"Kinematik, Liesche Kreisgeometrie und Geraden-Kugel-Transformation," Elemente der Mathematik **8** (1953), 4-13.

Kinematics, Lie's circle geometry, and the line-sphere transformation ([1])

By Karl STRUBECKER, Karlsruhe

Translated by D. H. Delphenich

I. Planar kinematics

One can, following STUDY ([2]), represent motions (e.g., rotations, parallel displacements) in the plane by four homogeneous parameters $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$ with bilinear composition.

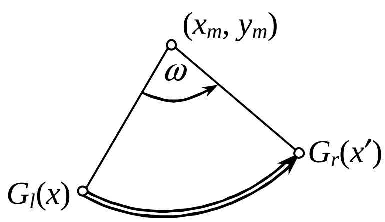

Figure 1.

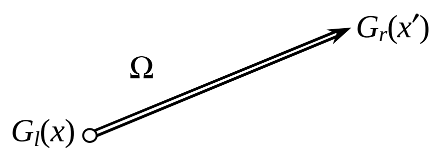

Figure 2.

If (x_m, y_m) is the center of rotation and ω is the angle of that rotation (Figure 1) then one will have:

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = -\cot\frac{\omega}{2} : x_m : y_m : 1.$$

$\alpha_3 = 0$ gives (Figure 2) parallel translation by the vector $(-2\alpha_2 / \alpha_0 , 2\alpha_1 / \alpha_0)$ whose length is Ω. In the homogeneous, Cartesian coordinates $x_1 : x_2 : x_3 = x : y : 1$, the equations of motion then read:

$$\begin{aligned}
(\alpha_0^2 + \alpha_3^2)\, x_1' &= (\alpha_0^2 - \alpha_3^2)\, x_1 + 2\,\alpha_0\,\alpha_3\,x_2 + 2\,(\alpha_1\,\alpha_3 - \alpha_0\,\alpha_2)\,x_3\,, \\
(\alpha_0^2 + \alpha_3^2)\, x_2' &= -2\,\alpha_0\,\alpha_3\,x_1 + (\alpha_0^2 - \alpha_3^2)\, x_2 + 2\,(\alpha_0\,\alpha_1 + \alpha_2\,\alpha_3)\,x_3\,,
\end{aligned}$$

([1]) Lecture presented on 19 September 1951 at the conference of the DMV in Berlin.

([2]) EDUARD STUDY:

a) "Über Systeme komplexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen," Monats. Math. Phys. **1** (1890), 283-355.

b) "Von den Bewegungen und Umlegungen," Math. Ann. **39** (1891), 441-566.

$$(\alpha_0^2+\alpha_3^2)\,x_3' = \qquad\qquad (\alpha_0^2+\alpha_3^2)\,x_3,$$

and the bilinear composition formulas are:

$$\begin{aligned}
\alpha_0'' &= \alpha_0\alpha_0' \quad * \quad * \quad - \alpha_2\alpha_3', & \alpha_2'' &= \alpha_0\alpha_2'+\alpha_2\alpha_0'+\alpha_3\alpha_1'-\alpha_1\alpha_3',\\
\alpha_1'' &= \alpha_0\alpha_1'+\alpha_1\alpha_0'+\alpha_2\alpha_3'-\alpha_3\alpha_2', & \alpha_3'' &= \alpha_0\alpha_3'+\alpha_3\alpha_0' \quad * \quad * \quad .
\end{aligned}$$

One can, like STUDY, write these formulas in an especially elegant way, when one introduces a system of higher complex numbers that take the form:

$$\alpha = \alpha_0\, e_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3,$$

or

$$x = \qquad x_1\, e_1 + x_2\, e_2 + x_3\, e_3,$$

resp., which are a limiting case of Hamilton's quaternions, and shall be referred to as the system of STUDY quaternions, whose four units e_0, e_1, e_2, e_3 define the following product table:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	0	0	$-e_2$
e_2	e_2	0	0	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$.

The product formula then reads simply:

$$\alpha'' = \alpha\,\alpha',$$

and the equations of motion themselves read:

$$x' = \alpha^{-1}\, x\, \alpha,$$

in which:

$$\alpha^{-1} = \frac{\alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3}{\alpha_0^2+\alpha_3^2} = \frac{\bar\alpha}{\alpha\,\bar\alpha} = \frac{\bar\alpha}{N(\alpha)}$$

is the reciprocal quaternion to α.

