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Direction fields and teleparallelism in n-dimensional manifolds

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Introduction

1. The n-dimensional manifolds that will be considered in this paper will be closed and continuously differentiable (1). The question of whether a non-singular, continuous direction field exists on such a manifold is answered by the following well-known theorem (2):

Theorem A_1 . *A singularity-free, continuous direction field exists on the manifold* M^n *iff the Euler characteristic of* M^n *has the value* 0 *(§ 5, no. 2).*

Therefore, on the one hand, amongst all closed and orientable surfaces, the ones with the topological type of the torus are the only ones that admit the existence of a continuous direction field (3); on the other hand, one can endow any manifold of odd dimension – in particular, any three-dimensional manifold – with a continuous direction field (§ 6, no. 1).

However, since one would not expect that all manifolds of odd dimension behave precisely the same way in relation to the continuous direction fields that exist on them, the contradiction that was formulated just now (e.g., between n = 2 and n = 3) compels one to look for a refinement of the original question. The following question is closely related: Let an n-dimensional manifold M^n and a number m from the sequence 1, 2, …, n be given. *Is there a system of m direction fields on* M^n *that are linearly independent at every point of* M^n*?*

This question, which is answered by Theorem A_1 for m = 1, and which commands special and self-evident interest for $m = n - 1$ and $m = n$ (cf., no. 5 of this introduction), defines the subject of the present paper. Indeed, the question will not be answered completely, in the sense of presenting the generalization of Theorem A_1 to a necessary and sufficient condition for the existence of a system of m independent direction fields – in the sequel, referred to briefly as an "m-field." Rather, some theorems will be proved that, on the one hand, serve to resolve the problem in many special cases, and which, on the other hand, represent new contributions to the general topology of closed manifolds.

(1) Cf., chap. XIV, § 4 of *Topologie* (v. 1) of Alexandroff and Hopf (J. Springer, Berlin, 1935). This book, whose terminology we will follow in this paper, will be briefly referred to as "AH" in the sequel.
(2) AH: chap. XIV, § 4, Theorem III.
(3) Poincaré, Journal de Liouville (4) **I**, pp. 203-208.

2. Before we formulate the most important theorem, we recall a theorem that is related to Theorem A_1 and is likewise well-known ([4]):

Theorem B_1 . *There exists a direction field on any manifold* M^n *that is singular (i.e., discontinuous) at no more than finitely many points. The number of these singularities, when counted with the correct multiplicities ("indices"), is independent of the particular field: It is always equal to the characteristic of* M^n *(§ 5, no. 2).*

We shall prove the following generalization of this theorem:

Theorem B_m . *For any* m $(1 \leq m \leq n)$, *there exist m-fields on any* M^n *whose singularities (i.e., points of discontinuity for the individual direction fields or points of linear dependency for the various fields) define a complex of dimension at most* $m - 1$. *With a correct enumeration of the multiplicities of the singularities, it is a cycle, and the homology class of this cycle is independent of the particular m-field: It is a distinguished element of the* $(m - 1)^{th}$ *Betti group* ([4a]) *of* M^n *(§ 4, no. 4, 5).*

We shall call this homology class F^{m-1} the "*m^{th} characteristic class*" of M^n. In the case of $m = 1$, it is the zero-dimensional homology class that consists of a point of M^n, multiplied by the Euler characteristic.

Theorem A_1 will now be generalized, in a certain sense, by way of the following theorem:

Theorem A_m . *There exists an m-field on* M^n *whose singularities define a complex of dimension at most* $m - 2$ *iff* $F^{m-1} = 0$ *(i.e., the zero element of the* $(m - 1)^{th}$ *Betti group of* M^n*) (§ 4, no. 5).*

It follows from this immediately that:

Theorem A'_m. *In order for a singularity-free m-field to exist on* M^n, *it is necessary that:*

$$F^0 = F^1 = \ldots = F^{m-1} = 0.$$

However, this condition might not be sufficient.

3. This suggests the problem of determining the characteristic classes F^{m-1} ($m = 1, 2, \ldots$) for a given M^n. In the case $m = 1$, the determination of F^{m-1} is equivalent to the determination of the Euler characteristic of M^n, and on the basis of the Euler-Poincaré formula:

$$\sum (-1)^r a^r = \sum (-1)^r p^r,$$

([4]) AH: chap. XIV, § 4, Theorem I.

([4a]) The coefficient domain to which these Betti groups relates is defined in § 4, no. 3 (cf., also AH: chap. V).

in which the a^r refer to the numbers of r-dimensional cells in a decomposition of M^n and p^r means the r^{th} Betti number of M^n, one can express it in two different ways: namely, in terms of the a^r and in terms of the p^r.

The first of these two possibilities seems to be capable of being carried over to an arbitrary m (§ 5, no. 3, footnote 22); however, the more important question is whether one can also represent the class F^{m-1} in a way that corresponds to the representation of the characteristic on the right-hand side of the Euler-Poincaré formula, and thus in terms of known topological invariants of M^n. Moreover, if the answer to this question, which was unknown to us up till now, is in the negative then that would teach us something new: F^{m-1} would be a *new topological invariant* of a manifold.

There exists yet another relationship between the class F^{m-1} and the Euler characteristic, in another regard: The intersection number of F^{m-1} with an $(n - m + 1)$-dimensional manifold that is embedded in M^n is congruent (mod 2) to the characteristic of that manifold, as long as the embedding fulfills certain requirements that are formulated in § 6, no. 2.

4. The determination of F^{m-1} for a given manifold is achieved in some cases with the help of Theorem B_m alone; on the basis of that theorem, one indeed needs to construct only a *special m*-field that is constructed so neatly that one can specify the complex by means of its singularities. In this way, we will treat the $(4k + 1)$-dimensional projective spaces as an example; it will be shown that:

Theorem *C. For the (4k + 1)-dimensional real projective space P^{4k+1}, F^1 is the class that contains the projective line, so it is therefore non-zero (§ 6, no. 3).*

This theorem, as well as in the fact that there is a continuous direction field on any odd-dimensional manifold, includes the fact that:

Theorem *C′. There is a continuous direction field on P^{4k+1}, so for any pair of fields there exist points at which the directions of the two fields are either equal or opposite.*

This property of projective spaces allows one to prove certain algebraic theorems whose proofs seem to be unknown, up to now, when one works with the usual algebraic lemmas (§ 6, no. 3).

5. The question of whether an n-field exists on an M^n deserves a special and self-evident interest; namely, the existence of such a field is equivalent to the idea that one can introduce a *teleparallelism* on M^n, or, as we also say, that M^n is "*parallelizable.*" Therefore, we call M^n parallelizable when one can decompose the totality of *all* directions in M^n into mutually disjoint, single-valued, and continuous direction fields that we call "parallel fields," such that the following condition is fulfilled: If $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_k$ are directions at a point p of M^n and $\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_k$ are the same directions at another arbitrary point p', as deduced from some parallel fields, then the linear independence of the $\mathfrak{v}'_i$

follows from the linear independence of the $\mathfrak{v}_i$. We will briefly call directions "parallel" when they are taken from the same parallel field.

In fact, one easily sees that parallelizability is identical to the existence of an n-field: If an n-field exists then one calls two directions $\mathfrak{v}$, $\mathfrak{v}'$ at the points p and p', resp., "parallel" in the event that their components relative to the directions of the n-field at p and p', resp., agree with each other, up to a positive factor; one has then introduced a teleparallelism. On the other hand, if a teleparallelism is defined then one distinguishes n linearly-independent directions at a fixed point; the directions that are parallel to these directions at the remaining points of M^n then define an n-field.

Non-orientable manifolds are not parallelizable. On the other hand, one easily shows that the existence of an n-field on an orientable manifold already follows from the existence of an $(n - 1)$-field. With that, the examination of parallelizability is completely converted into the examination of $(n - 1)$-fields. It is therefore no restriction when we assume that $m < n$ in what follows. Theorem A'_m yields:

Theorem *D*. *The vanishing of all characteristic classes $F^0, F^1, \ldots, F^{n-2}$ is necessary for the parallelizability of M^n.*

Here, as well, – confer Theorem A'_m – one should not assume that the condition is sufficient.

Since a *group manifold* ([5]) is certainly parallelizable, Theorem *D* yields a necessary condition for a given manifold M^n to be able to be made into a *group space*.

6. All manifolds for which the Euler characteristic is non-zero are certainly non-parallelizable – like, e.g., the spheres of even dimension – so one indeed also has $F^0 \neq 0$; neither are the projective spaces of dimension $4k + 1$ that were mentioned in Theorem *C*. By a product construction, one can further prove:

Theorem *E*. *For any dimension n that is different from 1 and 3, there are n-dimensional (closed and orientable) manifolds that are non-parallelizable (§ 6, no. 2).*

For $n = 1$, there is a single closed manifold, namely, the circle; it is trivially parallelizable. The question of parallelizability is then first open only for $n = 3$, and there one has:

Theorem *F*. *Any three-dimensional closed and orientable manifold* ([5a]) *is parallelizable (§ 5, no. 3).*

This remarkable special position of dimension three once again points to the difficulty in the search for a classification of three-dimensional manifolds; the attempt to divide the orientable three-dimensional manifolds into parallelizable and non-parallelizable ones would then fail.

([5]) AH: Introduction, § 3, no. 17; there, you will also find references.

([5a]) In addition, the manifold must fulfill certain differentiability assumptions (cf., § 5 and Appendix I).

7. The theorems that were stated in this introduction will be formulated and proved in §§ 4-6; §§ 1 and 2 have a preparatory character. In § 1, only the definition in no. 1 and the results of no. 4 are important for the remaining part of the paper. In Appendix I, the determination of the class F^1 for three-dimensional, orientable manifolds will be discussed in detail that was only suggested in § 5, no. 3. Appendix II subsequently arises; in it, it will be proved that a manifold with an odd characteristic that lies in Euclidian space cannot be represented by regular equations ([6]).

I have already reported on the individual partial results of this paper in other places (Verh. der schw. naturf. Gesellschaft, 1934, pp. 270; furthermore, Enseignement mathématique, 1934, 1, pp. 6).

At this point, I would like to thank Herrn Prof. H. Hopf for the impetus to do this work and for his enduring interest in its progress, as well as for his worthwhile advice at decisive moments.

§ 1. The manifolds $V_{n,m}$.

1. Definitions. In the sequel, we shall call an ordered, normalized orthogonal system $\sigma_{n,m}$ of m vectors $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_m$ that contact a point in n-dimensional Euclidian space R^n an *m-system in* R^n. In this, let m be constrained by the inequalities:

$$0 < m < n. \tag{1}$$

$V_{n,m}$ is defined to be the set of all m-systems $\sigma_{n,m}$ at a fixed point of R^n. If one introduces a notion of neighborhood into this set in a natural way then $V_{n,m}$ becomes a topological space whose points v are the m-systems $\sigma_{n,m}$.

$V_{n,1}$ is homeomorphic to the $(n-1)$-dimensional sphere S^{n-1} that it traced out by the endpoints of the vector $\mathfrak{v}_1$. However, if $m > 1$ then we displace the vectors $\mathfrak{v}_2, \ldots, \mathfrak{v}_m$ of $\sigma_{n,m}$ parallel to the endpoint of the vector $\mathfrak{v}_1$. Therefore, $V_{n,m}$ can also be described as the set of all $(m-1)$-systems in R^n that are tangential to S^{n-1}. In particular, $V_{n,2}$ is the set of directed line elements on S^{n-1}.

One can arrive at another representation of the space $V_{n,m}$ by *stereographic projection*, which we will briefly denote by V in what follows: If one projects S^{n-1} from its North Pole onto its equatorial space R^{n-1} then a system $\sigma_{n,m-1}$ that contacts the sphere at a point p goes to an $(m-1)$-system $\sigma_{n-1,m-1}$ in R^{n-1} that contacts the image point p_1 to p. $\sigma_{n-1,m-1}$ is established uniquely by its contact point p_1 and the $(m-1)$-system that is parallel to $\sigma_{n-1,m-1}$ of a $V_{n-1,m-1} = V_1'$ that is embedded in R^{n-1}. A point v of V is thus given by a point p_1 of R^{n-1} and a point v_1 of V_1'. We briefly write:

$$v = p_1 \times v_1 . \tag{2}$$

([6]) One can also confer AH: Introduction, § 1, no. 7.

This representation breaks down only for those systems $\sigma_{n,\,m-1}$ that contact the North Pole. In order to also treat these systems, we project S^{n-1} onto R^{n-1} from the South Pole. Analogous to (2), one gets:

$$v = p_2 \times v_2\,. \tag{2}$$

v_2 is a point of the set V_2' that features in place of V_1' under the second projection. If we denote the equatorial sphere of S^{n-1} by S^{n-2} then the two points p_1 and p_2 go to each other under the transformation by means of reciprocal radii in S^{n-2}.

Formula (2) describes a relationship between V and V_1'; i.e., between $V_{n,\,m}$ and $V_{n-1,m-1}$. By iteration, we obtain a relation between spaces of the sequence:

$$V_{n,\,m}\,,\ V_{n-1,\,m-1}\,,\ \ldots,\ V_{n-k,\,m-k},\ \ldots,\ V_{n-m+1,\,1} = S^{n-m}. \tag{4}$$

One can infer the following conclusions from this:

I. Any point of $V_{n,\,m}$ possesses a neighborhood that is homeomorphic to the interior of a Euclidian ball.

II. $V_{n,\,m}$ is connected. (Due to (1), S^{n-m} is connected.)

III. One has the recursion formula for the dimension $\mu_{n,\,m}$ of $V_{n,\,m}$:

$$\mu_{n,\,m} = \mu_{n-1,\,m-1} + (n-1), \tag{5}$$

so

$$\mu_{n,\,m} = m \cdot \left(n - \frac{m+1}{2}\right). \tag{6}$$

2. Decomposition of $V_{n,\,m}$. For our first projection, S^{n-2} bounds the closed ball E_1 in R^{n-1}. We define:

$$K_1 = E_1 \times V_1'\,. \tag{7}$$

Analogously, for the second projection, one has:

$$K_2 = E_2 \times V_2'\,. \tag{8}$$

V is then the set union of K_1 and K_2:

$$V = K_1 + K_2\,. \tag{9}$$

If one iterates this decomposition of $V_{n,\,m}$ for the sequence (4) then it follows inductively that:

VI. $V_{n,\,m}$ is a polyhedron.

It now follows from I-IV that:

Theorem 1. *$V_{n,m}$ is a closed manifold.*

We call the manifolds of the sequence (4) the *manifolds that are associated with $V_{n,m}$*. For the intersection of K_1 and K_2, one gets:

$$\text{For the first projection:} \qquad K_1 \cdot K_2 = S^{n-2} \times V_1', \tag{10}$$

$$\text{For the second projection:} \qquad K_1 \cdot K_2 = S^{n-2} \times V_2'. \tag{11}$$

We would like to derive the properties of the Betti groups of V from our decomposition (9) of the manifold V by induction on the sequence of associated manifolds. For $r > 0$, we understand $B^r(K)$ to mean the r-dimensional Betti group of the complex K, while for $r = 0$, it is the group of 0-dimensional integer homology classes that contain only reducible cycles. (A 0-dimensional cycle is reducible when the sum of its coefficients vanishes ([7])). We call algebraic subcomplexes of:

$$\begin{array}{cccccc} V = K_1 + K_2, & K_1, & K_2, & K_1 \cdot K_2, & V_1', & V_2' \\ C, & C_1, & C_2, & C_{12}, & C_1', & C_2', \text{ resp.} \end{array}$$

Cycles will always be denoted by z or Z.

