"Über Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen" Monatsth. f. Math. u. Physik **1** (1890), 283-354.

On systems of complex numbers and their application to the theory of transformation groups.

By **E. Study** in Marburg (Hessen)

Translated by D. H. Delphenich

Contents

The author has concerned himself with the theory of systems of complex numbers and their relationship to the theory of transformations groups in two treatises that appeared in the Notices of the Göttingen and Leipzig Societies for Science (*) The following includes a summary and partly reworked presentation of these investigations. Only a few things have be omitted, and they chiefly relate to the work of other mathematicians (**).

§ 1. Basic concepts.

For the concept of a so-called "extensive" or "complex" quantity that is composed of n "basic numbers" or "principal units," we refer to the first chapter in H. Grassmann's *Ausdehnungslehre* (1862 edition). We now concern ourselves with those complex quantities whose multiplication obeys the so-called *distributive law*, which is expressed in the formulas:

$$a\,(b + c) = ab + ac,$$
$$(a + b)\,c = ac + bc.$$

The totality of all of the extensive quantities:

$$a = a_1\,e_1 + a_2\,e_2 + \ldots + a_n\,e_n\,,$$
$$b = b_1\,e_1 + b_2\,e_2 + \ldots + b_n\,e_n\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

that are defined by the basic numbers $e_1, \ldots, e_n$ with real or ordinary complex coefficients $a_i, b_k, \ldots$ will be called a *system of complex numbers* when it satisfies the following conditions:

1. *The product of any two of the extensive quantities must be again regarded as an extensive quantity with the same principal units.*

From the distributive law, the necessary and sufficient condition for this to be true is that there exist n^2 relations of the form:

$$e_i\,e_k = \sum_{s=1}^{n} \gamma_{iks} e_s\,, \tag{1}$$

in which the coefficients γ_{iks} represent ordinary real or complex numbers.

2. *Any three of the extensive quantities must fulfill the so-called associative law for multiplication, which is expressed in the formula:*

(*) Gött. Nachr. 1889, no. 9 (pp. 237, *et seq.*). Sächs. Ber. 1889, 6 May (pp. 177, *et seq.*). To some extent, Scheffers also arrived at the results that are recorded in these treatises. Sächs. Ber. 3 June 1889 (pp. 290, *et seq.*).

(**) Gött. Nachr., pp. 237, *et seq.*, pp. 240, rem. pp. 265, *et seq.*, Sächs. Ber., pp. 177, rem. pp. 220, rem. pp. 227, *et seq.*

(2) $$(ab)\, c = a\, (bc).$$

The necessary and sufficient condition for this is the existence of all relations of the form:

(3) $$(e_i\, e_k)\, e_j = e_i\, (e_k\, e_j) \qquad (i, k, j = 1, \ldots, n);$$

i.e., the existence of the following systems of quadratic identities for the constants γ_{iks} that were introduced in (1):

(4) $$\sum_{s=1}^{n} \gamma_{iks}\gamma_{sjt} = \sum_{s=1}^{n} \gamma_{kjs}\gamma_{ist} \qquad (i, k, j, t = 1, \ldots, n).$$

3. *A quantity e^0 must exist among the extensive quantities that satisfies the two equations:*

(5) $$e^0\, x = x, \qquad x\, e^0 = x,$$

independently of x.

In this, it is only required that a quantity e^0 exist that satisfies the $2n$ equations:

$$e^0\, e_i = e_i\, e^0 = e_i \qquad (i = 1, \ldots, n).$$

If one sets the unknown quantity $e^0 = \sum \alpha_i\, e_i$ then that will yield the following $2n^2$ equations for the coefficients $\alpha_1, \ldots, \alpha_n$:

$$\sum_{i=1}^{n} \alpha_i \cdot \gamma_{iks} = \begin{cases} 0 \;\; (k \neq s), \\ 1 \;\; (k = s), \end{cases}$$

$$\sum_{i=1}^{n} \alpha_i \cdot \gamma_{kis} = \begin{cases} 0 \;\; (k \neq s), \\ 1 \;\; (k = s). \end{cases}$$

These equations must be compatible with each other and yield a single system of solutions. This would still not follow from the conditions that were stated in (1) and (2). One does not have the theorem that the solubility of one of these systems of equations would imply that of the other one, either. For example, one might have:

$$e_1^2 = e_1, \qquad e_1\, e_2 = e_2\,, \qquad e_2\, e_1 = 0, \qquad e_2^2 = 0.$$

The demands (1) and (2) are fulfilled here, and one will also have:

$$e_1\, (x_1\, e_1 + x_2\, e_1) = x_1\, e_1 + x_2\, e_1$$

identically; however, there is no quantity that is linearly derivable from e_1 and e_2 that will satisfy the second of the requirements that were expressed in (3).

At the same time, the condition (3), strictly speaking, requires a bit too much: It would already suffice to just require that the system of equations:

$$e^0 x = x \qquad x\,\eta^0 = x$$

possess solutions e^0, η^0, at all. It would then follow immediately that:

$$e^0 = e^0\,\eta^0 = \eta^0;$$

both solutions would then be identical, and both of the systems of equations would possess only this one solution.

The requirement (3) is completely subsumed by the other one that the equations:

$$(6) \qquad ax = b, \qquad ya = b$$

should be soluble for x and y "in general"; i.e., except for special values of a, or, as one can also say, that *division* is admissible in the system of complex numbers. In fact, in order for this to be true, it is necessary that neither of the determinants:

$$(7) \qquad \left| \sum_i \gamma_{iks} a_i \right|, \qquad \left| \sum_k \gamma_{iks} a_k \right|$$

vanish identically. Now, if the assumption (3) is fulfilled then one can take $a = e^0$; one will then obviously obtain two non-vanishing determinants. However, the one can also prove the converse immediately. Namely, if the determinants (7) are non-zero for a certain system of values $a_1, \ldots, a_n$ then one will, in particular, also be able to solve the equations:

$$a\xi = a, \qquad \eta a = a$$

for ξ and η. However, it will then follow that for every value of x, one will have:

$$a\xi x = ax, \qquad x\eta a = xa.$$

However, by assumption, the equations:

$$ax' = ax, \qquad x'' a = xa$$

will have only one solution x', x''. It will then follow that for all values of x:

$$\xi x = x, \qquad x\eta = x,$$

and, as we remarked already, this will imply that $\xi = \eta$.

From now on, we will refer to the quantity e^0 that was defined in (3) *as the "number one" of the system, and, when no misunderstanding can arise, express its characteristic property by the symbolic equation* $e^0 = 1$.

We are justified in doing this, since the number e^0 will obviously play the same role in an arbitrary system of complex numbers as unity does in the ordinary number system.

The solutions of the equations (6) can be written in a simpler form, when one has previously solved the simpler equations:

$$ax = 1, \qquad xa = 1. \tag{8}$$

The following theorem relates to this:

The determinants (7) *of the two equations* (8) *have the number a as a simple factor as functions of the coefficients* $a_1, \ldots, a_n$. *If they are non-zero then both equations will have the same solution, which we shall denote by* a^{-1}. *In the other case, there is no number in the system whatsoever that will satisfy one of the equations.*

Namely, let the number a of the system be such that the equation $ax = 1$ possesses a solution x, and let $a^0 = 1, a^1, \ldots, a^{k-1}$ ($k < n + 1$) be linearly-independent powers of the number a, while a^k is expressible as a linear combination with numerical coefficients:

$$a^k = a_1\, a^{k-1} + \ldots + a_k\, a^0 . \tag{9}$$

It is then necessary that $a_k \neq 0$; otherwise, one could derive another form from the relation (9) by multiplying by x, for which a smaller number would enter in place of the number k; $a^0, \ldots, a^{k-1}$ would not be linearly-independent then. If we now multiply a^k by x then that will give:

$$x = \frac{1}{a_k}\,[a^{k-1} - a_1\, a^{k-2} - \ldots - a_{k-1}\, a^0]; \tag{10}$$

with that, we will obtain a uniquely-defined expression for x, and indeed an expression that also clearly satisfies the other equation $xa = 1$.

Moreover, it will already be completely sufficient to show that, along with a solution x^0 of the equation $ax = 1$, there will always likewise exist a solution x of the equation $xa = 1$. It will then follow from the associative law of multiplication that $x_0 = (x_1\, a)\, x_0 = x_1\, (a\, x_0) = x_1$; there will then exist only *one* solution of the two equations, and their determinants will always be simultaneously non-zero.

We will refer to the number a, when its determinant is non-zero, as a *general* number of the system, in contrast to the *special numbers*, for which the symbol a^{-1} will no longer have any meaning.

If the numbers a and b are general then one can, with no further ado, derive the solutions to all equations of the form:

$$axb = c \tag{11}$$

from the solution $x = a^{-1}$ to the equations $ax = 1$ and $xa = 1$. One will then have:

$$x = a^{-1}\, c\, b^{-1}.$$

If one of the numbers a, b is special – say, a – then not just the equations $ax = 1$ and $xa = 1$ will be insoluble, but along with them, also all equations of the form (11), when c is a general number. Namely, if one sets – say – $ax_0b = c$ then x_0bc^{-1} will be a solution of the equation $ax = 1$, which is impossible. In order to decide whether the equation or the system of equations $axb = c$ is soluble for a special number c, one might investigate the sub-determinants of the matrix that arises from this system of equations. If that yields the existence of a solution then it can naturally not be determined: In our case, a will be what one calls a *divisor of zero.*

The totality of all numbers in the system that satisfy the equation $ax = 0$, and likewise, the totality of all numbers that satisfy the equation $ya = 0$, along with the number one in the system, will already define a system of complex numbers, in its own right.

Once we have restricted the domain in which the operation that is opposite to "multiplication" can be performed, we can assume that all calculation operations with the complex numbers of a given system will be valid that consist of addition (including subtracting), multiplication, and division in the units an arbitrary number of times. In order to perform the latter operation, which will be represented by the symbolic equation $x' = x^{-1}$, we have only to exclude the domain of special numbers x.

In order to give an example of the rule above, we will define the two determinants that belong to the system of complex numbers:

	e_1	e_2	e_3	e_4
e_1	e_1	·	·	e_4
e_2	·	e_2	e_3	·
e_3	·	e_3	·	·
e_4	·	e_4	·	·

In this table, the i^{th} horizontal row and the k^{th} vertical row will be filled in with the value of the product $e_i\ e_k$; for example, $e_3\ e_1 = 0$, $e_2\ e_3 = e_3$. If one now computes the determinants above then one will find that:

$$\left| \sum_i \gamma_{iks} a_i \right| = a_1^2 a_2^2 ,$$

$$\left| \sum_k \gamma_{iks} a_k \right| = a_1 a_2^3 ;$$

one will then obtain two entire functions with the same simple factors a_1 , a_2 .

By far, it is, moreover, frequently the case that the two determinants (7) become virtually identical, which is what happens for quaternions, for example.

§ 2. Classification of systems of complex number.

The fundamental problem in the theory of systems of complex numbers must be defined as: *Determine all systems of complex numbers with n principal units.* One cannot, however, pose the problem as: "Find the most general system of complex numbers with n principal units." It would not be *a priori* clear then – and, as a closer examination will show, also not correct – that the system of constants γ_{iks} that satisfy the conditions (1), (2), (3) (§ 1) will define an irreducible manifold: It will, moreover, decompose into different separate domains whose most general representatives are to be considered as equivalent and equally general, insofar as none of them can be obtained from the other ones by passing to some limit. However, since we feel completely ignorant of the nature of this domain, nothing will remain for us to do but to look at the small values of the number n in order to immediately determine the systems that exist.

As a natural classification principle, we shall henceforth appeal to the idea that all systems should belong to a class when they *go over to each other under linear transformations.*

From any of the systems of complex numbers that one defines, one can derive infinitely many other ones, for which one introduces new basic numbers $\overline{e}_0, \ldots, \overline{e}_{n-1}$ in place of the basic numbers $e_1, \ldots, e_n$ by way of a linear transformation with a non-vanishing determinant, so that all of the numbers in the system can be represented as linear functions of $\overline{e}_0, \ldots, \overline{e}_{n-1}$.

All of the properties of the systems that are obtained in this way will obviously be known, just like those of the system $e_1, \ldots, e_n$. We will therefore consider it to not be essentially different from the system $e_1, \ldots, e_n$, and will place it, along with the latter system, into one and the same "type."

However, there is a second, especially remarkable, way of deriving from a system of complex numbers, another one that has just as many principal units. If we imagine any system of complex numbers that is represented by a quadratic multiplication table, such as the table that is defined in page 5, then we will obviously again obtain a system of complex numbers when we switch any two elements in the square that lie symmetrically to the diagonal. In this way, e.g., the system that was given on pp. 5 will go to the following one:

	e_1	e_2	e_3	e_4
e_1	e_1	·	·	·
e_2	·	e_2	e_3	e_4
e_3	·	e_3	·	·
e_4	e_4	·	·	·

In fact, the operation that was performed will have only the effect of inverting the order (sequence) of all multiplications, such that every product $ab\ldots ef$ will then be replaced with the corresponding product $fe\ldots ba$. However, the characteristic properties of a system of complex numbers that were described in (1), (2), (3) will obviously not be disturbed. All that will happen is that all of the constants γ_{iks} will be switched with the corresponding constants γ_{kis}, from which, nothing will change in the relations (4) of § 1.

We would like to refer to the system thus-obtained as the *reciprocal system* to the given one. In many cases, it can also be obtained from the given one by introducing new basic numbers, and indeed, it can coincide with the given one completely (namely, when the given system obeys the co-called *commutative law* of multiplication); however, it can also be different from the given one, as in the example that was given. Namely, if the two multiplication tables that were given on pages 5 and 7 can be converted into each other by the introduction of new basic numbers then the two associated determinants $\left|\sum_i \gamma_{iks} a_i\right|$ and $\left|\sum_i \gamma'_{iks} a_i\right| = \left|\sum_i \gamma_{iks} a'_i\right| = \left|\sum_k \gamma_{iks} a'_k\right|$ will have to go to each other at the same time. However, this is impossible, since the one is a product of two squares, while the other one is a product of a first and third power.

Just as one has in the individual cases, in any event, the reciprocal system to a given one is known to be identical to it in all of its properties. We will therefore be justified in counting it as likewise having the same type.

We can now make the problem that was posed in the beginning of this paragraph more precise: *Give all of the different types of systems with n basic numbers.*

One will obviously then come to the problem of representing each type by those representatives that will given the clearest possible multiplication table, which would be similar to the one that was given on pp. 5.

In place of the concept of a system of complex number that was defined to be fundamental here, one can also choose a somewhat different concept to be fundamental, and then once more pose a similar problem.

Namely, if one restricts oneself to those systems of complex numbers for which the constants γ_{iks} in the relations have *real* values then one will also find oneself dealing with only the number systems of a class that can be permuted within itself by linear transformations with real coefficients. It does not seem to me preferable to take such a standpoint from now on, since the algebraic character of the problem would be corrupted by it. It is, however, completely worthwhile to distinguish the systems within a given type that are associated with real values of the constants γ_{iks}, and which can be permuted with each other by means of linear transformations with real coefficients. We would like to call them *different forms of the same type.*

Finally, one can also alter the problem in such a way that one prescribes a certain domain of rationality in which the quantities γ_{iks} and the transformation coefficients will come from. However, we will not go further into this question.

If one has a system with n basic numbers and $n > 1$ then a number k that lies between the limits of 2 and n must exist with the property that the k^{th} power of any number a of the system will be expressible in terms of the pervious powers $a^0 = 1$, a^1, a^2, …, a^{k-1}, while the latter are linearly-independent of a for a sufficiently general choice of a (cf., § 1, formula 9).

We will arrange the number systems for a given value of n in increasing order of k.

In §§ 3-6, the types that are present in the cases n = 2, 3, 4 will be determined, along with their forms, and clearly summarized. Systems of the same type will bear the same Roman numeral. Some forms will be presented twice, since under some circumstances another canonical form (i.e., multiplication table) will seem preferable, depending upon whether one is dealing with the determination of types or the determination of the various forms of the same type. In such cases, the second form of a multiplication table will characterized by an appended notation a). The remaining forms of the same type will then follow with the notation b), c), …

§ 3. Systems with two basic numbers.

Here, one will necessarily have $n = 2$.
Let a be linearly-independent of $a^0 = 1$, and let:

$$a^2 = a_1 a^1 + a_2 a^0,$$

in which a_1, a_2 are ordinary real or complex numbers. Now, if the quadratic equation:

$$\lambda^2 = a_1\lambda + a_2$$

has two separate roots λ_0, λ_1 then one will introduce the new basic numbers:

$$e_0 = \frac{a-\lambda_1}{\lambda_0-\lambda_1}, \qquad e_1 = \frac{a-\lambda_0}{\lambda_0-\lambda_1}.$$

If it has a double root λ then one will take the new basic numbers to be these:

$$e_0 = (a - \lambda)^0 = 1, \qquad e_1 = a - \lambda.$$

One will thus obtain the only two types of systems that have two basic numbers, which are illustrated in Tables I and II. Distinct real systems obviously belong to just the first type, and indeed there will be two forms, corresponding to the two possibilities that the quadratic equations $\lambda^2 = a_1\lambda + a_2$ has two real or two conjugate-imaginary roots, resp. In the second case, one will obtain Table Ib) by introducing the new basic numbers:

$$\overline{e}_0 = e_0 + e_1, \qquad \overline{e}_1 = (e_0 - e_1)\, i,$$

which defines the system of ordinary complex numbers. Table I itself can serve as the type of the number systems of the first class. However, it seems preferable to write this table once more in the form Ia) by the introduction of the new basic numbers:

$$\overline{e}_0 = e_0 + e_1, \qquad \overline{e}_1 = e_0 - e_1 .$$

In Tables Ia) and Ib), we will then have two analogous canonical forms for the two forms of type I (*).

(*) The determination of systems with real basic numbers was achieved by Weierstrass and Cayley. S. Stolz, *Vorlesungen über allgemeine Arithmetik*, Bd. II, § 5 and Cayley, Proc. Lond. Math. Soc. XV (1883-84), pp. 186, *et seq.* Scheffers was kind enough to make me aware of this paper by Cayley.

I. Ia). Ib)

$$\begin{array}{c|cc} & e_0 & e_1 \\ \hline e_0 & e_0 & \cdot \\ e_1 & \cdot & e_1 \end{array} \qquad \begin{array}{c|cc} & e_0 & e_1 \\ \hline e_0 & e_0 & e_1 \\ e_1 & e_1 & e_0 \end{array} \qquad \begin{array}{c|cc} & e_0 & e_1 \\ \hline e_0 & e_0 & e_1 \\ e_1 & e_1 & -e_0 \end{array}$$

$$\text{II.} \quad \begin{array}{c|cc} & e_0 & e_1 \\ \hline e_0 & e_0 & e_1 \\ e_1 & e_1 & \cdot \end{array}$$

As we said, Table Ia) will go to I under the real substitution:

$$\bar{e}_0 = e_0 + e_1, \qquad \bar{e}_1 = e_0 - e_1,$$

when one, in turn, drops the overbar again. Similarly, Table Ib) will go to Ia) under the substitution:

$$\bar{e}_0 = e_0\,, \qquad \bar{e}_1 = e_1\, i \qquad (i = \sqrt{-1}).$$

§ 4. Systems with three basic numbers.