$N(\alpha) = \alpha_0^2+\alpha_3^2 = 0$ characterizes the singular motions.

II. Kinematic map of the line space and quasi-elliptic geometry

If one interprets the four homogeneous parameters α_i as homogeneous, Cartesian point coordinates in space, when one sets:

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = z : x : y : 1,$$

such that α_0 then corresponds to the (vertical) z-direction and $\alpha_3 = 0$ represents the plane at infinity then *any motion* α in space will be associated with an *image point* $A(\alpha)$. In Lie's way of expressing things, space will be the *parameter space* of a planar motion.

In 1911, W. BLASCKE ([1]) and J. GRÜNWALD ([2]) realized this map of planar motions to spatial points, which E. MÜLLER and E. KRUPPA ([3]) later referred to as the *kinematic map*, by a simple *geometric construction*.

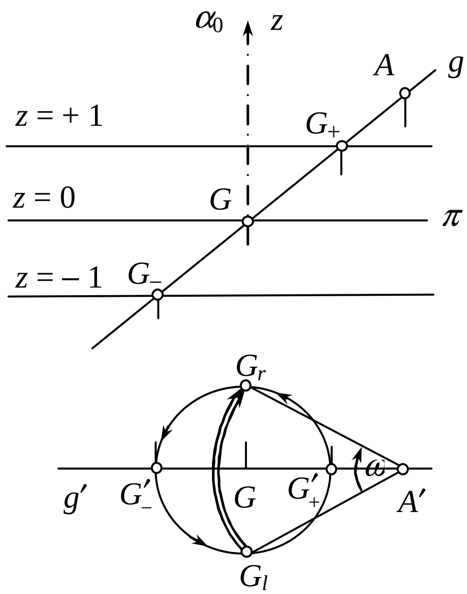

Figure 3.

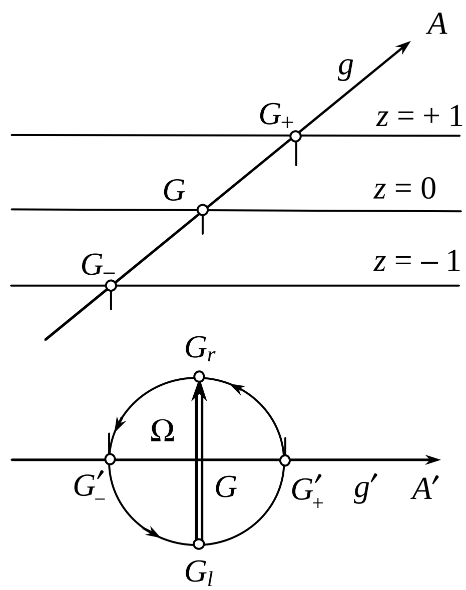

Figure 4.

In fact, if one draws (Figures 3 and 4) the ∞^2 lines g of a bundle through the image point $A(\alpha)$ in space, and one cuts it with the planes $z = -1$ and $z = +1$ at the points G_- and G_+, and one further looks for their base planes G'_-, G'_+ on the plane π (viz., $z = 0$), and one pivots these points G'_-, G'_+ around their midpoint G through $+\pi/2$ then one will get two points G_l and G_r as the "*kinematic image*" *of the line* g, which correspond precisely to the ones whose image point was $A(\alpha)$ under the motion α.

([1]) WILHELM BLASCHKE, "Euklidische Kinematik und Nichteuklidische Geometrie, I and II," Z. Math. Phys. **60** (1911), 61-91, 203-204. On that, cf., the brief presentation of W. BLASCHKE in: F. KLEIN: *Vorlesungen über höhere Geometrie* (Springer, Berlin, 1926), § 81, and the thorough presentation in the booklet of W. BLASCHKE, *Ebene Kinematik*, Hamburger math. Einzelschriften 25. Heft (Leipzig and Berlin, 1938).

([2]) JOSEF GRÜNWALD, "Ein Abbildingsprinzip, welches die ebene Geometrie und Kinematik mit der raumlichen Geometrie verknüpft," Sitz.-Ber. Akad. Wien, Math.-naturw. Kl., Abt. IIa, **120** (1911), 677-741.