We now make the following basic assumption:

$$\text{let } B^r(V_{n-1,\,m-1}) = 0 \text{ for a fixed } r \text{ with } 0 \leq r < n-2. \tag{$\mathbf{J}_1$}$$

One then has ([8]), for an arbitrary $(r + 1)$-dimensional sub-cycle z^{r+1} of $V_{n,m}$:

$$z^{r+1} = z_1^{r+1} + z_2^{r+1}. \tag{12}$$

(z_1^{r+1} is a sub-cycle of K_1 and z_2^{r+1} is a sub-cycle of K_2 .)

Proof: It follows from ($\mathbf{J}_1$) that $B^r(V_1') = 0$, so one also has ([9]) $B^r(S^{n-2} \times V_1') = 0$; it then follows from (10) that:

$$B^r(K_1 \cdot K_2) = 0. \tag{13}$$

Now let $z^{r+1} = C_1 - C_2$ be any decomposition of z^{r+1} into two algebraic $(r + 1)$-dimensional sub-complexes of K_1 and K_2 . Taking the boundary yields $\dot{C}_1 = \dot{C}_2$; this common boundary lies in K_1, as well as in K_2 , so it is a z_{12}^r. It follows from (13) that z_{12}^r

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([7]) AH: chap. IV, § 4, no. 7, and furthermore, chap. V, § 1, no. 5.

([8]) This theorem is a special case of an addition theorem in combinatorial topology; cf., AH: chap. VII, § 2, especially no. 5.

([9]) For Betti groups of product complexes, see AH: chap. VII, § 3.

$= \dot{C}_{12} \cdot C_1 - C_{12}$ and $C_2 - C_{12}$ are cycles z_1, z_2, resp., and one has $z^{r+1} = z_1 - z_2$, with which (12) is proved.

Under the sharper assumption:

$$\text{Let } B^r(V_{n-1,\,m-1}) = 0 \text{ for a fixed } r \text{ with } 0 \leq r < n-3, \tag{$\mathbf{J}_2$}$$

one then obtains the isomorphism:

$$B^r(V_{n,\,m}) \approx B^r(V_{n-1,\,m-1}). \tag{14}$$

Proof: From the theorem on the Betti groups of product complexes, it follows that:

$$B^r(K_1) = B^{r+1}(E_1 \times V_1') = B^{r+1}(E_1 \times V_{n-1,\,m-1}) \approx B^{r+1}(V_{n-1,\,m-1}).$$

Analogously, one obtains, with consideration of the fact that $r + 1 < n - 2$:

$$B^{r+1}(K_1 \cdot K_2) = B^{r+1}(S^{n-2} \times V_1') = B^{r+1}(S^{n-2} \times V_{n-1,\,m-1}) \approx B^{r+1}(V_{n-1,\,m-1}), \tag{15}$$

and therefore:

$$B^{r+1}(K_1 \cdot K_2) \approx B^{r+1}(K_1).$$

This isomorphism can be realized if one associates a homology class of $K_1 \cdot K_2$, whose representative cycle is z_{12}^{r+1}, with the homology class of z_{12}^{r+1} in K_1. From that, we infer the following conclusions:

a) A cycle of $K_1 \cdot K_2$ is contained in any $(r + 1)$-dimensional homology class of K_1 (or K_2).

b) From the homology $z_{12}^{r+1} \sim 0$ in K_1 (or K_2), it follows that:

$$z_{12}^{r+1} \sim 0 \text{ in } K_1 \cdot K_2 .$$

If one associates a homology class of $K_1 \cdot K_2$, whose representative cycle is Z_{12}^{r+1}, with the homology class of Z_{12}^{r+1} in $K_1 + K_2$ then a homomorphic map of $B^{r+1}(K_1 \cdot K_2)$ into $B^{r+1}(K_1 + K_2)$ comes about. This map is an isomorphism, in the event that:

1. A cycle of $K_1 \cdot K_2$ is contained in any $(r + 1)$-dimensional homology class of $K_1 + K_2$.

2. The homology $Z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$ follows from the homology $Z_{12}^{r+1} \sim 0$ in $K_1 + K_2$.

1. follows from (12) and a).

2. is verified in the following way:
$Z_{12}^{r+1} \sim 0$ in $K_1 + K_2$ means that $Z_{12}^{r+1} = \dot{C}$. A decomposition $C = C_1 - C_2$ of C gives $Z_{12}^{r+1} = \dot{C}_1 - \dot{C}_2$. This possible only when $\dot{C}_1 = z_{12}^{r+1}$ and $\dot{C}_2 = \overline{z}_{12}^{r+1}$. Since $z_{12}^{r+1} \sim 0$ in K_1, one gets from b) that $z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$, and likewise $\overline{z}_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$, and therefore also $Z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$. With that, we have $B^{r+1}(K_1 + K_2) \approx B^{r+1}(K_1 \cdot K_2)$.

Our proof then gives:

Lemma. *Under the assumption* ($\mathbf{J}_2$), *an (r + 1)-dimensional homology basis for* $K_1 \cdot K_2$ *is also a homology basis for* $V = K_1 + K_2$.

The following theorem can now be proved easily:

Theorem 2. *For* $0 \leq r < n - m - 1$, one has $B^r(V_{n,m}) = 0$.

The proof proceeds by complete induction on the sequence of associated manifolds; thus, let it be already proved that:

$$B^{r+1}(V_{n-1,\,m-1}) = 0 \qquad \text{for } 0 \leq r < n - m - 1.$$

It further follows from Theorem 1 that $B^0(V_{n-1,\,m-1}) = 0$, so one also has $B^r(V_{n-1,\,m-1}) = 0$. Since $m > 1$ was assumed, ($\mathbf{J}_2$) is true, and therefore (14), and therefore Theorem 2. The induction will be anchored on the manifold $V_{n-m+1,\,1} = S^{n-m}$, for which Theorem 2 is trivial.

Theorem 3. *For* $m > 2$, *one has* $B^{n-m}(V_{n,m}) \approx B^{n-m}(V_{n-1,\,m-1})$.

Proof: From Theorem 2, ($\mathbf{J}_2$) is true for $r = n - m - 1$. (14) then gives the assertion.

3. Topology of $V_{n,2}$. $B^{n-m}(V_{n,\,m})$ can be determined from Theorem 3 when $B^{n-m}(V_{n-m+2,\,2})$ is known; therefore, the $(n-2)$-dimensional Betti group of a manifold $V_{n,\,2}$ shall be calculated in this section. The sequence of associated manifolds consists of only an $(n - 2)$-dimensional sphere in this case. We use our first projection for the representation of $V_{n,\,2}$; V_1' is then a sphere $S_1'^{n-2}$. Let the two spheres S^{n-2} and $S_1'^{n-2}$ be equally oriented, so we also denote the cycles that are provided by these orientations by S^{n-2} and $S_1'^{n-2}$. If s is an arbitrary, but chosen once and for all, point of S^{n-2}, and s_1' is a point of $S_1'^{n-2}$ then, from (10), the two cycles $z_{12} = s \times S_1'^{n-2}$ and $S^{n-2} \times s_1'$ define an $(n-2)$-dimensional homology basis for $K_1 \cdot K_2$. (The case of $n = 3$ is represented in Fig. 1) Any $(n-2)$-dimensional cycle Z_{12} of $K_1 \cdot K_2$ thus satisfies a homology:

$$Z_{12} \sim \alpha\, z_{12} + \beta\, \overline{z}_{12} \qquad \text{in } K_1 \cdot K_2, \tag{17}$$

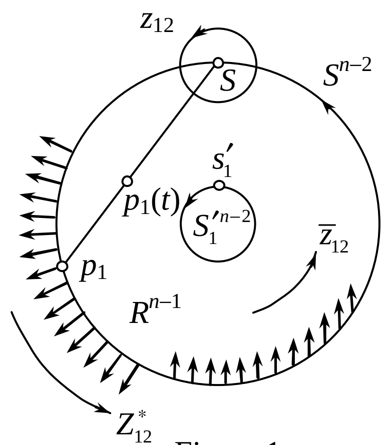

Figure 1.

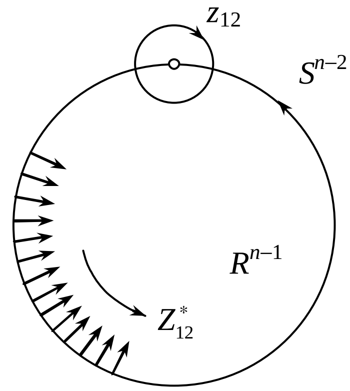

Figure 2.

where α and β are well-defined numbers. We now pose the problem of determining the homologies (17) that Z_{12} fulfills in K_1 or K_2 . We first solve this problem for a special cycle Z^*_{12} that is defined in the first projection as the field of exterior normal vectors on S^{n-2}. For this cycle, (17) reads:

$$Z^*_{12} \sim z_{12} + \overline{z}_{12} \qquad \text{in } K_1 \cdot K_2 . \tag{17*}$$

Proof: Z^*_{12} fulfills a homology:

$$Z^*_{12} \sim \alpha^* z_{12} + \beta^* \overline{z}_{12} \quad \text{in } K_1 \cdot K_2 . \tag{17**}$$

The determination of the unknowns α^* and β^* is achieved in the following way: One associates a point $p_1 \times v_1$ of $K_1 \cdot K_2$ [see (2)] with the point v_1 of $V_1' = S_1'^{n-2}$; this continuous map f of $K_1 \cdot K_2$ into $S_1'^{n-2}$ induces a homomorphic map of the Betti groups of $K_1 \cdot K_2$ into the Betti groups of $S_1'^{n-2}$ that transforms (17**) into the homology $f(Z^*_{12}) \sim \alpha^* \cdot f(z_{12}) + \beta^* \cdot f(\overline{z}_{12}) = \alpha^* \cdot S_1'^{n-2}$. The fact that $f(Z^*_{12}) \sim S_1'^{n-2}$ yields $\alpha^* = 1$; one finds that $\beta^* = 1$ in an analogous way.

Relative to K_1, Z^*_{12} fulfills the homology:

$$Z^*_{12} \sim z_{12} \qquad \text{in } K_1 . \tag{18*}$$

The proof is by continuous variation of Z^*_{12}: One lets an arbitrary point $p_1 \times v_1$ of Z^*_{12} run through the path that is suggested by the following schema:

$$p_1 \times v_1 , \qquad p_1(t) \times v_1 , \qquad s \times v_1 . \tag{D}$$

In this, t is a deformation parameter that ranges from 0 to 1; $p_1(t)$ moves uniformly and rectilinearly from p_1 to the point s of S^{n-2}.

By performing the transformation through reciprocal radii on S^{n-2}, Figure 1 becomes Figure 2, where one finds, in an analogous way:

$$Z_{12}^{*} \sim (-1)^n \cdot z_{12} \qquad \text{in } K_2 . \tag{19*}$$

For the arbitrary cycle Z_{12} , we now have, from (17) and (17*), that $Z_{12} \sim (\alpha - \beta) \cdot z_{12} + \beta \cdot Z_{12}^{*}$ in $K_1 \cdot K_2$, and thus also in K_1 ; it then follows from (18*) that $Z_{12} \sim \alpha \cdot z_{12}$ in K_1 . Analogously, with the use of (19*), one gets: $Z_{12} \sim [\alpha - \beta + (-1)^n \cdot \beta] \cdot z_{12}$ in K_2 . This then yields the following solution to our problem:

From the fact that:

$$Z_{12} \sim \alpha z_{12} + \beta \bar{z}_{12} \qquad \text{in } K_1 \cdot K_2 ,$$

it follows that:

$$Z_{12} \sim \alpha z_{12} \qquad \text{in } K_1 \tag{18}$$

and

$$Z_{12} \sim [\alpha - \beta + (-1)^n \beta]\, z_{12} \text{ in } K_2. \tag{19}$$

We now infer some consequences from these formulas:

Theorem 4. *The* (n – 2)-*dimensional Betti group of* $V_{n,2}$ *is cyclic and has order* 0 *for even n and order* 2 *for odd n*.

In this, we understand a cyclic group of order 0 to mean a free cyclic group.

Proof: From (7), our cycle z_{12} defines an (n – 2)-dimensional homology basis in K_1 ; however, since K_1 and K_2 are mapped to each other topologically by our transformation through reciprocal radii, z_{12} is also a homology basis for K_2 . Furthermore, from (12) [the assumption (**J**$_1$) is fulfilled for $r = n - 3$], any (n – 2)-dimensional cycle of $V_{n,2}$ can be written as the sum of a cycle in K_1 and a cycle in K_2 . From these facts, it follows that the homology class of z_{12} in $V_{n,2}$ generates the group $B^{n-2}(V_{n,\,2})$, so that group is cyclic; in order to establish its order, we must determine the order of z_{12} . Thus, let, say, $\gamma \cdot z_{12} \sim 0$ in $V_{n,\,2}$ – i.e., $\gamma \cdot z_{12} = \dot{C}$. A decomposition $C = C_1 + C_2$ of C then gives $\gamma \cdot z_{12} = \dot{C}_1 + \dot{C}_2$. This is possible only for $\dot{C}_1 = Z_{12}$ and $\dot{C}_2 = \bar{Z}_{12}$. We then find that:

$$\gamma \cdot z_{12} = Z_{12} + \bar{Z}_{12} \qquad \text{with } Z_{12} \sim 0 \text{ in } K_1 \text{ and } \bar{Z}_{12} \sim 0 \text{ in } K_2 . \tag{20}$$

If we assume that n is perhaps odd then it follows from $Z_{12} \sim 0$ in K_1, by means of (18), that $Z_{12} \sim \beta \cdot \bar{z}_{12}$ in $K_1 \cdot K_2$. By substituting this into (20), we find the homology $\gamma \cdot z_{12} \sim 2\bar{\beta} \cdot z_{12} + (\beta + \bar{\beta}) \cdot \bar{z}_{12}$ in $K_1 \cdot K_2$. This homology is possible only for $\gamma = 2 \cdot \bar{\beta}$; it then follows that γ is even from the fact that $\gamma z_{12} \sim 0$ in $V_{n,2}$. The order of z_{12} is then at least 2; the fact that it is exactly 2 follows from a consideration of $-\bar{z}_{12}$. Namely, from (18), one has $-\bar{z}_{12} \sim 0$ in $V_{n,2}$, and from (19), $-\bar{z}_{12} \sim 2 \cdot z_{12}$ in $V_{n,2}$. One then has, in fact, that $2z_{12} \sim 0$ in $V_{n,2}$. Since the case of even n can be examined analogously, Theorem 4 is proved.

