}} .
$$

However, if one brings the first two relations into the form:

$$
\begin{array}{r}
\left(\varphi_{10}-\Pi\left(q_{1}\right)\right) \Phi_{1}+\sum_{h} \varphi_{h 0} \cdot \Phi_{h}=0 \\
\left(\varphi_{11}-1\right) \Phi_{1}+\sum_{h} \varphi_{h 1} \cdot \Phi_{h}=0
\end{array}
$$

then one will easily see that it is fulfilled identically by:

$$
\begin{array}{llll}
\varphi_{10}=\Pi\left(q_{1}\right), & \varphi_{20}=0, & \ldots, & \varphi_{n 0}=0, \\
\varphi_{11}=0, & \varphi_{21}=0, & \ldots, & \varphi_{n 1}=0,
\end{array}
$$

while the remaining $n(n-1)$ functions:

$$
\varphi_{\kappa \mu}\left(q_{\kappa}\right) \quad\binom{\kappa=1,2, \ldots, n}{h=2,3, \ldots, n}
$$

will still be arbitrary, except for the restriction that $\Phi_{1}, \Phi_{2}, \ldots, \Phi_{n}$ must not vanish identically. If that is the case then the quadratic differential form:

$$
d s^{2}=\Phi_{1} \sum_{\kappa} \frac{d q_{\kappa}^{2}}{\Phi_{1}}
$$

and the force function $\Pi=\Pi\left(q_{1}\right)$ will belong to the Hamilton-Jacobi differential equation:

$$
\frac{1}{2} \sum_{\kappa} \frac{\Phi_{\kappa}}{\Phi_{1}}\left(\frac{\partial W}{\partial q_{\kappa}}\right)^{2}-\left(\Pi\left(q_{1}\right)+\alpha_{1}\right)=0
$$

which can be integrated by separation of variables. For the associated dynamical problem, the integral equations of the differential equations of motion then read:

$$
\begin{aligned}
& \int \frac{d q_{1}}{\sqrt{2 \Pi\left(q_{1}\right)+\sum_{\lambda} 2 \varphi_{1 \lambda} \cdot \alpha_{\lambda}}} \quad=\tau-t, \\
& \int \frac{\varphi_{1 \mu} d q_{1}}{\sqrt{2 \Pi\left(q_{1}\right)+\sum_{\lambda} 2 \varphi_{1 \lambda} \cdot \alpha_{\lambda}}}+\sum_{h} \int \frac{\varphi_{h \mu} d q_{1}}{\sqrt{2 \sum_{\lambda} 2 \varphi_{h \lambda} \cdot \alpha_{\lambda}}}=\beta_{\mu} .
\end{aligned}
$$

One might now set $n=2$. One will then have:

$$
d s^{2}=d q_{1}^{2}+b_{2} d q_{2}^{2} .
$$

One then deals with the motion of a point on a surface when the lines $\Pi=$ const. or the level lines are geodetic lines on the surface. Should this problem admit separation of variables, then one would need to have:

$$
\varphi_{10}=\Pi\left(q_{1}\right), \quad \varphi_{20}=0, \quad \varphi_{11}=0, \quad \varphi_{21}=0,
$$

while $\varphi_{12}$ would be an arbitrary function of $q_{1}$, and $\varphi_{22}$ would be an arbitrary function of $q_{2}$. One will now have $\Phi_{1}=\varphi_{22}, \Phi_{2}=\varphi_{12}$, and one would then need to have $b_{2}=\varphi_{22} / \varphi_{12}$. However, by introducing a function of $q_{2}$ in place of $q_{2}$, one can always arrange that one has:

$$
d s^{2}=d q_{1}^{2}+\frac{d q_{2}^{2}}{\varphi_{12}\left(q_{1}\right)}
$$

from the outset, such that a problem will arise that can be interpreted as the problem of motion of a point on a surface of revolution when the force function is constant along the parallel circles. The problem in $n$ variables can then be referred to as a generalization of the aforementioned Jacobi problem for two-fold manifolds. At the same time, that will imply a remarkable generalization of those surfaces that are developable to surfaces of revolution, because in the domain of $n$ variables, they will then correspond to the manifolds for which the square of the linear element can be put into the form:

$$
d s^{2}=\Phi_{1} \cdot \sum_{\kappa} \frac{d q_{\kappa}^{2}}{\Phi_{\kappa}} .
$$