([3]) EMIL MÜLLER, *Vorlesungen über Darstellende Geometrie*, I Band: *Die linearen Abbildungen*, revised by ERWIN KRUPPA (Deuticke, Leipzig and Berlin, 1923).

It is occasionally convenient to think of the plane p, which is regarded as the locus of the left image points G_l and the right image points G_r, as being divided into two *sheets*, and then speaking of the plane π_l of the left image points and the plane π_r of the right ones.

It then follows that the left and right images of two lines g, h that intersect at $A(\alpha)$ will have equal distances between them:

$$\overline{G_l\,H_l} = \overline{G_r\,H_r}\,.$$

A pencil of rays will have two congruent, linear point sequences as its kinematic image, a *point* (i.e., a pencil of rays) will have a *motion*, and a *plane* (i.e., a ray field) will have a *transfer* $G_l \to G_r$. The image of a real point A (i.e., $\alpha_3 \neq 0$) will be a rotation, and the image of a point at infinity (i.e., $\alpha_3 = 0$) will be a parallel displacement. The point at infinity O of the vertical α_0-axis will have the identity motion $G_l \equiv G_r$ for its image.

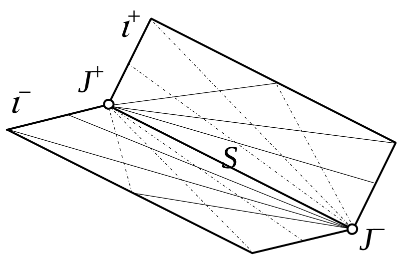

Figure 5

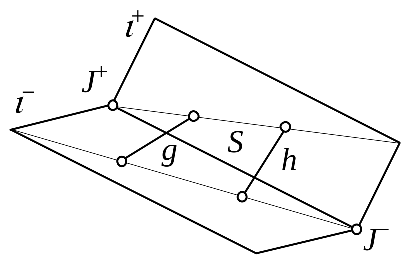

Figure 6

The singular motions $\alpha_0^2 + \alpha_3^2 = 0$ – i.e., $\alpha_0 \pm i\alpha_3 = 0$ – correspond (Figure 5) to the points of a conjugate complex pair ι^+, ι^- of planes: $z = \pm\, i$, the singular transfers likewise correspond to planes through the absolute points J^+, J^- of the line at infinity s of the image plane π (i.e., $z = 0$), such that a self-dual singular structure in the kinematic parameter space will be distinguished that consists, in total, of two conjugate complex planes ι^+, ι^-, and conjugate complex points J^+, J^-, and that (when regarded as a locus of lines) will carry two distinguished pairs of restricted pencils of rays of "generators," namely, the left generators (J^+, ι^+), (J^-, ι^-), and the right generators (J^+, ι^-), (J^-, ι^+).

If one distinguishes this structure as the *absolute structure* of a projective metric then the space will take on a *quasi-elliptic structure.* It consists of a limiting case of elliptic space – viz., the so-called *quasi-elliptic space* – whose geometry is very similar to that of elliptic space. For example, there are also Clifford parallels, Clifford displacement, etc., here.

The lines g, h, for example, are left-parallel in the Clifford sense when they have the same left kinematic image points $G_l = H_l$ in common; they then intersect (Figure 6) the metric structure at points with the same pairs of left generators.

Lines that are right-parallel in the Clifford sense are defined analogously.

All of the mutually left- (right-, resp.) parallel lines define a ray net, namely a:

Left net	Right net

with the representation:

$$\begin{Vmatrix} \beta_1 & \beta_2 & \beta_3 \\ p_{01}+p_{23} & p_{02}+p_{31} & p_{03} \end{Vmatrix} = 0$$	$$\begin{Vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ p_{01}-p_{23} & p_{02}-p_{31} & p_{03} \end{Vmatrix} = 0,$$

that is generally elliptic, while in the case $\beta_3 = 0$ ($\alpha_3 = 0$, resp.) it is parabolic, and its guiding lines are two conjugate complex generators of the:

Left family	Right family

of the absolute structure, which will coincide with the line at infinity s in the parabolic case.

