On motions and transfers

(Parts I and II)

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Foreword

The study of *motions* in space belongs to the circle of ideas surrounding elementary geometry, to the extent that one does not (as in kinematic geometry) follow the positions that a moving body successively assumes, but only observes the initial and final positions, with no concern for the intermediate positions. By contrast, one has the geometric properties of two or more congruent systems. Alongside them, one places the study of *transfers* (as we will say briefly), namely, transformations of space that take a figure, not to a congruent one, but to a mirror-image (i.e., symmetric) figure (*).

More extended investigations of the transfers in three-fold extended space do not exist up to now, as far as I know. However, **Chasles** has devoted a rigorous presentation to just the corresponding transformations of plane geometry, whose theory can also be derived from the theory of motions in space. Those motions are much better known, since the theory of such things has been founded and developed by a series of some of the most distinguished mathematicians; we mention only the names of **Joh. Bernoulli, d'Alembert, Euler, Möbius, and Chasles.**

Möbius gave a momentous impulse to them with his important discoveries in the theory of infinitely-small motions; however, we also have **Chasles** to thank for the first comprehensive, general theory of *finite* motions (**). These investigations have received only part of the admiration that they are due; one finds many of them in the textbooks of mechanics today (***). However, up to the present day, only the theory of the infinitely-small motions has experienced any further development. It was only in recent times that the beautiful papers of **H. Wiener** brought new attention to the theory of finite motions. However, those considerations have a different character from those of Chasles. The two theories still do not define a unified whole.

Meanwhile, since the appearance of Chasles's investigations, several disciplines that are closely connected with the theory of motions have developed or even essentially perfected them, such that it is on that basis that one can take up the subject anew at this point in time.

We mean the general theory of transformation groups, and some special theories of the recent conception of non-Euclidian geometry, the theory of orthogonal substitutions

(*) The term "transfer" seems justified when one refers to plane geometry, in which one can link the word with an intuitive picture. Admittedly, that picture breaks down in space, since an intuition for a four-fold extended manifold that includes ordinary space is not at our command. If one would like to employ only intuitive terminology then one would, perhaps, prefer the word "eversion," which the author has, in fact, proposed. Meanwhile that expression is not applicable to the symmetric transformations of the straight line and the plane without some restrictions, and in any case, it is in those cases that it first seems not at all intuitive. One must then endow entirely corresponding things with different names, which is obviously inconvenient. Therefore, after careful consideration, we have preserved the first-mentioned expression.

(**) "Propriétés géométriques relatives au movement infiniment petit d'un corps solide libre dans l'espace," Comptes rendus **16** (1843), 1420.

"Propriétés relatives au déplacement fini quelconque, dans l'espace, d'une figure de forme invariable," Comptes rendus **51** (1860), pp. 855, 905; **52** (1861), pp. 77, 189, 487.

The (otherwise simple) proofs that were lacking from **Chasles** were extended by **Brisse**, Liouville J. (2) **15** (1870); **19** (1870); (3) **1** (1875). One also finds the main theorems of **Chasles's** theory (in a freer representation) in the book by **Schönflies** on *Die Geometrie der Bewegung* (Leipzig, 1886).

(***) Cf., especially, **Schell**, *Theorie der Bewegung und der Kräfte*, v. I.

and its generalization, the theory of linear transformations of a quadratic form, which is, in turn, closely related to the geometry of reciprocal radii, and finally, the study of the so-called systems of complex numbers.

The mutual connection of the topics that we have touched upon is well-known in some directions; however, large gaps remain to be filled in. Namely, one has exploited it only very little for the individual theories, although some beginnings already exist in regard to it.

The author now intends to set down his thoughts in several treatises, the first two of which will be submitted for publication. The matter shall be divided according to the methods that are applied: Only if one develops each closed circle of ideas, if possible, with the tools that are peculiar to it can one justify the not-be-misjudged diversity of the individual ways of imagining things, despite all of the agreement. In the first section, we will then treat the geometric properties of the motions and transfers in Euclidian space, while are either supported by analysis, or we shall choose a detour through non-Euclidian space or the theory of second-degree surfaces: Those things must temporarily remain in the background, although many times they have pointed to our guiding viewpoint.