For that reason, the integral equations to which the generalization of the Jacobi problem will lead are especially simple ones, because the dependency of the variable $q_{1}$ on time is determined from the equation:

$$
\int \frac{d q_{1}}{\sqrt{2 \Pi\left(q_{1}\right)+\sum_{\lambda} 2 \varphi_{1 \lambda} \cdot \alpha_{\lambda}}}=\tau-t
$$

for them, and that equation can be examined independently of the remaining integral equations, and that is precisely the reversal problem for $n=1$ that was mentioned on pp. 12. Methods are developed in the treatises that were cited there with whose assistance one can exhibit the individual branches of the reversal function $q_{1}(t)$, and one then needs only to ascertain the modifications that the reversal function would suffer when one varies the initial conditions.

If one imagines that the moving point always begins its motion from a well-defined starting point $\left(\bar{q}_{1}, \bar{q}_{2}, \ldots, \bar{q}_{n}\right)$, but the direction and magnitude of its initial velocity change, so the quantities $\bar{q}_{1}^{\prime}, \bar{q}_{2}^{\prime}, \ldots, \bar{q}_{n}^{\prime}$ can be regarded as variable, then every branch of the reversal function that one now obtained will belong to motions of the point that run between two reversal fields for which the associated values of $q_{1}$ might be denoted by, say, $Q_{\rho}$ and $R_{\rho}(r=1,2,3, \ldots, r)$. Due to their mechanical significance, I would like to call those quantities briefly the first (second, resp.) reversal values. The number $r$ of branches of $q_{1}(t)$ depends upon the number of real roots of the equation:

$$
P\left(q_{1}\right)=\Pi\left(q_{1}\right)+\sum_{\lambda} \varphi_{1 \lambda}\left(q_{1}\right) \cdot \alpha_{\lambda}=0,
$$

in which the quantities $\alpha_{1}, \ldots, \alpha_{n}$ are regarded as functions of $\bar{q}_{1}^{\prime}, \ldots, \bar{q}_{n}^{\prime}$. One easily convinces oneself that $\alpha_{1}, \ldots, \alpha_{n}$ are linear functions of ${\vec{q}_{1}^{2}}^{2}, \ldots, \bar{q}_{n}^{2}$, and that one can then write the equation $P=0$ in the form:

$$
P\left(q_{1}\right)=P_{0}\left(q_{1}\right)+\sum_{\lambda} \bar{q}_{\lambda}^{\prime 2} \cdot P_{\lambda}\left(q_{1}\right)=0,
$$

which will exhibit the dependency on the initial conditions.
In order to ascertain how the reversal function $q_{1}(t)$ changes under continuous changes in $\bar{q}_{1}^{\prime}$, $\ldots, \vec{q}_{n}$, I shall appeal to a method of investigation that can also be of use in other considerations. The variables $\bar{q}_{1}^{\prime}, \ldots, \bar{q}_{n}^{\prime}$ shall be regarded as the coordinates of a point in a planar n-fold manifold $\mathfrak{M}_{n}$. It suffices to assign positive values to those quantities here, since only their squares will come under consideration. If one considers $q_{1}$ to be a parameter in the equation $P=0$ then it will represent a family of $\infty^{1}(n-1)$-fold manifolds. For $n=1$, those fields are conic sections in the plane, for $n$ $=3$, they are second-order surfaces in space, and in general they can be called second-order fields. It suffices to assign only values that come under consideration for the motion to the parameter $q_{1}$. The considerations that were discussed in the first section will be important here since they show that the motion will be subject to certain limits that cannot be exceeded from the outset as a result of the pole and the level field $\Pi=-\infty$. One can then bound a subset of the given manifold inside of which the entire course of motion must play out, which is a subset that I will refer to as the domain of motion. Upon introducing that terminology, one can say briefly that the only fields $P=$

0 that will be considered in what follows will be the ones whose parameter $q_{1}$ belongs to a location in the domain of motion.