For this net, there is a one-parameter continuous group of collineations, under which the point x in space will be displaced along the rays of the net (and therefore rectilinearly!), and which, because they will thus necessarily leave the absolute structure of the quasi-elliptic space fixed, one will then refer to as quasi-elliptic *Clifford displacements*. More precisely, one speaks of left-displacements (right-displacements, resp.) according to whether the path-lines of the displacement are left-parallel (right-parallel, resp.).

If one also composes the homogeneous coordinates x_i of the Study quaternions in space, when one sets:

$$x = x_0\, e_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3\,,$$

and if:

$$\alpha = \alpha_0\, e_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3\,,$$

and

$$\beta = \beta_0\, e_0 + \beta_1\, e_1 + \beta_2\, e_2 + \beta_3\, e_3$$

mean arbitrary Study quaternions then these quasi-elliptic Clifford:

Left displacements	Right displacements

will read:

$x' = \beta\, x,$	$x' = x\, \alpha.$

They will define the commutative, three-parameter groups:

$\mathfrak{G}_3^l$	$\mathfrak{G}_3^r\,,$

which will collectively yield the $\mathfrak{G}_6$ of quasi-elliptic motions:

$$x' = \beta\, x\, \alpha\,.$$

Now, how does one express such a Clifford displacement when it is applied to the line (g) in space in the planes π_l and π_r (which cover the plane π) of the left and right kinematic image points (G_l) and (G_r)?

The answer gives the so-called *fundamental theorem of the kinematic map:*

The left image field π_l will experience a motion under a left displacement, while the right one π_r will remain fixed.	The right image field π_r will experience a motion under a right displacement, while the left one π_l will remain fixed.

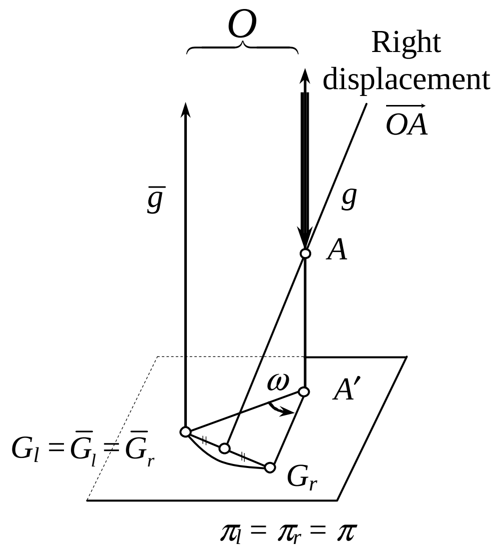

Figure 7.

Application: the bundle of vertical lines $\bar{g}$ through the infinitely-distant point O on the x_0-axis will be mapped to the identical image pair $\bar{G}_l = \bar{G}_r$. (Figure 7). It will then follow that:

If one brings (Figure 7) the point O ($x = e_0$) to A ($x = \mathrm{e}_0$ $\alpha = \alpha$) by a right displacement (α) then $\bar{g}$ will go to g, so the left image field π_l will remain fixed ($G_l = \bar{G}_l$), although the right one π_r will experience a *motion* ($G_r = \bar{G}_r$), namely, the one that belongs to the point $A(\alpha)$ and is the image of the right displacement $\overrightarrow{OA}$.

The kinematic image of the point A (e.g., rotation, translation) *is then identical with the image of the right displacement that takes O to A.*

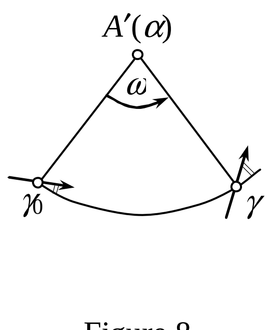

Figure 8.

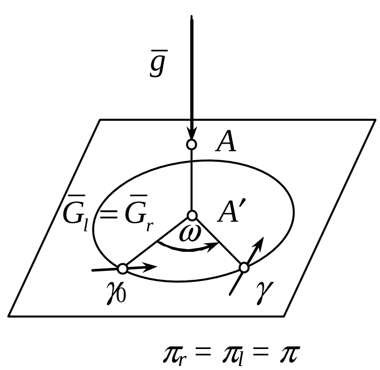

Figure 9.