It is likewise shown that z_{12} is a basis cycle for the group $B^{n-2}(V_{n,2})$. (This will be important later.) We shall then given a definition of z_{12} that is independent of the decomposition of $V_{n,2}$. To this end, one considers all 2-systems $\sigma_{n,2}$ of $V_{n,2}$ (no. 1) that coincide in their first vector. The endpoints of the second vectors of this system will run through an $(n-2)$-dimensional sphere, which we think of as oriented. The system $\sigma_{n,2}$ then defines an $(n-2)$-dimensional cycle that call $z_{n,2}$. It is clear that $z_{n,2}$ can be identified with z_{12}; we then find the following:

Lemma: *The cycle $z_{n,2}$ is the basis element for the $(n-2)$-dimensional Betti group of $V_{n,2}$.*

The manifold $V_{n,2}$ is orientable. We will prove this later. From Theorems 2 and 4, one can then determine all Betti groups of $V_{n,2}$ with the help of the Poincaré duality theorem. One then obtains the following result:

Theorem 5. *For even n, the non-zero Betti numbers of $V_{n,2}$ are: $p^0 = p^{n-2} = p^{n-1} = p^{2n-3} = 1$; no torsion is present. For odd n, one also has $p^{n-2} = p^{n-1} = 0$, but an $(n-2)$-dimensional torsion of order 2 also enters in.*

Furthermore, the relations (18) and (19) allow us to determine the continuous maps of an at most $(n-2)$-dimensional sphere into $V_{n,2}$. One has, in fact:

Theorem 6. *Two continuous maps of an at most $(n-2)$-dimensional sphere into $V_{n,2}$ are homotopic if they have the same homology type* ([10]) ([10a]).

We preface the proof with some preliminary considerations. Let, perhaps, f be a given continuous map of the sphere S_0^r $(r \leq n-2)$ into $V_{n,2}$, and let v_0 be an arbitrary point of S_0^r. If, as in no. 1, we think of $V_{n,2}$ as the set of all vectors in R^n that are tangent to S^{n-1} then we can assume for all homotopy investigations that the image vector of point v_0 does not contact S^{n-1} at the North Pole. (If this were not true then, since $r < n-1$, one could always make it so by a continuous change in f.) No image vectors are then lost under the transition to our first projection, and one has, from (2), that $f(v_0) = p_1 \times v_1$. Furthermore, one can actually assume that only the points $s \times v_1$ (see Fig. 1) can appear as image points. (In fact, the continuous map $v_0 \to p_1 \times v_1$ can be changed into a map that has the desired property by the deformation process (**D**) (beginning of this no.)) We then assume that:

$$f(v_0) = s \times v_1. \tag{21}$$

We call the map $\varphi(v_0) = v_1$ of S_0^r into the associated manifold $S_1'^{n-2}$ to $V_{n,2}$ the *associated map* φ to the map f. Now, if $\bar{f}$ is a second map of S_0^r into $V_{n,2}$ and $\bar{\varphi}$ is its associated map then one has:

([10]) AH: chap. VIII, § 3.

([10a]) This theorem is a generalization of the theorem on the classification of sphere maps (AH: chap. XIII, § 2).

$$\text{The homotopy of } f \text{ and } \bar{f} \text{ follows from the homotopy of } \varphi \text{ and } \bar{\varphi}. \tag{22}$$

This follows simply from the fact that multiplication by the fixed point s is a topological map of $S_1'^{n-2}$ into $V_{n,2}$.

We now go on to the proof of Theorem 6. There are three cases to consider:

Case 1. $r < n - 2$. From Theorem 2, we must show that any map of S_0^r into $V_{n,2}$ is homotopic to zero, so the image of S_0^r can be contracted to a point. However, from (22), this is a consequence of the fact that since $r < n - 2$, the associated map is homotopic to zero.

Case 2. $r = n - 2$ and n is even. Let f and $\bar{f}$ be the two maps of which we spoke in Theorem 6. If we understand $S_0^r = S_0^{n-2}$ to also mean the cycle that this sphere represents with a chosen orientation then the assumption of Theorem 6 says that $f(S_0^{n-2}) \sim \bar{f}(S_0^{n-2})$ in $V_{n,2} = K_1 + K_2$. From (21), $f(S_0^{n-2})$ and $\bar{f}(S_0^{n-2})$ are cycles in $K_1 \cdot K_2$, so they fulfill the homologies (17): $f(S_0^{n-2}) \sim \alpha z_{12}$, $\bar{f}(S_0^{n-2}) \sim \bar{\alpha}\, z_{12}$ in $K_1 \cdot K_2$; one then has $\alpha \cdot z_{12} \sim \bar{\alpha}\, z_{12}$ in $V_{n,2}$. From Theorem 4, this is possible only if $\alpha = \bar{\alpha}$, and one finally gets that $f(S_0^{n-2}) \sim \bar{f}(S_0^{n-2})$ in $K_1 \cdot K_2$. We map this homology to $S_1'^{n-2}$ by assigning the point $p_1 \times v_1$ in $K_1 \cdot K_2$ to the point v_1. One thus finds that $\varphi(S_0^{n-2}) \sim \bar{\varphi}(S_0^{n-2})$ in $S_1'^{n-2}$. The two maps φ and $\bar{\varphi}$ of S_0^{n-2} into $S_1'^{n-2}$ thus have the same mapping degree, from which their homotopy follows. (22) concludes the proof.

Case 3. $r = n - 2$ and n is odd. Theorem 4 then gives only that $\alpha \equiv \bar{\alpha}$ (mod 2). Let $\bar{\alpha} = \alpha - 2k$, perhaps. The proof above will also work in this case if we can show that our map f with $f(S_0^{n-2}) \sim \alpha\, z_{12}$ in $K_1 \cdot K_2$ can be changed continuously into a map f_2 with $f_2(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in $K_1 \cdot K_2$ that satisfies the condition (21). To that end, let F be an arbitrary map of S_0^{n-2} into S^{n-2} of degree k. Next, f will be changed into a map f_1 according to the following schema:

$$f(v_0) = s \times v_1\,, \qquad F(v_0, 1 - t) \times v_1\,, \qquad F(v_0) \times v_1 = f_1(v_0).$$

$F(v_0, t)$ again moves uniformly and rectilinearly from $F(v_0)$ to s. The cycle $f_1(S_0^{n-2})$ again lies in $K_1 \cdot K_2$ and satisfies the homology $f_1(S_0^{n-2}) \sim \alpha\, z_{12} + k\, \bar{z}_{12}$ there, which one proves analogously to (17*). From (19), one has $f_1(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in K_2. One now goes to Figure 2 by means of the transformation through reciprocal radii, and changes f_1 there by the deformation process that is analogous to (**D**). The result is a map f_2 with $f_2(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in K_2 and $f_2(S_0^{n-2}) \sim \delta\, z_{12}$ in $K_1 \cdot K_2$ that satisfies the condition (21). As for the unknown δ, one easily finds from (19) that $\delta = \alpha - 2k$. With that, Theorem 6 is proved completely.

It then follows from Theorems 2 and 6 that:

Theorem 7. *For $n > 3$, the manifold $V_{n,2}$ is simply-connected, and thus orientable.*

As a non-simply-connected manifold, the manifold $V_{3,2}$ then occupies a special place in the $V_{n,2}$, which we will later (§ 5, no. 3) exploit in our investigation of the parallelizability of three-dimensional manifolds. We mention that $V_{3,2}$ is homeomorphic to the three-dimensional projective space P^3. To prove this, one observes that $V_{3,2}$, as the set of line elements on a two-dimensional sphere, is homeomorphic to the group of Euclidian rotations of that sphere. Such a rotation is, however, determined uniquely by four homogeneous parameters.

4. Topology of $V_{n,m}$. The union of the results of sections 2 and 3 allows the derivation of further topological properties of the $V_{n,m}$. One proves the following theorem by induction on the sequence (4) of associated manifolds – which is now, however, broken by the manifold $V_{n-m+2,\,2}$ – in which one always assumes that $m > 1$:

1. The Betti group $B^{n-m}(V_{n,m})$ is cyclic of order 0 for even $n - m$ and of order 2 for odd $n - m$.

The proof follows from Theorems 3 and 4. In order to find a basis cycle for $B^{n-m}(V_{n,m})$, one considers all m-systems $\sigma_{n,m}$ in $V_{n,m}$ (no. 1) whose first $(m - 1)$ vectors are given as fixed. The endpoints of the latter vectors of this system run through an $(n - m)$-dimensional sphere that we regard as being oriented. The systems $\sigma_{n,m}$ then define an $(n - m)$-dimensional cycle $z_{n,m}$.

2. $z_{n,m}$ is a basis cycle for $B^{n-m}(V_{n,m})$.

The proof follows from the two lemmas in no. 2 and no. 3.

3. Two continuous maps of an at most $(n - m)$-dimensional sphere into $V_{n,m}$ are homotopic when they have the same homology type.

To prove this, if f and $\bar{f}$ are two maps then one defines the associated maps φ and $\bar{\varphi}$ into $V_{n-1,m-1}$ in a manner that is analogous to no. 3. The homotopy of f and $\bar{f}$ then follows from the homotopy of the associated maps.

From 3, $V_{n,m}$ is simply-connected for $m < n - 1$, so it is also orientable. $V_{n,n-1}$ is homeomorphic to the group of Euclidian rotations of an $(n - 1)$-dimensional sphere and, as a group manifold, it is therefore orientable. For this manifold, one has, moreover:

4. The fundamental group of $V_{n,n-1}$ is a cyclic group of order 2 ($n > 2$).

The proof of this differs from that of 3 in only inessential ways. (In order to anchor the induction, one observes that 4. follows for $V_{3,2}$ from its homeomorphism with projective space.)

In conclusion, we would like to derive some properties of $V_{n,m}$ from these theorem that will be needed in what follows:

Theorem 8. *The continuous image of an at most* $(n - m - 1)$*-dimensional sphere in* $V_{n,m}$ *(m arbitrary) can be contracted to a point.*

Proof is from 3. and Theorem 2.

Theorem 9. *If f is a continuous map of an orientable sphere* S_0^{n-m} *into* $V_{n,m}$ *then one has the homology:*

$$f(S_0^{n-m}) \sim \alpha z_{n,m} \qquad \text{in } V_{n,m}.$$

If $n - m$ *is even or* $m = 1$ *then* α *is determined uniquely, and two maps with the same value of* α *are homotopic.*

However, if $n - m$ *is odd and* m *is different from* 1 *then* α *is determined only* (mod 2) ([11]); *two maps that are associated with values of* α *that are congruent* (mod 2) *are homotopic.*

§ 2. The open manifolds $V_{n,m}^*$.

1. Definitions. In this section, we would like to freely make the restriction to orthogonal and normalized m-systems. We define: An ordered system $\sigma_{n,m}^*$ of m linearly-independent vectors $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_m$ that contact a point of R^n is called an *affine m-system* in R^n. We now call the systems $\sigma_{n,m}$ of § 1 *orthogonal m-systems*, in order to distinguish them from the affine m-systems; m again fulfills the inequalities:

$$0 < m < n. \tag{1}$$

The set of all affine m-systems that contact R^n at a fixed point is called $V_{n,m}^*$. A system $\sigma_{n,m}^*$ is given by the $n \cdot m$ components of its vectors, so it can be regarded as a point in an $(n \cdot m)$-dimensional numerical space. In this way of looking at things, $V_{n,m}^*$ becomes a sub-domain of the numerical space, so it is an open manifold.

2. Retraction mapping. For any m-system of $V_{n,m}^*$, we replace the vector $\mathfrak{v}_i$ with the vector:

([11]) Therefore, we can assume in what follows that α has the value 0 or 1 in this case.

$$\mathfrak{v}_i' = \mathfrak{v}_i - (\mathfrak{v}_i \cdot \mathfrak{v}_j)\, \mathfrak{v}_j\,. \tag{2}$$

i and j are chosen to be fixed, but different from each other; $(\mathfrak{v}_i \cdot \mathfrak{v}_j)$ means the scalar product of $\mathfrak{v}_i$ and $\mathfrak{v}_j$. This produces a continuous map f of $V_{n,m}^*$ into itself; we denote the image set by $f(V_{n,m}^*)$. By considering the family of maps:

$$\mathfrak{v}_i'(t) = \mathfrak{v}_i - t\,(\mathfrak{v}_i \cdot \mathfrak{v}_j)\, \mathfrak{v}_j \qquad (0 \le t \le 1),$$

one recognizes that f is a deformation; i.e., it belongs to the class of the identity. If one replaces the vector $\mathfrak{v}_k$ in any system $\sigma_{n,m}^*$ in $V_{n,m}^*$ for a definite value of k with the vector:

$$\mathfrak{v}_k' = \frac{\mathfrak{v}_k}{|\mathfrak{v}_k|} \tag{3}$$

then this gives another continuous map g of $V_{n,m}^*$ into itself. g is also a deformation, as the family of maps:

$$\mathfrak{v}_k'(t) = [t + (1-t)\cdot|\mathfrak{v}_k|]\frac{\mathfrak{v}_k}{|\mathfrak{v}_k|} \qquad (0 \le t \le 1)$$

yields. The two maps f and g leave the manifold $V_{n,m}$ invariant, which is indeed a subset of $V_{n,m}^*$.

One can once more perform a deformation of type (2) [(3), resp.] with $f(V_{n,m}^*)$ [$g(V_{n,m}^*)$, resp.], and ultimately construct a deformation that maps $V_{n,m}^*$ onto $V_{n,m}$ continuously by composing finitely many deformations of this type. This follows from the well-known fact of analytic geometry that any affine system $\sigma_{n,m}^*$ in $V_{n,m}^*$ can be orthogonalized by finitely many steps of type (2) and (3). We call the deformation F the *retraction mapping* ([12]) of $V_{n,m}^*$ onto $V_{n,m}$.

3. Topology of $V_{n,m}^*$. With the help of our retraction, we can now carry over the results of § 1, no. 4 to the open manifold $V_{n,m}^*$:

Theorem 10. $V_{n,m}^*$ *is completely homology-equivalent to* $V_{n,m}$ *; i.e., one has for an arbitrary r:* $B^r(V_{n,m}^*) \approx B^r(V_{n,m})$; *furthermore, all of the results that were proved for* $V_{n,m}$ *in § 1, no. 4 are also true for the manifold* $V_{n,m}^*$.