A certain number of those fields $P=0$ will go through each point $\mathfrak{M}_{n}$ that have a simple relationship to the number $r$ of branches that the reversal function $q_{1}(t)$ possesses when the initial values of $d q_{1} / d t, \ldots, d q_{n} / d t$ have precisely the values $\bar{q}_{1}^{\prime}, \ldots, \bar{q}_{n}^{\prime}$ of the coordinate of the point of $\mathfrak{M}_{n}$ in question. If $P$ initially vanishes for no real value of $q_{1}$ then $q_{1}(t)$ will possess only one branch, and one will have $r=1$. However, it $P=0$ has real roots then it will be necessary to further consider the values $q_{1}= \pm \infty$ and add their roots in the event that $P$ changes it sign there. Moreover, as I would like to emphasize, that will be true only when the values $q_{1}= \pm \infty$ belong to points of the domain of motion. The number of roots of $P=0$, when taken in that sense, is always even as long as one excludes multiple roots. One can now say that the number $r$ of branches is equal to one-half the number of real roots of $P=0$ when one adds that a branch of that function can possibly go from a finite point to infinity and back to a finite point.

In that way, any point of $\mathfrak{M}_{n}$ is associated with a whole positive number $r$, and the points that belong to the same $r$ will define a connected region that is separated by certain boundary structures. In that way, the places for which $r=1$ can be distinguished by whether $P=0$ has no real roots or two of them, and the boundary between those two regions is counted as the boundary structure. In the following presentation, for the sake of simplicity, that special boundary structure always be left out of consideration. However, one can easily see how the following theorems would be modified in order for them to be true in full generality.

A change in the number of branches can occur only when either two previously-separate branches merge together or a branch splits into two new ones. The one case requires that two real roots must coincide, while the other requires the appearance of a new root, so the coincidence of two complex roots. In both cases, the equation:

$$
\frac{\partial P}{\partial q_{1}}=0
$$

must then be fulfilled. One will then get the boundary structure when one defines the envelope of the fields $P=0$. Moreover, part of that envelope might not belong to the boundary structure, which probably needs no explanation.

If one has separated $\mathfrak{M}_{n}$ into the regions that were defined above with the help of the envelope of the fields $P=0$ then it will be easy to show how the reversal function $q_{1}(t)$ changes when the initial conditions $\bar{q}_{1}^{\prime}, \ldots, \vec{q}_{n}^{\prime}$ vary continuously, so when the representative point in $\mathfrak{M}_{n}$ describes a continuous curve. If one starts from a point $A$ that belongs to the number $r$, and therefore $r$ pairs of reversal values, then that reversal value cannot initially change continuously when one traverses the curve. Its decrease or increase can go to an increase or decrease, resp., only when the curve contacts one of the fields $P=0$. If the curve reaches part of the envelope that belongs to the boundary structure then the motion of point will asymptotically approach the level field for which $q_{1}$ has the parameter value of the field in question. That is because the occurrence of a double root of $P=0$ will imply that $t$ will become infinitely large, so one will have $d q_{1}=0$ during the entire
motion, that is, the motion will proceed in precisely that level field, which is a possibility that we will come back to soon. The number $r$ will change upon going through the envelope. That will happen in such a way that the reversal value will then change discontinuously, namely it will jump suddenly when two branches merge and drop suddenly when a new branch appears. It is clear how one can continue that argument to further traversals of the curve.