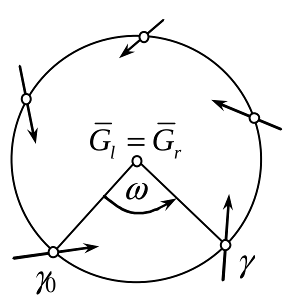

Figure 10.

The rotational angle ω (the translation segment Ω, resp.) is therefore equal to twice the quasi-elliptic displacement length $2\overrightarrow{OA}$.

III. Kinematic map of lines to turbines. Lie's circle geometry.

One can (Figure 8) characterize the *position of the plane* π by the *position of an oriented line element* γ *with respect to a given basic element* γ_0 *(i.e., an Ur-element).* STUDY called γ a "*soma,*" and γ_0, the "*Ur-soma*" *(or "proto-soma").* Any motion α (viz., $\pi_l \to \pi_r$) takes the basic element γ_0 (which should lie in π_l) to an oriented line element γ (in π_r), which, conversely, determines the motion α uniquely (by its position with respect to γ_0), and thus the motion α is mapped to the image point $A(\alpha)$ in a one-to-one way. This invertible, single-valued "*kinematic map of the oriented line element* γ *to the spatial point* $A(\alpha)$" is based upon the following ([1]):

From the fundamental theorem, the point groups (x), $(\dot{x})$ in space, which arise from each other by a *right displacement* α, will thus have images (γ), $(\dot{\gamma})$, resp., in the plane π figure of oriented line elements that emerge from each other by a *motion*, and indeed by the motion α of π_l to π_r that corresponds to right displacement.

Which figures of oriented line elements correspond to the point A of a line g under our kinematic map?

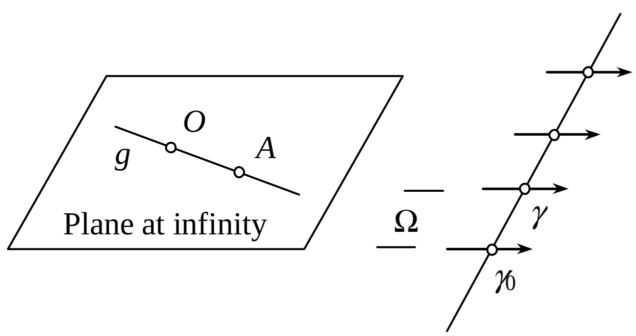

Figure 11.

1. Let the line $\bar{g}$ contain the point O, and:

a) Let there be a real – thus vertical – line (Figure 9). One will then have $\bar{G}_l = \bar{G}_r$, so the points of g will map to the rotations of π_l to π_r around the fixed point $\bar{G}_l = \bar{G}_r = A'$, by which the oriented basic element γ_0 will describe a rotational family of oriented line

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([1]) The line elements γ, as representatives of planar motions, in the sense of EDUARD STUDY's *Geometrie der Dynamen* (Teubner, Leipzig, 1903), are referred to as (positive) *somas*. Since the kinematic map analogously takes the *planes* in quasi-elliptic space to *transfers* of the plane π in a one-to-one way, which, following STUDY, one can identify with the negative somas γ' in the plane π (after choosing the Ur-soma γ_0), that will yield *a similar map of the negative somas of p to the planes* of quasi-elliptic space. From a presentation of FRANK LÖBELL, one can represent positive and negative somas by line elements that are oriented merely on their left (i.e., positive) or right (i.e., negative) edge by a half-arrow, with which, figures will arise that LÖBELL referred to as *right (left, resp.) hooks.* However, due to the required brevity, we cannot go further into this important set of circumstances.

elements (γ) – viz., a *Kasner turbine through* γ_0 [1] – (Figure 10), for which the rotational angle $\omega = 2\overline{OA}$, so it is equal to twice the quasi-elliptic displacement distance.

b) If the line $\bar{g}$ is a *line at infinity* through O (Figure 11) then its points $\bar{A}$ will map to a fixed direction of *translations*, whereby the oriented basic element γ_0 will describe a *line turbine through* γ_0, and the translation distance $\Omega = 2\overline{OA}$ will again be equal to twice the quasi-elliptic displacement distance. If $A' = [\bar{g}\, s]$ is the (infinitely distant) intersection point of the line $\bar{g}$ with the image plane π then the translation direction will be normal to the direction of the point at infinity A'.

