"Über einen allgemeinen Satz der geometricscher Optik und einige Anwendungen," Phys. Zeit. **4** (1902-03), 114-117.

On a general theorem of geometric optics and some applications ([1])

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The second communication concerns a general theorem of geometrical optics, and indeed one that comes from the physically-realizable domain in its own right. The theorem has a geometric nature, insofar as it includes only spatial elements, except for the refraction exponents, and a physical nature, insofar as it can be proved using the assumption (and in fact, the only one) that the paths of the pencil considered are the shortest (or longest) paths between two points.

The theorem splits into two sub-theorems, according to whether one is dealing with a planar or spatial pencil. However, it shall be expressly emphasized from the outset that one should understand "planar" pencils to means not only the ones that remain in the same plane during the entire course of their motion, but also pencils with planar aperture angles.

I. The theorem for "planar" pencils. An infinitely-thin plane pencil with the aperture angle dw emanates from a point, which should possess the linear width ds' at an arbitrary point of its path, which has been constructed from the principle of least time. Conversely, we then let a second infinitely-thin pencil emanate from a point with the line element ds' in the plane that goes through ds' and the path to the second point, and indeed, in such a way that the axes coincide. Let the angle of that pencil at the second point and the linear width at the first point be dw' and ds, resp.; the theorem then reads ([†]):

$$n\, ds\, dw = n'\, ds'\, dw'.$$

n and n' are the refraction exponents of the medium at the first and second point, resp.

If the line element (which is now denoted by dl) is not perpendicular to the pencil, but defines the angle w with its normal, then the theorem will read:

$$n \cos w\, dw\, dl = n' \cos w'\, dw'\, dl'.$$

([1]) A more thorough presentation will be given later.

([†]) Translator's note: A minor editorial revision of the notation has been made here and in the following equations for the sake of clarity and emphasis.

II. Theorem for spatial pencils. If we replace the two plane pencils with two spatial ones with the aperture angles $d\omega$ and $d\omega'$, and denote their cross-sections at the two points by dq and dq', resp., then the corresponding theorem will read:

$$n^2\, d\omega\, dq = n'^2\, d\omega'\, dq'.$$

If the normal to the surface element (which is now denoted by df) defines an angle ϑ with the pencil then the last equation will now take the form:

$$n^2 \cos\vartheta \cdot d\omega \cdot df = n'^2 \cos\vartheta' \cdot d\omega'\, df'.$$

Proof

The proofs of the two theorems follow easily from the principle of least time; e.g., when one follows a path that is similar to the one that **Kirchhoff** and **Clausius** followed in their well-known treatises.

If one defines a rectangular, planar, coordinate system around the starting point of the pencil whose xz-plane coincides with the pencil plane and whose z-axis points in the direction of the light ray, and further lets c denote the speed of light in empty space and lets T denote the time that the light takes between the two points, then one will immediately get:

$$n\,\frac{dw}{ds'} = n'\,\frac{dw'}{ds'} = c\,\frac{\partial^2 T}{\partial x\,\partial x'}$$

or

$$n\, dw\, ds = n'\, dw'\, ds'.$$

If we preserve the origin and position of the z-axis for the coordinate system for a spatial pencil then we will find analogously that the ratios of the aperture angles and cross-sections are:

$$n^2\,\frac{d\omega}{dq'} = n'^2\,\frac{d\omega'}{dq} = c^2\left\{\frac{\partial^2 T}{\partial x\,\partial x'}\cdot\frac{\partial^2 T}{\partial y\,\partial y'} - \frac{\partial^2 T}{\partial x\,\partial y'}\cdot\frac{\partial^2 T}{\partial x'\,\partial y}\right\},$$

or

$$n^2\, d\omega\, dq = n'^2\, d\omega'\, dq'.$$

It is remarkable that **Kirchhoff** and **Clausius** have overlooked the theorems above, and especially for spatial pencils, or at least they have refrained from formulating them. For **Clausius**, this is perhaps explained by the fact that, from the outset, he started with the relationship between the elements of the pencil at conjugate points. In any case, the aforementioned authors had the apparatus that was necessary for the proof of the theorem so well in hand that Professor **Abbe** – who, as he communicated to me, had known of the theorem that relates to spatial pencils for a long time – was almost of the opinion that the theorem belongs to **Kirchhoff** or **Clausius**.