([12]) This concept goes back to K. Borsuk; cf., AH: chap. VIII, § 6.

Proof: The retraction map F induces a map of $B^r(V^*_{n,m})$ to $B^r(V_{n,m})$. In order to prove that this homomorphism is an isomorphism, it suffices (since any r-dimensional homology class of $V_{n,m}$ appears trivially as an image class) to show that its kernel consists of only the zero class. Therefore, let, say, z^r be a cycle of $V^*_{n,m}$ and $F(z^r) \sim 0$ in $V_{n,m}$, hence, also in $V^*_{n,m}$. Since $F(z^r)$ goes to z^r under deformation, one has $F(z^r) \sim z^r$ in $V^*_{n,m}$, so, in fact, $z^r \sim 0$ in $V^*_{n,m}$.

The second assertion of Theorem 10 can now be proved easily with the help of our retraction.

Remark. All *positively-oriented* n-systems that contact a fixed point of R^n define a manifold that is homeomorphic to the group A_n of all proper affine maps of R^n. From our analysis, it easily follows that A_n is completely homology-equivalent to $V_{n,\,n-1}$ and that the fundamental group of A_n is a cyclic group of order 2 for $n > 2$.

§ 3. Vector fields in Euclidian space. Characteristic.

1. Characteristic of an *m*-field on a sphere. In this section, we understand E^{r+1} to mean an $(r + 1)$-dimensional curved cell that is embedded in the Euclidian space R^n and S^r to mean the boundary sphere of E^{r+1}. If we denote a point of S^r by p then we can establish the points of the cell E^{r+1} by means of a polar coordinate system ρ, p. (ρ is a number that runs from 0 to 1, the point $(0, p)$ is the origin of the coordinate system, and $(1, p)$ is identical with p.)

If an *affine* m-system $\sigma(p)$ of R^n is attached to every point of S^r then we speak of an *m-field* $\mathfrak{F}$ on S^r. The examination of this field is the objective of this paragraph. To that end, we choose a set of vectors $V^*_{n,m}$ that is embedded in R^n and associate the point p of S^r with the m-system of $V^*_{n,m}$ that is parallel to $\sigma(p)$. A map f of the sphere S^r into the manifold $V^*_{n,m}$ is given by this association that we call a *mapping by parallel m-systems.* We further call the field $\mathfrak{F}$ *continuous* when f is continuous; this will always be assumed in what follows. We define a continuous field on the cell E^{r+1} and the associated mapping by parallel m-systems in an analogous way.

Now, this immediately suggests the question: Under what conditions can a continuous field $\sigma(p)$ that is given on S^r be extended to a continuous field $\sigma(\rho, p)$ on E^{r+1}? [i.e., $\sigma(1, p) = \sigma(p)$.] If the dimension r of our sphere is less than $n - m$ then Theorem 8 (10) shows that this process is always possible. In fact, if $f(S^r)$ is then homotopic to zero in $V^*_{n,m}$ then ([13]) f can be extended to a continuous map of E^{r+1} into $V^*_{n,m}$. However, if $r = n - m$ then the sphere is $(n - m)$-dimensional (and oriented), and it follows from Theorem 9 (10) that the desired process is possible iff the number α that is associated with our map f by parallel m-systems vanishes.

([13]) AH: chap. XIII, § 1, Lemma II.

This number α is called the *characteristic* of the m-field $\mathfrak{F}$ on the sphere $S^r = S^{n-m}$. One then finds that:

Theorem 11. *A continuous m-field that is given on the boundary of a cell can be continuously extended into its interior:*

a) *If the dimension of the sphere is less than n – m.*
b) *If the sphere is* (n – m)*-dimensional and the characteristic of the field on it is* 0.

Extension through central projection:

A boundary field can always be extended into the interior of the cell E^{r+1} by the definition: "$\sigma(\rho, p)$ is parallel to $\sigma(p)$." We call this process *extension through central projection* from the point (0, p). However, the continuity of the extended field will then generally break down at the center of projection. Moreover, if an arbitrary, not-necessarily-continuous m-field is given on the boundary sphere S^r, and we denote the set of its discontinuities by M, then the field that is extended by central projection into the cell E^{r+1} is discontinuous at all points of the cone over M with the center of projection for its vertex.

2. Remarks on the calculation of the characteristic. In many cases, it proves to be useful to calculate the characteristic in some other way than by means of the mapping by parallel m-systems: Let a continuous field $\mathfrak{B}$ of *positively-oriented* n-systems $\beta(\rho, p)$ be given on the cell E^{n-m+1}. Such a field is called a *basis field* on the cell E^{n-m+1}. ("Positively-oriented" means oriented the same as the system $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ of basis vector in R^n.) In order to calculate the characteristic of an m-field $\sigma(p)$ that is given on S^{n-m}, we proceed as follows: Let $\mathfrak{v}_\mu$ (μ = 1, 2, …, m) be a vector of $\sigma(p)$ and let $v_{\mu i}$ (i = 1, 2, …, n) be its components *relative to the basis* $\beta(1, p)$. If one now associates every vector $\mathfrak{v}_\mu$ with the vector that contacts the origin of R^n and has the components $v_{\mu i}$ relative to $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ then this produces a continuous map f' of S^{n-m} into the $V^*_{n,m}$ at the origin of R^n. From Theorem 9 (10), a number α' is associated with this map; we prove that α' is the characteristic of the given m-field on S^{n-m}.

To that end, we construct a continuous family $\beta_t(p)$ ($0 \leq t \leq 1$) of basis fields on S^{n-m} such that $\beta_0(p) = \beta(1, p)$ and $\beta_1(p)$ is parallel to $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$. (To construct this family, one defines, say, for $0 \leq t \leq \frac{1}{2}$: $\beta_t(p)$ is parallel to $\beta(1 - 2t, p)$; the systems $\beta_{1/2}(p)$ are then parallel to each other and can easily be made parallel to $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ by a deformation in the interval $\frac{1}{2} \leq t \leq 1$.) A map f'_t of S^{n-m} into $V^*_{n,m}$ that is continuous and continuously varying in t belongs to every basis field $\beta_t(p)$. f'_0 is our f', while f'_1 is identical with the map f through parallel m-systems. f and f' are then homotopic; the assertion the follows from this.

Calculation of the characteristic by recursion:

Our new method of calculation of the characteristic is very useful when one is dealing with the following situation:

a) The cell E^{n-m+1} lies in an n'-dimensional plane $R^{n'}$ of R^n. ($n' < n$) $R^{n'}$ will be spanned by, perhaps, the basis vectors $\mathfrak{e}_{n-n'+1}, \dots, \mathfrak{e}_n$.

b) Suppose that the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_{n-n'}$ of the system $\sigma(p)$ are not contained in $R^{n'}$; they then define an $(n - n')$-system in R^n, and all of these systems define an $(n - n')$-field on S^{n-m}. We assume that this field can be extended to an $(n - n')$-field $\bar{\sigma}(\rho, p)$ on E^{n-m+1}.

c) Let the vectors $\mathfrak{v}_{n-n'+1}, \dots, \mathfrak{v}_m$ of $\sigma(p)$ be contained in $R^{n'}$; they then define an m'-system $\sigma'(p)$ in $R^{n'}$. ($m' = m - n + n'$).

$\sigma(p)$ and $\sigma'(p)$ then possess characteristics α and α' on S^{n-m}. One then has:

$$\alpha \equiv \alpha' \pmod 2.$$

(One can actually prove the equality of α and α' for certain orientation assumptions; for our purposes, however, it suffices to have congruence mod 2.)

Outline of proof: One chooses a basis field $\beta'(r, p)$ on the cell E^{n-m+1} in $R^{n'}$. This basis field will be extended by $\bar{\sigma}(\rho, p)$ to a basis field $\beta(\rho, p)$ in R^n. One calculates the desired characteristics relative to this basis field, where one suitably lets the basic cycle $z_{n,m}$ of $V^*_{n,m}$ (§ 1, no. 4) run through the orthogonal m-systems that contact the origin of R^n whose first $(m - 1)$ vectors are $\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_{m-1}$.

3. Characteristic of a field-pair on a cell. If two continuous m-fields $\sigma_0(\rho, p)$ and $\sigma_1(\rho, p)$ are given on our cell E^{r+1}, and if, moreover, a continuous family $\sigma_t(p)$ of m-fields in constructed on the boundary sphere S^r for $0 \leq t \leq 1$ that satisfies the boundary conditions $\sigma_0(p) = \sigma_0(1, p)$ and $\sigma_1(p) = \sigma_1(1, p)$ then we speak of a *field-pair* in E^{r+1}. A field-pair thus consists of two fields on a cell that are coupled on the boundary by a continuous family.

We would now like to examine the conditions under which this continuous coupling can be extended into the interior. A continuous family of m-fields $\sigma_t(\rho, p)$ shall then be constructed in E^{r+1} that satisfies the requirement $\sigma_t(\rho, p) = \sigma_t(p)$. This investigation can be carried out with the help of Theorems 8 and 9, with consideration given to Theorem 10, if the dimension $r + 1$ of the cell is at most $n - m$:

Let T be, say, the (oriented) unit interval that the parameter t runs through. We then construct the cylinder Z, *in abstracto*, which is defined as the topological product $T \times E^{r+1}$, and we denote its points by $t \times (\rho, p)$. We further associate the point $0 \times (\rho, p)$ of Z

with the system that is parallel to $\sigma_0(\rho, p)$ that contacts the origin in R^n [and analogously for $1 \times (\rho, p)$] and associate the point $t \times (\rho, p)$ with the system that is parallel to $\sigma_t(p)$ that contacts the origin in R^n. With that, a continuous map of the boundary of Z into $V^*_{n,m}$ is given. If $r + 1 < n - m$ then f can be extended to a continuous map of the entire cylinder into $V^*_{n,m}$, from which, the extension of our continuous coupling is also constructed. However, if $r + 1 = n - m$ then the extension is possible iff the number α that is associated with f according to Theorem 9 (10) vanishes, so we call it the *characteristic of the field-pair* on E^{r+1}. In order to calculate this characteristic, the cylinder boundary must be oriented; since Z is a product, an orientation can be given by an orientation of the cell E^{r+1}. One then has:

Theorem 12. *The boundary family that belongs to a field pair can be extended into the interior:*

a) *If the dimension of the cell on which the pair lies is less than* $(n - m)$.

b) *If this dimension is* $(n - m)$ *and the characteristic of the field-pair on the (oriented) cell is* 0.

We then give a relation between the characteristic of a field and a field-pair. Let two arbitrary continuous m-fields $\mathfrak{F}$ and $\mathfrak{F}'$ be given on the sphere S^{n-m} with the characteristics α and α', resp. Furthermore, let S^{n-m} be decomposed into the cells E_i^{n-m}, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be coupled by a continuous family of fields on the complex K of the $(n - m - 1)$-dimensional cells of this cell decomposition. With that, a field-pair is given on any cell E_i^{n-m}, whose characteristic we denote by α_i. (Let the cells E_i^{n-m} be coherently oriented with respect to the orientation of S^{n-m} that was employed for the calculation of α and α'.) One then has:

$$\alpha' = \alpha + \sum_{(i)} \alpha_i \qquad \text{for even } n - m \text{ or } m = 1.$$

$$\alpha' \equiv \alpha + \sum_{(i)} \alpha_i \pmod 2 \qquad \text{for odd } n - m \text{ and } m \neq 1. \tag{C}$$

These formulas define the foundation for the following analysis; it is easy to prove:

One constructs the orientated product complex $T \times S^{n-m} = T \times \sum E_i^{n-m} = \sum Z_i$, where the Z_i are constructed over E_i^{n-m} and with the cylinder that was employed in the proof of Theorem 12. Taking the boundary gives the relation:

$$(1 \times S^{n-m}) - (0 \times S^{n-m}) = \sum \dot{Z}_i . \tag{R}$$

The cylinders Z_i define a cell decomposition of $T \times S^{n-m}$; if one maps each $\dot{Z}_i$ into $V^*_{n,m}$, as in the proof of Theorem 12 then a continuous map F is given from the complex of $(n - m)$-dimensional cells of this cell decomposition into $V^*_{n,m}$, and it follows from (R) that:

$$F(1 \times S^{n-m}) - F(0 \times S^{n-m}) = \sum F(\dot{Z}_i) \text{ in } V^*_{n,m},$$

and from the definition of α, α', and α_i that:

$$\alpha' z_{n,m} - \alpha z_{n,m} \sim \sum \alpha_i z_{n,m} \quad \text{in } V^*_{n,m}.$$

The assertion follows from this homology and Theorems 9 and 10.

Here, we must mention the following special case of a field-pair: We call a field-pair with $\sigma_0(1, p) = \sigma_t(p) = \sigma_1(1, p)$ a *field-pair with rigid boundary values;* it consists of two continuous m-fields that are given on the cell E^{r+1} and coincide on the boundary S^r. (The connecting boundary family coincides with the common boundary values of the two fields.) It now follows from Theorem 12 that: The first field of a given field-pair with rigid boundary values on E^{n-m} can be deformed into the second field *while preserving its boundary values* iff the characteristic of the pair vanishes on E^{n-m}.

4. Fields and field-pairs with given characteristics. We need a topological lemma for what follows:

Let S^k be a k-dimensional sphere that is decomposed into the two k-dimensional cells E and E', and let P be a connected polyhedron. A continuous map f_1 of E into the polyhedron P can be extended to a continuous map of S^k that belongs to a given mapping class of S^k into the polyhedron P.

Proof: Let F_0 be any map of S^k into the polyhedron P that belongs to the given class, and let f_0 be the map that F_0 induces on E. We construct a continuous family of maps f_t $(0 \leq t \leq 1)$ that connects f_0 to f_1. (Such a family can be found, since P is connected.) The family f_t can be extended to a family of maps F_t of S^k into the polyhedron P ([14]); F_1 is our desired map.

If one identifies S^k with our cylinder Z over a cell E^{n-m} (no. 3) in this lemma and identifies P with the manifold $V^*_{n,m}$ then this easily yields:

Theorem 13. *A continuous m-field that is given on the cell E^{n-m} can be extended through a second field on that cell to a field-pair with rigid boundary values (no. 3) and a given characteristic.*

([14]) AH: chap. XIII, § 1, lemma Ia.

§ 4. Vector fields on manifolds

1. ***m*-fields, parallelizability.** We now move on to the study of m-fields on a closed n-dimensional and differentiable manifold M^n. For this, we must temporarily make the case distinction of Theorem 9:

Case 1. $n - m$ is even or $m = 1$. M^n is then orientable.

Case 2. $n - m$ is odd and $m \neq 1$. M^n can then also be non-orientable.