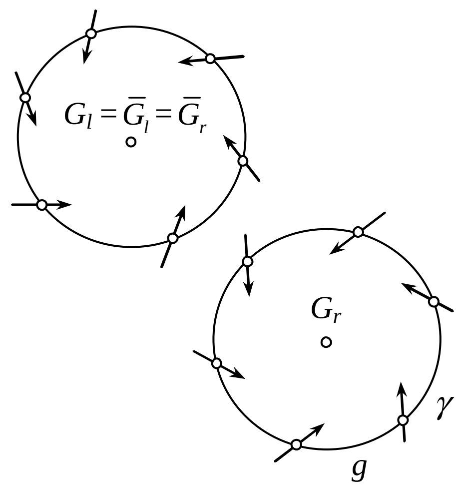

Figure 12.

2. If *g is an arbitrary line:*

a) That does not meet the absolute line s then one can (Figure 7) convert it into such a line $\bar{g}$ through O by a right displacement. The left image point G_l of g thus remains unchanged ($G_l = \bar{G}_l$), while the right one suffers a motion (viz., a rotation) $G_r \to \bar{G}_r$ that, from the fundamental theorem, will take the image figure of the oriented line elements of g to those of $\bar{g}$. It then follows (Figure 12):

The image of an arbitrary line g (that does not intersect the absolute line $s = [\iota^+\ \iota^-]$) *is a Kasner turbine with the right kinematic image point* G_r *of g as its midpoint that is congruent to that turbine that the basic element* γ_0 *describes under a rotation around the left kinematic image points* G_l *of g.*

(1) EDUARD KASNER, "The group of turns and slides and the geometry of turbines," Amer. J. Math. **33** (1911), 193-202.

b) If g is an arbitrary line that intersects the absolute line $s = [\iota^+ \; \iota^-]$ (i.e., it is horizontal) then one will analogously obtain an arbitrary *line turbine* as the kinematic image of the point A of g. (Figure 13).

If (G_l, G_r) are the two (infinitely-distant) kinematic image points of the horizontal line g, and if γ is the image element of an arbitrary point of the line g then the image line turbine of g will then arise by translating γ normal to the direction G_r, and is thus congruent to the line turbine that arises when one displaces the basic element γ_0 normal to the direction G_l .

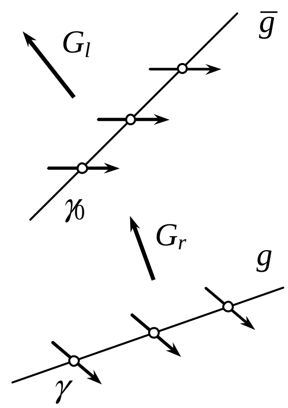

Figure 13.

Intersecting lines g_1, g_2 always correspond to *contacting* turbines – i.e., ones that have an oriented line element (viz., the image of the intersection point $A = [g_1 \; g_2]$) in common (Figure 14).

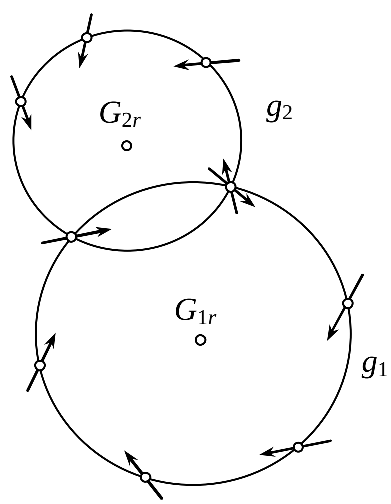

Figure 14.

Special case (Figure 15): If G_l lies on the normal n_l of the basic element γ_0 then the line g will belong to a *thread* (viz., a *"left thread"*) $\mathfrak{G}_l$. If γ_0 lies on the zero point of the x-axis then the equation of this so-called *"auxiliary thread"* will be:

$$p_{01} + p_{23} = 0.$$

One gets the oriented line elements of *cycles* as the kinematic images of the (points of) lines g of this auxiliary thread $\mathfrak{G}_l$. *Intersecting lines g, h of the auxiliary thread* $\mathfrak{G}_l$ will have *contacting cycles* as their images.

