"Über Nicht-Euklisische und Linien-geometrie," Jahresbericht d. Deutschen Mathem.-Vereinigung, XI Heft 8 and 9 (1900), 313-342.

On non-Euclidian and line geometry.

By E. STUDY in Greifswald.

Lecture not presented ([1]).

Translated by D. H. Delphenich

I.

Generalities. The principle of duality.

The modern investigations of non-Euclidian geometry have their origin in the continuing, century-old – but always unsuccessful – attempts, to show that Euclid's fifth postulate – the so-called parallel axiom – is superfluous. One seeks to *prove* the statement that is included in that – i.e., its content – by purely logical tools from the remaining assumptions that were made in the Euclidian system of geometry and to deduce it from them alone. This question can be regarded as having been solved today. Three different mathematicians – namely, Gauss, Lobatchevski, and Johann [*sic*] Bolyai – which were, as we now know, independent of each other, constructed a system of concepts that started with one general assumption and seemed to contain no logical contradictions that is precisely non-Euclidian geometry. Thereupon, Beltrami, as well as Cayley and Klein, showed in various ways that one can also arrive at this general system of concepts by a special interpretation of the firmly-established formulas and theorems of ordinary analysis and geometry, and that any possibility that a contradiction might perhaps be found in the future is therefore excluded. With that, the aforementioned main question is, as we said, resolved. However, the interest of mathematicians in non-Euclidian geometry itself is not exhausted with that. Investigations into the systematic establishment of the basic concepts of mathematics are also sufficiently worthwhile – in particular, investigations in geometry into the question of whether a geometric theorem must be regarded as an axiom – although the material content of the individual disciplines is still *worthwhile*, on whose account alone that sort of investigation will have any purpose at all, and non-Euclidian geometry belongs to each discipline, moreover.

Now, it can generally seem at first as if the further development of non-Euclidian geometry has only a secondary interest, as if the geometer, whose heart is in the study of the space in which we live, does not necessarily need to worry about that. Indeed, the indistinctness of our presentation of space has a place, not just for the system that is

([1]) Except for some meaningless formal alterations and some additional remarks that are enclosed within [], this paper is a verbatim transcription from the Festschrift of the Greifswald Philosophical faculty on the fifty-year doctoral jubilee of Heinrich Limpricht. (Greifswald, 20, III, 1900.)

referred to as non-Euclidian geometry as such, along with its subordinates, but also for many enveloping conceptual constructions. Such speculations would, however, be able to take on an immediate practical meaning only when certain procedures compel us to subject the conventional, and undoubtedly nearest, system of geometry, at least, to a more detailed examination. However, up to now, no facts are known that would also make it only obvious that any other system of geometry corresponds to observed phenomena better than Euclidian. Therefore, up to now, the physicist, as well as the geometer, has regarded his study as an analysis of the empirically-given space, while his theoretical statements might be based upon the system of Euclidian geometry.

Admittedly, non-Euclidian geometry will also find a place in the circle of ideas of this geometer. However, the position in which this geometric discipline is found lies not in the roots of the geometric study. The geometry of surfaces of constant negative curvature, in which the theorems of Beltrami on planar non-Euclidian geometry find their realization, can hardly attract a higher degree of interest for our geometer than the theory of other special classes of surfaces, like, e.g., the theory of minimal surface that is so rich in beautiful theorems, and Cayley's "absolute metric," which non-Euclidian geometry realizes in a three-fold extended space, will be presented to any geometer as a section of the theory of second-degree surfaces, and thus, as a very special situation. Thus, if our geometer wished to explain that it would be impossible for him to deal with all particular questions of geometry, and that he was not especially interested in this precisely this situation, then he would be well within his rights.

However, in mathematics one cannot be careful enough when dealing with value judgments that arise from the use of the concepts "special" and "general." We fear that our geometer (who is, moreover, not merely a complete fiction) might have committed a mistake. (Namely, he has not observed that for infinite manifolds the totality can be mapped onto a subset.) The conception of geometry as a science of experience is only one of many possible ones, and the standpoint of the empiricist is, in no way, the most insightful standpoint in regard to geometry. He would then not be justified in his single-minded view of the fact that the mathematical sciences are intertwined with each other in various, and often surprising, ways in such a way that, in reality, they define an indivisible whole. Although it is possible, and indeed, more desirable, that any discipline be developed with the tools that are specific to it and independently of the other disciplines, one would be deprived of one of the most important tools for analysis, which would not consider the manifold connections between the various disciplines. When applied to Euclidian and non-Euclidian geometry, this truly self-evident – but at the same time, unheeded – truth leads to the following somewhat paradoxical conclusion that *in some situations, the knowledge of non-Euclidian geometry cannot be deprived of a more penetrating understanding of a very elementary portion of Euclidian geometry.*

The author believes that the fact that things actually work that way can be shown by an example that is especially educational, due to its simplicity. Along with the so-called force parallelogram, one can also present two other constructions that are equivalent, but still essentially different from the parallelogram, with whose help, the combination of two forces that act upon the same point can likewise be performed. These constructions, each of which, like the parallelogram, can define the starting point for further developments, can also be made understandable to a non-professional with no effort. Despite its simplicity, and despite uncommonly numerous versions of the situation, the detour through non-Euclidian geometry is, in fact, necessary for its discovery.

In the following, we now wish to demonstrate some up-to-now apparently unknown or not sufficiently appreciated facts that, in part, likewise establish a connection between Euclidian and non-Euclidian geometry, and are, moreover, also suitable in other respects for shedding some light on the systematic and heuristic meaning of non-Euclidian geometry. We will thus confine ourselves to some less characteristic theorems. Partly due to nature of things, a thorough presentation with further applications that present themselves in greater numbers, but necessitate the development of a sizable formal apparatus, must be postponed to a later publication.

We begin with some remarks on the so-called *principle of duality.*

As is known, one finds conspicuous analogies in Euclidian geometry between theorems in which the concept of the distance between two points appears and ones in which the concept of the angle between two planes appears. Now, as one knows, these analogies find their explanation in properties of non-Euclidian space and in the fact that Euclidian geometry represents a limiting case of non-Euclidian geometry. One can generalize this remark to the theorem that the concepts of non-Euclidian geometry are associated with each other in pairs that can emerge from each other by a trivial transformation – viz., the *absolute correlation* – and that various concepts of Euclidian geometry are, or can be, obtained from such associated concepts by passing to the limit. This remark, which already includes very fruitful methods of analysis in it, now admits various extensions.

First, it must not at all be taken to be true that the concepts and theorems of non-Euclidian geometry are associated with each other pair-wise. In general, one can only say that they are associated group-wise, where the number of the statements that are united in a group can be any arbitrary number. Thus, in the aforementioned example of the composition of forces, three different constructions belong together, each of which will produce another theorem under passage to the Euclidian space.