We call M^n *differentiable* if the following condition is fulfilled: M^n is endowed with a system of neighborhoods that is chosen once and for all, and which we will call *elements* in the sequel. Each element is homeomorphic to a Euclidian space R^n and is equipped with a Cartesian coordinate system. The coordinate transformation that is induced on the overlap of two coordinate systems shall be continuously differentiable and possess a nowhere-vanishing, and in Case 1, positive functional determinant.

With these assumptions, one can define vectors on M^n and apply the conceptual structures and theorems of § 3 to it; One must only replace the Euclidian space with an element in M^n, which is reasonable.

If an m-system is attached to every point of M^n then we speak of an *m-field* on M^n; this field is called *continuous* in the event that it is continuous on every element. If there are continuous μ-fields on M^n but no continuous $(\mu + 1)$-fields then we call μ the *degree of parallelizability* of M^n; A manifold with $\mu = n$ will be referred to as a *parallelizable manifold.* The basis for this terminology is easy to see: If $\mu = n$ then there is a continuous basis field (§ 3, no. 2) on M^n. If we establish an arbitrary vector on M^n with the contact point p by its components relative to the basis that is given at p then two vectors are called *parallel* when the possess positively-proportional components. With that, a continuous teleparallelism is constructed on M^n, from which it follows, for example, that the manifold of directed line elements in M^n is homeomorphic to the topological product of M^n with an $(n - 1)$-dimensional sphere. Examples of parallelizable manifolds are easy to give: The product of two parallelizable manifolds is again parallelizable, so the n-dimensional torus (i.e., the product of n circles) provides an example of a parallelizable M^n. We further remark that one can calculate characteristics by parallel translation of all the distributed vectors to a fixed point of M^n, precisely as one does in Euclidian spaces (§ 3, no. 2).

The central problem of this paper, towards whose solution some steps will be made in what follows, is the determination of the degree μ of a given manifold. We are justified in calling this problem a topological one, since two manifolds that correspond by means of a map that is one-to-one and differentiable in both directions will obviously have the same degree.

2. Frameworks and framework-pairs. Let a fixed cell decomposition of M^n be established for the following considerations; we denote an r-dimensional, oriented cell by

x^r and the cell that is dual to x^r in the dual decomposition ([15]) by ξ^{n-r}. Let the cell decomposition be sufficiently fine that the star of x^r (which is the totality of all cells that have points in common with x^r) lies completely in some element of M^n. In Case 1 (no. 1), we would further like to orient the dual cell ξ^{n-r} to x^r as is customary in orientable manifolds ([15]); in Case 2, orientations will play no role whatsoever.

Now, a *framework* is a continuous m-field that is defined on all cells of a sub-complex K of the *dual* cell-decomposition. If K is homogeneously ρ-dimensional ([16]) then we also briefly speak of a *ρ-dimensional framework*. In the case that is most important for us, K is the complex of all ρ-dimensional cells of the dual cell-decomposition; a framework that belongs to this complex is called an *r-dimensional framework that is defined everywhere on the manifold M^n*. In the sequel, it will always be assumed that the cells of K are at most $(n - m)$-dimensional.

One then has:

Theorem 14. *Any framework on M^n can be extended to an $(n - m)$-dimensional framework that is defined on all of the manifold M^n.* $(0 < m < n)$.

Proof: Let ξ^ρ be the cells of K and let $\bar{\xi}^\rho$ be the cells of the dual cell-decomposition that do not belong to K. One now attaches an arbitrary m-system to every vertex $\bar{\xi}^0$. With that, an m-field is given on the boundary of every cell $\bar{\xi}^1$, which, from Theorem 11, can be extended continuously into the interior. Now, the m-field that is given on the boundary of every cell $\bar{\xi}^2$ can again (in the event that $m < n - 1$) be extended into the interior of the cell. (Theorem 11) One proves the theorem by pursuing the construction further. It follows from this that:

Corollary. *There exists an $(n - m)$-dimensional framework that is defined on all of M^n.*

Such a framework will always be denoted by $\mathfrak{G}$.

We will understand $\dot{\mathfrak{G}}$ to mean the framework that $\mathfrak{G}$ induces on the complex of $(n - m - 1)$-dimensional cells of the dual cell decomposition, while an arbitrary $(n - m - 1)$-dimensional framework that is defined on all of M^n will be denoted by $\mathfrak{g}$.

Two frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ define a *framework-pair* when a continuous family $\mathfrak{g}_t$ $(0 \leq t \leq 1)$ of frameworks $\mathfrak{g}$ is given with $\mathfrak{g}_0 = \dot{\mathfrak{G}}_0$ and $\mathfrak{g}_1 = \dot{\mathfrak{G}}_1$. $\mathfrak{G}_0$ and $\mathfrak{G}_1$ are then connected to each other on the complex of $(n - m - 1)$-dimensional cells of M^n by a continuous family.

Theorem 15. *Two arbitrary frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ can always be combined into a framework-pair.*

([15]) cf., Seifert-Threlfall: *Lehrbuch der Topologie* (B. G. Teubner, 1934), and furthermore, AH: chap. XI, § 1, § 68.

([16]) cf., AH: chap. IV, § 1, no. 2.

The proof proceeds analogously to that of Theorem 14. One then connects the two *m*-systems that are given by $\mathfrak{G}_0$ and $\mathfrak{G}_1$ at a vertex ξ^0 of the dual cell decomposition by means of a continuous family of *m*-systems and then extends this connection to the higher-dimensional cells using Theorem 12.

3. Preliminary remarks on characters in Λ^{n-r}. We next choose a *coefficient ring J* that will serve for the definition of algebraic complexes in M^n ([17]), and, in fact, let *J* be the *ring of whole numbers* in Case 1 (no. 1) and the *ring of residue classes* (mod 2) in Case 2. We denote algebraic sub-complexes of the *x*-cell decomposition by *C* and algebraic sub-complexes of the ξ-cell decomposition by Γ. All $(n - m)$-dimensional complexes Γ^{n-r} define a group Λ^{n-r} that contains the group Z^{n-r} of $(n - r)$-dimensional cycles and the group H^{n-r} of $(n - r)$-dimensional boundaries as subgroups. The difference group $Z^{n-r} - H^{n-r}$ is, as is well-known, the $(n - r)$-dimensional Betti group B^{n-r} of M^n.

A *character* χ in Λ^{n-r} is a homomorphic map from Λ^{n-r} to the coefficient ring *J*. Therefore, if Γ_1 and Γ_2 are complexes in Λ^{n-r} and α is an element of *J* then one has:

a) $\chi(\Gamma_1 + \Gamma_2) = \chi(\Gamma_1) + \chi(\Gamma_2)$; b) $\chi(\alpha\Gamma_1) = \alpha\chi(\Gamma_1)$.

From these two facts, it follows that:

c) A character χ is known when its values for the complex defined by a basis of Λ^{n-r} are given. (e.g., all cells ξ^{n-r} define such a basis.)

d) If *C* is an *r*-dimensional sub-complex of the *x*-cell decomposition that is chosen to be fixed then a character χ in Λ^{n-r} will be generated by setting:

$$\chi(\Gamma) = \phi(C, \Gamma).$$

(In this, Γ is an arbitrary complex of Λ^{n-r} and ϕ means the intersection number of the complexes in parentheses.)

e) Any character in Λ^{n-r} can be generated by a complex *C* in the way that is suggested by d). *C* is determined uniquely, and is called the *complex that is associated with* χ.

The proof best proceeds by giving *C* explicitly. One has:

$$C = \sum_{(j)} \chi(\xi_j^{n-r})\, x_j^r .$$

In this, the summation is extended over all *r*-dimensional cells x_j^r.

([17]) AH: chap, IV.

Our next objective is to determine the properties of the generated complex C from the properties of the characters χ:

f) C is a cycle iff χ vanishes in the group H^{n-r}; i.e., if for every $(n - r + 1)$-dimensional complex Δ of the ξ-cell decomposition one has:

$$\chi(\dot{\Delta}) = 0. \tag{I}$$

The proof follows from the fact that for any character χ and an arbitrary Δ one has the relation:

$$\chi(\dot{\Delta}) = \phi(C, \dot{\Delta}) = \pm\, \phi(\dot{C}, \Delta).$$

g) Between two characters χ_0 and χ_1 in Λ^{n-r} and a character $\dot{\chi}$ in Λ^{n-r-1}, there exists the following relation:

$$\chi_1(\Gamma) - \chi_0(\Gamma) = \dot{\chi}(\dot{\Gamma}), \tag{II}$$

so between the associated complexes C_0, C_1, and D, there exists the relation:

$$C_1 - C_0 = \pm\, \dot{D}.$$

Proof: For an arbitrary $(n - r)$-dimensional complex Γ, one has:

$$\begin{aligned}\phi(C_1 - C_0, \Gamma) &= \phi(C_1, \Gamma) - \phi(C_0, \Gamma) = \chi_1(\Gamma) - \chi_0(\Gamma) \\ &= \dot{\chi}(\dot{\Gamma}) = \phi(D, \dot{\Gamma}) = \pm\, \phi(\dot{D}, \Gamma).\end{aligned}$$

Since Γ was arbitrary, the assertion follows from this that:

h) Let a set of characters χ_i in the group Λ^{n-r} be given, each of which satisfies the condition (I), and any two of which fulfill a relation of the form (II). One then has:

α) The given set determines a character χ^* in the Betti group B^{n-r} whose elements (these are homology classes) we denote by Ξ.

β) The complexes that are associated with χ_i are, from f), cycles and lie in a single r-dimensional homology class A.

γ) One has: $\chi^*(\Xi) = \phi(A, \Xi)$.

Proof:

Of α): From the existence of (II), it then follows that all characters χ_i in the cycle group Z^{n-r} coincide, and thus induce a single character in that group. Due to (I), this character has the same value for homology cycles, moreover, so it actually determines a single character χ^* in the Betti group B^{n-r}.

Of β): Due to (II), one has the assertion g), from which, it follows that the cycles that are associated with two characters of the given set are homologous.

Of γ): This follows directly from the definitions of χ^* and A.

4. The characters that are determined by frameworks; main theorems. Now, let an $(n - m)$-dimensional framework $\mathfrak{G}$ that is defined on all of the manifold M^n be given. (cf., corollary to Theorem 14.) We now define a character χ in Λ^{n-m+1} by giving the values of χ for the cells ξ^{n-m+1}, as in no. 3, c): Let $\chi(\xi^{n-m+1})$ be the characteristic of the continuous m-field that is given by $\mathfrak{G}$ on the boundary sphere $\dot{\xi}^{n-m+1}$ of ξ^{n-m+1}. (Naturally, the orientation $\dot{\xi}^{n-m+1}$ is the one that was employed in the calculation of this characteristic. One further observes that the characteristic is an element of J.) The character thus defined is called the *character χ that is associated with* $\mathfrak{G}$.

In addition, we consider all $(n - m)$-dimensional frameworks $\mathfrak{G}_i$ that are also defined on all of M^n. The characters χ_i that are associated with them define a set like the one that we considered in no. 3, h). We assert that this set fulfills the assumption of no. 3, h).

Proof: Let, say, $\mathfrak{G}_0$ and $\mathfrak{G}_1$ be two frameworks, and let χ_0 and χ_1, resp., be the associated characters. We next show that a character $\dot{\chi}$ exists in Λ^{n-m} such that χ_0 , χ_1, and $\dot{\chi}$ fulfill the relation (II) of no. 3. Due to no. 3, c), it suffices to define $\dot{\chi}$ for the cells ξ^{n-m}. To that end, we couple the frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ into a framework-pair using Theorem 15; let $\dot{\chi}(\xi^{n-m})$ be the characteristic of the field-pair that is induced on ξ^{n-m} by this framework-pair. Due to formula (C) of § 3, no. 3, one has:

$$\chi_1(\xi^{n-m+1}) - \chi_0(\xi^{n-m+1}) = \dot{\chi}(\dot{\xi}^{n-m+1}) .$$

The relation (II) now follows from no. 3a) and b), in fact. Furthermore, we have to show that each of our characters satisfies the condition (I) of no. 3. If we apply the relation (II) that we just proved to the complex $\dot{\Delta}$ then this yields:

$$\chi_1(\dot{\Delta}) = \chi_0(\dot{\Delta}),$$

so it suffices to prove (I) for a single character that is induced by a special framework $\mathfrak{G}_0$. Moreover, due to a) and b), it suffices that Δ be a cell ξ^{n-m+2}. We now construct $\mathfrak{G}_0$ as follows: Let the m-systems of $\mathfrak{G}_0$ be parallel to each other on the boundary $\dot{\xi}^{n-m+2}$. (This definition makes sense, since ξ^{n-m+2} lies in an element (no. 1).) From Theorem 14, such a framework can always be found. For the associated character χ_0, one now has, trivially: $\chi_0(\xi^{n-m+2}) = 0$, with which all parts of (I) are proved.

From the assertion of no. 3, h), it now follows that:

The character χ that is associated with a framework $\mathfrak{G}$ has a cycle for its associated complex, which will be called the *singular cycle* of $\mathfrak{G}$, and from no. 3, e), it is given by:

$$z = \sum_{(j)} \chi(\xi_j^{n-m+1})\, x_j^{m-1} .$$

All of the characters χ_i determine a character χ^* in the $(n - m + 1)$-dimensional Betti group B^{n-m+1}, which we will call χ^{n-m+1} in the sequel ([18]). One further has:

Theorem 16. (First main theorem). *The singular cycles of all* $(n - m)$*-dimensional frameworks* $\mathfrak{G}$ *that consist of m-systems and can be defined on the entire manifold* M^n *lie in a single (m – 1)-dimensional homology class; it is called the* characteristic homology class F^{m-1}. *If* Ξ *is an arbitrary* $(n - m + 1)$*-dimensional homology class then one has:*

$$\chi^{n-m+1}(\Xi) = \phi(F^{m-1}, \Xi).$$

In the next paragraph, we shall see that the character χ^{n-m+1} represents a *generalization of the Euler characteristic.*

To these immediate consequences of the discussion in no. 3, we must add a somewhat deeper theorem:

Theorem 17 (Second main theorem). *Any cycle that is contained in the characteristic class* F^{m-1} *is the singular cycle of a framework.*

Proof: Let, say, z be the given cycle in the class F^{m-1}. We choose an arbitrary, but fixed, initial framework $\mathfrak{G}_0$ with the singular cycle z_0. From Theorem 16, z_0 also lies in F^{m-1}, so one has $z \sim z_0$, and therefore $z - z_0 = \dot{D}$. Our framework $\mathfrak{G}_0$ induces an m-field $\mathfrak{F}_0$ on the cell ξ^{n-m}, which we extend by means of another field $\mathfrak{F}_1$ to a field-pair with rigid boundary values (§ 3, no. 3) whose characteristic on ξ^{n-m} possesses the value $\phi(D, \xi^{n-m})$ (Theorem 13). The m-field $\mathfrak{F}_1$ that is thus constructed on all cells ξ^{n-m} combines into a framework $\mathfrak{G}_1$. $\mathfrak{G}_0$ and $\mathfrak{G}_1$ together define a framework-pair that gives rise to a character $\dot{\chi}$ as in the beginning of this section. By construction, one has $\dot{\chi}(\xi^{n-m}) = \phi(D, \xi^{n-m})$; i.e., the complex that is associated with $\dot{\chi}$ is the complex D.