We thus have thus arrived at an exceptionally simple constructive (descriptive-geometric) presentation of any of SOPHUS LIE's *celebrated contact transformations, by which the rays g of a thread will be mapped to the oriented Lie circle (i.e., cycle)* ([1]). *The ten-parameter continuous group* $\mathfrak{G}_{10}$ *of projective automorphisms of the auxiliary thread* $\mathfrak{G}_l$ *will thus correspond to the* $\mathfrak{G}_{10}$ *of Lie circle transformations.*

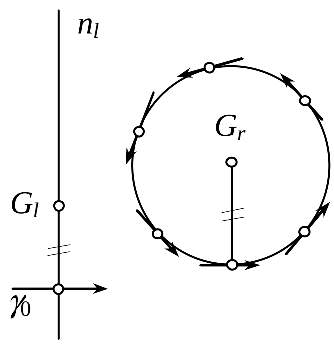

Figure 15.

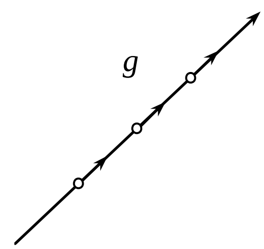

Figure 16.

Special case (Figure 16): The rays g of the auxiliary thread $\mathfrak{G}_l$ that cut the absolute structure s (which likewise lies in $\mathfrak{G}_l$), and which define a parabolic net (*"auxiliary net"* $\mathfrak{N}$), correspond to a *line cycle* – i.e., a *spear.* The seven-parameter continuous group $\mathfrak{G}_7$ of the projective automorphisms of the auxiliary net will then taken to the $\mathfrak{G}_7$ of *Laguerre spear transformations*.

IV. Euclidian line-sphere transformations

Cyclography ([2]) teaches us (Figure 17 and Figure 18) that oriented line elements γ in the plane π should be regarded as the images of isotropic lines a or, after a reality displacement (viz., multiplication of the z-coordinate by i), as the images of lines that are inclined above the image plane π by a rotation of 45° "to the left, as seen from above," and thus cut a certain one-piece circle at infinity "C," with the equation:

$$t = 0 \quad \Big| \quad \text{Plane at infinity}$$

([1]) SOPHUS LIE, *Geometrie der Berührungstransformationen* (Teubner, Leipzig, 1896).
([2]) EMIL MÜLLER, *Vorlesungen über Darstellende Geometrie*, II Band: *Die Zyklographie*, revised by J. L. KRAMES (Deuticke, Leipzig and Vienna, 1929).

$$x^2 + y^2 - z^2 = 0 \mid \text{Circle at infinity "}C\text{".}$$

The second-order surfaces of rotations (hyperboloids of rotation of one sheet) then function as spheres (i.e., "*C*-spheres") through the circle at infinity C with the equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

(r = throat radius = "radius of the sphere").

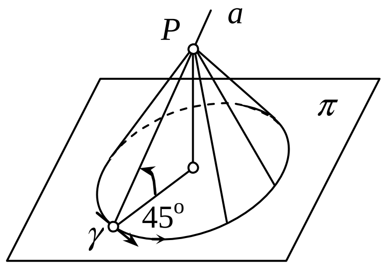

Figure 17.

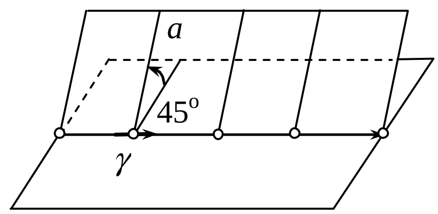

Figure 18.

The oriented line elements of a *turbine* are then cyclographic images of the generators of a family of such *"spheres,"* which is *oriented* by distinguishing a family of generators. *Contacting turbines* (Figure 14) are cyclographic images of *contacting, oriented "spheres."*

By composing the kinematic map of the lines in space to turbines and the cyclographic map of turbines to oriented (C)-spheres, we have thus obtained, all tolled, a conceivably simple descriptive-geometric construction of Lie's celebrated contact transformations ([1]) *that maps the lines g in (quasi-elliptic) space to the oriented spheres k of (quasi-Euclidian) space.*

Intersecting lines then correspond to contacting spheres.