Next, let $n = 3$, such that one will have:

$$a^3 = a_1\, a^2 + a_2\, a^1 + a_3\, a^0,$$

while, for a general choice, a^2 will not already be expressible in terms of a^1 and a^0.

We will distinguish three cases, according to whether the cubic equation:

$$\lambda^3 = a_1\, \lambda^2 + a_2\, \lambda + a_3\,,$$

has three distinct roots λ_0, λ_1, λ_2, one double root λ_0 and a simple root λ_2, or finally a triple root λ_0 for a general choice of the number a.

In the first case, one simply sets:

$$e_0 = \frac{(a-\lambda_1)(a-\lambda_2)}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)}\,,$$

and correspondingly for e_1 and e_2, with a cyclic permutation of the indices. Since e_0, e_1, and e_2 are linearly-independent, on the basis of the assumption that was made, and satisfy the relations $e_i^2 = e_i$, $e_i\, e_k = 0$, in addition, which are verified immediately, one will then obtain the canonical form I that is listed below.

If λ_0 and λ_1 coincide then e_0 and e_1 will become infinite, and the transformation to the canonical form I will be impossible. However, one can remark that $e_0 + e_1$ and $(\lambda_0 - \lambda_1)$ e_0 will remain finite. In the limit, one will have these expressions:

$$\bar{e}_0 = -\frac{(a-\lambda_1)(a-2\lambda_0+\lambda_2)}{(\lambda_0-\lambda_2)^2},$$

$$\bar{e}_1 = -\frac{(a-\lambda_0)(a-\lambda_2)}{(\lambda_0-\lambda_2)},$$

together with the limiting value for e_2:

$$\bar{e}_2 = \frac{(a-\lambda_0)^2}{(\lambda_2-\lambda_0)^2},$$

which are again linearly-independent numbers, and one will then obtain the canonical form II.

Finally, if λ_2 also coincides with λ_0 and λ_1 then this form will also be impossible. However, $\bar{e}_0 + \bar{e}_2$, as well as $(\lambda_0 - \lambda_2)$ $\bar{e}_0 + \bar{e}_1$ and $(\lambda_0 - \lambda_2)$ $\bar{e}_1$, will again remain finite, and one will have these expressions in the limit:

$$\bar{\bar{e}}_0 = 1, \qquad \bar{\bar{e}}_1 = (a - \lambda_0), \quad \bar{\bar{e}}_2 = (a - \lambda_0)^2,$$

which are the basic numbers of a new type III.

Furthermore, let $k = 2$, such that any number a of the system will fulfill an equation of the form:

$$a^2 = a_1\, a^1 + a_2\, a^0.$$

If we think of the number a as being expressed here in terms of any three basic numbers $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2$ in the form $\lambda_0 \bar{e}_0 + \lambda_1 \bar{e}_1 + \lambda_2\, \bar{e}_2$ then a_1 will become a homogeneous, linear function of λ_0, λ_1, λ_2 that does not vanish when one sets a equal to the number one for the system. One can then associate the unity e_0 with two basic numbers e_1', e_2' in such a way that the coefficient a_1 that is constructed for an number of the form $\lambda_1 e_1' + \lambda_2\, e_2'$ will vanish. The coefficient a_2 will then become a second-degree homogeneous, linear function of λ_1 and λ_2. It can possess two distinct zeroes. One can then introduce two new basic numbers e_1 and e_2 in place of e_1' and e_2' that satisfy the conditions:

$$e_1^2 = 0, \qquad e_2^2 = 0,$$

and the same thing will happen when a_2 vanishes identically. However, if a_2 is a square then one can choose the basic numbers in such a way that:

$$e_1^2 = 1, \qquad e_2^2 = 0.$$

We next consider the second case.

If we make the Ansatz:

$$e_1\, e_2 = a_0\, e_0 + a_1\, e_1 + a_2\, e_2$$

then by forming the products $e_1\,(e_1\, e_2)$ and $(e_1\, e_2)\, e_2$, we will recognize that one must have:

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2^2 = 1.$$

If we first take $a_2 = 1$, so $e_1\, e_2 = e_2$ then it will follow from the associative law that $e_2\, e_1 = -\, e_2$; we will then obtain Table IV. The other assumption $a_2 = -\,1$ would yield the reciprocal system, but it will lead back to the first one immediately under the substitution $\bar{e}_1 = -\, e_1$.

If we further assume that:

$$e_1^2 = 0, \qquad e_2^2 = 0$$

then the associativity law will immediately imply that $e_1\, e_2 = e_2\, e_1 = 0$. The case that was previously presented as possible, in which the coefficient a_2 vanished identically, can therefore not actually occur; we now obtain multiplication table V.

We have thus found, in total, five different systems of complex numbers that correspond to the assumption that $n = 3$. There is no difficulty in also determining all of their forms.

Once more, first set $k = 3$, so we can have made an imaginary substitution only when the cubic equation:

$$\lambda^3 = a_1\, \lambda^3 + a_2\, \lambda + a_3$$

that is characteristic of the system has two conjugate-imaginary roots.

This can occur only in case I. When λ_0 and λ_1 are the conjugate-imaginary roots, e_0 and e_1 will become conjugate-imaginary. By the substitution:

$$\bar{e}_0 = e_0 + e_1, \quad \bar{e}_1 = i\,(e_0 - e_1),$$

we will then once more obtain a system with real constants γ_{iks} , which is represented in Table Ib).

Of the systems that were found in the case $k = 2$, the system V obviously cannot have two different forms. Should the system IV be capable of a second real form then this could arise only when one sets $e_1^2 = -\,1$, $e_2^2 = 0$, in place of $e_1^2 = 1$, $e_2^2 = 0$. However, that would then yield imaginary values for some of the constant γ_{iks} .

We will then have to list the following tables for the case of $n = 3$:

I.

	e_0	e_1	e_2
e_0	e_0	·	·
e_1	·	e_1	·
e_2	·	·	e_2

Ia).

	e_0	e_1	e_2
e_0	e_0	e_1	·
e_1	e_1	e_0	·
e_2	·	·	e_2

Ib).

	e_0	e_1	e_2
e_0	e_0	e_1	·
e_1	e_1	$-e_0$	·
e_2	·	·	e_2

Table Ia) is not essentially different from Table I, and will go over to it under the real substitution:

$$\bar{e}_0 = e_0 + e_1, \quad \bar{e}_1 = e_0 - e_1, \quad \bar{e}_2 = e_2 .$$

This is only due to the cited analogy with the following table Ib).

Table Ib) will go to Ia) under the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_1 = i\, e_1, \qquad \bar{e}_2 = e_2 .$$

II.

	e_0	e_1	e_2
e_0	e_0	e_1	·
e_1	e_1	·	·
e_2	·	·	e_2

III.

	e_0	e_1	e_2
e_0	e_0	e_1	e_2
e_1	e_1	e_2	·
e_2	e_2	·	·

IV.

	e_0	e_1	e_2
e_0	e_0	e_1	e_2
e_1	e_1	e_0	e_2
e_2	e_2	$-e_2$	·

V.

	e_0	e_1	e_2
e_0	e_0	e_1	e_2
e_1	e_1	·	·
e_2	e_2	·	·

The system IV will go to its reciprocal system under the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_1 = -e_1, \qquad \bar{e}_2 = e_2 .$$

§ 5. Systems with four basic numbers.

First, let $k = 4$, so:

$$a^4 = a_1\, a^3 + a_2\, a^2 + a_3\, a^1 + a_4\, a^0.$$

If the bi-quadratic equation:

$$a^4 = a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4$$

has four separate roots λ_0, λ_1, λ_2, λ_3 for a general assumption on the number a then one can take the basic numbers to be e_0, e_1, e_2, e_3, where, e.g.:

$$e_0 = \frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)(\lambda_0-\lambda_3)};$$

one will then obtain the canonical form I that is listed below.

In the case of a double root λ_0 and two simple roots λ_2, λ_3, one takes the basic numbers to be the limiting values of $e_0 + e_1$, $(\lambda_0 - \lambda_1)\, e_0$, e_2, e_3, namely:

$$\bar{e}_0 = \{3\lambda_0^2 - 2\lambda_0 (a + \lambda_2 + \lambda_3) + (a\lambda_2 + a\lambda_3 + \lambda_2\lambda_3)\} \frac{(a-\lambda_2)(a-\lambda_3)}{(\lambda_0-\lambda_2)^2(\lambda_0-\lambda_3)^2},$$

$$\bar{e}_1 = \frac{(a-\lambda_0)(a-\lambda_2)(a-\lambda_3)}{(\lambda_0-\lambda_2)(\lambda_0-\lambda_3)},$$

$$\bar{e}_2 = \frac{(a-\lambda_0)^2(a-\lambda_3)}{(\lambda_2-\lambda_0)^2(\lambda_0-\lambda_3)},$$

$$\bar{e}_3 = \frac{(a-\lambda_0)^2(a-\lambda_2)}{(\lambda_3-\lambda_2)^2(\lambda_0-\lambda_2)}.$$

The associated multiplication table will be II.

Furthermore, if λ_3 also coincides with λ_2 then one will take the basic numbers to be the limiting values of $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2 + \bar{e}_3$, $(\lambda_2 - \lambda_3)\, e_2$:

$$\bar{e}_0' = -\frac{(2a-3\lambda_0+\lambda_2)(a-\lambda_2)^2}{(\lambda_0-\lambda_2)^3},$$

$$\bar{e}_1' = \frac{(a-\lambda_0)(a-\lambda_2)^2}{(\lambda_0-\lambda_2)^2},$$

$$\bar{e}_2' = -\frac{(2a-3\lambda_1+\lambda_0)(a-\lambda_0)^2}{(\lambda_2-\lambda_0)^3},$$

$$\bar{e}_3' = \frac{(a-\lambda_2)(a-\lambda_0)^2}{(\lambda_2-\lambda_0)^2};$$

one will then get Table III.

By contrast, if λ_0 is a triple root and λ_3 is a single root then by introducing the limiting values of $\overline{e}_0 + \overline{e}_2$, $(\lambda_0 - \lambda_2)\ \overline{e}_0 + \overline{e}_1$, $(\lambda_2 - \lambda_3)\overline{e}_1$, $\overline{e}_3$ one will obtain:

$$\overline{\overline{e}}_0 = 1 - \frac{(a-\lambda_0)^2}{(\lambda_3-\lambda_0)^3},$$

$$\overline{\overline{e}}_1 = -\frac{(a-\lambda_0)(a-\lambda_3)(a-2\lambda_0+\lambda_3)}{(\lambda_0-\lambda_3)^3},$$

$$\overline{\overline{e}}_2 = \frac{(a-\lambda_0)^2(a-\lambda_3)}{(\lambda_0-\lambda_3)},$$

$$\overline{\overline{e}}_3 = \frac{(a-\lambda_0)^3}{(\lambda_3-\lambda_0)^3},$$

as the basic numbers in Table IV.

Finally, if the root λ_3 also coincides with λ_0 then:

$$\overline{\overline{e}}_0 + \overline{\overline{e}}_3,\ (\lambda_0 - \lambda_2)\ \overline{\overline{e}}_0 + \overline{\overline{e}}_1,\ (\lambda_0 - \lambda_3)\overline{\overline{e}}_1 + \overline{\overline{e}}_3,\ (\lambda_0 - \lambda_2)\ \overline{\overline{e}}_3$$

will again converge to finite limits, namely, to:

$$(a - \lambda_0)^0, \qquad (a - \lambda_0)^1, \qquad (a - \lambda_0)^2, \qquad (a - \lambda_0)^3.$$

By introducing then as limiting numbers, one will obtain Table IV.

We now turn to the second principal case $k = 3$.

Here, any general, sufficiently-chosen number a, along with its square and $a_0 = 1$ will together define a system of three numbers that corresponds to the types I, II, III that were presented for $n = 3$. Since only four linearly-independent units are present, any two such three-dimensional domains will have a two-dimensional domain in common – viz., a system of two numbers (which will then belong to any system of three numbers, in the same way).

We now first assume that the system of three numbers that belongs to a generally-chosen number a is a system of the first type. The two-dimensional system that is defined by the intersection of system I with another system will then be (triply) well-defined. Namely, if one expresses the idea that the square of a number $\lambda_0\, e_0 + \lambda_1\, e_1 + \lambda_2\, e_2$ of system I should already be linearly representable in terms of the number itself and $e_0 + e_1 + e_2 = 1$ then it should follow that λ_0, λ_1, λ_2 are roots of a quadratic equation such that two of these three quantities must be equal to each other.

Therefore, there are only three mutual-equivalent sub-domains in our number system that have the desired property; one of them is determined by the numbers $\overline{e}_0 = e_0 + e_1 + e_2 = 1$, $\overline{e}_1 = e_0 + e_1$. If one introduces these, together with $\overline{e}_2 = e_0$, as new basic numbers and appends another basic number $\bar{e}_3$ that is coupled with $\bar{e}_0$ and $\bar{e}_1$ by the same relations as $\bar{e}_2$ then part of the multiplication rules that we seek will already be known. It will be represented by the formulas:

$$(A) \qquad \begin{aligned} &e_0 = 1, \qquad e_1^2 = e_1, \qquad e_2^2 = e_2, \qquad e_3^2 = e_3, \\ &e_1\, e_2 = e_2\, e_1 = e_2, \qquad e_1\, e_3 = e_3\, e_1 = e_3\,. \end{aligned}$$

Secondly, let the system of three numbers that is determined by a^0, a^1, a^2 be a system of the second type, and also for a completely general choice of a. One can then associate the number $\bar{e}_0 = 1$ with a number $\bar{e}_1$ in two essentially different ways that will both collectively define a system of two numbers: $\bar{e}_1 = e_1$ and $\bar{e}_1 = e_2$.

In the first case, one takes $\bar{e}_0$, $\bar{e}_1$, and $\bar{e}_2 = e_2$ to be the new basic numbers, and appends a third basic number $\bar{e}_3$ that has the same relationship to $\bar{e}_0$ and $\bar{e}_1$ as $\bar{e}_2$. One will then obtain the multiplication rules:

$$(B) \qquad \begin{aligned} &e_0 = 1, \qquad e_1^2 = 0, \qquad e_2^2 = e_2\,, \qquad e_3^2 = e_3\,, \\ &e_1\, e_2 = e_2\, e_1 = 0, \qquad e_1\, e_3 = e_3\, e_1 = 0. \end{aligned}$$

In the other case, the following choice of basic units $\bar{e}_0 = 1$, $\bar{e}_1 = e_2$, $\bar{e}_2 = e_1$ will yield the formulas:

$$(C) \qquad \begin{aligned} &e_0 = 1, \qquad e_1^2 = e_1, \qquad e_2^2 = 0, \qquad e_3^2 = 0, \\ &e_1\, e_2 = e_2\, e_1 = 0, \qquad e_1\, e_3 = e_3\, e_1 = 0. \end{aligned}$$

Finally, for the most general choice of a, the system that is determined by a^0, a^1, a^2 might belong to the third type. One can then extend the number $\bar{e}_0 = e_0 = 1$ by another number $\bar{e}_1$ in essentially one way such that $\bar{e}_0$ and $\bar{e}_1$ collectively make up a system of two numbers; this system will be determined by $\bar{e}_0 = 1$ and $\bar{e}_1 = e_2$. If one then writes $\bar{e}_2$ for e_1 and, appends a third basic number $\bar{e}_3$, and drops the overbar then one will obtain a fourth system of formulas:

$$(D) \qquad \begin{aligned} &e_0 = 1, \qquad e_1^2 = 0, \qquad e_2^2 = e_1, \qquad e_3^2 = e_1, \\ &e_1\, e_2 = e_2\, e_1 = 0, \qquad e_1\, e_3 = e_3\, e_1 = 0. \end{aligned}$$

We shall go through the four assumptions (A), (B), (C), (D) individually. We will add to them, in the most general way, relations of the form:

$$\begin{aligned} e_2\, e_3 &= \alpha_0\, e_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3\,, \\ e_3\, e_2 &= \beta_0\, e_0 + \beta_1\, e_1 + \beta_2\, e_2 + \beta_3\, e_3\,, \end{aligned}$$

and determine the constants α and β according to the associative law:

$$(A) \qquad e_0 = 1, \quad e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_3^2 = e_3,$$
$$e_1\, e_2 = e_2\, e_1 = e_2, \qquad e_1\, e_2 = e_3\, e_1 = e_2\,.$$

In addition, if we let:

$$e_2\, e_3 = \alpha_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3$$

then it will follow upon multiplying by e_1 that $\alpha_0 = 0$. Furthermore, when one forms the products $e_2\,(e_2\, e_3)$ and $(e_2\, e_3)\, e_3$, one will get:

$$\alpha_1 = \alpha_1\, \alpha_2 = \alpha_1\, \alpha_3, \quad \alpha_2^2 = \alpha_2, \quad \alpha_3^2 = \alpha_3,$$
$$\alpha_1 + \alpha_2\, \alpha_3 = 0;$$

i.e., one will have either:

$$(a) \qquad \alpha_1 = -1, \quad \alpha_2 = 1, \quad \alpha_3 = 1,$$

or

$$(b) \qquad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 1,$$

or, what amounts to the same thing:

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = 0,$$

or finally:

$$(c) \qquad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0.$$

$$(a) \qquad e_2\, e_3 = -e_1 + e_2 + e_3\,.$$

If:

$$e_3\, e_2 = \beta_1\, e_1 + \beta_2\, e_2 + \beta_3\, e_3$$

then one will find, by the use of the equation $(e_2\, e_3)\, e_3 = e_2\,(e_3\, e_3)$, or the other one $e_3\,(e_2\, e_3) = (e_3\, e_3)\, e_3$, that $\beta_3 = \beta_2 = -\beta_1$; i.e., one will have either:

$$\beta_1 = -1, \quad \beta_2 = 1, \quad \beta_3 = 1$$

or

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 0.$$

At this point, the first assumption will be useless for us, because it leads back to system I under the substitution $\bar{e}_0 = e_0 - e_1$, $\bar{e}_1 = e_2 + e_3 - e_1$, $\bar{e}_2 = e_1 - e_3$, $\bar{e}_3 = e_1 - e_2$.