We have seen that the relation (II) of no. 3 exists between the characters χ_0 of $\mathfrak{G}_0$ and χ_1 of $\mathfrak{G}_1$ and the character $\dot{\chi}$, so the assertion of no. 3, g) is true; i.e., $z_1 - z_0 = \pm \dot{D}$, if we denote the singular cycle of $\mathfrak{G}_1$ by z_1. The given cycle z is then a singular cycle of $\mathfrak{G}_1$, with which, Theorem 17 is proved.

([18]) The character χ^{n-m+1} is, *ex definitone*, independent of the choice of framework; it is given by the geometric properties of M^n.

The meaning of the characteristic class F^{m-1} for the problem of this paper is based in the following Theorem:

Theorem 18 (Existence theorem). *The exists an* $(n - m + 1)$*-dimensional framework that is defined on the entire manifold* M^n *iff the characteristic class* F^{m-1} *is the zero class.*

Proof: a) Let an $(n - m + 1)$-dimensional framework that is defined on all of M^n be given. It induces a framework $\mathfrak{G}$ on the complex of cells ξ^{n-m}, and thus an m-field $\mathfrak{F}$ on each cell boundary $\dot{\xi}^{n-m+1}$. Since $\mathfrak{F}$ is extended into the interior of the cell ξ^{n-m+1}, its characteristic vanishes on $\dot{\xi}^{n-m+1}$, so the character χ that is associated with $\mathfrak{G}$ also vanishes, and one has $z = 0$ for the singular cycle z of $\mathfrak{G}$, so $z \sim 0$ precisely.

b) Let the characteristic class F^{m-1} be the zero class. From Theorem 17, there is a framework $\mathfrak{G}$ whose singular cycle is the zero cycle. The characteristic χ that is associated with $\mathfrak{G}$ then vanishes; however, from Theorem 11, the field that is induced by $\mathfrak{G}$ on $\dot{\xi}^{n-m+1}$ can be extended into the interior.

5. Fields with singularities. Our endeavors to construct a continuous m-field on the manifold M^n step-wise by frameworks are obstructed by the existence of the class F^{m-1}; however, we can always find m-fields whose continuity is broken at certain "singular" points. In order to not go into dimension-theoretic difficulties, we would like to consider only m-fields that satisfy the following assumption: If a cell x^{r-1} of our x-cell decomposition contains a singular point in its interior then it consists of nothing but singular points. All of these cells define an absolute complex K^{r-1} – viz., the *singularity complex* of the field in question. [The number $(r - 1)$ means the dimension of the highest-dimensional cell in this complex.] Now, a field with the singularity complex K^{r-1} obviously induces an $(n - r)$-dimensional framework that is defined on all of M^n. However, the converse is also true: Every $(n - r)$-dimensional framework that is defined on all of M^n is associated with an m-field on M^n with a singularity complex K^{r-1}. In order to see this, one extends the m-field that is given by the framework on the cells ξ^{n-m} by central projection (§ 3, no. 11) into the higher-dimensional cells ξ^{n-m+k}. If one then chooses the projection center to be the intersection point of ξ^{n-m+k} with the dual cell x^{r-k} then the necessary cone construction can be performed simplicially on a common subdivision U of the x and ξ-cell decompositions. With this relationship between frameworks and singular fields, it now follows from the corollary to Theorem 14 and from Theorem 18 that:

Theorem 19. *There always exists an m-field with an* $(m - 1)$*-dimensional singularity complex on a manifold* M^n*; the necessary and sufficient condition for the existence of an m-field with an at most* $(m - 2)$*-dimensional singularity complex is the vanishing of the characteristic class* F^{m-1}.

Since any singular *m*-field an ($m - 1$)-dimensional singularity complex on M^n uniquely determines an ($n - m$)-dimensional framework $\mathfrak{G}$ that is defined on all of M^n, we can briefly call the singular cycle that is associated with $\mathfrak{G}$ (no. 14) the *singular cycle* of the given field. One then has:

Theorem 19a. *The singular cycle of an m-field with the* ($m - 1$)*-dimensional singularity complex* K^{m-1} *is an algebraic sub-complex of* K^{m-1} *in the subdivision that it induces through U; it measures the multiplicities of the* ($m - 1$)*-dimensional singularities and represents the characteristic class* F^{m-1}.

In order to prove this, one employs the explicit representation of the singular cycle $z = \sum_{(j)} \chi(\xi_j^{n-m+1})\, x_j^{m-1}$ and Theorem 11.

§ 5. Determination of the characteristic classes in special cases.

1. Differential simplicial decompositions. A simplicial decomposition K of a given manifold is called *differentiable* when any simplex of K, along with its perimeter, lies in an element of M^n (§ 4, no. 1) and is either a Euclidian simplex ([19]) or the image of a Euclidian simplex by means of a topological map that is continuously differentiable in both directions in this element.

For what follows, we will need the barycentric subdivision ([20]) $\bar{K}$ of such a simplicial decomposition K. If we denote the center of mass of an r-dimensional simplex of K by a_r then the simplexes $x^s = (a_{r_0}, a_{r_1}, \ldots, a_{r_s})$ are the simplexes of $\bar{K}$. ($r_0 < r_1 < \ldots < r_s$) and ($s = 0, 1, \ldots, n$). Now, let $\bar{K}$ be our x-cell decomposition of § 4; we denote the dual cell ξ^{n-s} of x^s by $\xi^{n-s} = \xi_{(r_0 r_1 \ldots r_s)}$.

2. Single vector fields. In this number, we concern ourselves with the theory of 1-fields (in the sequel, we briefly refer to them as *vector fields*) on a manifold M^n. This theory has already been developed for some time ([21]), and the concluding results go back to H. Hopf.

Theorem 19 then shows that there is always a vector field $\mathfrak{F}$ with a 0-dimensional singularity complex in M^n; $\mathfrak{F}$ is then singular at only finitely many vertices x_i^0 of the x-cell decomposition. We understand the *index* j_i of the singularity x_i^0 to mean the characteristic of the 1-field that is given by $\mathfrak{F}$ on the boundary $\dot{\xi}_i^n$ of the cell ξ_i^n that is dual to x_i^0. (We find ourselves in Case 1 of § 4, no. 1; M^n is therefore *orientable*, and the

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([19]) AH: chap. III, § 1, no. 1.
([20]) AH: chap. III, § 2, no. 3.
([21]) AH: chap. XIV, § 4.

cells ξ_i^n are *coherently oriented*.) A simple argument gives the singular cycle of $\mathfrak{F}$ (§ 4, no. 5) as:

$$z = \sum_{(i)} j_i \, x_i^0 \,. \tag{1}$$

The characteristic class F^0 will then be represented by the cycle $x^0 \sum j_i$ (x^0 is an arbitrary, but fixed, vertex of the x-cell decomposition). The index sum $\sum j_i$ is called the *algebraic number of singularities.* If one denotes the n-dimensional homology class that is represented by the sum of all cells ξ^n by Ξ^n then, from Theorem 16, this yields for the character χ^n in the Betti group B^n :

$$\chi^n(\Xi^n) = \phi(F, \Xi^n) = \sum j_i \,. \tag{2}$$

Since Ξ^n is the single basis element for B^n, (2) determines the character χ^n completely.

It now follows from Theorem 16 and 18 that:

Theorem 20. *The algebraic number of singularities is the same for all vector fields on M^n; one then has vector fields that are continuous at all points of M^n iff this number vanishes.*

One further has:

Theorem 20*a*. *For a suitable orientation of M^n, the algebraic number of singularities of any vector field on M^n is equal to the Euler characteristic $\chi(M^n)$ of M^n.*

This theorem is equivalent to the following assertion:

The characteristic class F^0 can be represented by $x^0 \cdot \chi(M^n)$. Moreover, the formula:

$$\chi^n(\Xi^n) = \chi(M^n) \tag{3}$$

also says precisely the same thing. We will prove the theorem for the simplest case of $n = 2$ in this latter form. We carry out the proof under the assumption that M^2 possesses a differentiable simplicial decomposition. (Theorem 20*a* is still true without this assumption.) We then construct a *special* one-dimensional framework $\mathfrak{G}$ that consists of 1-systems on the barycentric subdivision of the dual cell decomposition whose associated character χ we determine. The part of $\mathfrak{G}$ that lies in a simplex (a_0, a_1, a_2) (no. 1) of the barycentric subdivision $\bar{K}$ is depicted in Fig. 3. From this figure, it is clear that the vectors of $\mathfrak{G}$ that lie on the boundary of a cell of type $\xi_{(0)}$ (no. 1) point to the exterior of $\xi_{(0)}$, and on the boundary of a cell of type $\xi_{(2)}$, they point to the interior of $\xi_{(2)}$. [In Fig. 3, the parts of three cells that lie in (a_0, a_1, a_2) are suggested by $\xi_{(0)}$, $\xi_{(1)}$, $\xi_{(2)}$.] For a suitable orientation, one finds that the characteristic of the field that is induced by $\mathfrak{G}$ on the boundary of a cell $\xi_{(r)}$ has the value $(-1)^r$, so one has:

$$\chi(\xi_{(r)}) = (-1)^r \qquad (r = 0, 1, 2), \tag{4}$$

and the singular cycle z of $\mathfrak{G}$ will be:

$$z = \sum(-1)^r a_r, \tag{4a}$$

where the summation is taken over all vertices of $\bar{K}$. If one denotes the number of cells of type $\xi_{(r)}$ by a_r then that would yield for the character $\chi^n = \chi^2$:

$$\chi^2(\Xi^2) = \chi(\sum \xi^2) = \sum \chi(\xi^2) = \sum \chi(\xi_{(0)}) + \sum \chi(\xi_{(1)}) + \sum \chi(\xi_{(2)}) = a_0 - a_1 + a_2 .$$

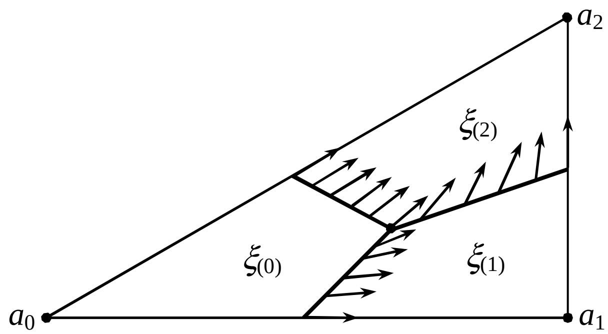

Figure 3.

However, by definition, $a_0 - a_1 + a_2$ is the Euler characteristic $\chi(M^2)$; with that, (3) is proved in the special case $n = 2$. Theorem 20*a*) can be proved for *n*-dimensional manifolds in an analogous way.

Formula (3) confirms the fact that was mentioned in § 4 that the character χ^{n-m+1} can be regarded as a generalization of the Euler characteristic.

If follows from Theorems 20 and 20*a*) that:

Corollary. *There exists a continuous vector field on the manifold M^n iff the Euler characteristic $\chi(M^n)$ vanishes* ([21a]).

This theorem is true for *non-orientable manifolds*, but this is not directly provable by our methods. Our argument can also be carried out for non-orientable manifolds in the event that we introduce the ring of residue classes (mod 2) in place of the ring of whole numbers (§ 4, no. 3). If we understand Ξ^n in this case to mean the *n*-dimensional homology class that is represented by the sum of the (unoriented) cells ξ^n then one has:

$$\chi^n(\Xi^n) \equiv \chi(M^n) \qquad (\text{mod } 2). \tag{3a}$$

([21a]) Cf., AH: chap. XIV, § 4, Theorem 3.

3. Three-dimensional manifolds. We now examine the parallelizability (§ 4, no. 1) of three-dimensional manifolds. One has the important result:

Theorem 21. *Any orientable, three-dimensional, closed manifold that admits a differentiable simplicial decomposition is parallelizable.*

Before we give the proof of this theorem, we mention that it follows from the considerations of § 4, no. 1 that:

Corollary. *If a three-dimensional manifold M^3 fulfills the assumptions of Theorem 21 then the manifold of its directed line elements is homeomorphic to the topological product of M^3 with a two-dimensional sphere.*

The proof of Theorem 21 proceeds in four steps:

I. Determination of the characteristic class F^1.

We can satisfy ourselves with the following hints for the solution of this problem, since in Appendix I we have rigorously determined the characteristic class F^1 for three-dimensional, orientable manifolds under somewhat different assumptions and by other methods.

F^1 is the characteristic class of the 2-fields, so we must set $m = 2$ and $n = 3$. We are then in Case 2 of § 4, no. 1; J is then the ring of residue classes (mod 2). In order to determine F^1, one can, in analogy to no. 2 (Fig. 3), construct a *special* 1-dimensional framework $\mathfrak{G}$ that is defined on all of M^3, and which is coupled with the barycentric subdivision $\bar{K}$. I will not go into the somewhat tedious construction of this framework that is composed of 2-systems here; one finds for the associated character χ that:

$$\chi(\xi_{(r_0,r_1)}) = 1, \tag{5}$$

such that the singular cycle z of $\mathfrak{G}$ is given by ([22]):

$$z = \sum \left(a_{r_0}, a_{r_1}\right). \tag{5a}$$

This cycle (mod 2) thus consists of *all edges of the barycentric subdivision* $\bar{K}$. One can now show that z always bounds in an *orientable* manifold M^3, while this does not

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([22]) Formulas (4a) and (5a) are closely related to the conjecture that for arbitrary n and m the characteristic class F^{m-1} can be represented:

in Case I of § 4, no. 1 by $\sum (-1)^{r_0+r_1+\cdots+r_{m-1}}\left(a_{r_0}, a_{r_1}, \ldots, a_{r_{m-1}}\right)$

and in Case II, by $\sum \left(a_{r_0}, a_{r_1}, \ldots, a_{r_{m-1}}\right)$.

The summation is therefore taken over all $(m-1)$-dimensional cells of $\bar{K}$; the complexes above are, in fact, cycles of the coefficient ring J.

necessarily need to be the case in a non-orientable manifold ([23]). This then yields that in an orientable M^3 the characteristic class F^1 is always the zero class.

II. There exists a framework $\mathfrak{H}$ that is defined on all of M^3 and consists of 2-systems.

Since, from I, the class F^1 vanishes in our orientable M^3, this fact is a direct consequence of the existence Theorem 16.