Second, it must be remarked that one can conceivably carry out the passage to the limit itself in several different ways, in such a way that several different conceptions of Euclidian geometry can emerge from a single conception of non-Euclidian geometry. We clarify this with an example:

In a non-Euclidian space of positive curvature, the manifold of rotations – i.e., all motions of a rigid body that leave all points of a line at rest – is identical with the manifold of all motions that leave all planes through a line at rest. Such an identity does not exist in Euclidian geometry: The manifold considered *decomposes* in the limit into the manifold of rotations, in the usual sense of the word, and the manifold of "screw reversals," which are screw motions for which the angle of rotation is equal to two right angles. The analogy between the following two theorems that belong to Euclidian geometry ([1]) will find its explanation in this, and in the principle of duality:

"If one represents a motion in all possible ways as a sequence of two rotations, one of whom has a constant angle of rotation, then	If one represents a motion in all possible ways as a sequence of a rotation and a screw reversal whose pitch is constant, then

([1]) Here, we consider only the so-called general case, where it merely involves the basic ideas. We will communicate a precise formulation in another place.

the locus of the associated rotational axes will be a "cyclic" ray complex,	the locus of the associated screw axes will be a "cyclic" ray complex,

namely, a line complex that is different from the tangent complex to a cone whose plane complex curves are circles, and whose complex cone is a cone of rotation.

That complex will also be preserved:

when one subjects all tangents of a circle to all screws around a line that cuts a diameter of the circle perpendicularly.	when one subjects all generators of a cone of rotation to all screws around a lines whose shortest distance from the axis of the cone meets this axis at the vertex of the cone.

Finally, it is worthy of mention that associated theorems can become partially illusory under the passage to the limit, or they can assume a form that is so different that their intrinsic kinship is no longer recognizable. An example of the first remark is defined by the non-existence of one of last two analogous theorems, in which two screw reversals would appear in place of the two rotations, or in place of the rotation and the screw reversal. The second remark will be clarified by the following well-known theorem of planimetry:

If one stretches a string of constant length around an ellipse from a point outside of that ellipse then the locus of the point is an ellipse that is confocal to the given one. (Graves's theorem)	If one lets a line move such that it cuts out a piece of constant surface area from an ellipse then this line will envelope an ellipse that is similar and concentric to the given one.

Many investigations into the metric properties of second-order surfaces and other figures that are not inessential in content and interest will be achieved by a thorough consideration of such quasi-dualistic relationships. Its true fecundity, however, will be first shown by such considerations once the algebraic and differential geometry of non-Euclidian space, which is still in its beginnings, is better understood.

We now turn to the exposition of a second connection between Euclidian and non-Euclidian geometry that is of an entirely different kind. We thus find it preferable to distinguish manifolds of positive curvature from ones of negative curvature. We direct our immediate attention to just the so-called elliptic and hyperbolic spaces; as for the spherical and pseudo-spherical space forms that are to be distinguished, the things that will be said can be carried over with mostly only minor alterations. For the sake of simplicity of representation, we will set the curvature measure of elliptic space equal to positive unity, and likewise choose the curvature measure of hyperbolic space to equal the negative unity. The fact that the established restriction to real figures is inessential in all of the theorems to be posed in § 2, as well as in some of the ones that are posed later, is self-explanatory; therefore, an entrance into imaginary figures would necessitate much more involved arguments. – In each of the two methods that will be described in § 2 and

§ 3, one will recognize an extension of Gauss's *method of spherical images* to non-Euclidian space.

II.

Line geometry in elliptic space.

As one knows, the concept of parallels has two analogues in triply-extended non-Euclidian space.

One of them occurs as a real figure only in spaces of negative curvature, while the other one – which was discovered by Clifford – has analogues only in spaces of positive curvature. One calls both of these two figures "parallels." That does not seem suitable for us. One can deduce no adjective in the German language for "Cliffordian parallels." However, one can still not avoid the consideration of imaginary figures in much deeper analyses. One thus comes to the awkward position of referring to two different things by one and the same word. On these grounds, we shall allow ourselves to propose another terminology: We will call the "Cliffordian" parallels *paratactic lines*. Leaning on the usual terminology, we will further distinguish the two different types of parataxy as *left-handed* and *right-handed*. This distinction will then correspond to the distinction between the two invariant three-parameter subgroups of the group of motions in elliptic space as *left-handed* and *right-handed displacements* in such a way that displacements will be called "left-handed" when their one-parameter subgroups have left-handed paratactic lines for their path curves. – We can, moreover, allow a new concept to emerge from the concept of straight line by a process that we refer to as *orientation*, which consists in the determination of the value of a certain square root, namely, the concept of a *spear*.

A spear is a line with a distinguished "positive" direction. It is distinguished from the unoriented line by the fact that the goniometric tangent of the distance between two points taken from the spear has a well-defined sign, as long as these two points are taken in a particular sequence. If a line were oriented, so one would be dealing with a spear, then one could rationally separate the orientations of all lines that are – e.g. – paratactically left-handed to it. Thus, two new concepts arise from the concept of paratactic line, namely, those of the (left-handed or right-handed) *syntactic* and *anti-syntactic spears*, which do not need to be specified in greater detail. Finally, it is recommended that one introduce a special word for the figure that consists the absolute polar of a line. This figure, when seen from any point of (elliptic) space, takes the form of the image of a cross with a right angle. We thus call it a *line cross*.

With these preparations, the following two theorems will be understandable:

I. *The manifold of all real line-crosses in elliptic space can be mapped, in a one-to-one manner, to the manifold of all real spear-pairs that one can define from a ray of two bundles (e.g.) in Euclidian space, in such a way that the left-handed displacements will be associated with the rotation of the one (viz., left) bundle and right-handed displacements will be associated with the other (viz., right) bundle.*

II. *The manifold of all real spears in elliptic space can be mapped, in a uniquely invertible way, to the manifold of all real point-pairs that one can define from points of* *two spheres of radius one in Euclidian space in such a way that the left-handed displacements will be associated with the rotations of the one (left) sphere surface and the right-handed displacements will be associated with the rotations of the other (right) sphere* ([1]).

If one introduces the line cross or the spear as the space element in elliptic (or spherical) space then the facts of the associated non-Euclidian geometry will be put into a certain invertible relationship with certain facts of Euclidian geometry.

Due to the vast scope of the situation, we will be forced to restrict ourselves here to making the import of these theorems tangible by way of some examples. We would next like to explain Theorem II by juxtaposing a number of concepts of elliptic geometry with the associated concepts from the geometry of our two sphere surfaces. We will assume that the structures in question are analytic and real.

The two normal distances between two spears (the stationary distances between points).	One-half the sum of and one-half the difference between the spherical distances between two point-pairs (determined up to sign and mod π).
Point as locus of spears.	Congruence relation (map) of the two spheres.
Plane as locus of spears.	Symmetric relationship between the two image spheres.

Each point (plane, resp.) in elliptic space will then correspond to a real (imaginary, resp.) ternary orthogonal transformation. The homogeneous coordinates of the point (plane, resp.) are identical to the *Euler parameters* of orthogonal transformation for a suitable choice of coordinates.