In the second case, we introduce two new units $\bar{e}_0 = e_1$, $\bar{e}_3 = e_0 - e_1$, in place of e_0 and e_1, which are determined such that one will have $\bar{e}_0^2 = \bar{e}_0$, $\bar{e}_3^2 = \bar{e}_3$, $\bar{e}_0 + \bar{e}_3 = 1$. Furthermore, one introduces two new basic numbers $\bar{e}_1$ and $\bar{e}_2$, in place of e_2 and e_3, which are chosen such that $\bar{e}_0$, $\bar{e}_1$, and $\bar{e}_2$, when taken by themselves, will define the fourth type of associated system of three units. To that end, we determine $\bar{e}_2$ such that

$\bar{e}_2^2 = 0$, namely, $\bar{e}_2 = -e_1 + e_2 + e_3$, and then determine $\bar{e}_1$ such that $\bar{e}_1^2 = \bar{e}_0$, namely, $\bar{e}_1 = e_2 - e_3$. We will then obtain multiplication table VI.

(*b*) $e_2 e_3 = e_3$. If one introduces a new basic number $\bar{e}_3$ in place of e_3 by means of the substitution $\bar{e}_3 = e_3 + e_1 - e_2$ then one will get back to the case (*a*) that was treated above.

(*c*) $e_2 e_3 = 0$. If one does not wish to return to the cases (*a*), (*b*) then one must assume that $e_3 e_2$ is equal to zero in any case. However, that would be superfluous, since, by the substitution:

$$\bar{e}_0 = e_0 - e_1, \qquad \bar{e}_1 = e_1 - e_2 - e_3, \qquad \bar{e}_2 = e_2, \qquad \bar{e}_3 = e_3,$$

one would, in turn, obtain Table I.

(*B*)
$$e_0 = 1, \qquad \bar{e}_1^2 = 0, \qquad \bar{e}_2^2 = e_2, \qquad \bar{e}_3^2 = e_3,$$
$$e_1 e_2 = e_2 e_1 = 0, \qquad e_1 e_3 = e_3 e_1 = 0,$$
$$e_2 e_3 = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 .$$

It would follow upon multiplication by e_1 that $\alpha_0 = 0$; furthermore, upon multiplication by e_2 and e_3:

$$\alpha_1 = \alpha_1\alpha_2 = \alpha_1\alpha_3, \qquad \alpha_2\alpha_3 = 0,$$
$$\alpha_2^2 = \alpha_2, \qquad \alpha_3^2 = \alpha_3,$$

i.e., one will have either:

(*a*) $\qquad \alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 1,$

or

$\qquad \alpha_1 = 0, \qquad \alpha_2 = 1, \qquad \alpha_3 = 0,$

or

(*b*) $\qquad \alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 0.$

(*a*) $\qquad e_2 e_3 = e_3 .$

Let:

$$e_3 e_2 = \beta_2 e_2 + \beta_3 e_3 .$$

It will then follow upon multiplication by e_3 that:

$$\beta_2 + \beta_3 = 1;$$

i.e., one will have either $\beta_2 = 0$, $\beta_3 = 1$ or $\beta_2 = 1$, $\beta_3 = 0$. The first assumption is inadmissible, since one would obtain Table II under the substitution $\bar{e}_0 = e_0 - e_3$, $\bar{e}_1 = e_1$, $\bar{e}_2 = e_2 - e_3$, $\bar{e}_3 = e_3$. All that will remain then is the assumption that $e_2 e_3 = e_3$, which we can also replace with $e_2 e_3 = e_3$, $e_3 e_2 = e_3$, in order to go to the reciprocal system. Here, e_0, e_2, e_3 will define a system of three numbers that belong to the fourth type. It will go to its canonical form under the substitution $\bar{e}_0 = e_0$, $\bar{e}_1 = e_0 - 2e_3$, $\bar{e}_2 = e_3 - e_2$; if we append $\bar{e}_3 = e_1$ then we will obtain Table VII.

Starting from the first of the assumptions that were made under (*a*), we have arrived at a system that it is reciprocal to the system VII. Had we started with the second assumption, we would likewise have obtained system VII itself. Neither system – viz., system VII or its reciprocal – can be transformed into the other one (while this is still possible for system VI, which can be derived from system VII by a passage to the limit that is easy to give).

In fact, let $\bar{e}_0, \ldots, \bar{e}_3$ be the basic numbers of the system that is reciprocal to VII, so its multiplication table will emerge from Table VII by permuting the horizontal and vertical rows. Should $\bar{e}_0, \ldots, \bar{e}_3$ then be linearly derivable from $e_0, \ldots, e_3$ with numerical coefficients then it would next follow that $\bar{e}_0 = e_0$. Furthermore, since one should have $\bar{e}_1^2 = \bar{e}_0$, $\bar{e}_2^2 = 0$, $\bar{e}_3^2 = 0$, $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ must have the form:

$$\bar{e}_1 = \pm e_1 + \lambda_2 e_2, \qquad \bar{e}_2 = \mu_2 e_2 + \mu_3 e_3, \qquad \bar{e}_3 = \nu_2 e_2 + \nu_3 e_3.$$

Since one should have $\bar{e}_1 \bar{e}_2 = -\bar{e}_2$, only the lower sign in the first of these expressions will be admissible. However, the absurd result that $\nu_2 = \nu_3 = 0$ would follow from the equation $\bar{e}_1 \bar{e}_3 = \bar{e}_3$. (Cf., the remarks on page 8, as well. The table that was given on page 6 will go to VII under the substitution:

$$\bar{e}_0 = e_1 + e_2, \quad \bar{e}_1 = e_1 - e_2, \quad \bar{e}_2 = e_1, \quad \bar{e}_3 = -e_3.)$$

(*b*) If $e_2\, e_3 = 0$ then it will also follow that $e_1\, e_2 = 0$. This would once more be inadmissible, since one would obtain Table II under the substitution $\bar{e}_0 = e_0 - e_2 - e_3$.

(*C*)
$$e_0 = 1, \qquad e_1^2 = e_1, \qquad e_2^2 = 0, \qquad e_3^2 = 0,$$
$$e_1\, e_2 = e_2\, e_1 = 0, \qquad e_1\, e_3 = e_3\, e_1 = 0,$$
$$e_2\, e_3 = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3.$$

It will then follow forthwith upon multiplying by e_1, e_2, and e_3 that:

$$e_2\, e_3 = e_3\, e_2 = 0.$$

If one introduce basic numbers:

$$\bar{e}_0 = e_0 - e_1, \quad \bar{e}_1 = e_2, \quad \bar{e}_2 = e_3, \quad \bar{e}_3 = e_1$$

in place of $e_0, \ldots, e_3$ then one will obtain Table VIII.

(*D*)
$$e_0 = 1, \qquad e_1^2 = 0, \qquad e_2^2 = e_3^2 = e_1,$$
$$e_1\, e_2 = e_2\, e_1 = 0, \qquad e_1\, e_3 = e_3\, e_1 = 0,$$
$$e_2\, e_3 = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3.$$

Upon multiplying by e_1, e_2, and e_3, one will find:

$$\alpha_0 = \alpha_2 = \alpha_3 = 0,$$

such that one can then set:

$$e_2\, e_3 = \alpha\, e_1, \quad e_3\, e_2 = \beta\, e_1.$$

If one introduces a new basic number $\bar{e}_3 = e_3 - \alpha\, e_2$, in place of e_3, then one will get:

$$e_2\, \bar{e}_3 = 0, \qquad \bar{e}_3\, e_2 = (\beta - \alpha)\, e_1\,, \qquad \bar{e}_3^2 = (1 - \alpha\beta)\, e_1\,.$$

We now distinguish three cases:

(*a*) $\beta - \alpha \neq 0,$

(*b*) $\beta - \alpha = 0, \quad 1 - \alpha\beta \neq 0,$

(*c*) $\beta - \alpha = 0, \quad 1 - \alpha\beta = 0.$

(*a*) One introduces the new basic numbers $e_0' = e_0$, $e_1' = e_2$, $e_2' = e_2 - 2\,\dfrac{\bar{e}_3}{\beta - \alpha}$, $e_3' = e_1$, in place of e_0, e_1, e_2, $\bar{e}_3$, of which, the third one is determined such that $e_2\, e_2' = -\, e_2' e_2 = e_1$.

One will then obtain Table IX.

The numerical parameter c that enters here cannot be eliminated by the introduction of new basic numbers.

(*b*) $e_3\, e_3' = e_3' e_2 = 0$, $\bar{e}_3^2 = c\, e_1$. Here, if one makes the substitution $e_0' = e_0$, $e_1' = e_2$, $e_2' = \dfrac{1}{\sqrt{c}}\bar{e}_3$, $e_3' = e_1$ then one will obtain the number system X.

(*c*) $e_3\, e_3' = e_3' e_2 = 0$, $\bar{e}_3^2 = 0$. This assumption will give Table XI.

With that, the assumption that $k = 3$ is also dealt with. All that we have left to present are the number systems for which $k = 2$.

Here, we distinguish between the number systems for which, along with $e_0 = 1$, two basic numbers e_1 and e_2 are given in such a way that one of the products $e_1\, e_2$, $e_2\, e_1$ of e_0, e_1, e_2 is linearly-independent (*A*) of the other one (*B*), for which $e_0 = 1$ and two numbers a and b already define a system of three units in itself.

(*A*) Here, with no loss of generality, we can assume that either:

(*a*) $e_1^2 = 1, \qquad e_2^2 = 1,$

or

(*b*) $e_1^2 = 0, \qquad e_2^2 = 0.$

(*a*) Let $e_1^2 = e_2^2 = 1$. If we then express the idea that:

$$(\lambda_1 e_1 + \lambda_2 e_2)^2$$

should be linearly-expressible in terms of $(\lambda_1 e_1 + \lambda_2 e_2)$ and $e_0 = 1$ then it will follow that:

$$e_1 e_2 + e_2 e_1 = c,$$

in which c means a numerical constant.

Here, we introduce a new basic number:

$$\bar{e}_2 = \lambda e_1 + \mu e_2 ,$$

in place of e_2 and seek to determine the constants λ and μ in such a way that one will still have $\bar{e}_2^2 = 1$, but on the other hand, one will have $e_1 \bar{e}_2 + \bar{e}_2 e_1 = 0$. However, this will always be possible, except when $c^2 = 4$. First, let $c^2 \neq 4$. We can once more replace $\bar{e}_1 = e_1$ and $\bar{e}_3$ with new basic numbers by the substitution $e_1 = i\bar{e}_1$, $e_2 = i\bar{e}_2$. We will then have the relations $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$. If we set $e_1 e_2 = e_3$, and then determine the expressions for the remaining unit products from the associative law then we will obtain Table XIII, which is the well-known system of quaternions.

Secondly, set $c = 2$. We then take the basic numbers to be:

$$\bar{e}_0 = 1, \qquad \bar{e}_1 = ie_1, \qquad \bar{e}_2 = e_1 + \lambda e_2, \qquad \bar{e}_3 = \bar{e}_1 \bar{e}_2 ,$$

in which we now determine λ such that one will have $\bar{e}_2^2 = 0$; this will yield $\lambda = -1$. One will then obtain Table XIII.

Thirdly, let $c = -2$. One will then come back to the case $c = +2$ that was just treated by the substitution $\bar{e}_2 = \dfrac{e_1 - e_2}{2}$.

(b) Let $e_1^2 = e_2^2 = 0$. As above, it will then follow from the condition $k = 2$ that:

$$e_1 e_2 + e_2 e_1 = c.$$

If $c \neq 0$ then we can take $c = 1$, with no loss of generality. We can then immediately go back from this case to case (a) by the substitution:

$$\bar{e}_1 = e_1 + e_2 , \qquad \bar{e}_2 = i\,(e_2 - e_1).$$

However, if $c = 0$ then we will obtain the new Table XIV.

(B) Any two numbers a, b, together with $e_0 = 1$, define a system of three units. Here, there are two possibilities: Either this number system belongs to the fourth type that was presented for $n = 3$ for an arbitrary choice of a, b, or it is a system of the fifth type.

In the case, we can assume that the system of e_0, e_1, e_2 has already been brought into the canonical form III (cf., pp. 13). If e_3 is any number that is linearly-independent of e_0, e_1, e_2 then e_0, e_1, e_3 will likewise define a system of the fourth type. We can therefore take a new basic number $\bar{e}_3$ that satisfies the equation $\bar{e}_3^2 = 0$ in place of e_3 in this system. In that way, the system e_0, e_1, e_3 will go to its canonical form. If we again write $\bar{e}_3$ in place of e_3 then e_0, e_2, e_3 will now define a system of the fifth type that will already be in its canonical form since $e_2^2 = e_3^2 = 0$. With that, the expressions for all of the products of the basic units e_0, e_1, e_2, e_3 will already be known; we will then get Table XV.

Finally, in the second case, we can choose e_1, e_2, e_3 such that we will have $e_1^2 = e_2^2 = e_3^2 = 0$. The last Table XVI will then arise.

The fact that the types that were enumerated here are actually all distinct – i.e., that it would not be possible to derive another table in the same sequence from Tables I-XVI by the introduction of new basic numbers – hardly seems necessary to emphasize, in particular.

One can also easily extend the argument that was just described that one will obtain all of the different forms of any type. Meanwhile, we will go into the treatment of that problem, but only give the result. A form – viz., the second of system X – was overlooked by me in the original version of this report. From the remarks that we made in our examination of the case $n = 3$, we do not especially need to write down the substitutions by which the distinct real forms Ia), …, Ic), IIa), and IIb) could arise from the basic forms of their types I and II.

I.

	e_0	e_1	e_2	e_3
e_0	e_0	·	·	·
e_1	·	e_1	·	·
e_2	·	·	e_2	·
e_3	·	·	·	e_3

Ia).

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	e_0	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	e_2

Ib)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	$-e_0$	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	e_3

Ic)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	$-e_0$	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	$-e_3$

II.

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	·	·	·
e_2	·	·	e_2	·
e_3	·	·	·	e_3

IIa)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	·	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	e_2

IIb)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	·	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	$-e_2$

III

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	·	·
e_1	e_1	·	·	·
e_2	·	·	e_2	e_3
e_3	·	·	e_3	·

IIIa)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_3	e_2
e_2	e_2	e_3	·	·
e_3	e_3	e_2	·	·

IIIb)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	e_3	·	·
e_3	e_3	$-e_2$	·	·

The form IIIa) arises from II under the real substitution:

$$\bar{e}_0 = e_0 + e_2\,, \qquad \bar{e}_1 = e_0 - e_2\,, \qquad \bar{e}_2 = e_1 + e_3\,, \qquad \bar{e}_3 = e_1 - e_3\,.$$

Table IIIb) arises from IIIa) by the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_1 = ie_1, \qquad \bar{e}_2 = e_2, \qquad \bar{e}_3 = ie_2.$$

IV

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	·
e_1	e_1	e_2	·	·
e_2	e_2	·	·	·
e_3	·	·	·	e_3

V

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_2	e_3	·
e_2	e_2	e_3	·	·
e_3	e_3	·	·	·

VI

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	·
e_1	e_1	e_0	e_2	·
e_2	e_2	$-e_2$	·	·
e_3	·	·	·	e_3

VIII

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_2	e_3
e_2	e_2	$-e_2$	·	·
e_3	e_3	e_3	·	·

The system VI will go to its reciprocal under the substitution $\overline{e}_1 = -e_2$; the system VII cannot go to its reciprocal system.

VIII

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	·
e_1	e_1	·	·	·
e_2	e_3	·	·	·
e_3	·	·	·	e_3

IX

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	e_3	·
e_2	e_2	$-e_3$	e_3	·
e_3	e_3	·	·	·

Table IX will represent infinitely many types, corresponding to the different values of the parameter c. Any of these systems will go to its reciprocal under the substitution $\overline{e}_2 = -e_2$.

X

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	·	·
e_2	e_2	·	e_3	·
e_3	e_3	·	·	·

Xa)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	·	·
e_2	e_2	·	$-e_3$	·
e_3	e_3	·	·	·

Table Xb) will arise from X under the substitution $\overline{e}_2 = ie_2$.

XI

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	·	·
e_2	e_2	·	·	·
e_3	e_3	·	·	·

XII

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

XIIb)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_3	e_2
e_2	e_2	$-e_3$	$-e_0$	$-e_1$
e_3	e_3	$-e_2$	e_1	$-e_0$

System XII, which is well-known under the name of the *quaternions*, will go to system XIIb) under the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_1 = ie_1, \qquad \bar{e}_2 = ie_2, \qquad \bar{e}_3 = -e_3 .$$

XIIIb) will also emerge from XIII under the same substitution.

Systems XII-XV will each go to their reciprocals under the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_1 = -e_1, \qquad \bar{e}_2 = -e_2, \qquad \bar{e}_3 = -e_3 .$$

XIII

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	·	·
e_3	e_3	e_2	·	·

XIIIb)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_3	e_2
e_2	e_2	$-e_3$	·	·
e_3	e_3	$-e_2$	·	·

XIV

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	·	e_3	·
e_2	e_2	$-e_3$	·	·
e_3	e_3	·	·	·

XV

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_2	e_3
e_2	e_2	$-e_2$	·	·
e_3	e_3	$-e_3$	·	·

XVI

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	·	·	·
e_2	e_2	·	·	·
e_3	e_3	·	·	·

§ 6. Special systems with *n* basic numbers.

The extension of the determination of the number systems with n basic numbers that was just given in some simple cases to an undetermined value of the number n can have its difficulties. Nonetheless, a seemingly extended class of such systems can be determined in general. They are the systems for which the number k that was defined in § 2 has the largest possible value n. It is obvious that they will obey the commutative law of multiplication. Their totality defines an irreducible manifold whose most general, and likewise simplest, representative can be considered to be a system with the following multiplication rules (*):

$$e_i^2 = e_i\,, \qquad e_i\, e_k = 0 \qquad (i \neq k,\ i, k = 1, \ldots, n).$$

The totality of all systems of the stated type will be given by the following theorem (**):

In order to find all distinct systems with n basic numbers for which the powers $A^0, A^1, \ldots, A^{n-1}$ of any number A are, in general, linearly-independent of each other, one must represent the number n as a sum of whole numbers in all possible ways.

If:

$$n = \alpha + \beta + \ldots + \mu$$

is such a decomposition of n then one will arrange the n basic numbers in groups $\alpha, \beta, \ldots, \mu$, and denote them by:

$$a_0, \ldots, a_{\alpha-1}, b_0, \ldots, b_{\beta-1}, \ldots, m_0, \ldots, m_{\mu-1}\,.$$

One will then set the products of any two basic numbers from different groups equal to zero, and assume a rule for the multiplication of two numbers within the same group, which might be stated, for example, as:

$$\begin{aligned} a_i\, a_j &= a_{i+j} && (i + j \leq \alpha - 1), \\ a_i\, a_j &= 0 && (i + j > \alpha - 1), \end{aligned}$$

for the first group.

Any two number systems that are determined in this way will be distinct; i.e., it will not be possible to take one of them to another one by the introduction of new basic numbers. However, any system of n numbers that satisfies the stated condition can be taken to one of the systems that was presented by a suitable choice of basic numbers.