I.

On the elementary theory of motions and transfers.

The present section is connected with the circle of ideas that surrounds **Chasles**'s investigation: It includes a new conception of Chasles's theory, as well as extensions and elaborations in certain directions. We must then have many more details to go into than were originally intended.

For example, Chasles presented the theorem that the midpoints of the chords that connect the corresponding points of two congruent point-fields in space lies in a plane that he called the "middle plane" of the two point-fields. He then developed a series of important theorems in which that middle plane played a role. However, for special positions of those points it can happen that the midpoints of the chords do not fill up a plane at all, but only a straight line, or they might even be united into a single point. The definition of the middle plane then became unusable in these cases, which Chasles did not address or properly exclude. It is also impossible to correct Chasles's theorems in such a way that one could call *any* plane through the chord midpoints considered a "middle plane."

In order to establish the domain of validity of Chasles's theorems, one must then carry out a whole new examination of them, and that will be all the more necessary since the theory of general motions is, in part, founded upon just those exceptional cases. Naturally, I say that only in order to justify my own starting point, and not perhaps to rebuke the greater geometers that first created the theory of motions in some truly wondrous works, and could not devote the same attention to all of the details in the wealth of new things. One must once more go back to the very beginning, if only to arrive at a unified plan.

Therefore, the repetition of many known facts seems inescapable, but they will appear in a new context. We have also restricted ourselves in other respects. Only those things will be presented thoroughly that seem especially suited to the problem of shedding some light on certain theorems that stand at the center of the theory. Several topics that must not be missing for the completion of the total picture (e.g., metric relations, congruent pencils of planes and bundles in space, the decomposition of a motion into two unscrewings, et al.) shall be discussed later on.

Apart from the thorough consideration of the exceptional cases, we might emphasize some aspects of the theory that are new in comparison to Chasles's theory:

1. The introduction of the concept of an *unscrewing* as concept that is on a par with that of rotation and is a main component of the theory.

2. A very large part of the properties of the transformations that are denoted by $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, etc.

3. The revelation of the peculiar parallelism that exists between entire series of theorems and that goes back to the principle of duality only indirectly.

4. The law of progression from geometry in domains of rank n to geometry in domains of rank $(n + 2)$ that is generally first suggested in the present section and will feature in many theorems.

5. Finally, the entire theory of transfers in triply-extended space.

At the center of our investigations stands the splitting of a motion or transfer into *two* such transformations with prescribed special properties. Thereby, we shall come, not so much to the foundation of the individual theorems that can result in manifold ways and actually never raise any difficulties, as to an insight into the intrinsic connection that exists between the various specialized theories. Insofar as we shall treat involutory motions and transfers, in particular, our work shall touch upon the investigations of **H. Wiener** ([*]) that were cited. Knowledge of the widely-published theory of motions up to now was very useful to the author. How far the agreement with the theory of transfers can go will await further communications from H. Wiener ([**]).

We must especially emphasize that the following arguments have an *elementary* character, for the most part. The tools of the ancients are completely sufficient to found the most important theorems in the geometry of motion. If we nevertheless often employ concepts from group theory and projective geometry then we shall do that only for the sake of brevity, and not to explain known things anew. In truth, we shall mostly use the theorems of elementary geometry whose proofs do not require the concept of projectivity; e.g., we shall speak of dualistic transformations. Exceptions to that are found at only a few places whose content is indeed required for an overview, but not absolutely necessary.

A prior knowledge of the geometry of motion will not be required for the experienced readers to understand what follows. Still, it would be good if the reader had some familiarity with **H. Wiener**'s theory, especially as far as its simple foundations are concerned. We shall come to speak of those things only in a rather complicated context, and can treat them only quite briefly.