Tangents to an oriented curve (or spears of a cone).	Pair of isometric point-sequences (pair of mutually-mapped curves with equal length arcs between corresponding points).
Congruence of left-hand syntactic spears.	Points of the left sphere (linked with all points of the right sphere).

It then follows that the geometry of a congruence of syntactic spears is completely identical to spherical geometry.

([1]) We further mention that the author has already published (Leipz. Ber., v. 9, January, 1899) an application of Theorem I. He communicated Theorem II in a talk that was given at a naturalist meeting in Munich (Sept, 1899).

Special congruences of translations that are generated in two ways by displacing a surface that is composed of syntactic spears.	Pairs of curves (i.e., each point of a curve in the left sphere is linked with each point of a curve in the right sphere).
Likewise special congruences that consist of ∞^1 surfaces that are composed of syntactic spears.	Association between the points of a curve in the one sphere and ∞^1 curves in the other sphere.
Congruences of a general sort.	Reciprocal map of both spheres by a point transformation.
Normal congruence to a family of parallel surfaces of zero curvature.	*Curve-pair* (cf., above)
Normal congruence of a general kind.	*Real, surface-preserving map.*
Translation complex that consists of ∞^1 left-hand-syntactic congruences (cf., above).	Curve on the left sphere (linked with all points of the right sphere).
Complex whose spears can be distributed into ∞^1 congruences of translations.	Association between ∞^1 curves in both sphere surfaces.
Line or spear complex of a general kind.	*Contact transformation.*

This summary, brief and fragmentary as it is, will still show us what a wealth of new problems arise from Theorems I and II. They will yield new classification principles for algebraic surfaces, complexes, etc. Problems of differential geometry that are not so simply accessible by other means can admit a simple and immediate solution. A closely-related consequence of our theorem is, e.g., the theorem that a single surface of zero curvature can be drawn through any analytical strips whose normals are not paratactic; one further finds that one can integrate the differential equation of these surfaces in closed form. Even further properties of these special classes of surfaces – whose study we can thank Bianchi for ([1]) – can be derived in this way. The *stereometric addition of motors*, which is a process for the geometric composition of forces ([2]), was found by the author in just that way. It also yields important consequences for the theory of transformation groups. We can pose the following problem, among others:

Find all transformations of straight lines in elliptic space that always emerge from some line of a normal net of a line.

([1]) "Sulle superficie a curvature nulla in geometria ellitica," Annali di Matematica, **24** (1896).
([2]) Confer the book *Geometrie der Dynamen*, Leipzig, 1901.

One gets the solution from our Theorem I with no calculation. One maps the manifold of line crosses in the given way onto the ray-pairs of two bundles, or, more intuitively, onto the point-pairs of two planes. If one then subjects each of these bundles, or each of these planes, to an arbitrary collinear transformation then a transformation of the desired property will arise in elliptic space, and likewise when one associates the one plane with the other one, and conversely. All transformations in question will be found in this way. They then define a so-called mixed group with 16 parameters whose continuous subgroup is semi-simple. The theorems of projective geometry on two planes find a surprising, if not entirely simple, interpretation in the kind of line geometry that belongs to this group (which is naturally completely different from Plückerian line geometry). This new kind of line geometry is, however, only one among many that arise in similar ways. One can extend the group of circle transformations, the groups of conformal and surface-preserving transformations, and the group of all point transformations in the same way. The latter extension includes, e.g., the solution to the problem:

Find all transformations of straight lines in elliptic space that can always arise from the normal congruence of a family of parallel surfaces of zero curvature.

Another infinite group is then defined in such a way that it converts nothing but normal congruences into other ones. Its transformations can be likewise determined in other ways; one can also give the intersection of both groups, and much more.

The possibility of another sort of application of non-Euclidian geometry opens up when one remarks that (except for trivial exceptional cases) each pair of mutually developable surface in *Euclidian space* that is determined up to its form is linked with a line complex (spear complex) in non-Euclidian space that is completely-determined, up to a motion, and whose complex lines can be associated with planar pencils.

Finally, we would like to mention another fact that seems to us to be of fundamental significance to a deeper algebraic foundation of the non-Euclidian geometry. As a first main problem in the theory of invariants for the general projective group in space, one can consider the exhibition of types of invariants in an unbounded system of linear and alternating forms that represent points, planes, and linear complexes when they are set equal to zero. This as-yet-unsolved problem can be replaced with a simpler one that pertains to various subgroups of the general projective group. In particular, in the case of the group of a second-order surface with a non-vanishing discriminant – hence, also in the case of the group of motions in elliptic space – it is sufficient to consider just one kind of linear form in addition to the alternating forms. However, this problem leads back to a problem in the theory of binary forms by an imaginary transformation. It consists in the determination of simultaneous invariants of certain forms with variables in two different binary domains that are transformed into each other by linear transformations that have equal discriminants, but are still independent. The forms themselves have two types of characteristics: Firstly, bilinear forms that connect the variables in both domains (corresponding to the points or planes in non-Euclidian space), and secondly, pairs of quadratic forms (corresponding to linear complexes), such that the six coefficients of a pair are to be regarded as a single system of homogeneous quantities. The solution to this problem raises no special difficulty.

The possibility of the suggested reduction is a peculiarity of non-Euclidian geometry. The development of a theory of invariants of Euclidian motions is a problem that is not of equal difficulty.

We shall now turn to a discussion of hyperbolic geometry. We will consider it from a different standpoint that we had also been able to take, *mutatis mutandis*, relative to the theorems that were presented in the current §, moreover. Later on, we will treat some of the topics that were not touched upon, or only brushed up against, in the current § in greater detail.

III.

A geometric interpretation of the so-called imaginary geometry in the plane or on the sphere surface.

So-called imaginary figures are introduced into geometry for the same purpose as imaginary quantities in analysis. With their assistance, one arrives at a simpler representation, namely, a synopsis of many otherwise disparate theorems to be stated and proved into a single expression, which thus gives a simpler overview, *in toto*. In its beauty, the system of geometry, thus-extended, is superior to the geometry that works only with real figures to such a degree that attempts could not fail at the association of real – and thus the simplest possible – geometric forms with the "imaginary points," etc., the theorems of that imaginary geometry are additionally again subject to a real-geometric, indeed, intuitive, interpretation. In the case of the geometry on a singly-extended manifold, these attempts were accompanied by distinguished success. The Gaussian number plane and the various kinds of Riemann surfaces, with the help of which, one studies imaginary geometry on algebraic curves, are in general usage nowadays.