III. There exist continuous 2-fields on M^3.

In order to prove this, we show that the 2-field $\mathfrak{F}$ that is given by $\mathfrak{H}$ on the boundary $\dot{\xi}^3$ of a cell ξ^3 can be continuously extended into the interior of ξ^3. Since ξ^3 lies in an element (§ 4, no. 1), we must therefore prove the following theorem: A continuous 2-field $\mathfrak{F}$ that is given on the boundary sphere S^2 of a 3-dimensional cell E^3 that lies in Euclidian space R^3 can be continuously extended into the interior of E^3.

The following statement is equivalent to this theorem: The map of S^2 into the manifold $V_{3,2}^*$ by parallel 2-systems (§ 3, no. 1) that is associated with $\mathfrak{F}$ is homotopic to zero. Our statement III can thus be expressed in the following form: Any continuous map of a 2-dimensional sphere S^2 into $V_{3,2}^*$ is homotopic to zero. Now, since, from § 2, no. 2, the closed manifold $V_{3,2}$ is a deformation retract of $V_{3,2}^*$, it suffices to prove this assertion for maps of S^2 into $V_{3,2}$. However, since $V_{3,2}$ is homeomorphic to the projective space P^3 (§ 1, no. 3), and since any map of S^2 into P^3 is, in fact, homotopic to zero, we have proved the assertion III.

IV. There exist continuous 3-fields on M^3.

The fact that the existence of continuous 3-fields follows from the existence of continuous 2-fields on an *orientable* M^3 is easily proved.

§ 6. Theorems on characteristic cohomology classes. Applications.

1. Order of the characteristic class. In this section, we pose the problem of determining the order of a non-vanishing characteristic class. This problem is meaningful only in Case 1 of § 4, no. 1, for which the coefficient ring J is the ring of whole numbers. We will solve it for even $(n - m)$.

We preface the following analysis with a subsidiary consideration that relates to the manifolds $V_{n,m}^*$ (§ 2) for which $n - m$ is even. Namely, we shall examine the topological map φ of $V_{n,m}^*$ to itself that comes about when one replaces the m^{th} vector $\mathfrak{v}_m$ in any m-system of $V_{n,m}^*$ with its opposite vector $-\mathfrak{v}_m$. On the $(n - m)$-dimensional sphere that is

([23]) Cf., problem 187 in the Jahresbericht der deutschen Mathematikervereinigung, Band 45, pp. 22.

provided by the basis cycle $z_{n,\,m}$ of § 1, no. 4 for a fixed orientation, φ is the diameteral map; since this sphere possesses an even dimension, this yields:

$$\varphi(z_{n,\,m}) = -\,z_{n,\,m}\ . \tag{1}$$

With those preparations, a framework $\mathfrak{G}$ that consists of m-systems will be constructed on the given manifold M^n by employing the notations and assumptions of § 4, and we let $n - m$ be even. We thus find ourselves in Case 1 of § 4, no. 1, and the coefficient ring J is therefore the ring of whole numbers. The framework $\mathfrak{G}$ induces an m-field on the boundary of any $(n - m + 1)$-dimensional cell ξ whose characteristic $\chi(\xi)$ is established by means of the map f of $\dot{\xi}$ into $V_{n,m}^{*}$ by parallel m-systems (§ 3, no. 1).

If one now replaces the m^{th} vector on any m-system of $\mathfrak{G}$ with its opposite vector then a new framework $\bar{\mathfrak{G}}$ arises that is associated with the characteristic $\bar{\chi}(\xi)$ and the map $\bar{f}$. Obviously, $\bar{f}$ arises from the composition of f and φ; it then follows from (1) that: $\bar{\chi}(\xi) = -\,\chi(\xi)$. The relation:

$$\bar{\chi} = -\,\chi \tag{2}$$

then exists between the characters χ and $\bar{\chi}$ that belong to $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp. χ, as well as $\bar{\chi}$, then induce the character χ^{n-m+1} in the $(n - m + 1)$-dimensional Betti group; it then follows from (2) that: $\chi^{n-m+1} = -\,\chi^{n-m+1}$, so ultimately $\chi^{n-m+1} = 0$.

It would be incorrect to conclude the vanishing of the characteristic class F^{m-1} from the vanishing of χ^{n-m+1}; this conclusion is only permissible when no $(m - 1)$-dimensional torsion is present in M^n.

If we set, say, $m = 1$ then we find that $\chi^n = 0$ for manifolds of odd dimension; however, from § 5, formula (3), it follows from this that the characteristic of an orientable manifold of odd dimensions vanishes ([24]). From the corollary to Theorem 20 it then follows, moreover, that any orientable manifold of odd dimension possesses a continuous vector field.

Theorem 22. *If M^n is orientable, $(n - m)$ is even, and the class F^{m-1} is not the zero class then that class has order 2.*

Proof: We have to show: For even $(n - m)$, one always has $2 \cdot F^{m-1} = 0$. Now, from (2), the relation $z = -\,\bar{z}$ exists between the singular cycles z and $\bar{z}$ of the frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp., that were employed above. Since both of these cycles lie in F^{m-1}, one has $F^{m-1} = -\,F^{m-1}$; this was to be proved.

Corollary. *If $(n - m)$ is even and no $(m - 1)$-dimensional torsion is present in M^n then F^{m-1} is the zero class.*

([24]) To my knowledge, J. Hadamard was the first to derive the vanishing of the Euler characteristic of a manifold of odd dimension from the theory of vector fields. Cf., Tannery: *Introduction à la théorie des fonctions* (Paris, Hermann, 1910), t. II, note by Hadamard, no. 42, pp. 475.

2. An intersection theorem. In what follows, Cases 1 and 2 of § 4, no. 1 will no longer be distinct; *all consideration will be based upon the ring of residue classes* (mod 2) *as the coefficient ring J*, and M^n can be either an orientable or non-orientable manifold.

In order to bring our theory to a definite conclusion, we must find manifolds in which non-zero characteristic classes exist; only then will the theorems of § 4 contain non-trivial statements. The analysis of this section will serve to resolve this problem.

We call a ν-dimensional manifold M^ν that is embedded in the given manifold M^n a *hypersurface* when the following conditions are fulfilled:

a) Let M^ν be the image of a differentiable parameter manifold by means of a topological and continuously-differentiable map of this parameter manifold into M^n.

b) M^ν admits a cell decomposition that is a sub-complex of the ξ-cell decomposition (§ 4, no. 2) of the manifold M^n.

Due to a), vectors on M^ν are also vectors on M^n, and the totality of all vectors on M^ν that contact a point p of M^ν defines a ν-dimensional vector structure on M^n. If the vectors in a $(n - \nu)$-system on M^n that contact p do not belong to this structure then we call the system *foreign* to M^ν. If a continuous field of $(n - \nu)$-systems exists on M^n that are foreign to M^ν then we say that M^ν possesses an *external* $(n - \nu)$-field ([25]). If $\nu = n - 1$ then this simply means that M^ν is two-sided in M^n.

Due to b), M^ν is a cycle (mod 2) of the ξ-cell decomposition that represents a ν-dimensional homology class Ξ^ν of M^n and a ν-dimensional homology class $\bar{\Xi}^\nu$ in M^ν. One has:

Theorem 23. *If a hypersurface M^ν that lies in M^n possesses an external $(n - \nu)$-field then the intersection number of the characteristic class $F^{n-\nu}$ of M^n with M^ν is the (mod 2) reduced Euler characteristic of M^ν.*

Before we prove this theorem, we introduce the following relations: Let $\bar{\xi}$ be the cells of the ξ-cell decomposition that induce a cell decomposition of M^ν using b); a $(\nu - 1)$-dimensional framework that is defined on all of M^n and consists of $(n - \nu + 1)$-systems will be denoted by $\mathfrak{G}$, and associated character in the group Λ^ν of M^n (§ 4, no. 4), by χ. A $(\nu - 1)$-dimensional framework that is defined on all of M^ν and consists of 1-systems will be denoted by $\bar{\mathfrak{G}}$, and the associated character in the group Λ^ν of M^ν, by $\bar{\chi}$. The characters χ determine the character χ^ν (§ 4, no. 4) in the ν-dimensional Betti group of M^n, while the characters $\bar{\chi}$ determine the character $\bar{\chi}^\nu$ in the ν-dimensional Betti group of M^ν in an analogous way.

We then prove the following:

([25]) A hypersurface with an external $(n - \nu)$-field that lies in an orientable manifold is orientable.

Lemma. If there exist two frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ such that for every cell $\bar{\xi}^\nu$ the relation:

$$\chi(\bar{\xi}^\nu) \equiv \bar{\chi}(\bar{\xi}^\nu) \qquad \text{(mod 2)} \tag{2}$$

is fulfilled then the assertion of Theorem 23 is true.

Proof: By summing over all cells $\bar{\xi}^\nu$, one gets from (2) that:

$$\chi^\nu(\Xi^\nu) \equiv \bar{\chi}^\nu(\bar{\Xi}^\nu) \qquad \text{(mod 2).} \tag{3}$$

From Theorem 16, the left-hand side of (3) is the intersection number of $\phi(F^{n-\nu}, \Xi^\nu)$, while, from § 5, formula (3a), the right-hand side is congruent to the Euler characteristic of M^ν. With that, we have proved the lemma.

In order to prove Theorem 23 now, we have to construct the frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ that satisfy the assumption of the lemma: First, $\bar{\mathfrak{G}}$ is chosen arbitrarily. Furthermore, the system of $\mathfrak{G}$ on the cells $\bar{\xi}^{\nu-1}$ shall be the system of external $(n - \nu)$-fields, extended by the vectors of $\bar{\mathfrak{G}}$; in the remaining part of M^n, $\mathfrak{G}$ will be constructed arbitrarily with the use of Theorem 14. (2) is, in fact, fulfilled with this choice of $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, as one easily confirms by applying the process of calculating the characteristic by recursion (§ 3, no. 2).

We shall not go into the closely-related generalizations of Theorem 23, but merely apply this theorem to the solution of the problem that was posed at the start of this paragraph:

Theorem 24. *For a given n and m with $n \equiv m - 1$ (mod 2), there exists a manifold M^n in which the characteristic class F^{m-1} is not the zero class.*

Addendum. *If $n \equiv m - 1$ (mod 4) then there is indeed an orientable M^n in which F^{m-1} does not vanish.*

The following remarks suffice for the proof of these theorems:

1. The assumption of Theorem 25 is fulfilled when M^n is the topological product of M^ν and an arbitrary $(n - \nu)$-dimensional manifold.

2. If the assumption of Theorem 23 is fulfilled and if M^ν possesses an odd Euler characteristic then it follows from this theorem that the class $F^{n-\nu}$ does not vanish in M^n.

3. There exist manifolds of even dimension that have odd characteristics, and there exist orientable manifolds with dimensions that are divisible by 4 and have odd characteristics. One now sets $m - 1 = n - n$ and constructs M^n as a product manifold.

By a special choice of m, it follows easily from the Addendum that:

Theorem 25. *For any dimension n that is not equal to 1 or 3, there exists an orientable, but not parallelizable, n-dimensional manifold.*

(One observes that, from Theorem 19, the vanishing of all characteristic classes is a necessary condition for parallelizability.)

3. Examples and applications. Let $x_0, x_1, x_2, \ldots, x_n$ be coordinates in an $(n + 1)$-dimensional number-space R^{n+1}, and let $\mathfrak{p}$ mean the position vector $(x_0, x_1, x_2, \ldots, x_n)$ in that space. Let m vector fields $\mathfrak{v}^\mu$ $(\mu = 1, 2, \ldots, m)$ be given in R^{n+1}, and for every μ, let the components v_i^μ $(i = 0, 1, 2, \ldots, n)$ of the vector $\mathfrak{v}^\mu$ be homogeneous functions of first degree of the independent variables $x_0, x_1, x_2, \ldots, x_n$. We project this vector field from the origin of R^{n+1} onto the n-dimensional projective space P^n that completes R^{n+1} into an $(n + 1)$-dimensional projective space. From our homogeneity condition, it follows that in order for m vector fields in P^n to define an m-field in the sense of § 4, no. 1, the $(m + 1)$ vectors $\mathfrak{p} = \mathfrak{v}^0, \mathfrak{v}^1, \mathfrak{v}^2, \ldots, \mathfrak{v}^m$ would have to be linearly independent at all points of R^{n+1}, except for the origin.

We shall employ this convenient representation for the vector fields in projective spaces in the sequel in order to discuss the characteristic classes of n-dimensional projective spaces. So, for example, for $n = 3$ and $m = 3$, the vectors:

$$\begin{aligned} &\mathfrak{v}^0(\ \ x_0,\ \ x_1,\ \ x_2,\ \ x_3) \\ &\mathfrak{v}^1(-x_1,\ \ x_0,\ -x_3,\ \ x_2) \\ &\mathfrak{v}^2(-x_2,\ \ x_2,\ \ x_0,\ -x_1) \\ &\mathfrak{v}^3(-x_3,\ -x_2\ \ x_1,\ \ x_0) \end{aligned} \tag{I}$$

provide a continuous 3-field in 3-dimensional projective space P^3, with which the parallelizability of P^3, and therefore the 3-dimensional sphere, is established by example. One can also find an analogous example in dimension 7 that parallelizes P^7 and the 7-dimensional sphere ([26]).

We now examine the case $n = 5$, $m = 2$, so we concern ourselves with 2-fields in P^5. The three vectors:

$$\begin{aligned} &\mathfrak{v}^0(\ \ x_0,\ x_1,\ \ x_2,\ \ x_3,\ \ x_4,\ x_5) \\ &\mathfrak{v}^1(-x_1,\ x_0,\ -x_3,\ \ x_2,\ -x_5,\ x_4) \\ &\mathfrak{v}^2(-x_2,\ x_2,\ \ x_0,\ -x_1,\ \ 0,\ \ 0) \end{aligned} \tag{II}$$

are linearly-independent only for $x_0 = x_1 = x_2 = x_3 = 0$, so except for the projective line P^1 that is given by $x_0 = x_1 = x_2 = x_3 = 0$, they provide *two* linearly-independent vector fields on P^5 that that we again denote by $\mathfrak{v}^1$ and $\mathfrak{v}^2$, for the sake of simplicity. We now construct a ξ-cell decomposition of P^5, with the use of the notations of no. 2, in which the

([26]) Cf., H. Hurwitz: "Über die Komposition der quadratischen Formen von beliebig vielen Variabeln" (Math. Werke, Band II, pp. 565-571, especially pp. 570, where one finds the matrix that is analogous to I.)