I have found the symbolic notation $x\ \{S\}\ y$ that H. Wiener introduced to say that the object x is taken to the object y by the operation S to be very convenient; it will thus be applied many times. It has the advantage over the usual notation $y = Sx$ or the $(x)\ S = (y)$ that **S. Lie** employed that many such formulas can be chained together sequentially; e.g.:

$$x\ \{S\}\ y\ \{T\}\ z,$$

from which, it will follow that:

$$x\ \{ST\}\ z.$$

It will correspond to a natural advance in thinking when we draw a distinction between the geometry of the line or plane geometry and the geometric properties of a line or plane that lies in space. In the former case, we regard the line or plane as a domain that is closed in itself, from which we would like to fail to emerge, in the sense that it is forbidden for us to leave the usual three-fold extended space. If we then consider, in succession, the geometry of lines, planes, and space – or, in the terminology of the analysts, the geometry of domains of rank two, three, and four, resp. – then we will have the start of a sequence of concepts that extends to infinity. In each of the aforementioned domains – or in domains of rank n, more generally – we consider the motions and transfers, namely, continuous families of transformations, each of which depend upon n

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([*]) "Die Zusammensetzung zweier endlichen Schraubungen zu einer einzigen" and "Zur Theorie der Umwendingen," Sächs. Ber. (1890), pps. 13 and 71. -

([**]) One will find a provisional notice in the Verhandlungen der Gesellschaft deutscher Naturforscher in the year 1890. – The author has addressed the same topic since the Spring of 1890.

$(n - 1) / 2$ parameters. The motions define a group, and the motions and transfers together likewise define a group.

The difference between motions and transfers corresponds to the difference between congruence and symmetry. However, we must remark that there is no difference between congruence and symmetry for figures that are contained in a planar region of rank $n - 1$. In a three-fold extended space, for example, a planar system can be made to coincide with a congruent system by a transfer, as well as a motion, and in fact, in a single way.

We now consider successively the cases that are simplest, and at the same time, the most important to us, for which $n = 2, 3, 4$.

§ 1.

On the motions and transfers on the straight line.

Any motion S on the straight line is a *displacement:* All points will be shifted in the same direction by a segment of constant length. The midpoints $\bar{x}$ of the segments that are bounded by two ordered points x, x' fill up the entire line and define a point sequence $(\bar{x})$ that is congruent to the point sequences (x), (x'). If we let $\mathfrak{T}_1$ denote the transformation that assigns the point $\bar{x}$ to the point x and let $\mathfrak{T}_2$ denote the transformation that assigns the point x' to the point $\bar{x}$ then we will have $\mathfrak{T}_1 = \mathfrak{T}_2$, $S = \mathfrak{T}_1\mathfrak{T}_2 = \mathfrak{T}_2\mathfrak{T}_1$. If we use the third notation T for the transformation $\mathfrak{T}_1 = \mathfrak{T}_2$, then we can also write the last equation as $S = T^2$.

The introduction of the various notations for one and the same transformation might seem pointless. Meanwhile, it will be required by the systems of notation for the higher-rank cases. It will then express a general law that seems unclear and inchoate now, but will become meaningful later on.

The motions on the straight line define a group of *commuting* transformations. That is a peculiarity of the case $n = 2$ that will not be found in the higher-rank cases.

The *transfers* on the straight line are all *involutory*; because of that, we once more glimpse a peculiarity of the case $n = 2$.

A point o will always be fixed by any transfer; the transfer is determined completely by its "center." Any two associated points x, x' will be spaced equally far from the center o.

Any motion S can be replaced with a sequence of two transfers $\{o\}$, $\{o'\}$ (viz., transfers whose centers are o, o', resp.) in ∞^1 ways. The distance between the centers o, o' is equal to one-half the magnitude of the displacement; i.e., one-half the segment xx' through which any point x on the line is shifted by the motion S.