The fact that an equally intuitive realization of imaginary geometry is already not to be found in the case of doubly-extended manifolds lies in the nature of things. A meaningful success for the attempts that were made along these lines has been achieved only for the imaginary structures that belonged to v. Staudt's theory of projective geometry. Another very noteworthy – in any event, important for the geometry of the conformal group – Ansatz that goes back to Chasles, and consists in the suggestion that an imaginary point in space should be replaced with a circle with a well-defined sense of traversal, remains completely undeveloped. Further attempts have had the objective of associating the ∞^4 imaginary points of a doubly-extended manifold with the elements of some likewise four-fold-extended manifold of geometric figures that lies in ordinary space. The presence of a relationship between the imaginary point and its image that is invariant – perhaps, under motions – will thus not be necessarily assumed. These attempts have had modest success, because they likewise remain stuck in the rudiments, partly because the constructions that are employed are devised in such a way that they would not be easy for a geometer to consider for their own sake. Constructions such as the representation of imaginary figures that was given by S. Lie give the impression of being artificial; they were contrived merely for one purpose ([1]).

In what follows, another attempt at associating figures with plane spherical geometry will be described briefly. The map to be performed is itself not completely new, as we would like to emphasize immediately. However, what might be new, as far as it relates to the current matters, is the type and extension of applications, and likewise the path along which the author arrived at that map, independently of older investigations.

We think of a real or imaginary point in the plane as being represented by a system of three homogeneous coordinates. For the time being, we will interpret the real components and the real coefficients of the imaginary components of these coordinates as Plücker coordinates of a *thread* (i.e., a linear line complex) in a triply-extended space. The association of imaginary points in a plane and the real threads in space thus given is infinitely-many-valued, since the projective coordinates of any point are determined only up to an arbitrary (complex) factor. Any point thus corresponds to an entire pencil of threads. Now, this pencil exhibits a peculiar behavior: A point in general position corresponds to a complex that contains two different – *always real* – pencils of lines, namely, the so-called *special complex,* which reduces to a straight line. Two such lines that determine the associated pencil are polar to each other, relative to a certain second-degree surface, which has real points, but no real generators, and which therefore – since projective conversions do not come into play – can be identified with a *sphere.* However, if the square sum of the projective coordinates of the point considered vanishes, so this point belongs to an (arbitrary, moreover) conic section of non-vanishing discriminant, then the entire pencil considered will consist of straight lines, and they will all contact the aforementioned sphere in one and the same point. We thus have before us an, in general (1-2)-valued, and in special cases (1-∞)-valued, map of the imaginary points of the plane to real lines in space.

A reciprocally-single-valued map can now be derived from this map, with one simple precaution. Namely, since, one of two mutually polar associated lines will meet the sphere, but not the other, one needs to consider only one of the two. One can further present a *notion of equivalence*, under which, all tangents to the sphere that contact it at one and the same point are considered to be identical, and not further distinguished. When we arbitrarily decide to consider only those lines that meet the sphere, we arrive at the following conceptualizations: We call the piece of a line between two points of the sphere ([2]) a *proper ray*, but any point of the sphere itself an *improper ray*, and thus, any ray with coincident endpoints. We have thus mapped the totality of real and imaginary points in the plane to the totality of "rays" in a one-to-one, invertible way; the improper rays correspond to the points of some irreducible conic section.

If we now consider three ratios, moreover, which we had previously regarded as point coordinates, to be line coordinates then a point and a line with equal coordinates will be a

([1]) The fact that the mind of a great mathematician was led to worthwhile discoveries by such considerations changes nothing. The path to discovery was certainly a detour in the case at hand.

([2]) As such, another word – perhaps "segment" – might be preferred. However, there is an analogous concept in Euclidian geometry that coincides with the ordinary concept of straight line only in the real domain. Therefore, little else remains here except to allow a split in the terminology to occur, namely, to use the only two available words "line" and "ray" in different senses. We then carry over this distinction, *mutatis mutandis,* to the situation that is treated in this article. Any kind of terminology will do mischief in such cases.

pole and polar relative to the aforementioned conic section. We thus obtain immediately the rays on the points of an image of the ray manifold of the real and imaginary straight lines in the plane with our map. This remark permits us a further conceptualization: We imagine that the entire ray manifold is doubly-covered, and distinguish between *rays of the first sheet*, which are associated with points, and *rays of the second sheet*, which correspond to straight lines. Rays of both sheets that lie over each other thus correspond to poles and polars relative to the distinguished conic section.

What relationship now exists between the images of a point and a line when these figures are united? The answer is not hard to find: The two image rays intersect, and each of them, when lengthened, will intersect the polar of the other one relative to the distinguished sphere, in addition. If we then choose this sphere to be the absolute surface of a Cayley metric, and we thus consider the interior of the sphere to be a so-called *hyperbolic space* then we can say that the two image rays will intersect perpendicularly. One should naturally understand the term *rectangular intersection* of a proper and an improper ray to mean the proper ray contains the point of the sphere that represents the improper line.

We would now like to summarize some of what we said by a closely-related consequence:

III. *The real and imaginary points and the lines in the plane can be mapped to the doubly-covered ray manifold in a one-to-one and invertible way. Points and lines in united position will correspond to rays of the first and second sheets that intersect rectangularly (in the sense that we explained). A collinear transformation of the plane will correspond to "contragredient" transformations of the rays of both sheets, under which, rectangular intersections of such rays will be preserved, but not intersections, in general, nor the coincidence of rays on different sheets, nor the rectangular intersection of rays on the same sheet, nor the property of a line being proper or improper. The aforementioned transformations define a continuous (but semi-simple) group of sixteen parameters that is simple in real domains.*

We would like to call these transformations *dual-collinear transformations in hyperbolic space*, and the ∞^{16} transformations that arise from exchanging rays on both sheets that lie over each other, *dual-correlative transformations.*

In the geometry of these groups, one first finds the construction of the linking of imaginary points by straight lines and the exploration of the intersection point for imaginary straight lines with their real counter-image in the construction of the common normal on the second sheet to two rays on the first sheet, and conversely. However, one can, moreover, give a real interpretation, as such, of *any* theorem of the projective geometry that originates in these constructions, such as the associated invariant and function theory, Bezout's theorem and Riemann-Roch theorem, no less than, say, Abel's theorem in its application to planar curve. The actual interest of this interpretation is, nevertheless, mainly contained in the fact that the figures to which one is led are ones that deserve to be considered, anyway.

We would like to clarify this last especially important point through some examples.

We first extend the group of transformations that was introduced by adding a symmetric (which exchanges both families of imaginary generators of the absolute surface) collineation (which is a reallocation in hyperbolic space). In this way, two new families will arise from the transformations that we enumerated, which we can summarize as *dual-projective transformations*, namely, ones that we will call *dual-anti-projective transformations* ([1]), and ones that we can distinguish as *dual anti-collineations* and *anti-correlations*. One then has the theorem:

The dual projectivities and anti-projectivities in hyperbolic space subsume all transformations of lines that allow a normal net of a ray to emerge from another one.

It can be further shown:

The image of an (irreducible) plane analytic curve is, when the curve is regarded as the locus of its points, a "synectic congruence," namely, the normal congruence of the first sheet of a family of parallel, analytic surfaces of curvature zero, and when the curve is regarded as the locus of its tangents, a synectic congruence of the second sheet.