4-dimensional projective space P^4 lies as the hypersurface $x_4 = 0$. The intersection point P of P^1 and P^4 lies in the interior of a cell $\bar{\xi}_0^4$ of the cell decomposition of P^4. Furthermore, two frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ shall be constructed that satisfy the assumptions of the lemma in no. 2: Let the vectors of $\bar{\mathfrak{G}}$ be the vectors $\mathfrak{v}^2$ on the cells $\bar{\xi}^3$, while the 2-systems of $\mathfrak{G}$ shall be the system $\mathfrak{v}^1$, $\mathfrak{v}^2$ on the cells $\bar{\xi}^3$; $\mathfrak{G}$ is arbitrary on the remaining cells ξ^3 of P^5 and can be constructed using Theorem 14. The characters χ and $\bar{\chi}$ that are associated with $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp., actually fulfill the congruence (2) that was required in the lemma:

$$\chi(\bar{\xi}^4) \equiv \bar{\chi}(\bar{\xi}^4) \qquad (\text{mod } 2).$$

In order to prove this, one observes that for any cell $\bar{\xi}^4$, except $\bar{\xi}_0^4$, the relation $\chi(\bar{\xi}^4) \equiv \bar{\chi}(\bar{\xi}^4) = 0$ exists, since $\mathfrak{G}$, as well as $\bar{\mathfrak{G}}$, can be continuously extended into the interior of the cell. One verifies the assertion for the cell $\bar{\xi}_0^4$ by calculating the characteristic by recursion (§ 3, no. 2); in order to be able to apply this method, it suffices that the cell $\bar{\xi}_0^4$ be foreign to the projective space $x_5 = 0$; the vectors $\mathfrak{v}^1$ whose contact points are points of $\bar{\xi}_0^4$ do not lie in P^4 then.

From the statement of the lemma, it now follows that the intersection number of the class F^1 of P^5 with the hypersurface P^4 is the (mod 2) reduced characteristic of P^4; however, this characteristic has the value 1. Therefore, the class F^1 is not the zero class, and will be represented by a projective line.

One achieves the determination of the class F^1 in projective spaces of dimension $4k + 1$ ($k > 0$) with the help of analogous vector fields; one finds:

Theorem 26. *The one-dimensional characteristic class in a real projective space of dimension* $(4k + 1)$ $(k > 0)$ *will be represented by a projective line; it is therefore impossible to find two linearly-independent continuous vector fields in these spaces.*

An algebraic application. We would like to relate our investigation of projective spaces to an algebraic problem that has a close connection with the older investigations ([27]).

We call $(m + 1)$ linearly-independent quadratic $(n + 1)$-sequences of real matrices:

$$A^{(\mu)} = \left(a_{ik}^{\mu}\right) \qquad \begin{pmatrix} \mu = 0,1,2,\ldots,m \\ i,k = 0,1,2,\ldots,n \end{pmatrix} \tag{1}$$

linearly-independent when any matrix $\sum A^{(\mu)} y_\mu$ that comes about through linear combination is non-singular, as long as only one of the real numbers y_μ is non-zero. One then has the following:

([27]) Cf., Hurwitz: Werke, Band II, pp. 565-571 and pp. 641-666; furthermore, Radon: Abh. math. Seminar der Univ. Hamburg, Band I, pp. 1-14.

Lemma. *If there are (m + 1) linearly-independent matrices (1) then there exists an everywhere-continuous m-field in projective space P^n.*

Proof: If B is any non-singular $(n + 1)$-rowed matrix then obviously the matrices $B A^{(\mu)}$ ($\mu = 0, 1, \ldots, m$) are also linearly-independent; since we can choose $B = (A^{(0)})^{-1}$, we can assume from now on that:

$$a_{ik}^{0} = \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k. \end{cases} \tag{2}$$

We now understand $\mathfrak{v}^{\mu}$, for $\mu = 0, 1, \ldots, m$, to mean the vectors of R^{n+1} whose i^{th} component ($i = 0, 1, 2, \ldots, n$) is given by:

$$v_i^{\mu} = \sum_{k=0}^{n} a_{ik}^{\mu} x_k ; \tag{3}$$

if one recalls (2) then $\mathfrak{v}^0$ is the position vector $\mathfrak{p} = (x_0, x_1, \ldots, x_n)$ in R^{n+1}. From no. 3, it follows that the statement of the lemma will be proved, as long as one can show that the $(m + 1)$ vectors $\mathfrak{v}^{\mu}$ are linearly-independent for $\mathfrak{p} \neq 0$.

Therefore, let $\sum_{\mu=0}^{m} y_{\mu}\mathfrak{v}^{\mu} = 0$ for a certain vector $\mathfrak{p} \neq 0$; i.e.:

$$\sum_{k,\mu} a_{ik}^{\mu} x_k y_{\mu} = 0 \qquad (i = 0, 1, 2, \ldots, n).$$

Since $\mathfrak{p} \neq 0$, the rank of the matrix $(\sum_{\mu} a_{ik}^{\mu} y_{\mu})$ is less than $(n + 1)$. Since the matrices $A^{(\mu)} = \left(a_{ik}^{\mu}\right)$ are linearly independent, this is possible only when all $y_{\mu} = 0$. This was to be proved.

The lemma now permits the following algebraic formulation of Theorem 26:

Theorem 27. *Any three quadratic* $(4k + 2)$*-rowed matrices are linearly independent* $(k \geq 0)$.

APPENDIX I

The one-dimensional characteristic class of an orientable three-dimensional manifold

In § 5, no. 3, we saw that that for a three-dimensional manifold M^3, the vanishing of the one-dimensional characteristic class F^1 is a necessary and sufficient condition for parallelizability. We further mentioned that for an orientable M^3 with a differentiable simplicial decomposition, F^1 is always the zero class, but left the reader responsible for the proof of this fact. It shall now be returned to under somewhat different differentiability assumptions.

1. A combinatorial lemma. The following lemma is interesting in its own right and is useful for the study of three-dimensional manifolds.

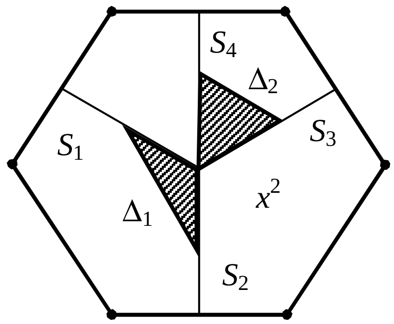

Figure 4.

Lemma. *Any cell decomposition of a three-dimensional manifold M^3 can be refined to a subdivision U such that any two-dimensional homology class (mod 2) of M^3 can be represented by a sub-cycle of U that consists of one or more disjoint two-dimensional manifolds.*

One must then show that any two-dimensional cycle z^2 of the given cell decomposition in U gives one or more disjoint surfaces that collectively define a cycle that is homologous to z^2. The proof proceeds in two steps:

1. z^2 is a cycle (mod 2), so an even number of polygons of z^2 meet along an edge of z^2. We now consider an edge ξ^1 of z^2 at which more than two (say, $2n$) polygons meet. Let ξ_1^0 and ξ_2^0 be the boundary points of ξ^1 and let x^2 be the dual cell to ξ^1 in the given cell decomposition of M^3. We denote the intersecting line segments of x^2 with the $2n$ polygons that meet at ξ^1 by $s_1, s_2, \ldots, s_{2n}$, where the numbering shall be given by the natural cyclic ordering of these line segments (see Fig. 4 for $n = 2$). Between two successive line segments s_{2k-1} and s_{2k} ($k = 1, 2, \ldots, n$), we now interpolate a small triangle Δ_k and construct the cone K_{k1} over the boundary of Δ_k that has its vertex at ξ_1^0. Analogously, K_{k2} will be constructed with its vertex at ξ_2^0. $K_{k1} + K_{k2}$ is a two-dimensional cycle that is homologous to zero, so $z^2 + \sum_{(k)}(K_{k1}+K_{k2})$ is a cycle homologous to z^2, in which ξ^1 is replaced with edges, each of which is incident with precisely two polygons of this new cycle. One naturally introduces a suitable sub-division of the given cell decomposition by carrying out this construction.

If all edges of z^2 at which more than two polygons met were removed by this construction then one would obtain a cycle $\bar{z}^2$ that would be homologous to z^2 and would consist of one or more disjoint pseudo-manifolds.

2. Let ξ^0 be an arbitrary vertex of $\bar{z}^2$. We construct a sub-division U in which the stars of the vertices ξ^0 are disjoint. Let S^2 be the boundary sphere of the star of ξ^0. The intersection of $\bar{z}^2$ with S^2 consists of some disjoint closed polygon perimeters that bound a sub-complex C^2 of S^2. We construct the cone K^2 that has its vertex at ξ^0 over the boundary C^2. $C^2 + K^2$ is a two-dimensional cycle that is homologous to zero, so $\bar{z}^2 + C^2 + K^2$ is a cycle that is homologous to $\bar{z}^2$, which we replace $\bar{z}^2$ with.

If one carries out this construction for every vertex then a cycle arises that is homologous to $\bar{z}^2$, as well as z^2, that consists of some disjoint two-dimensional surfaces.

3. Determination of the class F^1. We now determine the class F^1 of a given *orientable* manifold M^3 by comparing M^3 to a "standard manifold" M_0^3. M_0^3 is either the three-dimensional projective space P^3 or the topological product $T^3 = S^2 \times S^1$ of a sphere and a circle. Both standard manifolds are parallelizable. (The parallelizability of P^3 was proved in § 6, no. 3; from Theorem 23, the class F^1 is the zero class in T^3, so T^3 is parallelizable. One can, moreover, also give a continuous 3-field on T^3 directly.) The given manifold M^3 now fulfills the following *assumption:*

Any two-dimensional manifold that is embedded in M^3 without singularities possesses a neighborhood that can be mapped into a standard manifold topologically and continuously differentiably.

This assumption is only a differentiability assumption, since any two-dimensional manifold F that is embedded in M^3 without singularities possesses a neighborhood that can be mapped topologically into one of the standard manifolds. In order to show this, one constructs a manifold without singularities F' in P^3 or T^3 that is homeomorphic to F. (Three cases must be distinguished in the process of making this construction: a) F is orientable; F' can then be constructed in P^3 or T^3. b) F is not orientable and possesses an odd Euler characteristic; F' can then be constructed in P^3. c) F is not orientable and possesses an even Euler characteristic; F' can be constructed in T^3.) Now, since M^3 is orientable, F' is two-sided ([28]) in M_0^3, as long as F is two-sided in M^3, and likewise F' is one-sided in M_0^3 when F is one-sided in M^3; a topological map of F onto F' can then always be extended to a topological map of a neighborhood of F to a neighborhood of F'. With that, our assertion is proved.

We now consider the cell decomposition U of M^3 that was mentioned in the lemma, whose cells we denote by ξ^r; furthermore, let F now be a sub-cycle (mod 2) of U, in particular, that consists of the cells $\bar{\xi}^3$ of U. If we imagine that a continuous 2-field is constructed on the standard manifold M_0^3 then the map of a neighborhood of F into M_0^3, which exists by assumption, induces a continuous 2-field $\mathfrak{F}$ on that neighborhood. The 2-systems of $\mathfrak{F}$ that contact the points of the cells $\bar{\xi}^1$ define a one-dimensional framework (§ 4, no. 2) that, from Theorem 14, can be extended to a one-dimensional framework $\mathfrak{G}$ that is defined on all of M^3 and consists of 2-systems. The character χ (§ 4, no. 4) that is

([28]) On the relationships between the concepts of "orientable" and "two-sided," cf., Seifert-Threlfall, § 76.

associated with $\mathfrak{G}$ has the value 0 for every cell $\bar{\xi}^2$ if the 2-field that is induced by $\mathfrak{G}$ on $\dot{\xi}^2$ is continuously extended into the interior of $\bar{\xi}^2$. One then has $\chi(F) = 0$. In other words: The characteristic class F^1 has intersection number zero with F. Now, since F^1 has intersection number zero with any surface F, and on the other hand, from our lemma, any two-dimensional homology class (mod 2) can be represented by one or more two-dimensional manifolds F, F^1 has intersection number zero with any two-dimensional homology class, so from the Poincaré-Veblen duality theorem, it is the zero class (mod 2).

APPENDIX II

On the representation of hypersurfaces in Euclidian space by systems of equations ([29])

In this appendix, we deduce a consequence of the intersection theorem 23. In analogy to § 6, no. 2, we understand a ν-dimensional hypersurface that is embedded in n-dimensional Euclidian space to mean a sub-complex of the cell decomposition of R^n that is the topological image of a ν-dimensional parameter manifold by means of a topological continuously-differentiable map ($1 < \nu < n$).

Now, let $x_1, x_2, \ldots, x_n$ be Cartesian coordinates in R^n and let $(n - \nu)$ continuously-differentiable functions $f_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, n - \nu)$ of these coordinates be given. Now, the equations:

$$f_i(x_1, x_2, \ldots, x_n) = 0 \tag{1}$$

define a ν-dimensional hypersurface M^ν, and if the functional matrix of the functions f_i has rank $(n - \nu)$ at every point of M^ν then we will call M^ν a "hypersurface that is regularly representable by equations."

Theorem 28. *Any hypersurface that is regularly representable by equations has an even Euler characteristic.*

Proof: The gradients grad f_i of the functions f_i that contact the points of M^ν are disjoint to M^ν (§ 6, no. 2), and the gradients that contact a point of M^ν are, by assumption, linearly independent, so they define an $(n - \nu)$-system. Since this system varies continuously with its contact point, moreover, M^ν possesses an external $(n - \nu)$-field, in the sense of § 6, no. 2.

We close the Euclidian space R^n into the n-dimensional sphere S^n with an infinitely distant point. Our hypersurface M^ν that lies in S^n fulfills the assumption of Theorem 23, so, from that theorem, its characteristic is congruent (mod 2) to the intersection number of the characteristic class $F^{n-\nu}$ of S^n with M^ν. Since $F^{n-\nu}$ is trivially the zero class in S^n, this intersection number vanishes, with which our assertion is proved.

([29]) This Appendix came about as a follow-up to a question of H. Seifert.

It follows, in particular, from Theorem 28 that a hypersurface that is regularly representable by equations and homeomorphic to a real or complex plane cannot lie in any Euclidian space of any dimension ([30]).

([30]) The Euler characteristic of the real projective plane is 1, while that of the complex projective plane is 3 (cf., B. L. van der Waerden: "Topologische Begründung des Kalkuls der abzählenden Geometrie," Math. Ann. **102** (1929), 337-362, especially pp. 361.) The fact that the real projective plane cannot be regularly represented by equations in any R^n follows from the general theorem that any manifold that is regularly representable in R^n is orientable. (For the proof, cf., footnote 25.) This theorem was already proved by Poincaré (J. Ec. poly. (2), **I**, pp. 3). The representation of the projective plane in R^4 that was given in pp. 301 of the book by Hilbert and Cohn-Vossen on intuitive geometry (Berlin, J. Springer, 1932) is not regular.