Moreover, the basic numbers $a_0, \ldots, m_{\mu-1}$ in the given systems are not determined uniquely, except for only the simplest case $\alpha = \beta = \ldots = \mu = 1$, in which one will come back to the multiplication rules $e_i^2 = e_i$, $e_i\, e_k = 0$ that were given already. One

(*) One should confer the author's remarks in Gött. Nachr. 1889, pp. 264-267 concerning this oft-treated number system.

(**) According to this, a theorem that was stated by Poincaré [Comptes rendus de l'Ac. des Sciences, t. 99, (1884), pp. 740] can be corrected.

immediately recognizes that one will obtain the precisely the same multiplication rules when, e.g., one introduces the following basic numbers:

$$\begin{aligned}
\overline{a}_1 &= \lambda_1\, a_1 + \lambda_2\, a_2 + \ldots + \lambda_{\alpha-1}\, a_{\alpha-1}\,, \\
\overline{a}_2 &= (\lambda_1\, a_1 + \lambda_2\, a_2 + \ldots + \lambda_{\alpha-1}\, a_{\alpha-1})^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\overline{a}_{\alpha-1} &= (\lambda_1\, a_1 + \lambda_2\, a_2 + \ldots + \lambda_{\alpha-1}\, a_{\alpha-1})^{\alpha-1},
\end{aligned}$$

instead of $a_1, \ldots, a_{\mu-1}$.

Here, $\lambda_1, \ldots, \lambda_{\alpha-1}$ mean any sort of numerical values that are subject to only the restriction that λ_1 must be non-zero. The newly-introduced basic numbers are the most general ones that will produce the stated multiplication rules; the sub-domains a_0, $(a_1, \ldots, a_{\alpha-1})$, $(a_2, \ldots, a_{\alpha-1})$, …, $(a_{\alpha-1})$ will be determined uniquely, if not also the basic numbers themselves.

Several *distinct real* forms can emerge from the systems that emerge from the systems that correspond to the decomposition:

$$n = \alpha + \beta + \ldots + \mu$$

only when some of the numbers $\alpha, \beta, \ldots, \mu$ are equal to each other, and one if takes the number αm_1 times the first number that is different from α times m_2 times the first number that is different from the last two times m_3, etc., the one will indeed get:

$$\left[\frac{m_1+2}{2}\right]\cdot\left[\frac{m_2+2}{2}\right]\cdot\left[\frac{m_3+2}{2}\right]\ldots$$

different forms that can be written down immediately, as long as $[m]$ means the largest whole number that is less than or equal to m. For example, let $\alpha = \beta = \gamma$, but $\gamma \neq \delta \neq \varepsilon \neq \ldots \neq k$, so there will be two distinct real forms. The canonical form that was given in the theorem above can be used as the canonical form for the first one. However, it seems preferable to me to introduce 2α basic numbers instead of the basic numbers a_i, b_k by the substitutions:

$$\mathfrak{a}_i = a_i + b_i, \qquad \mathfrak{b}_i = a_i - b_i \qquad (i = 0, 1, \ldots, \alpha - 1),$$

with which, the following multiplication rules will arise:

$$\mathfrak{a}_i\ \mathfrak{a}_j = \mathfrak{a}_{i+j}\,, \qquad \mathfrak{b}_i\ \mathfrak{b}_j = \mathfrak{a}_{i+j}\,,$$
$$\mathfrak{a}_i\ \mathfrak{b}_j = \mathfrak{b}_{i+j}\,.$$

One will then obtain the corresponding table for the second real form simply by making the substitutions:

$$\bar{\mathfrak{a}}_i = \mathfrak{a}_i\,, \qquad \bar{\mathfrak{b}}_i = \sqrt{-1}\;\mathfrak{b}_i \qquad (i = 0, 1, \ldots, \alpha - 1)$$

in the form:

$$\mathfrak{a}_i\;\mathfrak{a}_j = \mathfrak{a}_{i+j}\,, \qquad \mathfrak{b}_i\;\mathfrak{b}_j = -\,\mathfrak{a}_{i+j}\,,$$
$$\mathfrak{a}_i\;\mathfrak{b}_j = \mathfrak{b}_{i+j}\,.$$

Naturally, all of the basic numbers in both tables will be set equal to zero when their index $i + j$ exceeds the largest admissible value of $\alpha - 1$.

I will provide the proofs of the assertions that were made in these paragraphs, with a somewhat weakened formulation of the theorem, in a second treatise that will treat recurring series and bilinear forms that are closely connected with our situation. Two different expressions for the basic numbers $a_0, \ldots, a_{\alpha-1}$ in terms of the numbers $A^0, \ldots, A^{\alpha-1}$ will be given there.

§ 7. On the theory of simply-transitive groups.

In the following paragraphs, the connection between systems of complex numbers and the theory of *transformation groups* will be developed. In particular, it shall be shown how one can make use of the theorems that were presented in §§ 3 – 6 in this theory. Before we do that, will include a section in which a number of – for the most part, new – theorems will be derived that relate to the so-called *parameter groups* that play an important role in the study of transformation groups, as is known. In order to better understand this, let it be first remarked that we will only speak of *continuous* groups throughout. If we speak of – e.g., – conformal transformations then we will mean by that only the ones for which the angle is not altered, and the terms "group of reciprocal radii," "groups of motions," "group of similarity transformations," "group of a second-degree surface" will be used in a similarly-restricted sense.

Regarding the theory of transformation groups, I will refer to the ground-breaking work *Theorie der Transformationsgruppen* (Leipzig, 1888) that was created by the collaboration of Fr. Engel and S. Lie; in the sequel, it will be simply referred to as (Lie). In particular, contents of chapters 16, 20, 21, 26, 27 will come under consideration by us.

From the theory that was developed by Lie, any continuous group, along with its parameter group, is composed the same as a simply-transitive group; on the other hand, two simply-transitive groups that are composed the same will be similar. Thus, any simply-transitive group will be similar with its parameter group. That means that one can introduce new variables (or also new parameters, if one prefers) into any given simply-transitive group that is not its own parameter group in such a way that it will become its own parameter group.

In fact, let a simply-transitive group G_1 be given that is generated by its infinitesimal transformations and is written in terms of the variables $x_1, \ldots, x_n$ and the parameters $y_1, \ldots, y_n$. Then let $\bar{y}_1, \ldots, \bar{y}_n$ be the parameters of a transformation of the group that takes an arbitrary, but chosen once and for all, point in general position $E(x_1^0, \ldots, x_n^0)$ to the point $x_1, \ldots, x_n$ such that the variables $\bar{y}_1, \ldots, \bar{y}_n$ will become independent functions of $x_1, \ldots, x_n$, due to the assumed property of the group G_1. We can therefore introduce $\bar{y}_1, \ldots, \bar{y}_n$ as new variables. However, the group will go to its parameter group under that. Then, let y $[y_1, y_2, \ldots, y_n]$, y', y'' be the parameter system of the transformations S, T, ST of the group G_1, so the general transformation of the parameter group of G_1 will then have the form:

$$y_i'' = \varphi_i (y_1, \ldots, y_n, y_1', \ldots, y_n') \qquad (i = 1, \ldots, n),$$

or, more briefly:

$$y'' = \varphi(y, y').$$

Here, one should think of $y_1, \ldots, y_n$ as independent variables, $y_1', \ldots, y_n'$, as parameters, and $y_1'', \ldots, y_n''$, as dependent variables. The transformation that was written down will be the transformation T' of the parameter group that corresponds to the transformation T of G_1. Now, however, from the above, y, y', y'' will just as much define a coordinate system for the three points $(E)S$, $(E)T$, $(E)ST$ that emerge from the arbitrarily chosen point E under the transformations S, T, ST, respectively, of G_1. Of these three points, the last one will be associated with the first one by way of the transformation T of G_1, and indeed, for any choice of the transformation S. However, we have just now seen that the point $y_1'', \ldots, y_n''$ will also correspond to the point $y_1, \ldots, y_n$ under the transformation T'. The transformations T and T' will then coincide.

From now on, we shall assume that the variables have already been chosen in such a way that the simply-transitive group G_1 will be its own parameter group.

Thus, any symbolic formula that demands the composition of a transformation S in terms of given transformations of G_1 – e.g., $S = S_1 S_2 S_3$ – will be capable of a second possible interpretation, but not in the same way for other groups. Namely, any transformation of a certain point in space that appears in the formula will now likewise correspond to the point whose coordinates are the parameters of the transformation. The formula thus likewise gives a dependency between different points of space.

We would now like to introduce a *symbolic notation* in order to be able to express such a dependency simply with formulas.

Let $x_1, \ldots, x_n$ be the parameter system of the transformation S, we let the symbol x represent the point that has the coordinates $x_1, \ldots, x_n$. Furthermore, if we let y and z be the points that correspond to the transformations T and ST in the aforementioned way then we will write symbolically:

$$z = xy\,;$$

this notation will then serve as an abbreviation for the system of equations:

$$z_i = \varphi_i (x_1, \ldots, x_n, y_1, \ldots, y_n) \quad (i = 1, \ldots, n).$$

If we let S, T, R be three arbitrary transformations of G_1 and let x, y, z be the corresponding point then we will likewise denote the point that is associated with the transformation STR symbolically by $(x\,y)\,z$ or $x\,(y\,z)$, or more simply by xyz; this symbol will thus represent a point whose coordinates are given by the expressions:

$$\varphi_i\,(\varphi_1(x, y), \ldots, \varphi_n(x, y), z_1, \ldots, z_n) = \varphi_i\,(x_1, \ldots, x_n, \varphi_1(y, z), \ldots, \varphi_n(y, z)).$$

We will further represent any point that is associated with a transformation that is composed from arbitrarily many transformations of G_1 in a similar manner, which does not particularly need to be explained in more detail.

Furthermore, let x' be the point that corresponds to the transformation S^{-1} (viz., the inverse transformation to S), so we set:

$$x' = x^{-1}.$$

This symbolic equation will serve as an abbreviation for a system of equations that one will obtain as follows: Solving the n equations:

$$y'_i = \varphi_i\,(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

will yield the equations:

$$y_i = \varphi_i\,(y'_1, \ldots, y'_n, x'_1, \ldots, x'_n)\,,$$

in which $x'_1, \ldots, x'_n$ are certain functions of $x_1, \ldots, x_n$:

$$x'_i = \psi_i\,(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n),$$

so the last system of equations will be equivalent to the symbolic formula $x' = x^{-1}$. This formula clearly represents an involutory transformation; naturally, one also has, conversely:

$$x_n = \psi_i\,(x'_1, \ldots, x'_n) \qquad (i = 1, \ldots, n)$$

Finally, as a logical continuation of the notation that was introduced, we will represent the point that corresponds to the identity transformation $S^0 = T^0 = \ldots = 1$ by the symbol x^0, or also by the symbol y^0, as desired, or finally, most simply by the symbol 1. Thus, from the symbolic notation $SS^{-1} = S^0 = 1$, will emerge the other one $xx^{-1} = x^0 = 1$, which say nothing by the fact that $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$ are the parameter system images of inverse transformations of the group G_1 . However, we can now write the identities:

$$x_i = \psi_i\,(\psi_1(x), \ldots, \psi_n(x)) \qquad (i = 1, \ldots, n),$$

simply as:

$$x = (x^{-1})^{-1},$$

etc. Obviously, all of the rules that are true for calculating with the symbols S, T will also be true for calculations using the symbols $x, y, \ldots$ For example, one will then have $(xy)^{-1} = y^{-1}\,x^{-1}$.

No matter how convenient the symbolic notations that were introduced might be, they also show that with their help one can derive a whole series of essential properties of simply-transitive groups – which have been noticed up to now, for the most part – in the simplest way, and make the formulas more intuitive. That will next yield a symbolic representation of the group G_1 itself:

1. *The symbolic equation:*

$$x' = xa, \tag{1}$$

in which x is thought of as variable, and a, as a parameter (viz., a parameter system), represents the most general transformation of the group G_1.

$$x' = xa^{-1} \tag{2}$$

is the inverse transformation to (1). $x' = x\,a\,b$ is the transformation of G_1 that arises from the composition of the transformations $x' = xa$ and $x' = xb$. $x' = xa^{-1}ba$ is the transformation of G_1 that arises from the transformation $x' = xb$ when one introduces new variables by means of the transformation $x' = xa$.

2. *The symbolic equation:*

$$x' = ax \tag{3}$$

gives the general transformation of the simply-transitive group G_2 that is reciprocal to G_1 (*).

3. *The reciprocal groups G_1 and G_2 are similar to each other by means of the involutory transformation $x' = x^{-1}$, such that, in fact, any transformation $x' = xa$ of G_1 will be associated with the transformation $x' = a^{-1}x$ of G_2 .*

In fact, if we let the transformations $x' = xa$ and $x' = x^{-1}$ be denoted by S and T then the transformation $S^{-1}T\,S = STS$ will be given by the symbolic equation:

$$x' = ((x^{-1})\,a)^{-1} = a^{-1}\,x;$$

however, this is a transformation of G_2 . Moreover, one likewise sees that the composed transformation $x' = xab$ of the transformations $x' = xa$ and $x' = xb$ in G_1 will go to the transformation $x' = (a\,b)^{-1}\,x = b^{-1}\,a^{-1}\,x$ of G_2, and thus to a transformation that one can obtain immediately by composing transformations of G_2 that are associated with the transformations $x' = xa$ and $x' = xb$. Above all, G_1 and G_2 are similar to each other by means of any transformation $x' = \alpha\,x^{-1}\beta$; in fact, this will take the transformation $x' = xa$ to $x' = \alpha\,a^{-1}\,\alpha^{-1}\,x$.

4. *The symbolic equation:*

$$x' = a\,x\,b \tag{4}$$

gives the general transformation of a group $G_{1,2}$. It will be obtained when one performs the general transformation of G_1 and the general transformation of G_2 one after each other. One immediately convinces oneself that G_1 and G_2 are invariant in this group, and

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(*) From now one, we will use the term "reciprocal groups" without the self-explanatory qualifier "simply-transitive."

furthermore, that the group $G_{1,2}$ has only $2n - m$ essential parameters when G_1 and G_2 have an m-parameter subgroup in common.

5. *The symbolic equation:*

$$x' = a^{-1}\, x\, a \tag{5}$$

gives the most general transformation of a continuous $(n - m)$*-parameter group* $\mathfrak{G}$ *that is characterized as a subgroup of* $G_{1,2}$ *by the fact that its transformations leave the point in general position* $x = 1$ *fixed.* As a result of a remark that was made in number 1, equation (5) tells one how the transformations of G_1 can be exchanged with each other by means of the transformations of G_1 itself.

Therefore, if one keeps a point in general position fixed in the group $G_{1,2}$ *that is generated by the composition of transformations from two reciprocal groups* G_1 *and* G_2 *then the largest continuous subgroup* $\mathfrak{G}$ *of* $G_{1,2}$ *that is determined in that way will be similar to the adjoint group of* G_1 *and* G_2 *.*

The transformation $x' = x^{-1}$ commutes with all of the transformations of the group $\mathfrak{G}$.

If $m = 0$ then the group $\mathfrak{G}$ will have yet another remarkable property. Namely, when it is written in the form (5), it will then again have the group G_1 as its *parameter group*. By contrast, if one writes $\mathfrak{G}$ in the equivalent form $x' = a\, x\, a^{-1}$ then one will obtain the group G_2 as a parameter group.

One can also immediately write down the continuous subgroup of $G_{1,2}$ that leaves another point b in general position fixed: One needs only to introduce new variables into the equations of the group $\mathfrak{G}$: $x' = a^{-1}\, x\, a$ or $x' = a\, x\, a^{-1}$ by means of the transformation $x' = xb$ or the other one $x' = bx$. One will then obtain two symbolic representations of the desired group:

$$x' = a^{-1}xb^{-1}ab \qquad \text{and} \qquad x' = ba^{-1}b^{-1}xa^{-1}. \tag{6}$$

Both notations are equivalent to each other; they will go to each other under the substitution $a = ba^{-1}b^{-1}$ or $a = b^{-1}ab$, resp.

Finally, to complete the statements, one might give the relationship that the infinitesimal transformations of the group $\mathfrak{G}$ have to those of G_1 and G_2, although we shall not use it later on.

6. *Let* $X_1 f, \ldots, X_m f, X_{m+1} f, \ldots, X_n f$ *be independent infinitesimal transformations of the group* G_1 *that are chosen such that any distinguished transformation of* G_1 *will have the form:*

$$\lambda_1 X_1 f + \ldots + \lambda_m X_m f\,.$$

Furthermore, should the expressions $X_i f$ *go to* $\bar{X}_i f$ *by means of the transformation* $x' = x^{-1}$ *then one would have:*

$$X_i f + \bar{X}_i f = 0 \qquad (i = 1, \ldots, m)$$

identically, although $X_{m+1} f + \bar{X}_{m+1} f$, …, $X_n f + \bar{X}_n f$ *are the infinitesimal transformations of the group* $\mathfrak{G}$.

We can also get this theorem with no calculation. Let:

$$x_i' = x_i + \xi_i\, \delta t \tag{7}$$

be an infinitesimal transformation of G_1 that might go to the infinitesimal transformation:

$$x_i' = x_i + \bar{\xi}_i\, \delta t \tag{8}$$

of G_2 by means of the transformation $x' = x^{-1}$. When both of them are performed one after the other that will give a transformation of $\mathfrak{G}$ [see formula (5) above]:

$$x_i' = x_i + (\xi_i + \bar{\xi}_i)\, \delta t. \tag{9}$$

Now, if (8) is, in particular, a transformation of G_2 that likewise belongs to the group G_1 then it will be the inverse transformation to (7), so $\bar{\xi}_i$ will then be equal to $-\xi_i$, identically.

The theorems that were derived are meaningful for the general theory of transformation groups, since a pair of reciprocal groups G_1, G_2 is indeed linked with every continuous group of transformations, namely, the parameter group of the given group and its reciprocal, resp. If one writes the transformations of the given group, in particular, in canonical form:

$$x_i' = x_i + \sum_{k=1}^{n} e_k \cdot X_k x_i + \dots \qquad (i = 1, \dots, n)$$

(cf., Lie, chap. 9, § 46, pp. 171) then the associated parameter group (viz., a *canonical parameter group*) g_1, its reciprocal g_2, and the adjoint group $\mathfrak{G}$ will be three of the type that we described. In this case, the transformation $x' = x^{-1}$ will take on an especially simple analytical expression: It will be nothing but the involutory perspective transformation:

$$e_k' = -e_k .$$

Therefore, from our general theorem, the canonical parameter group and its reciprocal group will be similar to each other by means of this transformation.

§ 8. Lemmas from the theory of linear transformations.

In addition to the theorems on reciprocal groups that were asserted in § 7, we will require some lemmas from the theory of bilinear forms for our further developments.

These theorems will be briefly summarized here. One will find a thorough derivation in one of the author's papers "Methoden zur Theorie der ternären Formen" that appeared in print through B. G. Teubner.