Conversely, any two irreducible, synectic congruences belong together, and these pairs are invariantly linked, not only by non-Euclidian motions and transfers, but by contragredient dual projectivities and anti-projectivities, in general.

They define the tangents to any two mutually-orthogonal (real) families of parallel geodetic lines in an arbitrary analytic and real surface of curvature zero, namely, their common focal surface ([2]).

A self-explanatory *exception* to all three theorems is defined by the distinguished conic section whose image – viz., the absolute surface – one can, moreover, add to the other one as an *improper synectic congruence*. It is further self-explanatory that the ∞^4 normal congruences of rays are excluded from the second and third theorem; they and only they are not paired with other ones. An exception to the third theorem – which is, moreover, inessential – is finally defined by *the* synectic congruences whose focal surfaces reduce to a curve. There is a five-fold infinitude of them. The curve in question is a real ray that lies in a plane with any two associated rays of both congruences, and will be constantly cut by these rays at an angle that is supplementary to a right angle.

Any irreducible synectic congruence that is not the normal congruence to an improper ray (or a family of so-called limit surfaces) mediates a proper-conformal map of the sphere surface, and conversely, any such map defines a synectic congruence.

That is, such a conformal map, in the ordinary sense of *Euclidian* geometry, of the simply or multiply-covered sphere surface or two pieces of the spherical surfaces for which the angle is not changed will be mediated by the two endpoints of the ∞^2 rays of the congruences.

([1]) We define these expressions by analogy to a terminology that was employed by C. Segre.

([2]) The evolute surface of a surface of curvature zero in hyperbolic space has only one *sheet* (*Mantel*) in the interior of the absolute surface.

If one then employs the spherical surface, as usual, for the representation of the value of a complex variable then one will immediately obtain a pair of Riemannian surfaces onto which the plane curve that belongs to the synectic congruence is mapped point-by-point. If the curve is algebraic then the Riemann surfaces will be closed and have the same genus as the curve. For the sake of brevity, we will not go into a thorough differentiation of the two kinds of algebraic synectic congruences.

The remark that one also obtains the same conformal map by connecting Hesse's transcription principle with the usual representation of the imaginary points of a binary domain by real points of a spherical surface can be extended to a second basis for Theorem III ([1]), and, at the same time, exhibit a path to an infinite series of similar and more general theorems. The theorem that is now *without exception*, namely, that a single surface of curvature zero goes through each real analytic strip (in the interior of hyperbolic space) (cf., § 2), proves to be a consequence of a known theorem of H. A. Schwarz on conformal maps.

A very closely-related, but unconnected, and especially surprising application of a mapping method is defined by the following remark: One imagines that the rays of two synectic congruences in the neighborhood of a suitable pair of rays are associated with each other in such a way that corresponding rays of one and the same ray will cut a third synectic congruence perpendicularly. We would like to say that the two congruences are related to each other "synectically" by this association and its analytic continuation.

If two synectic congruences are synectically related to one and the same third one then they will be related to each other synectically. That is, the common normals to corresponding rays again will define a synectic congruence (which is likewise synectically-related to the first two), or a subset of one.

The synectic relationship between two synectic congruences is invariant under not only dual collineations and anti-collineations, but, more generally, all transformations of ray space that take synectic congruences into other ones.

The transformations in question define an *infinite group* of "synectic" and "anti-synectic" transformations whose continuous subgroup is the image of the group of analytic point transformations in the plane.

Naturally, the applicability of our transcription principle also extends to metric geometry. Of several consequences that are related to it, we would like to cite just one of them that, in turn, exhibits a connection between Euclidian and non-Euclidian geometry.

The intersection of our group of dual collineations and anti-collineations with the group of collinear transformations, in the ordinary sense of the word, defines the motions and transfers in hyperbolic space. The group of real motions in hyperbolic space (which is long since known) likewise proves to be holomorphically isomorphic to the group of real and imaginary collinear transformations in the plane that fix a certain conic section, namely, the much-discussed distinguished conic section. We would now like to assume

([1]) In fact, one already comes close to Theorem III in this way. Cf., F. Klein, Math. Ann., **22** (1883), 246. Lindemann, *Vorlesungen über Geometrie*, II, pp. 613. The cited authors then remain within the circle of ideas of ordinary projective geometry. They have considered no transformations that take the points of the absolute surface to straight lines.

that the polar system to this conic section, along with the initially-employed coordinate system, is real in such a way that the conic section itself should have no real point. We further consider the conic section to be likewise the absolute curve of a Cayley metric, and go from elliptic to spherical geometry by introducing a double covering of the plane by way of a known process that was described in papers of F. Klein. On the other hand, we also think of the ray manifold in hyperbolic space as being doubly covered in a different way from what we did before by deriving a new concept from the concept of ray, namely, that of an *arrow*. The "proper arrow" shall be distinguished from the proper ray by the fact that its endpoints are given in a definite sequence, and that, as a result, the distance between two (real) points of the arrow that have a definite sequence likewise keep a definite value ([1]). The "improper" arrow shall be the same thing as the imaginary ray, such that manifold of all arrows will likewise define a continuum with a branching manifold in the absolute surface. On the basis of these determinations, one can now state another theorem:

IV. *The manifold of all arrows in hyperbolic space can be mapped in a one-to-one and invertible way to the manifold of all real and imaginary points of a sphere in Euclidian space in such a way that the real motions in hyperbolic space will be associated with the real and imaginary motions of the sphere.*

The real points of the spherical surface will thus be associated with the arrows of a bundle whose vertex lies in the interior of the absolute surface. Since we have likewise regarded the absolute surface as a sphere in the sense of the Euclidian geometry, the aforementioned vertex can be identified with the Euclidian center of this sphere. Furthermore, one can let both spheres – viz., the sphere in Euclidian space that is to be mapped and the absolute surface of the hyperbolic space – coincide. One can then further arrange the map so that any real point of the first sphere will be associated with the arrow of the aforementioned bundle that has its endpoint at that point. Arrows that correspond to conjugate-imaginary points whose rectilinear carrier lie symmetrically in relation to the indicated point, in the sense of hyperbolic geometry, and with the last convention that we encountered, also in the sense of Euclidian geometry.

A theorem that was formulated already now assumes the following form:

The real and imaginary points of an analytic curve on the sphere correspond to the arrow of an "oriented" synectic congruence, and conversely.

In particular, the intersection points of the sphere being mapped with the infinitely-distant plane in Euclidian space correspond to the improper arrow, while the real points correspond to the absolute surface. The two families of imaginary straight lines on the sphere correspond to the two different kinds of bundles of parallel arrows.

The last remarks can be seen, in our Theorem IV, to represent an extension of a known process that was given by F. Klein and employed several times. They likewise lead to another basis for Theorem IV itself. One draws a generator "of the first kind" for

---

([1]) The concept of arrow is closely related to the concept of a spear that was introduced in § 2, but we need another word for our purpose.

this surface through an imaginary point of the given sphere to be mapped; the real point of this line will be the endpoint of the desired arrow. One then draws a generator of the second kind through that point, makes it intersect the infinitely-distant plane, and once more draws a generator of the first kind through the intersection. The real point of this second line will be the initial point of the arrow to be constructed.