One will also arrive at the envelope of the fields $P=0$ when one asks the question "when can the moving point remain on the level field $q_{1}=\bar{q}_{1}$ during its entire motion?" Should that be the case, then the direction of the initial velocity would have to lie in the level field, so one would need to have $\bar{q}_{1}^{\prime}=0$. However, that is still not sufficient. In addition, the direction of the force that acts upon the point at time $t=\bar{t}$ must be perpendicular to the orthogonal trajectories, so $d \frac{d q_{1}}{d t}$ must also vanish for $t=\bar{t}$. However, that implies the condition:

$$
\frac{\partial P}{\partial q_{1}}=0
$$

which appeared above for the envelope. The representative points of $\mathfrak{M}_{n}$ that simultaneously lie in the field $\bar{q}_{1}^{\prime}=0$ and the envelope will then yield the directions and magnitudes of the initial velocities for which the motion can exist in the level field $q_{1}=\bar{q}_{1}$. Since the force function is constant on any level field, the point will describe a geodetic line in it, and indeed one with constant velocity.

It remains for us to bring the variables $q_{2}, q_{3}, \ldots, q_{n}$ into our field of view. We can initially regard all $n$ integral equations as one reversal problem and then obtain $q_{1}, q_{2}, \ldots, q_{n}$ as $(n-1)$-fold bounded periodic functions of time by applying the results of Section Two. In that special case, one will have:

$$
\omega_{11}=\omega_{11}, \quad \omega_{21}=0, \ldots, \quad \omega_{n 1}=0 .
$$

If the motion is periodic then is its period:

$$
2 \Omega=2 \omega_{11}=2 \cdot \int_{a_{1}}^{b_{1}} \frac{d q_{1}}{\sqrt{2 \Pi\left(q_{1}\right)+\sum_{\lambda} \varphi_{1 \lambda} \cdot \alpha_{\lambda}}}
$$

will then be equal to the period of the branch of the reversal function $q_{1}(t)$ in question.
The trajectory possesses certain properties in this case that I have derived for $n=2$ in my Inaugural Dissertation, and whose general validity can be demonstrated in the same way that was done there.

However, if one regards $q_{1}$ as a known function of time then on the basis of the foregoing considerations, the determination of $q_{2}, q_{3}, \ldots, q_{n}$ will require only that one must solve the reversal problem in $n-1$ variables:

$$
\sum_{h} \frac{\varphi_{h \mu}\left(q_{h}\right) d q_{h}}{\sqrt{\sum_{\lambda} 2 \varphi_{h \mu}\left(q_{h}\right) \cdot \alpha_{\lambda}}}=d t_{\mu} \quad(\mu=2,3, \ldots, n)
$$

If one would like to utilize the theorems of the second section in order to solve this reversal problem then the functions $\varphi_{h \mu}$ would not need to change sign in the region $\mathfrak{B}$. However, since that region $\mathfrak{B}$ depends upon the choice of the constants $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$, that assumption must be fulfilled for all systems of values $q_{2}, q_{3}, \ldots, q_{n}$ that occur inside of the domain of motion. Furthermore, a certain determinant must be non-vanish for the same system of values $q_{2}, q_{3}, \ldots$, $q_{n}$. However, if the functions:

$$
\varphi_{h \lambda}\left(q_{h}\right) \quad(h, \lambda=2,3, \ldots, n)
$$

are arranged such that they have well-defined finite values for all systems of values $q_{2}, q_{3}, \ldots, q_{n}$ in the domain of motion then the demand on that determinant will be fulfilled automatically. One can understand that remarkable theorem immediately when one actually forms the determinant. One will then find that $\Phi_{1}$ cannot vanish in the domain of motion. Now, one had:

$$
d s^{2}=d q_{1}^{2}+\Phi_{1} \cdot \sum_{h} \frac{d q_{h}^{2}}{\Phi_{h}}
$$

so the equation $\Phi_{1}=0$ can be fulfilled only when a level field reduces to a point, so at the poles of the manifold. One will then arrive at a new path to the introduction of the pole, and one will see that when one exceeds a pole, the single-valuedness of $q_{1}, q_{2}, \ldots, q_{n}$ as functions of time will generally break down since those points are branching places for the reversal problem. It is only when $d q_{2}, d q_{3}, \ldots, d q_{n}$ are continually equal to zero during the entire duration of the motion that the vanish of the determinant $\Phi_{1}$ will be irrelevant. Passing through a pole will then take place only when the point moves along a trajectory, which will again imply a theorem in the first section that related to the poles, but along an entirely-different path.