As is well-known, a linear transformation of an n-dimensional domain can be represented by setting equal to zero a bilinear form with two rows of any n (so-called "contragredient") variables:

$$S = (A\,X)(U\,P) = \sum a_{ik}\,x_i\,u_k\,.$$

Let:

$$T = (B\,X)\,(U\,Q) = \sum b_{ik}\,x_i\,u_k$$

be a second such form, so the transformation that arises by composing the transformations $S = 0$ and $T = 0$ (in this sequence) will be likewise represented in the same way by a simultaneous covariant of the forms S and T – viz., the *product* of S and T:

$$ST = (A\,X)\,(B\,P)\,(U\,Q) = \sum a_{ik}\,b_{kj}\,x_i\,u_j\,.$$

The form $(U\,X) = \sum x_i\,u_i$ will represent the identity transformation $S^0 = T^0$; it will satisfy the equations $S^0S = SS^0 = S$ identically. Furthermore, any form S whose discriminant does not vanish will be associated with a certain form S^{-1} that represents the inverse transformation to $S = 0$ when it is set equal to zero, etc. (*) A family – and in particular, a continuous group – of linear transformations will be determined by a certain system of relations between the coefficients a_{ik}. An r-parameter continuous group will contain r linearly-independent infinitesimal transformations that one can represent by r forms of the type:

$$S^0 + \delta t\,S,$$

in which S, in particular, means a form of vanishing linear invariant $(A\,P) = \sum a_{ii}$, so it is a so-called *normal form*. The normal form S is the "symbol" of the infinitesimal transformations $S^0 + \delta t\,S$ of the group. By the use of the symbol S, one will arrive at the expressions for a number of important properties of the group in a very simple way.

If $S_1, \ldots, S_r$ are the symbols of the independent infinitesimal transformations of the group then $\lambda_1\,S_1 + \ldots + \lambda_r\,S_r$ will be the symbol of the general infinitesimal transformation of the group. If one denotes the combination $S_i\,S_k - S_k\,S_i$ of the forms S_i, S_k (which is again a normal form) by (S_i, S_k) then there will exist $r\,(r-1)/2$ relations:

$$(S_i, S_k) = \sum c_{iks}\,S_s\,,$$

in which the constants c_{iks} will be characteristic constants for the *composition* of the group. If one introduces new variables into the symbol S by means of a finite transformation T of the group then one will again obtain a normal bilinear form of the same family that can be written simply as $T^{-1}\,S\,T$. Finally, the everywhere-convergent series:

(*) Cf., Frobenius, "Über lineare Substitutionen und bilineare Formen," Crelle's Journ., Bd. 84.

$$e^S = S^0 + S^1 + \frac{S^2}{2!} + \frac{S^3}{3!} + \ldots,$$

when S is the symbol for the general *infinitesimal* transformation of a group, will represent the general *finite* transformation of that group, and indeed by means of a bilinear form with determinant one.

If one has two different projective groups with the special property that all of the transformations of one of them commute with all of the transformations of the other one then there will exist the relation $(S, S') = 0$ between the symbols of the two-sided infinitesimal transformations S and S', etc.

§ 9. The simplest transformation groups that are connected with a system of complex numbers.

Groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ of the sort that we treated in § 7, are coupled to some system of complex numbers, and indeed, in a double way.

Let $a, b, \ldots, x, x', \ldots$ be numbers in an arbitrary given system – e.g., $x = x_0\, e_0 + \ldots + x_{n-1}\, e_{n-1}$. We can then regard $x_0, \ldots, x_{n-1}$ as *ratios*, and interpret them as the homogeneous coordinates of a point in an n-dimensional domain or an $(n - 1)$-dimensional space R. If we consider a, b, … as parameters, x, as independent variables, and x' as dependent ones then the latter will represent an actual transformation of the space R by way of the formulas:

$$x' = xa, \qquad x' = ax, \qquad x' = axb, \qquad x' = a^{-1}xa,$$

as long as the parameters a and b are general numbers in the system (§ 1). If one lets the numbers a, b vary then one will obtain a group of transformations that corresponds to each of them, and indeed the four groups thus-obtained will be related to each other in precisely the same way as the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that were investigated above, and may therefore be denoted similarly. It should only be remarked that they are written in homogeneous variables and that the aforementioned n numbers must therefore be now replaced with $n - 1$ numbers. At the same time, we would also like to write $m - 1$ for m, such that from now on $m - 1$ will mean the number of parameters of the largest subgroup that is common to G_1 and G_2, and the number of parameters in the four aforementioned groups will be:

$$n - 1, \qquad n - 1, \qquad 2n - m - 1, \qquad n - m.$$

The groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that we have been concerned with here are now groups of a special sort: They are projective groups. The group $\mathfrak{G}$ is, moreover, as we shall see later on, a so-called *linear, homogeneous group*. We can also likewise give another essential property of the aforementioned groups: A point in general position will not be fixed by any transformation of G_1 or G_2 besides the identity transformation. Therefore, a point a in general position can always be taken to another one b by a transformation of G_1

(G_2, resp.) – namely, by the transformation $x' = xa^{-1}b$ ($x' = ba^{-1}x$, resp.); as a result, the group $\mathfrak{G}$ will consist of the totality of *all* transformations of $G_{1,2}$ that leave the point $x = 1$ fixed.

Naturally, one can now immediately write down the families of bilinear forms that give representations of the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$, according to § 8. In regard to the groups G_1 and G_2 , we remark that the families of bilinear forms that belong to them will be *linear families;* i.e., ones in which the coefficients of the general form can be expressed as linear functions of n homogeneous parameters.

In fact, let $e_i\, e_k = \sum \gamma_{iks}\, e_s$, $x = \sum x_i\, e_i$, $y = \sum y_i\, e_i$, so one will have:

$$xy = \sum \gamma_{iks}\, x_i\, y_k\, e_s\,.$$

If one writes u_s here instead of e_s then a form will arise that will have three rows of variables, namely:

$$\sum_{i,k,s} \gamma_{iks} x_i y_k u_s\,.$$

If one considers $y_1, \ldots, y_{n-1}$ in this form as parameters then one will obtain a family of bilinear forms that will represent all of the transformations of the group G_1 when they are set equal to zero.

One can further remark that the multiplication of numbers in our system is precisely identical to the "multiplication" of the associated bilinear forms.

If one sets $y = e_0, e_1, \ldots, e_{n-1}$, in sequence, and one denotes the bilinear forms that are associated with the basic numbers e_i, e_κ, e_μ by:

$$S_i = \sum_{\lambda,\mu} \gamma_{\lambda i\mu} x_\lambda u_\mu\,, \qquad S_\kappa = \sum_{\mu,\nu} \gamma_{\mu\kappa\nu} x_\mu u_\nu\,, \qquad S_\mu = \sum_{\lambda,\mu} \gamma_{\lambda\mu\nu} x_\lambda u_\nu$$

then, due to the well-known relations:

$$\sum_{\mu} \gamma_{i\kappa\mu}\gamma_{\lambda\mu\nu} = \sum_{\mu} \gamma_{\lambda i\mu}\gamma_{\mu\kappa\nu}\,,$$

which follow from the associative law $(e_\lambda\, e_i)\, e_\kappa = e_\lambda\, (e_i\, e_\kappa)$, one will also have, in fact, the identity:

$$S_i\, S_k = \sum_{\mu} \gamma_{i\kappa\mu} S_\mu$$

for the composition of the bilinear forms S_i and S_k .

The so-called complex numbers are therefore nothing but an abbreviated notation for bilinear forms that represent certain groups of linear transformations; nothing stops us then from identifying the complex numbers e_i, in turn, with the associated forms S_i, since one can read off a formula in the one theory that reads the same as a formula in the other one theory by a mere change of notation.

Up to now, we have made no special assumptions about the choice of basic numbers $e_0, e_1, \ldots, e_{n-1}$. From now on, we would like to assume that there is one of them – say, e_0 – that is the number one in the system, such that one will have $S_0 = (U\,X) = \sum u_i\,x_i$. However, the other ones $e_1, \ldots, e_{n-1}$ will be chosen such that the linear invariants $\sum_{\lambda=0}^{n-1} \gamma_{\lambda i \lambda}$ of the associated bilinear forms S_i will vanish.

The constants γ_{iks} that enter into the multiplication rules will then take on simple values, in part. Namely, one will obviously have:

$$\gamma_{0ii} = \gamma_{i0i} = 1, \qquad \gamma_{0ik} = \gamma_{i0k} = 0 \qquad (i \neq k),$$

and in addition, one will have the relations:

$$(1) \qquad \gamma_{ik0} = \gamma_{ki0}\,,$$

for $i, k = 1, \ldots, n - 1$.

Namely, the linear invariant of the bilinear form $S_i\,S_k$ would be equal to that of $S_k\,S_i$, but one finds the value $n\gamma_{ik0}$ for the former and $n\gamma_{ki0}$ for the latter.

These remarks are useful because they show how one can read off the values of the constants c_{iks} that are characteristic of the composition of the group G_1 from the multiplication table of the system in a very simple way, and indeed also in the general case where the basic numbers $e_0, \ldots, e_n$ are not chosen in precisely the aforementioned special way.

Next, by substituting the values of the constants γ_{iks} that were written down above, one will get $(n - 1)(n - 2) / 2$ relations of the special form:

$$(2) \qquad e_i\,e_k - e_k\,e_i = \sum (\gamma_{iks} - \gamma_{kis})\,e_s \qquad (i, k, s = 1, \ldots, n - 1).$$

On the other hand, since the forms S_i are the symbols of the infinitesimal transformations of the group G_1, one will have:

$$(3) \qquad S_i\,S_k - S_k\,S_i = \sum c_{iks}\,S_s \qquad (i, k, s = 1, \ldots, n - 1).$$

If one compares the two formulas then it will follow that:

$$(4) \qquad c_{iks} = \gamma_{iks} - \gamma_{kis}\,.$$

If one now introduces new basic numbers in place of the basic numbers $e_0, \ldots, e_{n-1}$ by the substitution:

$$\bar{e}_0 = e_0, \qquad \bar{e}_s = \lambda_s\,e_0 + e_s$$

then one will obtain the following relations:

$$(5) \qquad \bar{e}_i\,\bar{e}_k - \bar{e}_k\,\bar{e}_i = \sum c_{iks}\bar{e}_s - \sum c_{iks}\lambda_s\,\bar{e}_0\,,$$

in place of the relations (2). Therefore, *congruences* with the modulus $\bar{e}_0$ will appear in place of *equations* (2). However, the basic numbers $\bar{e}_1, \dots, \bar{e}_{n-1}$ can be considered as completely arbitrary numbers in the system that only have to satisfy the one condition that the number one of the system cannot be derived from them linearly.

One can therefore read off the composition of the associated groups G_1, G_2 immediately from the multiplication table of a system of complex numbers without having to present the equations of these groups.

By way of example, in the case $n = 4$ of the system XIII, one will have:

$$e_2\, e_3 - e_3\, e_2 = 0,$$
$$e_3\, e_1 - e_1\, e_3 = 2\, e_2\,,$$
$$e_1\, e_2 - e_2\, e_1 = 2\, e_3\,.$$

Thus, the associated groups G_1, G_2 will have the composition:

$$(Y_2,\ Y_3) = 0, \qquad (Y_3,\ Y_1) = 2Y_2, \qquad (Y_1,\ Y_2) = 2Y_3\,,$$

or, after introducing $X_i f = \frac{1}{2} Y_i f$:

$$(X,\ X_3) = 0, \qquad (X_3,\ X_1) = X_2, \qquad (X_1,\ X_2) = X_3\,.$$

If we set $S = \xi_1 S_1 + \dots + \xi_{n-1} S_{n-1}$ then $e^S = 0$ will become the general transformation of the group G_1 (cf., pp. 33). From now on, with the use of the notation:

(6) $$\bar{\xi} = \xi_1\, e_1 + \dots + \xi_{n-1}\, e_{n-1}\,,$$

we can also write this transformation as:

(7) $$x' = x\, e^{\bar{\xi}}\,.$$

If we let the constants $\xi_1, \dots, \xi_{n-1}$ change in such a way that their ratios remain the same then we will obtain all of the transformations of the general one-parameter subgroup of G_1. If k of the powers of the number $\bar{\xi}$ – viz., $\bar{\xi}^0 = e_0$, $\bar{\xi}^1, \dots, \bar{\xi}^{k-1}$ – are linearly-independent then, under the transformations of the group, a point x in general position will go to the points of a curve that is contained in a planar domain of dimension k. Here, we will then have a simple interpretation of the number k that was used in § 2 for the classification of the systems of complex numbers: It is the dimension of the smallest linear manifold that circumscribes a general path of a sufficiently-chosen general one-parameter subgroup of the group G_1.

Another interpretation of the number k will come from a consideration of the transformation $x' = x^{-1}$. This will now obviously be a *Cremona* transformation of degree $k - 1$. [Cf., § 1, formula (10).] It will then be a projective transformation, in particular, as long as k has the value 2.

If the number x is general (cf., § 1) then the point x will be a point in general position relative to the transformations of G_1 and G_2 , and in addition, it will be a point at which the transformation $x' = x^{-1}$ will exhibit regular behavior. Such a point will likewise corresponds to a proper (i.e., non-degenerate) transformation of the groups G_1 and G_2 . By contrast, if the number x is special then the associated transformation of the groups G_1 and G_2 will degenerate; furthermore, the point x will be a singular place for the transformation $x' = x^{-1}$, and it will be continued from the transformations of the group G_1, as well as those of the group G_2 in less than $n - 1$ independent directions. One obtains the totality of all these points in special positions when one sets one of the determinants (7) that were defined more precisely in § 1 equal to zero. The theorem that one found there that the $(n - 2)$-fold extended point manifold that is invariant under the transformations of the group G_1 is identical to the associated manifold that corresponds to the group G_2 is, moreover, only a special case of a theorem that can be extended to arbitrary reciprocal groups. One effortlessly finds that the totality of all points that are fixed by the transformations of a simply-transitive group or any of its subgroups will define an invariant manifold for the transformations of the reciprocal group.

§ 10. A property of reciprocal projective groups.

We shall next make an application of what we have been saying up to now in which we shall come to understand a theorem that also possesses a certain interest outside of the theory of complex numbers.

Previously, we arrived at the group $\mathfrak{G}$ when we referred the finite transformations $y' = yx$ of the group G_1 to the point x of the space R (in a single-valued, invertible way), and now the permutation of these transformations through the transformations of G_1 itself will be, in turn, interpreted as a transformation of R. We have also emphasized that the group $\mathfrak{G}$: $x' = a^{-1}xa$ thus obtained is obviously similar to the adjoint group of G_1 .

However, we can apply the transformations of G_1 to the points of the space R in yet another remarkable way. Namely, if we set $x = e^{\xi}$ and now represent the transformations of G_1 in the form $y' = ye^{\bar{\xi}}$ [pp. 32, 33. nos. (6), (7)] then we can also associate the transformation of G_1, thus-written, with the point $\xi = e_0 + \bar{\xi}$ of R. Thus, any point ξ of the space R will also correspond to a certain transformation of G_1; conversely, however, a given transformation of G_1 will now be associated with a finite or infinite number of discrete points of the space R.

The group by which the points of the space R will now be permuted by means of the transformations of G_1 will obviously be the *adjoint* group of G_1 when one regards $\xi_1, \ldots, \xi_{n-1}$ as *Cartesian* coordinates.

We can write down their finite equations immediately. They will be represented by the formulas:

$$\xi' = e_0 + \bar{\xi}' = e_0 + a^{-1}\bar{\xi}a = a^{-1}(e_0 + \bar{\xi})\,a = a^{-1}\,\xi\,a\,; \tag{1}$$

i.e., we again obtain the group $\mathfrak{G}$.

We will now see directly that the groups G_1 and G_2 that were considered here actually give us the most general pair of reciprocal projective groups. We would like to temporarily assume that this has been proved, in order to not break up the continuity of the argument; by the addition of a remark that was made on pp. 34, we can then state the following theorem:

If two reciprocal groups G_1 and G_2 are both projective then the subgroup $\mathfrak{G}$ of the group $G_{1,2}$ that is generated by G_1 and G_2 that is determined by holding a point in general position fixed will be continuous and similar to the adjoint group of G_1 and G_2 by a projective transformation.

We will arrive at a new way of looking at this theorem, and likewise a remarkable extension of it, when we interpret the connection between the two maps that were employed as being itself a transformation of the space R. The first map of the point x corresponds to a transformation with a determinant of unity in the group G_1:

$$y' = yx = ye^{\bar{\xi}} = ye^{\xi - e_0},$$

while for the second transformation of the point ξ, the complex numbers x and $\xi = e_0 + \bar{\xi}$ will be connected by the equation:

$$x = e^{\xi - e_0},$$

at which point, we can also set the following:

$$\rho x = e^{\xi - e_0}, \tag{2}$$

since a common proportionality factor in all of the coordinates of the point x will make no difference, here.

This formula means a transformation of the space R that is not, in general, projective, and which associates any point ξ with a single point x, but will associate any point x in general position with a finite or infinite number of discrete points ξ. One likewise sees that the transformation (2) *commutes* with all of the transformations of the group $\mathfrak{G}$. Therefore, if one introduces the new variables $\xi_1, \ldots, \xi_{n-1}$ in place of the variables $\frac{x_1}{x_0}, \ldots, \frac{x_{n-1}}{x_0}$ in the equations of the group $\mathfrak{G}$ by the transformation (2) then not just the totality of all transformations of $\mathfrak{G}$, but also *any* transformation of this group will go to itself. However, two groups g_1 and g_2, which are not projective, in general, will emerge from the groups G_1 and G_2 by the transformation $x = e^{\xi}$ that will likewise have a simple meaning. If one regards $\xi_1, \ldots, \xi_{n-1}$, as above, as Cartesian coordinates then the group g_1 will obviously be nothing but the parameter group of the group G_1 that belongs to the canonical representation:

(3) $$\frac{x_i'}{x_0'} = \frac{x_i}{x_0} + \sum_{k=1}^{n-1} \xi_i X_k\left(\frac{x_i}{x_0}\right) + \dots$$

or the equivalent representation:

(4) $$x' = x\, e^{\bar{\xi}} .$$

With the use of a notation that was already introduced previously (§ 7, pp. 32), one will then obtain the following theorem:

If two reciprocal groups G_1 and G_2 are both projective then one can, by a suitable choice of projective coordinates, get a transformation that is, in general, not projective, and that will naturally take the group G_1 to its canonical parameter group g_1 and G_2 to its reciprocal group, and will likewise leave any transformation of the group $\mathfrak{G}$ fixed. With this choice of coordinates, the group $\mathfrak{G}$ will become the adjoint of the groups G_1, G_2.

If one wishes to also obtain the canonical parameter group in the canonical form then one must also naturally introduce new parameters into the equations of the group G_1 by means of the cited transformation.