In order to make the kind of application of our Theorem IV clear, we add the following: Two proper arrows, even when they do not intersect, subtend an *angle* with each other that is determined up to sign and mod 2π, whose sign depends upon the sequence of the two arrows and the direction that one has ascribed to their common normal. In addition to this angle, which is equal to the (non-Euclidian) angle between the two planes that link the given arrow with its common normal, one has a certain *distance*, namely, the distance to the base point of that normal, so the sign will depend upon the same convention. We now define the concept of the *dual angle* between both arrows for both quantities. We understand that to mean the former angle, increased by $i = \sqrt{-1}$ times the distance. If we then set the radius of our sphere equal to unity then we will get the theorem:

The dual angle between two arrows is equal to the spherical distance between the corresponding real or imaginary spherical points.

The multi-valuedness of the two mutually associated quantities is naturally the same.

The fact that one can give a real interpretation to all of spherical geometry in the imaginary domain is self-explanatory. An immediate consequence is, e.g., a theorem that was asserted by Fr. Schilling, by which the formulas of spherical trigonometry can be interpreted in hyperbolic spaces by assuming complex arguments ([1]).

Another application of our Theorem IV is the determination of all transformations of arrows that take "cyclic" congruences to other ones. A *cyclic congruence* is the image of a real or imaginary circle. Since this generally comes about by a rotation around a point, the cyclic congruence will be generated by screwing one arrow around another one. Exceptions are the reducible cyclic congruences, pairs of dissimilar bundles of parallel arrows (cf., *supra*), and the congruence of improper arrows that likewise must be regarded as cyclic.

The transformations in question define a twelve-parameter group with four different families of transformations, two of which come about under our map as the real and imaginary Möbius circle transformations. The continuous subgroup of this group will be obtained when one transforms the initial and final point of all arrows by two mutually independent proper circle transformations.

The map of the imaginary points of a sphere onto the arrows in hyperbolic space that was given by Theorem IV should not be confused with another one that one can derive from the aforementioned Chasles Ansatz, which, however, is arrived at much more simply, when one replaces any imaginary point of the sphere with the real points of the two associated generators of the sphere, which are associated with that point in some well-defined way. Both maps go to each other through a simple involutory

([1]) Cf., Math. Ann. **39** (1891), pp. 595, and the author's extension of it in Abh. der K. Sächs. Ges. d. W. **20** (1893), pp. 229.

transformation. The different images of one and the same point of the sphere are, in fact, catheti (*Katheten*) of a right-angled triangle that is inscribed in the sphere, in the sense of Euclidian geometry (the hypotenuse is a diameter of the sphere). In the event that the curve does not reduce to a generator of the sphere, the image of an analytic curve on the sphere will now become a congruence of arrows that mediates an *improper-conformal* map of the sphere; the image of an irreducible circle will be, e.g., the congruence that is determined by an improper circle transformation. If one assumes that the points of the curve considered are pair-wise conjugate-imaginary then the associated conformal map will become involutory. If the curve has a real evolution, moreover, then the map will go to the *conformal reflection in a real analytic curve* that was treated by H. A. Schwarz.

We shall not go further into this matter, and remark only that the arrow congruences that are defined by improper-conformal maps deserve a closer investigation in the sense of non-Euclidian geometry, as well as also in the sense of the theory of enveloping groups, which one can derive from the behavior of the map itself.

IV.

Examples of the map for non-analytic manifolds. Concluding remarks.

If one restricts oneself to *synectic* figures in the geometry of rays and arrows – i.e., ones whose properties under maps in the plane or on the sphere can be expressed by *analytic* equations – then one will consider a group of geometric theorems that already inherently have the simplicity and completeness that other realms of geometry first take on by the introduction of imaginary elements, and thus, a group of theorems that do not *demand* the introduction of imaginary elements. As a result, one will expand the circle of figures under investigation from the outset, and, as such, the introduction of imaginary elements can be *preferable* with the aforementioned restriction, since important facts – e.g., the property of the synectic congruences as congruences of translations – might otherwise escape notice. In the application to ordinary geometry to doubly-extended manifolds, one must naturally employ bi-complex quantities.

If we consider only real figures in ray space, as we shall also do in what follows, then the circle of geometric constructions thus circumscribed will encompass real hyperbolic geometry, among other things. One will generally be able to ask what kind of figures of plane geometry are then associated with the real figures in ray space. A question of this kind will be answered partially by the following theorem:

If one subjects the manifold of real points of a sphere or a suitable part of this manifold to an imaginary analytic proper surface-preserving transformation then the image that is exhibited from Theorem IV of the doubly-extended non-analytic manifold that thus arises will be an analytic, non-synectic, normal congruence in hyperbolic space, or a part of one.

Whether the converse theorem is also correct, or whether one can find *all* real, non-synectic, normal congruences of this kind can, in fact, only be decided on the basis of a special examination. Thus, much can be said of the fact that a normal congruence (which can be either non-synectic or synectic) can, in any event, have no *absolute* invariants under the action of the group of proper surface-preserving transformations.

We would like go further into a group of theorems that can be connected to known projective-geometric investigations.

We consider certain special line surfaces, congruences, and complexes, and then assume a standpoint that is different from the usual one. Namely, we will always look at only the part of such a manifold that enters the interior of hyperbolic space, and will likewise calculate those points of the absolute surface itself at which they will be contacted by lines of the manifold in question. The intended structure, which we will hereinafter regard as manifolds of (real) *rays* – with our definition of the word – are the *loci of the principal axes of linear families of threads (linear complexes) in hyperbolic space* ([1]).

We next set down a lengthy sequence of geometric theorems that refer to this structure.

We will call the always-real line surface – or for that matter, the part of such a surface that does not lie outside the sphere – that are composed of the principal axes of a *pencil* of non-coaxial linear complexes a *ray chain*.

There are ∞^7 such chains, each of which consists of the normals to a ray – viz., the *axis* of the chain – and is determined uniquely by any three mutually-distinct rays of it, such as three arbitrary normals to an arbitrary axis. They will then be permuted with each other through the dual-projective and anti-projective transformations, and they will all be equivalent to each other under the transformations of this group. Among them, one finds, in particular, all cylinders of rotation (i.e., cones with improper vertices), and furthermore, all pencils of rays that have their vertices in the interior of the absolute surface, and furthermore, any pair of pencils of rays that has its vertex in the interior of the absolute surface, and also any pair of pencils of rays in a perpendicular plane whose vertex is found on the poles of that plane that lie outside the absolute surface ([2]), and finally, any pencil of rays with improper vertices.