Halle a./S., January 1893.


[^0]:    (*) "Ueber die Differentialgleichungen der Dynamik und den Begriff der analytischen Aequivalenz dynamischer Probleme," J. reine angew. Math. 107 (1891), 319-348.

[^1]:    (*) Cf., the distinguished work of Herrn R. Lipschitz, "Untersuchung eines Problems der Variationrechnung, in welchem das Problem der Mechanik enhalten ist," J. reine angew. Math. 74 (1871), pp. 116 and "Bemerkungen zu dem Princip des kleinsten Zwanges," ibidem 82 (1877), pp. 316, as well as my aforementioned treatise, pp. 330.
    $\left(^{* *}\right)$ The concept of the equivalence of dynamical problems can be generalized, as $\mathbf{P}$. Appell showed in connection with my work in "Sur des transformations de mouvements," J. de Math. 110 (1892), pp. 37.
    $\left({ }^{* * *}\right)$ "Ueber die Integration der Hamilton-Jacobi’schen Differentialgleichung mittelst Separation der Variabeln," Habilitationsschrift, Halle, 1891.

[^2]:    (*) "Sulla teorica generale dei parametri differenziali," Memorie dell; Istituto di Bologna, (2) 8 (1869), pp. $549 . ~ . ~ . ~_{\text {(18 }}$.

[^3]:    (*) Similar considerations for the special case of $n=2$ were already developed in Section II of my Inaugural Dissertation: "Ueber die Bewegung eines Punktes auf einer Fläche," Berlin, 1885.

[^4]:    (*) "Eine charakteristische Eigenschaft der Flächen, deren Linienelement $d s$ durch $d s^{2}=$ $\left(\kappa\left(q_{1}\right)+\lambda\left(q_{2}\right)\right)\left(d q_{1}^{2}+d q_{2}^{2}\right)$ gegeben wird," these Annalen 35 (1889), pp. 91. I briefly referred to the surfaces in question as Liouville surfaces there. The term that is used here: "surfaces whose line element has the Liouville form" might be preferable to that one, since on the one hand, it relates to only the line element, and also the term Liouville surfaces was used by G. Darboux (Leçons sur la théorie générale des surfaces, t. 2, 1889, pp. 291) in a different sense.
    (**) C. G. J. Jacobi, Vorlesungen über Dynamik, edited by A. Clebsch, Lecture 26.
    (***) "Ueber die Bewegung eines Punktes," Inaugural Dissertation, Göttingen, 1877.
    ${ }^{\dagger}$ ) "Sur quelques cas particuliers où les équations du movement d'un point peuvent s'intégrer," Liouville's Journal 12 (1846), pp. 410.

[^5]:    (*) In the treatise: "Ueber die Bewegung eines schweren Punktes auf einer Rotationsfläche," Acta math. 11 (1888), pp. 303, Staude remarked that C. Neumann had already pointed out the bounded periodicity of hyperelliptic functions of two variables when both variables are linear functions of a third one ["De problemate quodam mechanico quod ad primam integralium ultraellipticorum classem revocatur," J. für Math. 56 (1850), pp. 46]

[^6]:    (*) The problem was reduced to quadratures by Jacobi: "De motu puncti singularis," Crelle's Journal 24 (1842), pp. 1. It is remarkable that the same problem is already found in Newton (Principia philosophiae naturalis mathematica, Lib. 1, Sec. 19, 1686.). The motion of a point on a surface could be treated for the first time there.
    (*) $^{* *}$ "Ueber verzweigte Bewegungen," Sitzungsberichte der Dorpater Naturforscher-Gesellschaft, Dec. 1887 and "Ueber die Bewegung eines schweren Punktes auf einer Rotationsfläche," Acta mathematica 11 (1888). I would like to expressly emphasize that Staude succeeded in presenting the concept of branched motion by investigations that were similar to the ones in my Dissertation (1885), but without knowing of them.