On the basis of that remark, we succeed in constructing the defining data (viz., magnitude of the displacement or center) of the composed transformation of two or more motions or transfers that are performed one after the other.

§ 2.

On the motions in the plane.

Any motion in the plane can be regarded as a rotation around a finite, or even infinite, point.

The rotations around infinitely-distant points and the ∞^2 *translations* define groups by themselves that are invariant subgroups of the group of all motions.

In addition, there are some special motions that must be underscored, namely, the involutory motions or *reversals*, which likewise number ∞^2: They are the rotations with an angle of rotation that equals two right angles or "reflections in the points of the plane."

The reversals alone do not define a group, but the reversals and translations together will again define a group.

We consider the chord xx' – viz., the connecting line of any two points x and x' that are associated with each other by a motion S, as well as the midpoint $\bar{x}$ of such a chord and the perpendicular $\bar{\bar{u}}$ to the chord xx' that is erected at $\bar{x}$, namely, the *normal* to the chord.

The midpoints to all chords xx' will either fill up the entire plane (in the general case) or they will all unite at one and the same point (for the reversals). The normals define the totality of rays through the center of rotation; any such ray is the common normal for ∞^2 chords xx'.

The point at infinity $\bar{\bar{x}}$ on a chord is, in a certain sense, analogous to its midpoint. In general, the points $\bar{\bar{x}}$ fill up the entire line at infinity; the only exceptions are the translations, under which, the point $\bar{\bar{x}}$ will naturally go to one and the same point at infinity.

We further consider the lines that bisect the angle between any two lines u, u' that are associated with each other by S. They exhibit a different sort of behavior.

The *angle-bisector of the first kind* $\bar{u}$ is characterized by the fact that the sequences of associated points on u and u' will produce congruent point-sequences (in the sense of § 1) when projected onto $\bar{u}$. $\bar{u}$ will run through the entire ray-field of the plane when one sets u, u' equal to all possible pairs of corresponding rays. The only exceptions are the reversals: All angle-bisectors of the first kind will coalesce into the line at infinity for them.

Conversely, *angle-bisectors of the second kind* $\bar{\bar{u}}$ arise from symmetric projections of the congruent point-sequences u, u'; $\bar{\bar{u}}$ will always run through the center of rotation. The totality of those lines will then define a pencil of rays, in general. However, the translations are exceptions to that: All angle-bisectors of the second kind will coalesce into the line at infinity for them.

The midpoints of chords and the angle bisectors can be linked to each other quite simply by the following theorem, which seems to have been unnoticed up to now:

Any motion S that is not a reversal can be linked with two commuting similarity transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$ that will produce the motion when performed in succession:

$$\mathfrak{T}_1\mathfrak{T}_2 = S = \mathfrak{T}_2\mathfrak{T}_1 .$$

Namely, let x, x′ be any pair of associated points, and let u, u′ be any pair of associated lines, such that:

$$x\,\{S\}\,x', \qquad u\,\{S\}\,u' .$$

The points x and x′ will then correspond to the midpoint $\bar{x}$ *of the chord xx′ under the transformations* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2^{-1}$*, resp., and likewise the lines u and u′ will correspond to the angle bisector* $\bar{u}$ *of the first kind of the ray-pair u, u′ under the transformations* $\mathfrak{T}_2$ *and* $\mathfrak{T}_1^{-1}$*, resp.* In symbols:

$$x\{\mathfrak{T}_1\}\,\bar{x}\,\{\mathfrak{T}_2\}x', \qquad u\{\,\mathfrak{T}_2\}\,\bar{u}\,\{\mathfrak{T}_1\}u'.$$

If the motion S goes to a reversal then the point transformation $\mathfrak{T}_1$ will degenerate, and $\mathfrak{T}_2$ will be undetermined, as the opposite of a degenerate transformation. If S is a translation then $\mathfrak{T}_1$ will likewise be a translation and coincide with $\mathfrak{T}_2$.