Secondly, we consider the congruence of principal axes of a real *net* (i.e., bundle) of linear complexes (whose individuals can nowhere be arranged into ∞^1 pencils of coaxial complexes). Such a congruence can consist of normals to an individual ray. We shall exclude these ∞^4 congruences, which can be regarded as degenerate forms of general congruences. We would like to call the remaining one *chain congruences.*

There are ∞^8 chain congruences, and they will likewise be permuted transitively amongst each other by the dual-projective and anti-projective transformations. A single chain congruence goes through any four rays, no three of which belong to the normal net

([1]) Cf., Sturm, *Liniengeometrie I*, in which the main facts were considered only in Euclidian space, and the literature that was cited there (pp. 175), which is, unfortunately, unavailable to the author, for the most part.

The ray chain, or for that matter, the line surface that it represents a piece of, is of degree four in the general case; it is the so-called projective generalization of the much-examined *cylindroid*. In the same sense, the chain congruence has, *inter alia*, order and class three, while the chain complex is quadratic.

([2]) Such a pair of pencils of rays is, from what was said, to be regarded as single algebraic (irrreducible) manifold here whose two components are connected by two improper rays.

of a ray. Such a congruence has a single ray in common with the normal net to a ray, in general, but an entire ray chain in common with the normal net to ∞^2 of the rays that are "adjoint" to the congruence. Any two rays of the congruence can be linked by a single ray chain that is contained in the congruence. The axes of these ∞^2 ray chains define a second chain congruence that has an invertible relationship with the first one, and which we can thus call its *reciprocal congruence.* The common normal to any two rays of the one congruence is a ray of the other congruence ([1]). The chain congruence can also be explained as the locus of common normals to the rays of two chains that have a ray in common, and indeed ∞^4 generators are possible. It is, moreover, the locus of the double elements for an involutory dual anti-collineation, and it determines one conversely.

One chain congruence, in particular, is any ray bundle that has its vertex in the interior of the absolute surface, and furthermore, the figure of all rays in a plane, combined with the figure of all rays that perpendicular to that plane ([2]). By contrast, by our definition, a pencil of rays with an improper vertex is not a chain congruence.

Thirdly, we consider the complex of the principal axes of a real spray (i.e., a fourth-order linear system) to be a linear complex whose individual members cannot be distributed into ∞^2 pencils of coaxial complexes. We call one such complex a *chain complex.*

There are ∞^7 chain complexes, and each of them is, as we would like to say, *reciprocal* to a certain ray chain. Namely, it consists in the totality of all normals to the rays of the chain. It then follows that the chain complexes are permuted transitively amongst themselves under dual projectivities and anti-projectivities.

The chain complex contains a "singular" ray, namely, the axis of the reciprocal ray chain. It contains three different kinds of ray chain: ∞^4 chains of the first kind are determined by any two rays of the complex that do not belong to the normal net of a ray of the reciprocal chain. By contrast, ∞^4 ray chains of the second kind are contained in any of these nets, but do not, in turn, contain the singular ray. The intersection of both manifolds can regarded as a third manifold that consists of ∞^3 chains, each of which is determined by two rays of a normal net that is contained in a complex and the singular ray of the complex.

The chain complex contains ∞^3 chain congruences that all have the singular ray of the complex in common with each other. Such a congruence is determined by three rays of the complex, none of which can be the singular ray, naturally, and which also cannot belong to the same normal net.

A chain complex generally has one chain of the first kind in common with the normal net to a ray. In special cases, only the singular ray, or even the entire normal net, belongs to the complex.

The multitude of these line-geometric theorems can obviously be regarded as *new.* All of them, and many others of their kind, can be obtained, however, *with no further*

([1]) The common normals to any two generators of the same kind of a rectilinear surface of order 2 that enters the interior of the absolute surface lie in a chain congruence (without this always being fulfilled completely, moreover), apart for some exceptional cases. In the limiting case of the Euclidian space, this yields the analogously-defined *transversal congruences* of E. Waelsch (Nova Acta Leopoldina, 1888, Bd. 52, no. 6).

([2]) This figure can also be regarded an (analytic) continuum.

assumptions, from a theory ([1]) that has been developed already on the basis of our Theorem III and the following simple remark:

Any ray chain is the image of a one-dimensional (v. Staudt) chain from projective geometry, any chain congruence is the image of a two-dimensional one, and finally, any chain complex is the image of a three-dimensional (so-called degenerate) chain.

We do not have to refer to this, in itself, interesting connection, moreover: The basis for parallel theorems emerges, in fact, by the same considerations.

Ordinary projective geometry escapes some of the stated theorems, due to the nature of things, namely, the invariance of the reciprocal relationships:

Ray chains	Chain complex
Chain congruence	Chain congruence
Chain complex	Ray chain

under contragredient dual-projective transformations.

Here, we break off this summary, which would lead to still more things, in order to consider *Euclidian geometry* in conclusion.

Applications to Euclidian geometry of the kind that would be of interest to the empiricist that we mentioned at the outset can be made in various ways using the relationships that were described.

For example, the theorem that was presented in § 2, by which, a connection exists between (non-synectic) normal congruences in elliptic space and the surface-preserving map of two spheres, immediately implies a method for the integration of the differential equations on which the surface-preserving map depends. The connection between curvature lines and isometric curves of the map that resides within it will be proved quite fruitfully on both sides. As is known, one can represent the spherical and pseudo-spherical geometry in Euclidian geometry in such a way that the concept of angle overlaps with the ordinary concept of angle. The straight lines then appear as circles that intersect a sphere in diameters in the first case and right angles, in the second. Triply-orthogonal systems go to other ones under this map, and special systems of the kind are obtained from the families of parallel surfaces in non-Euclidian space and its normal congruences that are associated with developable surfaces. Special orthogonal systems will be determined by proper-conformal maps between pieces of a sphere or a plane. These maps, and thus also the associated orthogonal systems, belong to each other pair-wise, apart for an exceptional case; they are linked with a third one that is determined by differentiation, etc.

Other applications can be obtained from passage to the limit that was mentioned in the beginning.

For example, the theorem on synectically-related synectic congruences (§ 3) does not generally take on a form that is as simple in Euclidian space. Parallel bundles of various

([1]) We refer to the investigations of chains and general structures of the kind that C. Segre carried out. ("Un nuovo campo di ricerche geometriche," Atti dell'Academia delle Science di Torino, v. 25, 1890; cf., also Math. Ann. **40** (1892), pp. 413, *et seq.*) The two-dimensional chains have already been examined by C. Juel (Acta Math., v. 14, 1890). These worthwhile papers seem to have attracted no attention whatsoever up to now.

normal congruences to developable surfaces enter in place of the synectic congruences. In general, the methods that were developed in § 3 can be applied to Euclidian space with certain, generally inessential, alterations ([1]). In this connection, one also infers new problems that are peculiar to Euclidian geometry. For example, the totality of all transformations of straight lines that permute the normal congruences of developable surfaces amongst themselves do not exhaust the group that defines the limiting case of our group of synectic and anti-synectic transformations. The transformations in question define a much more encompassing group that can, moreover, be represented by explicit formulas, and likewise their subgroup that additionally takes normal congruences of a general kind to other ones.