One effortlessly convinces oneself that the transformation $x = e^{\xi}$ must reduce to the *identity transformation* in the case where it actually is projective. The series:

$$e^{\bar{\xi}} = \bar{\xi}^0 + \bar{\xi} + \frac{\bar{\xi}^2}{2!} + \dots$$

will then always break off after the second term. This will happen, for example, when the group G_1 consists of the group of all transformations of space R.

In order to convert the group G_1 into its canonical parameter group in the general case, as well, one must solve $x = e^{\bar{\xi}}$, in which x means the parameter system of a transformation with determinant one in the group G_1, for $\bar{\xi}$. This can be accomplished by a process that can be characterized in general. As a result, we shall not go into the treatment of that problem, since we would then drift too far from the actual purpose of our investigation. It might suffice to explain the theorem that is presented by way of an example.

We choose a system with four basic numbers e_0, e_1, e_2, e_3 that obey the multiplication rules that were given in Table IX (cf., pp. 24). If we set:

$$\bar{\xi} = \xi_1\, e_1 + \xi_2\, e_2 + \xi_3\, e_3$$

then it will follow that:

$$e^{\xi} = e_0 + \xi_1\, e_1 + \xi_2\, e_2 + \left\{\xi_3 + \frac{\xi_1^2 + c\xi_2^2}{2}\right\} e_3 .$$

The transformation, whereby we have introduced new variables into the equations of the groups G_1, G_2, $\mathfrak{G}$, will then read, when take x_0 to be equal to one from now on:

(1) $$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad x_3 = \xi_3 + \frac{\xi_1^2 + c\xi_2^2}{2}.$$

We write the equations of the groups G_1 and G_2 most conveniently in inhomogeneous form, not just in terms of the variables alone, but also in terms of the parameters:

(2) $$\begin{cases} x_1' = x_1 + a_1, \qquad x_2' = x_2 + a_2, \\ x_3' = x_3 + a_3 + (a_1 + a_2)x_1 - (a_1 - ca_2)x_2, \end{cases}$$

(3) $$\begin{cases} x_1' = x_1, \qquad x_2' = x_2, \\ x_3' = x_3 + 2(a_2 x_1 - a_1 x_2). \end{cases}$$

If one introduces new variables into the latter equations by means of the transformation (1) then they will remain completely unchanged, as the theorem would demand. By contrast, we will obtain a new system of equations from the system of equations (2). If we simultaneously introduce new parameters by means of the substitution:

(1b) $$a_1 = a_1, \qquad a_2 = a_2, \qquad a_3 = a_3 + \frac{a_1^2 + ca_2^2}{2}$$

then the equations that emerge from (1) will assume the simple form:

(4) $$\begin{cases} \xi_1' = \xi_1 + a_1, \\ \xi_2' = \xi_2 + a_2, \\ \xi_3' = \xi_3 + a_3 + a_2\xi_1 - a_1\xi_2. \end{cases}$$

This is then the canonical parameter group of the group (2), and indeed, with the choice of parameters that we made, the canonical parameter group in its canonical form. In fact, one can immediately convince oneself by calculation that the group (4) is its own canonical parameter group. Now, we can also assert a theorem that was proved in § 7 with the example of the group (4): We likewise find that the group (4) is similar to its reciprocal by means of the transformation:

$$\xi_1' = -\xi_1, \qquad \xi_2' = -\xi_2, \qquad \xi_3' = -\xi_3.$$

The arbitrary constant c that enters into equations (2) drops out of equations (4) completely. Therefore, we can now also immediately convert the group (2) with the constant c into a group whose transformations, in turn, have the form (2), but contain a new constant d, instead of the constant c:

That will give the quadratic transformation:

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_3 = y_3 + \frac{c-d}{2} y_2^2,$$

as long as we also introduce new parameters by means of a corresponding transformation.

Moreover, we could also have foreseen his latter result. If the quantities c_{iks} that determine the composition of the group (2) are free of the parameter c then all of these groups will be similar to each other, and the parameter c cannot also enter into the equations of the canonical parameter groups.

Equations (4) are nothing but the equations of the group G_1 that belongs to the system of numbers:

$$e_0 = 1, \qquad e_1^2 = e_2^2 = e_3^2 = 0,$$
$$e_1\, e_2 = - e_2\, e_1 = e_3\,, \quad e_1\, e_3 = e_3\, e_1 = e_2\, e_3 = e_3\, e_2 = 0.$$

This number system is identical with the system that we gave on pp. 35 and denoted by XIV. One will infer the arguments of the next paragraphs, in conjunction with the ones in § 5, with no difficulty from the theorem that the group (4) and the group of all translations of space are the only three-parameter projective groups that coincide with their own canonical parameter groups.

§ 11. Projective groups whose transformations commute with given infinitesimal projective transformations.

We would now like to show that any two reciprocal projective groups can be related to a system of complex numbers in a way that is known to us *by distinguishing an arbitrary point in general position.*

We will then regard that theorem as a special case of the following general theorem:

The totality of all infinitesimal projective transformations that commute with the infinitesimal transformations of a given projective group will generate a projective group with the special property that its finite transformations can be determined by linear equations in the transformation coefficients.

In fact, let S' be the symbol of the given infinitesimal transformation and let S be the symbol of the desired one, so the condition for the commutativity – viz., $(S, S') = 0$ – will be linear in the coefficients of S. Let $S_1, \ldots, S_r$ be r independent normal forms that satisfy this condition, so all of the forms:

$$\lambda_0\, S^0 + \lambda_1\, S_1 + \ldots + \lambda_r\, S_r$$

will commute with not just the forms S', but also all powers and products of these forms; therefore, certainly all of the transformations in the family:

$$\lambda_0 S^0 + \lambda_1 S_1 + \ldots + \lambda_r S_r = 0$$

will commute with all transformations of the group that is generated by the infinitesimal transformations S'. However, the cited linear family will include the infinitesimal transformations $S_1, \ldots, S_r$, and will be, as a result, the group that it generates.

One can then immediately derive the finite transformations from the infinitesimal transformations of such a group without integration. (Cf., § 15, at the end.)

Now, if the reciprocal group G_2' to a simply-transitive projective group G_1' is likewise projective in an n-dimensional domain then it will follow immediately that the general finite transformation of G_1' (and likewise G_2') can be written in the form:

$$\lambda_0 S^0 + \lambda_1 S_1 + \ldots + \lambda_{n-1} S_{n-1} = 0.$$

We can now give a system of complex numbers that has a relationship to the group G_1' that is very simple, if also not exactly the desired one, namely, the bilinear forms S^0, $S_1, \ldots, S_{n-1}$. A certain group G_1 – namely, the *parameter group* of G_1' – will belong to it, which we will refer to as basic numbers $e_0, \ldots, e_{n-1}$. In order to substantiate the theorem to be proved, we will then still have to show that this group G_1 is not only similar to the given group G_1' – which is self-explanatory – but that it is similar under a *projective* transformation, in particular. However, that would follow immediately if we were to introduce the parameters of the transformation of G_1' that takes an arbitrary, but fixed, point E in general position to any point as the coordinates of the latter point in the n-dimensional domain. The new coordinates will then be naturally connected with the old ones by *linear* equations.

The stated theorem is proved with that. However, we can still go further.

If one introduces new variables into the equations of the group G_1' by mean of any linear transformation with a unity determinant then the relationship between the group and the (naturally simultaneously transformed) bilinear forms $S^0, S_1, \ldots, S_{n-1}$ will remain completely undisturbed. It will follow from this that all projective groups that are on an equal footing with G_1' inside the general projective group can be associated with one and the same number system $e_0, \ldots, e_{n-1}$. However, the group G_2' that is reciprocal to G_1' and the groups that are on an equal footing with it will also correspond to a number system that is not essentially different from $e_0, \ldots, e_{n-1}$, namely, the same system that one gets from the given one by switching all of the constants γ_{iks} with the corresponding constants γ_{kis}, which we have referred to as the "reciprocal" system to the system $e_0, \ldots, e_{n-1}$. Since, conversely, switching γ_{iks} and γ_{kis} will only change their roles in the groups G_1 and G_2 that belong to the system, and since furthermore the introduction of new basic numbers in place of $e_0, \ldots, e_{n-1}$ will only have the introduction of new variables and parameters (by linear transformations) into the groups G_1 and G_2 as a consequence, we can now state the theorem:

Any system of complex numbers with n principal units is, in a certain way, associated with two reciprocal projective groups of an n-dimensional domain (i.e., an n-dimensional *space), and conversely. Indeed, any two number systems of the same type will correspond to a pair of groups that are on an equal footing inside of the general projective group, but any two number systems of different types will also correspond to different types of pairs of reciprocal projective groups* (*).

If we assume that the two reciprocal groups coincide then that will yield:

Any system of complex numbers with n principal units for which the commutative law of multiplications is true will be associated with a transitive group of commuting linear transformations of an n-dimensional domain, and conversely. Indeed, any two of number systems will correspond to a certain type of group of the stated kind.

This theorem has not only a theoretical interest, but it is also of practical utility. It shows that one can find the different types of reciprocal projective groups in a simple and immediate way, as would be possible by applying the general methods of the theory of transformation groups. The fact that the reciprocal relationship that was established can also be extended to a real relationship, so two different forms of number systems of the same type will also correspond to two pairs of groups of the same type, which will still be different inside the group of all *real* linear transformations, and conversely, hardly needs to be emphasized, in particular.

§ 12. Examples.

A simple example of the connection between the systems of complex numbers and projective groups is provided by the system whose multiplication rules are:

$$e_i^2 = e_i \,, \qquad e_i\, e_k = 0 \qquad (i \neq k,\ i, k = 1, \ldots, n).$$

The associated group G_1 (= G_2) is the known group:

$$x_i' = a_i\, x_i \qquad (i = 1, \ldots, n).$$

As a second example, let us mention the system:

$$e_0 = 1, \qquad e_i\, e_k = 0 \qquad (i, k = 1, \ldots, n - 1).$$

The group G_1 that belongs to it is the group of all translations (i.e., parallel displacements) of the $(n - 1)$-fold extended manifold, or a group that is on an equal footing with it inside the general projective group of this domain:

$$x_0' = a_0\, x_0 \,,$$

(*) Poincaré (*loc. cit.*) was certainly the first to remark that the problem of complex numbers can be converted into the following one: "Find all of the continuous groups of linear substitutions in n variables whose coefficients are linear functions of n arbitrary parameters."

$$x_i' = a_0\, x_i + a_i\, x_0 \qquad (i = 1, \ldots, n-1).$$

We considered other examples in § 10. However, it is perhaps not superfluous to clarify the connection between complex numbers, bilinear forms, and groups of linear transformations that was treated in § 11 with a thorough and detailed example, as well.

The totality of all perspective transformations of the plane that possess a common perspectivity axis and have their centers on a second fixed line will obviously define a two-parameter, transitive group. One convinces oneself by a perfectly simple geometric argument that its reciprocal group will again be composed of linear transformations; namely, of all perspective transformations whose perspectivity axis is the second line and whose center lies on the first line. We will then have two groups G_1', G_2' here that will have the desired property.

If we choose the line $x_2 = 0$ to be the perspectivity axis and the line $x_3 = 0$ to be the locus of perspectivity centers of the group G_1' then we can represent its finite transformations by the following equations:

$$\begin{aligned} x_1' &= \lambda_0\, x_1 + \lambda_1\, x_2\,, \\ x_2' &= (\lambda_0 + \lambda_1)\, x_2\,, \\ x_3' &= \lambda_0\, x_2\,. \end{aligned} \tag{1}$$

The point $x_1 : x_2 : x_3 = \lambda_1 : \lambda_2 : 0$ will then be the center of this transformation. If we let (λ), (μ), (v) denote the parameter systems of the three transformations S, T, ST of G_1' then we will obtain the equations of the associated *parameter group* in the form:

$$\begin{aligned} v_0 &= \mu_0\, \lambda_0, \\ v_1 &= \mu_0\, \lambda_1 + \mu_1\, \lambda_1 + \mu_1\, \lambda_2, \\ v_2 &= \mu_0\, \lambda_2 + \mu_2\, \lambda_0 + \mu_2\, \lambda_3\,. \end{aligned} \tag{2}$$

Now, in order to determine the system of complex numbers that belongs to the group G_1', we next represent the transformations (1) as bilinear forms; i.e., we multiply the expressions for x_1', x_2', x_3' by u_1, u_2, u_3, resp., and set the sums of the products equal to zero. We will then obtain the general transformation of G_1' in the form:

$$S = \lambda_0\, S_0 + \lambda_1\, S_1 + \lambda_2\, S_2 = 0,$$

in which:

$$\left\{ \begin{aligned} S_0 = S^0 &= x_1\, u_1 + x_2\, u_2 + x_3\, u_3, \\ S_1 = x_3\, u_1, &\qquad S_2 = x_2\, u_2. \end{aligned} \right. \tag{3}$$

The bilinear forms S_0, S_1, S_2 will define a system of complex numbers whose multiplication rules will be represented by the formulas:

(4) $$e_0 = 1, \qquad e_1\, e_2 = 0, \qquad e_2\, e_1 = e_1, \qquad e_1^2 = 0, \qquad e_2^2 = e_2,$$

with the introduction of the new notations e_0, e_1, e_2.

If one sets:

$$\lambda = \lambda_0\, e_0 + \lambda_1\, e_1 + \lambda_2\, e_2 ,$$

and one defines μ and ν analogously then one can, with the help of this number system, summarize formula (2) in a single one:

(5) $$\lambda\,\mu = \nu.$$

When one regards λ as an independent variable, ν, as a dependent one, and μ as a parameter, the formulas (5) will then represent the group G_1 that belongs to the system (4).

In order to convert G_1' into G_1, we choose a point E that either lies on the line $x_2 = 0$ or the line $x_3 = 0$; say, 1 : 1 : 1. If we then let $y_0 : y_1 : y_2$ denote the parameters of the transformation $S = 0$ that take the point E to the point $x_1 : x_2 : x_3$ then y_0, y_1, y_2 and x_1, x_2, x_3 will be connected by a soluble linear substitution that can be written as the following transformation with determinant one:

(6) $$x_1 = y_0 + y_1 , \quad x_2 = y_0 + y_2 , \quad x_3 = y_0 .$$

Simultaneously, one will have:

(7) $$u_1 = v_1 , \qquad u_2 = v_2 , \qquad u_3 = v_0 - v_1 - v_2 ,$$

when v_0, v_1, v_2 mean the line coordinates that belong to the system y_0, y_1, y_2.

With the substitution (6), the equations of the group (1) will go to these:

(8) $$\begin{aligned} y_0' &= \lambda_0\, y_0 , \\ y_1' &= \lambda_0\, y_1 + \lambda_1\, y_0 + \lambda_1\, y_2 , \\ y_2' &= \lambda_0\, y_2 + \lambda_2\, y_0 + \lambda_2\, y_1 , \end{aligned}$$

and a comparison of these with the formula (2) will show that we do, in fact, have the group G_1 before us.

Furthermore, by means of the substitutions (6) and (7), the forms S_0, S_1, S_2 will go to:

(9) $$S_0 = y_0\, v_0 + y_1\, v_1 + y_2\, v_2 , \qquad S_1 = (y_0 + y_2)\, v_1, \qquad S_2 = (y_0 + y_2)\, v_2 .$$

The relationship between these expressions and the formulas (8) is obviously the same as the relationship between the expressions (3) and the formulas (1); i.e., S_0, S_1, S_2 are the coefficients of λ_0, λ_1, λ_2 in the expression $v_0\, y_0' + v_1\, y_1' + v_2\, y_2'$. However, the formulas (8) are now immediately the equations of the group G_1 that belongs to the number system that is determined by the bilinear forms (9).

The number system to which we have arrived here belongs to the fourth type that was presented in § 4 for the case of the assumption of three basic numbers. It goes into the canonical form IV under the substitutions:

$$e_0 = \bar{e}_0, \qquad e_1 = \bar{e}_2, \qquad e_2 = \frac{\bar{e}_0 + \bar{e}_1}{2}.$$

Because of this, new variables, and likewise new parameters, will be introduced into equations (8) for the group $y' = y\lambda$, where the new variables will be introduced by the substitutions:

$$x_0 = y_0 + \tfrac{1}{2}\, y_2, \qquad x_1 = \tfrac{1}{2}\, y_2, \qquad x_2 = y_1,$$

and the new parameters, by the *same* substitutions:

$$a_0 = \lambda_0 + \tfrac{1}{2}\lambda_2, \qquad a_1 = \tfrac{1}{2}\lambda_2, \qquad a_2 = \lambda_1 .$$

In this way, equations (8) will go to these:

$$(10) \qquad \begin{cases} x'_0 = a_0 x_0 + a_1 x_1, \\ x'_1 = a_0 x_1 + a_1 x_0, \\ x'_2 = (a_0 - a_1) x_2 + a_2 (x_0 + x_1), \end{cases}$$

which one can write more simply as $x' = xa$ with the use of the new basic numbers.

The perspectivity center of the transformation (10) will now have the coordinates $a_1 : a_1 : a_2$, while the axis will have the equation $x_0 + x_1 = 0$.

The general transformation of the reciprocal group G_2 will have the equation $x' = ax$, or when written out:

$$(11) \qquad \begin{cases} x'_0 = a_0 x_0 + a_1 x_1, \\ x'_1 = a_0 x_1 + a_1 x_0, \\ x'_2 = (a_0 + a_1) x_2 + a_2 (x_0 - x_1). \end{cases}$$

The associated perspectivity center will have the coordinates $- a_1 : a_1 : a_2$, while the axis will have the equation $x_0 - x_1 = 0$. The transformation $x' = x^{-1}$, or when written out, after dropping the common denominators $(x_0 - x_1)(x_0 + x_1)$ on the right-hand side:

$$(12) \qquad x'_0 = x_0, \qquad x'_1 = - x_1, \qquad x'_2 = - x_2,$$

will likewise be a linear transformation in the present case. It is the involutory perspective transformation whose center is the point $E(1 : 0 : 0)$, and whose axis $x_0 = 0$ will be harmonically separated from the point E by the perspectivity axes $x_0 + x_1 = 0$ and $x_0 - x_1 = 0$ that belong to the groups G_1 and G_2. The group $\mathfrak{G}$ will also consist of nothing but perspective transformations:

$$(13) \qquad \begin{cases} x_0' = (a_0 + a_1)x_0, \\ x_1' = (a_0 + a_1)x_1, \\ x_2' = 2a_2 x_1 - (a_0 - a_1)x_2, \end{cases}$$

whose common center will be the intersection of the lines $x_0 + x_1 = 0$ and $x_0 - x_1 = 0$, and whose axis $a_1\, x_2 - a_2\, x_1 = 0$ will contains all of the points E. In our special case, $\mathfrak{G}$ will then be similar to G_1 and G_2 by means of a dualistic transformation. Finally, the group $G_{1,2}$ will be the group of linear transformation that leave either of the two lines $x_0 + x_1 = 0$, $x_0 - x_1 = 0$ fixed.