Any motion can be decomposed in ∞^2 ways into two successive rotations, one of which is a reversal. If one decomposes S into a rotation followed by a reversal then the center of rotation will correspond to the center of the reversal under the transformation $\mathfrak{T}_1$. If one decomposes S into a reversal followed by rotation then the center of rotation will correspond to the center of reversal under the transformation $\mathfrak{T}_2^{-1}$. The angle of rotation will be the same each time; it is equal and opposite to the supplement of the angle of rotation of S. In particular, any reversal can be the composition of a reversal and a translation in succession in ∞^2 ways, and every translation can be decomposed into a succession of two reversals. Conversely, when two reversals are composed that will not give a motion, in general, but only a translation; a reversal and a translation will produce a new reversal when they are composed. (Cf., pp. 6, below.)

Any motion can be represented in ∞^1 ways as a succession of two reflections in the lines g_1, g_2, resp. The axes of reflection run through the center of rotation and subtend one-half the angle of rotation. They will then cross at right angles when the motion S is a reversal. If S is a translation then g_1 and g_2 will be parallel to each other and perpendicular to the direction of translation; their separation is equal to the one-half the magnitude of the translation, so it will be equal to one-half of the segment through which any point in the plane is displaced.

It is easy to construct the resultant rotation (i.e., its center and angle) that is the composition of two motions on the basis of these theorems. Only the translations occupy a special position in this, which one can, however, compose with no further analysis.

Any motion belongs to a well-defined one-parameter group of motions whose paths are circles around the center of rotation. Any finite motion will then be "generated" by a well-defined infinitely-small motion, to use the terminology of **S. Lie.**

In addition to these one-parameter groups of motions, a two-parameter group G_2 of commuting similarity transformations is determined by the motion S, to which, the transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ also belong, in addition to S: It is the group of all similarity transformations (without assigning an angle) under which the center of rotation remains fixed. If one associates any point in the plane with any other point then a transformation of G_2 will be determined by that. If one shifts the center of rotation to infinity then our group G_2 will go to the group of all translations.

§ 3.

On transfers in the plane.

Any transfer in the plane can be generated by a reflection in a line m and a translation in the direction of that line that precedes or follows it.

Among those transformations, one can distinguish the ∞^2 involutory transfers, which are the pure *reflections*, for which the translation to be applied will reduce to the identity transformation.

As in § 2, we consider a chord xx', its midpoint (which shall now be denoted by $\bar{\bar{x}}$), and its normal $\bar{u}$ (*).

The midpoints of the chords will always fill up a well-defined line, namely, the axis m of what we previously called a reflection, which will then be called the *center line* of the transfer.

The normals to the chords will either define the totality of all lines in the plane (in the general case) or they will all coalesce into one and the same line, namely, the center line (so, in fact, the transfer would be a reflection).

The points at infinity $\bar{x}$ of the chords will generally fill up the entire line at infinity; only when the transfer is a reflection will they all coalesce into the points at infinity that are perpendicular to the reflection axis.

The angle bisectors of two corresponding lines u, u' will again exhibit a different sort of behavior. The *angle bisector of the first kind* $\bar{u}$ is parallel to the center line m; it will coincide with it when the transfer is a reflection. The *angle bisector of the second type* $\bar{\bar{u}}$ is perpendicular to the center line. The totality of all lines $\bar{\bar{u}}$ always defines a pencil of lines.

As in § 2, the angle bisectors of the first and second kind also differ from each other by the fact that the projections of the corresponding point-sequences along u and u' onto the angle bisector of the first kind are congruent to u, while the projections onto the angle bisector of the second kind are symmetric to $\bar{\bar{u}}$, in the sense of § 1.

A dualistic transformation T is linked with any transfer S of the plane that is not a reflection, and which will generate the transfer when it is performed twice in a row:

$$S = T^2.$$

(*) Here, we appeal to another system of notation from the one in § 2, on grounds that will become clear later on (in § 13).