What seems especially noteworthy to us is the manifold of finite and infinite groups in line space that is defined by *simple* geometric properties that is produced by our considerations. Line geometry has space for an abundance of geometric and analytical theories, only one of which is Plücker line geometry, that have been predominantly stated in the context of algebraic investigations up to now ([2]). The author believes that the construction of such theories, to some of which we have shown a navigable path, could be called an advance in geometry.

Finally, we would like to refer, by way of appendix, to a connection that exists between some of the questions that we generally only brushed against and the geometry of contact transformations. Namely, one has the extraordinarily fruitful and important, due to its relationship with *geometrical optics*, theorem that any transformation of straight lines that takes normal congruences to other ones is one-to-one and invertibly linked with a family of ∞^1 contact transformations that convert parallel surfaces into other ones.

([1]) Cf., the author's *Geometrie der Dynamen*, where related investigations of other things were also carried out.

([2]) The general projective group in four-dimensional space (a group with 24 parameters) can also be mapped into line space in an apparently simple way. One chooses a general or special thread arbitrarily. Any two lines that do not belong to this thread (and are not polar related) then define a single second-degree surface that has a family of generators in the thread. The group in question includes the totality of all transformations that take three lines that belong to the same surface to other ones.

Line geometry yields still other remarkable groups indirectly. For example, one comes to such a group in a four-fold extended manifold when one starts with the (17-parameter) group of all transformations of Euclidian space that convert normal congruences of straight lines into other ones, and then introduces the pencil of parallels as its space element.

[The author has pursued some of the suggestions on Euclidian geometry that were made in the present article further in the meantime. Cf., the preliminary communication "Ein neuer Zweig der Geometrie," Jahresbericht XI, pp. 97-123, Nov, 1901. – One can also compare that with the previous attempts: Joh. Petersen, "Géométrie des droites dans l'espace non-euclidien," Kopenhagener Akademieberichte 1900, pp. 306, *et seq.*]

Appendix to the article: On non-Euclidian and line geometry.

By E. STUDY in Greifswald.

The connection between the theory of conformal maps and the surfaces of curvature zero in non-Euclidian space that was given in the cited article was already known at the time of publication (March, 1900). The theorem implied is found, in a somewhat different form and based in another way, in the German edition of the lectures on differential geometry of L. Bianchi (Leipzig, 1899) on page 640.

We will take this opportunity to explain the extension of Gauss's method of the spherical map to non-Euclidian geometry that was presented there by yet another application.

It is known that Gauss presented three different expressions for the curvature of a surface in his *Disquisitiones generales*, each of which can be employed as the definition of that quantity. The curvature appears once as the limiting value of the ratio of two surface areas, then as the reciprocal value of the product of the principal radii of curvature, and finally, as a bending invariant, it is expressed by the coefficients of the quadratic differential form that serves to represent the arc length element, and by certain derivatives of these quantities with respect to the curvilinear coordinates that are ordinarily denoted by u, v.

The fact that the third of these definitions can be carried over to non-Euclidian space without alteration is self-explanatory. Bianchi proved that the quantities thus-defined can then be likewise expressed through the principle curvature radii that belong to them. By contrast, as it seems, an analogue to the first, and in fact, original, Gaussian explanation for curvature is still lacking up to now. This analogue will now, as we supposed from the outset, also be easily obtained by the method of spherical images. Since they are real only for non-Euclidian spaces of positive curvature, we will *immediately* refer to just that assumption. Moreover, for the sake of ease of notation, we set the known curvature equal to positive unity. Furthermore, we speak only of *analytic* surfaces, and indeed, for the sake of brevity, only *real* ones.

We next recall the characteristic properties of the two spherical images of such a surface. The normal to the surface at a point of regular behavior will be oriented, so it is converted into a "spear;" i.e., its so-called positive direction will be assigned. One then draws two spears that are left-paratactic and right-paratactic to the normal through an arbitrarily-chosen point (or also through two different points), and as is described more precisely in *loc. cit.*, they will be associated with points of two spheres of radius one (in Euclidian space). Therefore, the normal congruence of the surface, and thus also the (oriented) surface itself, can be mapped to a family of ∞^2 point-pairs of the two spheres. These either trace out (namely, on the surface of curvature zero) two analytic curves or they mediate a proper surface-preserving map of the two spheres. The spherical images have no further special properties: Any curve-pair and *any* analytic proper surface-preserving map of the two spheres determine a unique family of parallel surfaces in non-Euclidian space (which can be easily determined by a quadrature); thus, it is naturally excluded that a curve or a point will appear in place of a surface of the family. The character of the map employed will not be influenced by the choice of the auxiliary point that is employed in the construction.

One now imagines a bounded surface patch of area F_1 (whose sign is determined by the positive direction of the surface normal) in a suitably-chosen neighborhood of a point in which the surface considered behaves regularly. A corresponding surface patch on any surface that is parallel to the given one will be defined immediately by the association of common normals, in particular, a surface patch of a certain area F_3 on *the* surface that has the given distance of $\pi / 2$ from it, so it is absolute-correlative to it.

If one now bends the given surface then not only does the surface area F_1 *remain preserved, as is self-explanatory, but also the value of the surface area* F_3 *will be unchanged.*

Furthermore, the sum $F_1 + F_3$ *does not change when one displaces both surfaces in the family of parallel surfaces while preserving its distance* $\pi / 2$.

Moreover, this sum measures the surface area of any of the two spherical images that belong to the family of parallel surfaces (or to their common normal congruence).

Finally, the curvature of the surfaces that are considered to be absolutely-correlative to each other at any pair of associated points, and thus the curvatures of the so-called first and third quadratic differential forms that belong to the given surface, will be represented by the two limiting values:

$$\lim \frac{F_1 + F_3}{F_1}, \quad \lim \frac{F_1 + F_3}{F_3}.$$

As one sees, the concept of the *total curvature* of a surface patch (with the associated consequences regarding geodetic polygons) can be carried over to non-Euclidian space unchanged.

The author has thought about going further into the properties of the spherical images in a paper that is still in preparation. For a suitable analytic representation of the so-called union of surface elements, the foundations of differential geometry of curves and surfaces can be given a common expression. The thoughts that are based in Lie's sphere geometry can be made more precise and developed further in the direction of metric geometry. In that way, the author arrived at, *inter alia*, the insight that the quadratic differential forms that characterize a surface, up to form, by the fundamental theorem of surface theory that Bianchi adapted to non-Euclidian space can be expressed in an essentially *well-defined* way by *linear* differential forms. The coefficients of these forms, which are, moreover, always imaginary in the ordinarily considered cases, are linked by equations that have, in a certain sense, a simpler structure than the equations that were cited by Gauss and Codazzi. It might indeed be considered to be an all-the-more-rewarding problem to develop the general theory of surfaces from the starting point that was indicated, which then offers the opportunity to derive many new theorems, one of which we have communicated here. A corresponding theory is also possible in the limiting case of Euclidian space, moreover.

Greifswald, 10/5/1902.