The theorems above (or, at least, the one that relates to spatial pencils) seem obvious, with no further assumptions, when they are considered from an energetic viewpoint (a minor argument will necessitate the introduction of the refraction exponents), and in fact, the theorems are first found energetically by a search for homologues of the general theorems of geometrical optics, on the one hand, and the theory of diffraction and interference, on the other.

Just the same, it would be much more proper to prove a so-to-speak geometrical theorem by mathematical tools to the greatest extent possible.

One can ask whether there are further such general theorems that are based upon the assumption of the principle of least light duration. Now, it can be easily recognized that there are no further theorems that include only the pencil elements that were used above, namely, the aperture angle and width (cross-section, resp.), as well as the refraction exponents.

Applications

I. To photometry. As is known, the laws of photometry seem very simple in their theoretical treatment, but in fact, there have been many errors made in their implementation, up to now. Now, it is a particular advantage of our theorems that they can be most easily applied to the question of photometry. Naturally, in order to employ them, one must fulfill the assumption that light follows the shortest paths in the domain in question. The fact that the theorems are so convenient to apply is based, on the one hand, upon their simple form, and on the other, and most importantly, upon the fact that one is not dealing with a special relationship – such as, e.g., the one between object and image – but with a general one between two spatial elements that are associated by the principle of shortest light duration.

At this point, I would like to prove the convenience of the applicability of the theorems to the problem of photometry by a simple example.

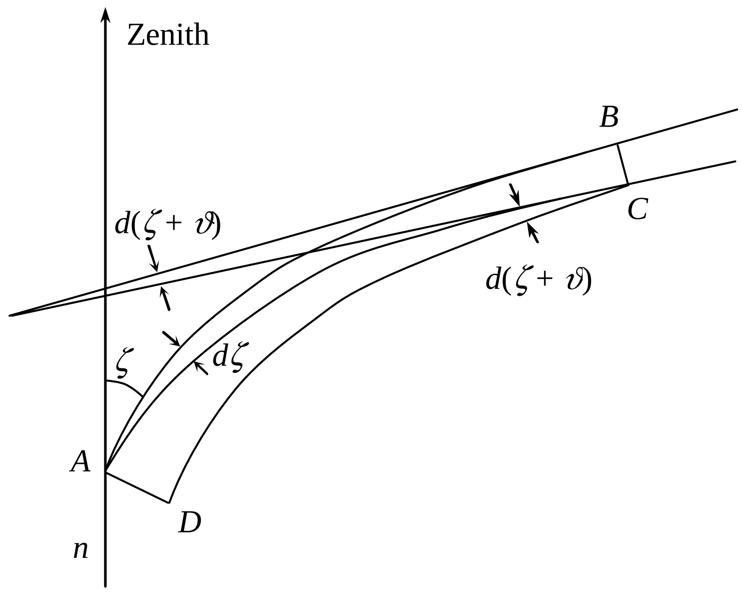
If we have a small, illuminated surface and an arbitrary optical system then we would like to answer the known question of what specific intensity is possessed by an arbitrary surface that lies after or between the system of lenses – i.e., how much light emanates from a unit area of this surface into the spatial aperture angle of measure 1. If one denotes the specific intensities of the light source and the surface by L and L', resp., the surface element of the ray angles by dq and $d\omega$, and further denotes the quantities that are associated with them by the principle of least light duration by $d\omega'$ and dq' then, if one ignores the losses that are due to reflection and absorption, one will have:

$$L\, d\omega\, dq = L'\, d\omega'\, dq'.$$

However, on the other hand, from the theorem above for spatial pencils, one will have:

$$n^2\, d\omega\, dq = n'^2\, d\omega'\, dq',$$

and one will then obtain the known result. The specific luminosity of the light source and any of the light from this intermediate surface will be directly proportional to the square of the refraction exponent. If one interpolates the surface at any other place in a body then the specific intensity will be raised by the ratio of the squares of the refraction exponents.