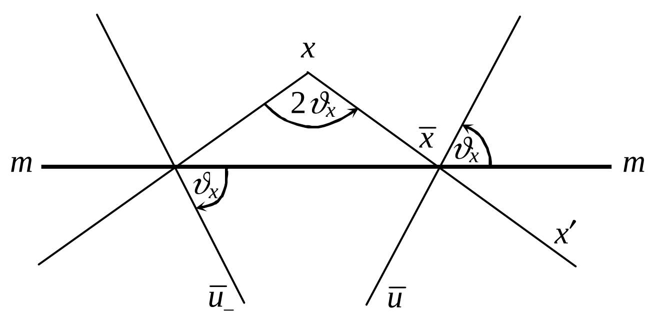
Namely, the point x corresponds to the normal $\bar{u}$ to the chord xx′, and that line will, in turn, correspond to the point x′ under one and the same dualistic transformation T. In symbols:

$$x\{T\}\,\bar{u}\,\{T\}x', \qquad\qquad x\{S\}x'.$$

The transfer can be represented by a rotation around the point and a subsequent reflection in the line $\bar{u}$, or also by a reflection in the line $\bar{u}$, followed by a rotation around the point x'. The angles of the rotations to be applied are equal and opposite in both cases. Thus, either the point x, or the line $\bar{u}$, or the point x' can be chosen arbitrarily. If $\bar{u}$ is perpendicular to the center line m then x and x' will come to lie along m, and conversely. If $\bar{u}$ is parallel to the center line then x and x' will lie at infinity, and conversely. If $\bar{u}$ coincides m then x and x' will fall upon the point at infinity that is perpendicular to m, and conversely.

One can easily find the angle of rotation $2\vartheta_x$ that is assigned to a point x in the plane. Let $\bar{u}_-$ be the line that corresponds to the point x under the transformation T^{-1}, S can be represented by a reflection in $\bar{u}_-$ and a subsequent rotation around x, and likewise by the rotation around x and a subsequent reflection through $\bar{u}$. Both times, $\bar{u}_-$ will overlap with $\bar{u}$. The lines $\bar{u}_-$ and $\bar{u}$ will then have a separation distance that equals x; they will subtend the angle $2\vartheta_x$.

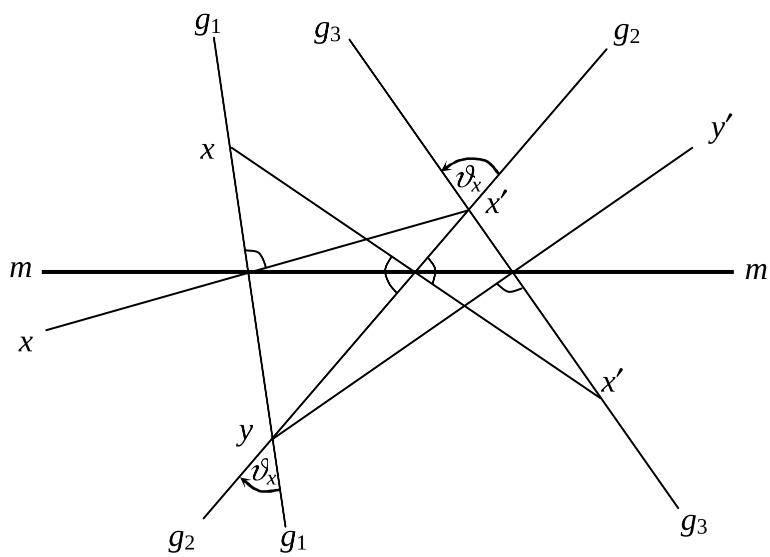
The angle of rotation $2\vartheta_x$ is constant for all points of a parallel to the center line m; points that are equally far from m on opposite sides will have equal and opposite angles of rotation. One constructs the center line m and the associated translation with no further analysis when the transfer S is given by the succession of a rotation and a reflection or a reflection and a rotation. (Cf., the figure.)