Here, we have treated the simplest example of a pair of reciprocal groups that leads to a system of complex numbers whose multiplication is not commutative. We have therefore likewise examined the only pair of non-coincident reciprocal projective groups that exists in the plane. There is a larger manifold of such groups in multiply-extended domains. Among them, one will find a pair of groups that can be regarded as the immediate generalization of the one that was treated here. Thus, e.g., the totality of all perspective transformations in a triply-extended space that have a perspectivity plane Σ_1 in common, and whose centers lie on a second fixed plane Σ_2 will define a simply-transitive group G_1 . The reciprocal group G_2 will be defined by all perspective transformations whose plane is the plane Σ_2, and whose centers lie on Σ_1 . Both groups will collectively generate a six-parameter group $G_{1,2}$, namely, the invariant subgroup of the group of all linear transformations that leave Σ_1 and Σ_2 fixed, and for which all points of the line of intersection of Σ_1 and Σ_2 will be fixed individually. The group $\mathfrak{G}$ will consist of all multiple (*geschart*) perspective transformations whose one perspectivity axis is the line of intersection of Σ_1 and Σ_2, while the other one goes through a fixed point that lies in either Σ_1 or Σ_2 . The associated number system will be system XV (§ 5, pp. 25).

Another well-known example of a pair of groups of the stated kind is defined by the two invariant subgroups of the group of the general second-degree surface. We will have to speak about this pair of groups later on.

§ 13. Further transformation groups that are coupled with a system of complex numbers.

Naturally, the relationships between systems of complex numbers and the theory of transformation groups are in no way exhausted by the theorems that we presented up to now.

Here, the theory of subgroups of a given group will be a theory that will be passed over, in order to treat the sub-systems that are included in a given system of complex numbers that includes the number one. We have already discussed one theorem that relates to this (in § 1). Another one states that the totality of all linearly-independent numbers c that satisfy the equation $cx = xc$ identically – and thus, in the sense of the foregoing developments, will represent the parameters of the $(m - 1)$-parameter subgroup that is common to G_1 and G_2 – will already define a closed (commutative) system of complex numbers by itself.

We have already further suggested that any system of complex numbers will be linked to four groups in a second way, and that those groups will relate to each other in the same way as the groups G_1, G_1, $G_{1,2}$, $\mathfrak{G}$ that were considered to begin with. One will obtain them when one regards the coefficients $x_0, \ldots, x_{n-1}$, etc., in the equations:

$$x' = xa, \qquad x' = ax, \qquad x' = axb, \qquad x' = a^{-1}xa,$$

not as ratios, as we did up to now, but as their absolute values, and thus interprets them as *Cartesian* coordinates.

The cited equations will then be the equations of four groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ of an n-dimensional space $\bar{R}$; the associated numbers of parameters will be n, n, $2n - m$, $n - m$, resp., where n and m have the meanings that were explained on pp. 31. However, these groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ will have *much more specialized* properties than the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that we considered up to now, and therefore it seems to me that they will also possess a more minor interest. One next sees immediately that the number m must have a value of at least one, and that the groups G_1 and $\mathfrak{G}$ can therefore always be only *merohedrally* isomorphic. Furthermore, they will be so-called linear homogeneous groups – i.e., projective groups whose transformations first of all fix the infinitely-distant manifold R_{n-1}, and secondly, a finite point, namely, the point 0, 0, …, 0.

Since the points of the infinitely-distant manifold will be permuted with each other under the transformations of $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$, and obviously under the transformations of the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that we treated in §§ 9-11, one will immediately recognize the reciprocal behavior of these two sequences of groups: One will determines the continuous groups G_1', G_2', $G_{1,2}'$, $\mathfrak{G}'$, which are composed in the same way as G_1, G_2, $G_{1,2}$, $\mathfrak{G}$, and belong to the so-called linear, homogeneous group, and transform the infinitely-distant manifold in the prescribed manner. The group $\mathfrak{G}'$ will then be immediately identical to the group $\mathfrak{G}$; however, the groups G_1, G_2, $G_{1,2}$ will arise from the groups G_1', G_2', $G_{1,2}'$ when one adds the transformations of the one-parameter distinguished subgroup of the general linear, homogeneous group. The theory of the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ can then be derived from that of the groups G_1', G_2', $G_{1,2}'$, $\mathfrak{G}'$ by entirely simple operations.

However, in addition, the groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ that belong to the given system of n basic numbers are already known completely when one has determined all systems of $n + 1$ basic numbers. From our general theorem, a system with $n + 1$ basic numbers must then be given, to which, the groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ must have the same relationship as the given system as the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$. One can also immediately write down any system of $n + 1$ basic numbers. One has to add only one further basic number η to the basic numbers $e_0, \ldots, e_{n-1}$ and establish that one should have $\eta^2 = \eta$, $e_i\,\eta = \eta\,e_i = 0$. For example, the number system VI, $n = 4$, arises from the number system IV, $n = 3$ in such a way; i.e., the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that belong to the first system are inside of the general projective group of the space on the same footing as the groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ that belong to the second system.

We have seen that any transformation of the group G_1 can be represented in the form $x' = x e^{\bar{\xi}}$ or in the inessentially-altered form $x' = x\, e^{\xi}$, where $\bar{\xi}$ and $\xi = e_0 + \bar{\xi}$ have the meaning that was explained on pp. 37 and 38. A corresponding representation of the group G_1 in the form $x' = x\, e^{\eta}$ will now likewise follow from this, where η is now a general number of the system. As a result, $x = e^{\eta}$ will be the expression for the transformation that takes the group $\bar{G}_1$ to its canonical form. If the transformations of $\bar{G}_1$ all commute with each other – i.e., the multiplication in the present number system obeys the commutative law – then the canonical parameter group will become a group of translations. In fact, in this case, the equations $x = x\, e^{\eta}$, $x' = x\, e^{\eta'}$, $x'' = x\, e^{\eta''}$, $\eta + \eta' = \eta''$ will immediately imply the further equation $xx' = x''$. Since this remark is useful for several applications, it might be again expressed as the theorem:

Any system of complex numbers whose multiplication follows the commutative law will be associated with a table of logarithms, with whose help one can completely convert the multiplication of two numbers of the system into an addition, as would happen in the system of common complex numbers through the ordinary logarithm (*).

For example, the logarithms for all number systems with four multiplicatively-commutating basic numbers will be given:

I. $\log x = \ln x_0 \cdot e_0 + \ln x_1 \cdot e_1 + \ln x_2 \cdot e_2 + \ln x_3 \cdot e_3$,

II. $\log x = \ln x_0 \cdot e_0 + \dfrac{x_1}{x_0} \cdot e_1 + \ln x_2 \cdot e_2 + \ln x_3 \cdot e_3$,

III. $\log x = \ln x_0 \cdot e_0 + \dfrac{x_1}{x_0} \cdot e_1 + \ln x_2 \cdot e_2 + \dfrac{x_3}{x_0} \cdot e_3$,

IV. $\log x = \ln x_0 \cdot e_0 + \dfrac{x_1}{x_0} \cdot e_1 + \dfrac{2x_0x_2 - x_1^2}{2x_0^2} \cdot e_2 + \ln x_3 \cdot e_3$,

V. $\log x = \ln x_0 \cdot e_0 + \dfrac{x_1}{x_0} \cdot e_1 + \dfrac{2x_0x_2 - x_1^2}{2x_0^2} \cdot e_2 + \dfrac{3x_0^2x_3 - 3x_0x_1x_2 + x_1^2}{3x_0^3} \cdot e_3$,

(*) Schur has also made a related remark. Math. Ann. XXXIII.

VIII. $\log x = \ln x_0 \cdot e_0 + \frac{x_1}{x_0} \cdot e_1 + \frac{x_2}{x_0} \cdot e_2 + \ln x_3 \cdot e_3$,

X. $\log x = \ln x_0 \cdot e_0 + \frac{x_1}{x_0} \cdot e_1 + \frac{x_2}{x_0} \cdot e_2 - \frac{x_1^2 - 2x_0x_3 + x_2^2}{2x_0^2} \cdot e_3$,

XI. $\log x = \ln x_0 \cdot e_0 + \frac{x_1}{x_0} \cdot e_1 + \frac{2x_0x_2 - x_1^2}{2x_0^2} \cdot e_2 + \frac{x_3}{x_0} \cdot e_3$,

XV. $\log x = \ln x_0 \cdot e_0 + \frac{x_1}{x_0} \cdot e_1 + \frac{x_2}{x_0} \cdot e_2 + \frac{x_3}{x_0} \cdot e_3$.

In all of these formulas, the Roman numeral means the number of the number system in question (cf. § 5); log x is the quantity that was denoted by η above.

From our discussion, if the theory of the groups $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_{1,2}$, $\bar{\mathfrak{G}}$ differs only slightly from the theory of the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ then the interpretation of x_0, .., x_{n-1} as Cartesian coordinates will also afford us a series of essentially new groups, by whose study, one can evaluate the algorithms of complex numbers to more or less advantage. If we exclude the trivial case of the group of translations then, e.g., the equations:

$$x' = xa + b, \qquad x' = ax + b,$$
$$x' = a^{-1}xa + b, \qquad x' = axb + c$$

will determine new groups for which one can immediately give a series of properties.

By way of example, one finds that the four given groups include the group $x' = x + a$ of all translations as an *invariant* subgroup, and that, furthermore, the groups $x' = xa + b$, $x' = ax + b$ will be invariantly included in the group $x' = axb + c$, and so forth.

In particular, if one takes x, a, b to be ordinary complex numbers then:

$$x' = ax + b$$

will be the four-parameter *group of similarity transformations in the plane*, and it is precisely that remark upon which *Bellavitus* based his so-called *calculus of equipollence.*

Furthermore, the known representation of the six-parameter *group of reciprocal radii in the plane* (*Möbius's* circle affinities) in the form:

$$x' = \frac{ax+b}{cx+d},$$

where a, b, c, d, x, x' mean ordinary complex numbers, belongs to them. If one takes a, …, x' in the same formula to be numbers of the second type in the case of $n = 2$ then one will obtain a remarkable six-parameter group that defines a limiting case of the group of reciprocal radii, and has the same composition as the group of motions in space.

Systems of complex numbers also give rise to the consideration of *infinite groups*. For example, if we let c_k denote a real or ordinary complex number, or even any number of a given system that commutes with all remaining numbers multiplicatively, and let x denote any number of the system then any convergent series of the form:

$$x' = \sum c_k x^k$$

will represent a transformation of a certain infinite group whose transformations all commute with the transformations of the group $\bar{\mathfrak{G}}$ (*). If we again take c_k, x, x' to be ordinary complex numbers then we will obtain the representation of the *group of conformal transformations* in terms of functions of one complex variable that was given by *Gauss*.

§ 14. The advantages of systems of complex numbers.

In many circles – namely, in Germany – the opinion is propagated that systems of complex numbers, or similar algorithms, have absolutely no advantages, except for just the ordinary complex numbers, and that one bases this opinion on the fact that nothing can be derived from them that could not be derived "just as well" without them.

It must be due to the fact that a tremendous misuse is already committed from the outset with the quaternions, which one chiefly considers. However, one should not suggest that a theory, to whose development such a spirit as *Hamilton* had devoted many years of his life, is something totally worthless. There are also some beautiful new theorems today that are due to the quaternions, precisely. I recall only the well-known formulas and constructions that are of use for the compositions of the rotations of a ball. If one restricts the application of the quaternion algebra to its most naturally-suggested domain and the terminology to only that which is most necessary then it will represent a preferable method that is indeed less necessarily unavoidable than any other method, but, at the moment, can, in no way, also be just as well done without.

Any disinclination seems to be based in the completely erroneous belief that the quaternions treat only a frivolous and arbitrary generalization of the laws of elementary calculation, to which one admittedly finds it easy to reconcile oneself. In reality, calculating with quaternions is a method of analytic geometry that is devised for an entirely well-defined purpose. *In its manner of application, it is completely comparable to that of the functions of one complex variable to the study of conformal transformations, in which the complex numbers also find the utility, not because of their property of being an extension of the real numbers, but as an analytic-geometric algorithm.* Whoever fundamentally condemns the use of quaternions, or similar algorithms, will then, when he would like to proceed logically, also have to cast aside all

(*) As an adjoint group, the group $\bar{\mathfrak{G}}$ will always be intransitive.

of the beautiful investigations of conformal maps, minimal surfaces, et al., in which those studies have experienced a veritable enrichment since the work of Gauss and Riemann. However, many do not seem to understand this line of reasoning. It is precisely the mathematicians, who have crusaded most zealously against quaternions and similar algebras, that make the most unhesitating use of the algebra of ordinary complex numbers in geometric investigations. They wrongly call upon the authority of Gauss in order to support the views that are debated here: We possess an authentic proof by Gauss that leaves no doubt about his having posed this problem (*). Why should it also be totally unacceptable to use something that makes things simpler when it is convenient and appropriate? (**)

I am of the opinion that one should not at all generally and *a priori* pass judgment on a method, but only in reference to a well-defined circle of wisdom, and after paying close attention to the special problems that exist within it.

However, that can give only a measuring stick for comparable assessments of different methods. Among some of the best of them, there will be ones that produce these results with the least total expenditure of mental labor. I say "total expenditure" in order to imply that the time and effort that goes into the learning of the method itself must be accounted for. In the cases that we have dealt with here, the effort was, moreover, very minor.

How the advantages that systems of complex numbers can bring will emerge from what was professed up to now – in case new realms of their application are not opened up – resides essentially in the fact that they allow certain groups of transformations to be first *represented* and second *determined* in a very simple way.

In the first case, the existing advantages become ever larger, the more immediate or – when the term is admissible – the more *natural* the connection between the group under scrutiny and the number system being used becomes. Likewise, one will also find that the field that the application of these algorithms opens up is very restricted.

There is then only a relatively small manifold of groups that are connected with systems of complex numbers, and among them, there are only comparatively few of them for which the connection in so intrinsic that its representation in terms of complex numbers defines a truly recommended method. However, whether the number system in question can arrive at a more significant meaning in such cases will, in turn, depend upon the place that the associated groups assume in the greater context of mathematical studies.

The use of a system of complex numbers will be especially close at hand in any case where one must examine four projective groups that have the same relationships as the groups G_1, G_2, $G_{1,2}$, $\mathfrak{G}$ that are known to us; a simpler representation of the essential properties of these groups than by the familiar formulas:

$$x' = xa, \qquad x' = ax, \qquad x' = axb, \qquad x' = a^{-1}xa, \qquad x' = x^{-1},$$

etc., will then be hard to imagine.

(*) Correspondence with *Schumacher*, Bd. 4, no. 833, pp. 147.
(**) I am pleased to see that *Dedekind* has also expressed similar views in regard to quaternions.

If one writes *a*, *b*, *c* for *x*, *a*, *x′* in the first of these formulas and one expands the formula $ab = c$ that then arises with the help of the multiplication table of the given system then the coefficients of the basic numbers on the left and right must be individually equal to each other. One will thus obtain a system of formulas that one refers to as the "multiplication theorem" of the number system in the case of quaternions. It is obvious that these formulas, in which one no longer sees any complex numbers, will substitute completely for the multiplication table of the system. One will then be able to place any formula in the theory of complex numbers alongside another one in which, in place of the multiplication table, the entirely "real" multiplication theorem of the number system in question will appear. One will have to perform such a conversion in many cases where one is dealing with applications of the theory of complex numbers. However, one would, as should be once more emphasized, deceive oneself if one would like to believe that anything would be gained by a translation of all of the individual steps into the calculation into the language of the so-called ordinary algebra. One would have, at most, exchanged simple and lucid formulas for involved and unclear ones. Precisely the opposite viewpoint seems to me to be more fruitful, by which the multiplication table of a system defined a preferable and convenient substitute for the associated multiplication theorem.

In the next two paragraphs, examples shall be pointed out of how one can appeal to the knowledge of all different systems of *n* basic numbers in order to find parameter representations of certain transformation groups for which the parameters of two transformations that are performed in succession can be combined with the parameters of the newly-arising transformation of the group.

§ 15. On the parametric representation of certain transformation groups.

If a given continuous transformation group has the same composition as a pair of reciprocal projective groups G_1, G_2 then one will be able to choose the parameters of the given group in such a way that one of these groups G_1, G_2 will be the parameter group. With that, one will have not just an especially simple composition of the parameters – viz., the homogeneous parameters of the composed transformation *ST* of two transformations *S* and *T* of the group will be bilinear functions of the parameters of *S* and *T* – but, as a result of our theorem about the group $\mathfrak{G}$ (cf., pp. 40), one will also, at the same time, have a likewise simple representation of the adjoint group that is connected with it. In order to survey the totality of all such parameter representations, the systems of complex numbers will now serve as the simplest means, whereby a special advantage will reside in the fact that one can immediately read off the composition of the groups G_1, G_2 that are associated with a given system from the multiplication table of the system, without having to exhibit the equations of this group (cf., pp. 37).

As is well-known, there is only one kind of composition of two-parameter groups that has non-commuting transformations. On the other hand, we know that in the plane only one pair of reciprocal groups exists with non-commuting linear transformations, and that these two groups will be on the same footing inside the general projective group (*). This yields the following remark, which is useful for applications:

If one considers parameter systems that emerge from each other by linear transformations to be not-essentially different then one will have the theorem that one can always make the finite transformations of a two-parameter group of non-commuting transformations depend upon three homogeneous parameters, and in only one way, such that the parameters of the composed transformation ST of two transformations S and T of the group will be bilinear functions of the parameters of S and T.

Similar theorems exist for certain compositions of groups with more parameters.

Here, we give an enumeration of all of the compositions of *three-parameter* groups that can exist for reciprocal projective groups. Their compositions that are real-distinct will be distinguished by appending the symbols *b*), *c*), … Roman numerals will give the corresponding number system with four basic numbers (§ 5). Systems whose associated groups G_1 and G_2 are not projective to each other, and which will then give rise to two different parametric representations, will be listed twice.

(1) $(X_2, X_3) = X_1, \quad (X_3, X_1) = X_2, \quad (X_1, X_2) = X_3,$
XII,

(1b) $(X_2, X_3) = -X_1, \quad (X_3, X_1) = -X_2, \quad (X_1, X_2) = X_3,$
XIIb,

(2) $(X_2, X_3) = 0, \quad (X_3, X_1) = X_2, \quad (X_1, X_2) = X_3,$
XIII,

(2b) $(X_2, X_3) = 0, \quad (X_3, X_1) = -X_2, \quad (X_1, X_2) = X_3,$
XIIIb,

(3) $(X_2, X_3) = 0, \quad (X_3, X_1) = 0, \quad (X_1, X_2) = X_3,$
IX, XIV,

(4) $(X_1, X_2) = X_2, \quad (X_1, X_3) = X_3, \quad (X_2, X_3) = 0,$
XV,

(5) $(X_1, X_2) = X_2, \quad (X_1, X_3) = 0, \quad (X_2, X_3) = 0,$
VI, VII, VII,

(6) $(X_2, X_3) = 0, \quad (X_3, X_1) = 0, \quad (X_1, X_2) = 0,$
I, II, III, IV, V, VIII, X, XI, XVI.