II. Application to the dioptrics of the atmosphere. A few years ago, **A. Gleichen** ([1]) examined how a cylindrical ray bundle that entered the atmosphere would be modified by it; the pencil was astigmatic. Some interesting consequences came out of that: The luminosity of stars is larger at the zenith than it would be with no atmosphere, and the luminosity would be the same at 60°, and then it decreases rapidly and amount to only 83 % at the horizon.

The calculation of the change in cross-section that such an infinitely-thin pencil of rays experiences does not result from a very simple process. However, the simple final formula that **Gleichen** arrived at seemed to me to prove that a simple proof must also exist.

A plane that goes through a star, which is thought of as infinitely-distant, and the center of the Earth, shall be called a *meridianal plane*, and any plane that is perpendicular to it, a *sagittal plane*. Now, as one easily sees, the determination of the ratios of the widths of the pencil in sagittal sections raises no difficulties at all, but the same is not true for the meridianal sections. From the theorem above that is valid for plane pencils, that determination can be reduced to the determination of the ratios of the ratios of the angles involved, namely, dw and dw'.

We draw (cf., the Fig.) an auxiliary light path from one end (e.g., the upper one) of the first linear cross-section to the other one (i.e., under the second one) and imagine that the center of the eye has been placed at the intersection of the light paths AB and AC. We call the apparent distance from the zenith to the light path $d\zeta$. Moreover, if we draw tangents to the upper light path and the auxiliary path at the place where the former enters the atmosphere then if one denotes the angle of refraction by ϑ then the angle between those tangents will be $d(\zeta + \vartheta)$. With that, the problem is essentially solved. One then easily obtained the formula that was given by **Gleichen** for the ratios of the cross-sections:

([1]) Verh. der deutschen Phys. Gesellschaft, II Jahrg., nos. 2 and 16, 1900.

$$\frac{q}{q'} = \frac{\sin(\zeta+\vartheta)}{n^2 \sin\zeta}\left[1+\frac{d\vartheta}{d\zeta}\right],$$

in which n means the refraction exponent at the position of the observer.

III. Application to the theory of mappings. In the theorems above, it was always assumed that the plane or spatial pencil considered possessed linear or planar cross-sections, resp., that had the same order of magnitude as the angle. However, one can also consider the case that was excluded up to now in which the rays intersect. To that end, one does not need to do anything but apply the theorem yet a second time, when one assigns the point of intersection to the second location. Here, we would like to consider only the simplest case of a spatial pencil for which the light rays that emanate homocentrically from the first point unite again homocentrically. As one sees, the theorems yield the following relations at the conjugate points:

$$n \cos w \cdot dw \cdot dl = n' \cos w' \cdot dw' \cdot dl',$$
$$n^2 \cos \vartheta\, d\omega\, df = n'^2 \cos \vartheta' d\omega' df'.$$

The theorems thus remain completely valid, although the relationships between the elements have changed: The angles, lines, and surface elements now correspond to others of the same kind.

One sees the following consequence, among others: Should a line element from any infinitely-thin sub-pencil of a plane pencil of finite aperture be mapped with the same magnification then one would have a kind of sine law. That law would not then seem to be specific to the centered line systems, but would have a more general sort of validity.

If a light ray that emanates normal to the first line element likewise falls upon the second line element normally then the sine law would be valid in its usual form. Similar considerations can be posed for spatial pencils, but we shall not go into them here.

Let us further remark: One obtains the sine law for centered systems immediately from the theorem for spatial pencils, which is something that needs only to be suggested. The geometric character of the sine law – as the condition for the map of a surface element that lies on the axis to give it the same magnification in all zones – is clear with no further assumptions, just like the proof that is given in the *Handbuch der Physik*.

Helmholtz has proved the sine law energetically, but with all reverence for **Helmholtz**, and despite his correct result, I must say that I do not regard the manner of proof to be correct. Mostly, he generalized the theorem about the ratio of the luminosities of the object and image that had been proved only for infinitely-thin pencils to one with finite apertures. In my opinion, one must prove the theorem on the luminosity ratio by a combination of the energy theorem and the sine law, and not, as **Helmholtz** did, derive the sine law from the energy theorem and the luminosity ratio.

(From the transcript that was reviewed by Dr. B. **Borchardt**)

(Received on 8 October 1902)