Lines of the auxiliary thread $\mathfrak{G}_l$ correspond, first, to a cycle, then (Figure 17) to a cyclographically-isotropic cone (tangent cone to the conic section C of the so-called "*C*-cone"). *Lines g* that *cut the absolute line s* correspond, first, kinematically to turbines, and then cyclographically to non-isotropic planes that are *oriented* by distinguishing one of their isotropic families. In particular, *lines of the auxiliary net* $\mathfrak{N}$ first correspond kinematically to spears and the (Figure 18) cyclographically to *isotropic planes* (viz., *C*-planes) that admit only one orientation, like the isotropic cone (i.e., *C*-cone).

([1]) SOPHUS LIE, "Über Komplexe, insbesondere Linien- und Kugelkomplexe, mit Anwendung auf die Theorie partieller Differentialgleichingungen," Math. Ann. **5** (1872), 145-156; *Gesammelte Abhandlungen*, Bd. 2, I, pp. 1-121.

A thorough historical overview was given by E. A. WEISS, "Die Geschichtliche Enwicklung der Lehre von der Geraden-Kugel-Transformation, I-VII," Deutsche Math. **1-3** (1935-1938).

LIE himself had still not oriented the spheres – i.e., his line-sphere-transformation was still not one-to-one. The two isotropic families of generators of the sphere correspond cyclographically to *"polar turbines"* whose oriented line elements lie symmetrically with respect to the carrier circle of the turbine (Figure 19); i.e., kinematically, they are lines g, $\bar{g}$ whose left image points G_l, $\bar{G}_l$ lie symmetrically with respect to the image line n_l of the auxiliary thread $\mathfrak{G}_l$ – i.e., lines g, $\bar{g}$ that are *null polar with respect to the auxiliary thread* $\mathfrak{G}_l$.

The necessity of orienting the spheres was first recognized (1897) by E. STUDY ([1]), who was also the first (1926) to give a complete analytic presentation ([2]) of the (Euclidian) line-sphere-transformation that was free of objections, and which is in agreement with our geometric model.

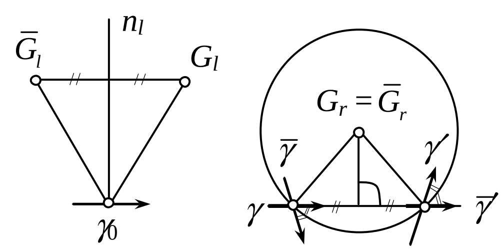

Figure 19.

One can base Lie's circle geometry on the sphere, and the *non-Euclidian line-sphere-transformation on geometric constructions* in the same way when one appeals to the *kinematics of the sphere* and its (*elliptic*) *parameter space*, as I already showed in 1930 ([3]).

The *Euclidian model* is already found in a *Vienna dissertation* of A. E. MAYER ([4]) that originated at the same time, which is still not available, and which was also not published.

In the winter and summer semesters of 1935/36, I myself have presented the situation thoroughly in a *Vienna lecture on "New Kinematics."* In the year 1948, W. BLASCHKE ([5]) published on it in the "Münchener Sitzungsberichten" and in 1949 in the "Rendiconti di matematica."

KARL STRUBECKER, Karlsruhe.

([1]) Cf., the thorough critical treatise of E. STUDY, "Über Lies Kugelgeometrie," Jber. Dtsch. Math.-Ver. **25** (1917), 96-113.

([2]) EDUARD STUDY, "Vereinfachte Begründungen von Lies Kugelgeometrie I," Sitz.-Ber. Preuss. Akad. Wiss., Berlin (1926), 360-380.

([3]) KARL STRUBECKER, "Zur nichteuklidischen Geraden-Kugel-Transformation," Sitz.-Ber. Akad. Wiss. Wien, Math.-naturw. Kl., Abt. IIa, **139** (1930), 685-700, and "Zur Geometrie sphärischer Kurvenscharen," Jber. Dtsch. Math.-Ver. **44** (1934), 184-198.

([4]) ANTON ERNST MAYER, *Die Kinematische Abbildung,* Dissertation, Techn. Hochschule, Vienna (1930).

([5]) W. BLASCHKE, "Kinematische Begründungen von S. Lies Geraden-Kugel-Transformation," Sitz.-Ber. Bayer. Akad. Wiss. (München, 1948), 291-297, and "Contributi alla cinematica," Rend. Mat. Applicazioni [V, no. 262], (1949), 268-280.