(*) As my friend Engel has informed me, Lie has proved a somewhat more general theorem, as a consequence of which, this pair of groups can be taken to either another projective group or itself under a non-projective transformation.

From the theorems that are included in this table, some things that are especial useful for the applications can be once more put into words.

We might first remark that the group of perspective similarity transformations of the plane has the composition (4). We can then say:

The parameters of the three-parameter group of all perspective similarity transformations of the plane, or a similarly-composed group, can be chosen in essentially only one way such that the associated parameter group and its reciprocal will be projective groups.

The geometric interpretation of these parameter groups is given on page 48.

Moreover, the composition (1) is that of the group of an (imaginary) conic section in the plane or the group of rotations of a sphere. With hindsight of the known properties of the *Euler* parameters (which we will also likewise derive), it follows from this that:

The Euler parameters of the rotations of a rigid body around a fixed point are determined completely, up to linear transformations, through the requirement that the associated parameter groups and their reciprocals should be projective groups (*).

However, an equivalent theorem is also true for the important degenerate case of the group of rotations of a sphere, namely, the group of motions of the (Euclidian) plane, which goes with the composition (2):

In the group of all motions of the plane (and the groups that are composed the same as it), one may introduce parameters, and indeed in essentially only one way, such that the associated parameter group and its reciprocal will be projective groups.

The group of motions in the plane, when regarded as a subgroup of the general projective group, will possess another degeneracy of the composition (3). In this case, the system of complex numbers will provide us with infinitely many parametric representations, which define a connected manifold, as one easily shows. However, among these parametric representations, one will also now find another uniquely distinguished one, namely, the one that belongs to the number system XIV. Among all of the systems that come under consideration, system XIV, will be, in fact, the only one that can be regarded as a limiting case of system XIII, and in addition, the associated parametric representation will have the peculiarity that it is a canonical parametric representation (cf., pp. 41).

One can easily give an entire class of groups that belong to parameter systems of the kind that are considered here.

Any group of linear transformations whose finite transformation can be characterized by linear equations in the transformation coefficients is composed the same as a simply-transitive projective group whose reciprocal group is likewise projective.

(*) This theorem can also be inferred from a theorem that was presented by Lie, which said that the group of a general second-degree surface cannot be taken to a projective group by any non-projective transformation.

In fact, under the given assumption, an r-parameter group can be represented by a linear family of $r + 1$ independent bilinear forms (**). One only needs to introduce them as the basic numbers of a number system in order to obtain the associated groups G_1 and G_2 of the group in question directly.

The number k that belongs to the number system is linked to an upper limit in such cases. It can attain the value m at most when the given group belongs to an m-dimensional region.

One group of the stated kind is, e.g., the *general projective group* of the n-dimensional region itself. The associated number system has n^2 basic numbers, namely, the special bilinear forms $e_{ik} = x_i\, u_k$. The associated multiplication rules are these:

$$e_{ij}\, e_{jk} = e_{ik}\,, \qquad e_{ij}\, e_{kl} = 0 \qquad (j \neq k).$$

The associated number k has the value n. Therefore, the transformation $x' = x^{-1}$ is not projective, except in the simplest case of $n = 2$ (cf., pp. 9). Nonetheless, the groups G_1 and G_2 that are associated with the system are similar by a projective transformation. In fact, the system goes to its reciprocal system:

$$\eta_{jk}\, \eta_{ij} = \eta_{ik}\,, \quad \eta_{kl}\, \eta_{ij} = 0 \qquad (j \neq k)$$

under the substitutions:

$$\eta_{ik} = e_{ki} \qquad (i, k = 1, \ldots, n).$$

We must refer to this number system, in the sense of our definition, which admits the number of principal units and then also the number k of a larger space, as a *special* one, although it naturally includes all remaining systems of complex numbers.

If one sets $n = 3$ then one will obtain the *nonions*, as they are called by the English and American mathematicians.

If one sets $n = 2$ then one will again arrive at the *quaternions*, and indeed in their *second* real form (*).

In order to obtain the canonical form XIIb), one needs only to introduce new basic numbers $e_0, \ldots, e_3$ by the substitutions:

$$\begin{aligned} e_0 &= e_{11} + e_{22}\,, & e_1 &= -\,e_{12} - e_{21}\,, \\ e_2 &= e_{22} - e_{11}\,, & e_3 &= e_{21} - e_{12}\,. \end{aligned}$$

The subgroups of the general projective group of the line all fall into the class that is considered here: As we know, the various types of one-parameter groups belong to different systems with two basic numbers, and the two-parameter group belongs to the number system IV, $n = 3$. Above all, the largest projective group of an n-dimensional region that leaves arbitrary given points and n–1-dimensional regions individually fixed will always belong to the stated class. Important groups of this kind are defined by the group of all similarity transformations of the plane and its three-parameter invariant subgroup that consists of the perspective similarity transformations. This latter group

(**) Cf., Leipz. Ber. 1889, pp. 220, rem.

(*) One can confer the splendid treatise of *Stephanos:* “Mémoire sur la répresentation des homographies binaires par des points de l’espace, etc.” Math. Ann., Bd. XXII, pp. 299, *et seq.*

belongs to a system of four basic numbers that we have already spoken of many times, namely, system XV.

Finally, the aforementioned degeneracy of the group of motions in the plane might be emphasized as likewise belonging to it. It can be defined by saying that its transformations leave fixed the figure of a line element, and indeed the points, as well as the line, when doubly-counted. The corresponding number system is system IX ($n = 4$), with the parameter value $c = -1$.

If a group G is representable by a linear family of bilinear forms then the same thing will be true for its dualistic group G'. The systems of complex numbers that correspond to the groups G and G' themselves belong to the same type, but are still reciprocal. Performing a dualistic transformation on the group G will then have the effect of switching the two projective parameter groups G_1 and G_2 that are coupled with them. By way of example, the system VII ($n = 4$) belongs to the projective group of the plane, whose transformations fix two straight lines, and on one of them, also one of the points that are close to the intersection point; the same system will belong to the dualistically contrapositive group, but in the reciprocal form.

By considering the infinitesimal transformations of a projective group, one can already decide whether their finite transformations can be represented by a linear family of bilinear forms.

Let $S_1, \ldots, S_r$ be the independent infinitesimal transformations of a projective group, which are written as normal bilinear forms, and furthermore, let S^0 be the bilinear form that represents that identity transformation. The characterization in question then consists of the existence of r^2 relations of the form:

$$S_i\, S_k = \gamma_{ik0}\, S^0 + \sum_{s=1}^{r} \gamma_{iks} S_s\,.$$

If this condition is fulfilled then:

$$\lambda_0\, S^0 + \lambda_1\, S_1 + \ldots + \lambda_r\, S_r = 0$$

will be the expression for the most general finite transformation of the group, but the bilinear forms $S^0, S_1, \ldots, S_r$ will immediately define the associated number system.

§ 16. The rotations of the sphere and the motions in the plane.

Of the parameter representations that we discussed in the previous paragraphs, the most important ones are the rotations of rigid body around a fixed point and the parametric representation of the motions in the plane. The former is well-known; the parametric representation was already studied by *Euler*. The formulas for the composition of the parameters were given by *Cayley* (*).

(*) Namely, in the convenient form (2) of this article (Cambridge Math. Journ., v. III, 1843). Cf., also *Rodrigues*: "Des lois géométric qui régissent the déplacements d'un système solide dans l'espace, …" Liouville Journ. de Math. **5** (1840), pp. 380, etc.

The analogous parametric representation for the motions in the plane seems to have escaped the attention of mathematicians up to now, and indeed that is undoubtedly due to the fact that one is accustomed to giving more weight to a simplest-possible representation of the motions than to a simple composition of the parameters of several motions that are performed in succession. Meanwhile, if one considers a measure of simplicity to be, not the brevity of the formulas, but their algebraic properties, then the ordinary analytical representation of the motions in the plane will be, in any event, *not* be the simplest one, since it requires transcendental functions, although an algebraic – *in fact, rational* – representation exists.

In order for the analogy between the two systems of formulas that we speak of to better emerge as formulas, we shall next given the well-known derivation of *Euler's* and *Rodrigues's* formulas from the theory of quaternions.

One obtains *Euler's* representation of the rotations of a rigid body immediately when one presents the equations of the group $\mathfrak{G}$ that belongs to the quaternions and interprets x_1, x_2, x_3 as rectangular coordinates:

$$(1)\qquad \left\{\begin{aligned} x_0' &= x_0, \\ Nx_1' &= (a_0^2 + a_1^2 - a_2^2 - a_3^2)x_1 + 2(a_1a_2 + a_0a_3)x_2 + 2(a_1a_3 - a_0a_2)x_3, \\ Nx_2' &= (a_0^2 - a_1^2 + a_2^2 - a_3^2)x_2 + 2(a_2a_3 + a_0a_1)x_3 + 2(a_2a_1 - a_0a_3)x_1, \\ Nx_3' &= (a_0^2 - a_1^2 - a_2^2 + a_3^2)x_3 + 2(a_3a_1 + a_0a_2)x_1 + 2(a_3a_2 - a_0a_1)x_2, \\ &\text{in which } N = a_0^2 + a_1^2 + a_2^2 + a_3^2. \end{aligned}\right.$$

The formulas for the composition of the parameters of two transformations of the group (1) that are performed in succession define the so-called multiplication theorem for quaternions:

$$(2)\qquad \left\{\begin{aligned} c_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3, \\ c_1 &= a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2, \\ c_2 &= a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1, \\ c_3 &= a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0. \end{aligned}\right.$$

If one replaces a with x and c with x' in this then one will obtain the equations of the parameter group that we denoted by G_1 ; by contrast, if one replaces b with x and c with x' then one will obtain the group G_2.

The group G_1 consists of all linear transformations that fix all individual elements of one family of lines of the imaginary sphere $N = 0$. Its real transformations are multiple-perspective transformations with conjugate imaginary axes (imaginary lines of the "second kind") that belong to the other generating family. One obtains the group G_2 from G_1 when one lets both of its families of lines change roles. The group $\mathfrak{G}$ consists of the totality of all rotations around the origin of the coordinates, or, as we can perhaps better say, in the present context, it consists of the totality of all linear transformation of space that fix all individual elements of a family of second-order surfaces $N - \lambda x_0^2 = 0$ that contact each other along an imaginary conic section (the so-called infinitely-distant sphere circle).

As is known, a simple connection exists between the ratios of the parameters a_0, a_1, a_2, a_3, the inclination angles ϑ_1, ϑ_2, ϑ_3 of the rotational axes above the coordinate axes, and the rotation angle φ: If we normalize the ratios a_1, a_2, a_3 such that the sum of their squares equals one unit then we will have the simple relations (*):

$$a_0 : a_1 : a_2 : a_3 \tag{3}$$

$$\cot\frac{\varphi}{2} : \cos\vartheta_1 : \cos\vartheta_2 : \cos\vartheta_3 .$$

If we then fix the parameters a_1, a_2, a_3 in formula (1) and let just a_0 vary then we will obtain the general one-parameter subgroup of the group $\mathfrak{G}$.

Starting with *Euler's* formulas, one now arrives, through a passage to the limit, at the corresponding parametric representation of the group of motions in the plane. Meanwhile, we would like to relate the derivation of this parametric representation, not directly to *Euler's* formula, but in agreement with the process of the development of system XIII in § 5, up to now, which also indeed defined a limiting case of the system XII of the system of quaternions that was considered just now.

The group $\mathfrak{G}$ that belongs to system XIII has the equations:

$$(4)\qquad \begin{cases} x_0' = x_0, \; x_1' = x_1, \\ Nx_2' = (a_0^2 - a_1^2)x_2 + 2a_0a_1x_3 + 2(a_2a_1 - a_0a_3)x_1, \\ Nx_3' = (a_0^2 - a_1^2)x_3 + 2(a_3a_1 + a_0a_2)x_1 - 2 - a_0a_1x_2, \\ \text{in which } N = a_0^2 + a_1^2. \end{cases}$$

If one sets $x_2 / x_1 = x$, $x_3 / x_1 = y$ here, and one interprets x and y as rectangular Cartesian coordinates then one will obtain the representation in question of the group of all motions of the plane in the form:

$$(4')\qquad \begin{cases} (a_0^2 + a_1^2)\, x' = (a_0^2 - a_1^2)\, x + 2a_0a_1y + 2(a_1a_2 - a_0a_3), \\ (a_0^2 + a_1^2)\, y' = (a_0^2 - a_1^2)\, y - 2a_0a_1x + 2(a_1a_2 + a_0a_3), \end{cases}$$

while the composition of the parameters will be given by the formulas:

$$(5)\qquad \begin{cases} c_0 = a_0b_0 - a_1b_1, \\ c_1 = a_0b_1 + a_1b_0, \\ c_2 = a_0b_2 - a_1b_2 + a_2b_0 + a_3b_1, \\ c_3 = a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0. \end{cases}$$

(*) *Rodrigues, loc. cit.*

The geometric meaning of the groups G_1, G_2 that belong to the system of formulas (5) will be, in turn, not difficult to grasp. One imagines two planes Σ_1 and Σ_2 in space, and furthermore, the part of their line of intersection that lies between two points S_1 and S_2 . This well-known figure determines, in all, four pencils of rays (Σ_i, S_k) that are defined by all lines that lie in the plane Σ_i and go through the point S_k . Any line of the pencil (Σ_1, S_1), together with any line of the pencil (Σ_2, S_2), will now determine a one-parameter group of multiple-perspective transformations whose axes are either of the two lines. The totality of all transformations thus-obtained will define a three-parameter, simply-transitive group G_1 whose transformations will fix the rays of the pencils (Σ_1, S_2) and (Σ_2, S_1) individually, while the pencils (Σ_1, S_1) and (Σ_2, S_2) will be transformed in a two-parameter way. The reciprocal group G_2 to G_1 will arise when one switches the pencils (Σ_1, S_1) and (Σ_2, S_2) with then pencils (Σ_1, S_2) and (Σ_2, S_1), resp. It will now be the one that is associated with the number system XIII, and indeed, in our case, the planes Σ_1, Σ_2, as well as the points S_1, S_2 , will be conjugate-imaginary. The axes of a real transformation of the groups G_1 or G_2 will be conjugate-imaginary lines of the second kind when the transformation does not exactly belong to the invariant two-parameter subgroup (corresponding to a translation in the plane). In this case, the two axes will unite into a real line that will be the line of intersection of the plane Σ_1, Σ_2 and the connecting line of the points S_1, S_2, or even better, they will be infinitely-close to these lines.

The group $\mathfrak{G}$ is also simple to define. Let Σ be the plane of the pencil (Σ_1, Σ_2), which is harmonically separated from an arbitrary point P that lies on either Σ_1 or Σ_2, or even any point P that is not included in the plane that pencil. One can then exhibit a group in Σ that is composed the same as the group of motions of the plane, which is determined by S_1 and S_2 when they are held fixed. The group $\mathfrak{G}$ is then already completely known as the group whose transformations permute the points of the plane Σ in a prescribed way, but in addition, the point P and all planes of the pencil (Σ_1, Σ_2) will remain individually fixed.

It is interesting to compare formula (4′) with the ordinary formulas of the transformation of rectangular coordinates systems in the plane.

If we regard the sense of rotation of the positive x-axis to the positive y-axis as being positive, call the angle of rotation φ, and call the coordinates of the center of rotation x_0, y_0 then a point x, y will be taken to another one by the motion of the plane thus-defined whose coordinates x', y' will be given by the formulas:

$$(6) \qquad \left\{ \begin{aligned} x' - x_0 &= (x - x_0)\cos\varphi - (y - y_0)\sin\varphi, \\ y' - y_0 &= (x - x_0)\sin\varphi + (y - y_0)\cos\varphi. \end{aligned} \right.$$

If one now normalizes the parameters in formulas (4′) such that one will have $a_1 = 1$ and one demands that the transformation (4′) be identical with the transformation (6) then that will yield a simple connection between the quantities x_0, y_0, φ, on the one hand, and the parameter ratios $a_0 : a_1 : a_2 : a_3$, on the other. Under the stated assumption, one will obtain simply (*):

(*) The general formula reads:

(7) $$\cot \frac{\varphi}{2} = - a_0, \qquad x_0 = a_2, \qquad y_0 = a_3 .$$

If one then fixes the parameters a_1, a_2 , a_3 in the formulas (4′) and one lets just a_0 vary then the general one-parameter subgroup of the group of motions will arise. The parameter system for which a_1 vanishes will correspond to the parallel displacements or translations.

The parameter group that corresponds to formulas (6) and its reciprocal, which both consist of transcendental transformations, will be taken to two projective groups G_1 and G_2 , resp.

The group of rotations of a sphere, as well as the group of motions in the plane, when regarded as subgroups of the general projective group, is capable of yet another real form. These newly-appearing forms correspond to the number systems XIIb and XIIIb, which differ from XII and XIII in their real behavior.

Here we shall not go further into that situation, and shall remark only that formulas that correspond to formulas (1), …, (7) will emerge from the latter by the imaginary substitutions:

$$\bar{x}_0 = x_0, \qquad \bar{x}_1 = - ix_1 , \qquad \bar{x}_2 = - ix_2 , \qquad \bar{x}_3 = - x_3 ,$$
$$\bar{a}^0 = a_0, \qquad \bar{a}_1 = - ia_1 , \qquad \bar{a}_2 = - ia_2 , \qquad \bar{a}_3 = - a_3 .$$

The examples considered, which one can easily increase, might suffice to prove the advantages of the systems of complex numbers in the treatment of certain situations in geometry.

Marburg, in November 1889.

Postscript. During the printing, I was only recently referred to a paper of *Scheffers* "Über die Berechnung von Zahlensystemen," which appeared in the meantime (Sächs. Ber., 1889, pp. 400, *et seq.*) In it, the problem of determining all systems of complex numbers was also solved for five principal units, and some worthwhile theorems on certain systems with more than five principal units were presented. The published tables do not afford a sufficient glimpse into the structure of the individual systems. However, in a series of letters, *Scheffers* disseminated a new, extended presentation of his investigations, in which this lack would be remedied. Meanwhile, one will find a new and – the author believes – important application of systems of complex numbers in a treatise "Über die Bewegungen des Raumes" that is currently in print (Sächs. Ber., 1890, pp. 341, *et seq.*).

$$\tan \frac{\varphi}{2} = - \frac{a_1}{a_0}, \qquad x_0 = \frac{a_2}{a_1}, \qquad y_0 = \frac{a_3}{a_1}.$$