If the point x falls along the center line m then the angle ϑ_x will be a right angle. *One can then represent any transfer in ∞^1 ways by a reflection and reversal that follows it.* The distance between the center of rotation and the axis of reflection, when measured in the correct sense, will both times be equal to one-half the magnitude of the translation that yields the transfer S in conjunction with the reflection through m.

Instead of replacing a transfer with a rotation and a reflection, one can also represent it a more symmetrically by *three successive reflections,* and in fact, in ∞^2 ways.

Let g_1, g_2, g_3 be three suitable axes of reflection, so the point of intersection y of g_1 and g_2 will correspond to the line g_3, and the line g_1 will correspond to the point of intersection z' of g_2 and g_3 under the dualistic transformation T. Moreover, g_1 will go through the point x that corresponds to the line g_2 under the transformation T^{-1}, and g_3 will go through the point x' that corresponds to g_2 under T. (Cf., the following figure.)



If the transfer is defined by three successive reflections with axes g_1, g_2, g_3, resp., then one will directly find that the feet of the altitudes that are dropped from y to g_3 and from z' to g_1 are two points on the center line m; along with that line, one likewise knows the associated magnitude of translation.

If the transfer S is a *reflection* then some simple alterations will be introduced into the cited theorems. The angle ϑ_x will equal zero when x is chosen to be outside the center line; it will be undetermined when x is a point of the center line. If one represents S by three reflections then their axes will intersect at a point of the center line; however, the replacement of S with three reflections is possible in ∞^3 ways, as before.

On the grounds of the cited theorem and constructions, it will be easy to *compose* several transfers and motions in every case; i.e., to construct the defining data of the composed transformation. If one would like to perform, e.g., two transfers S_1 and S_2 whose axes intersect at a finite point x in succession then one will represent S_1 most simply by a reflection and subsequent reversal about the point x and S_2 by the same reversal about x, followed by a reflection. The product S_1S_2 will then be a motion that is given in the simplest way by the succession of two reflections, etc.

If decompose a given *motion* S into two successive transfers and demand that only one of them, in particular, should be a reflection then we will come to a new conception of the similarity transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ that were considered in the previous paragraph.

In fact, if we decompose S into a reflection and a second previous (subsequent, resp.) transfer then the center line of that transfer will correspond to the axis of reflection in the transformation $\mathfrak{T}_2^{-1}$ ($\mathfrak{T}_1$, resp.).

§ 4.

Congruent point sequences and point fields in space.

Following the approach of **Chasles**, we pass along the theorems on congruent point sequences on lines and congruent point fields on planes that are equally valid for both theories to the investigation of motions and transfers in space.

In general, let x, x' be corresponding points of two congruent or symmetrically-equal figures in space, so we can consider the *midpoint of the chord* $\bar{x}$, the midpoint of the connecting line xx', and the *normal plane* to the chord xx'; i.e. the plane that is erected perpendicular to the chord xx' at its midpoint. We ask what the locus of the chord midpoints and the locus of the normal planes might be when the corresponding figures are congruent, but distinct, or what amounts to the same thing, symmetrically-equal point sequences or point fields.

We first compose two *congruent point sequences* g, g'.

The midpoints $\bar{x}$ *of the chords* xx' *that belong to two congruent point sequences* g, g' *either fills up a line* $\bar{g}$ *or coalesce into one and the same point.*

One can summarize the properties of the line $\bar{g}$ quite simply and intuitively when one introduces the concept of an *unscrewing*. That is what we call (following the pattern of the term "reversal") a screwing motion whose angle is equal to two right angles, and thus the sequence of a reversal (i.e., a rotation through the angle $2R$) and a translation in the direction of the reversal axis. If we now say: "The point sequence g can be made to coincide with the congruent point sequence g' by an unscrewing around the axis $\bar{g}$" then that will already be based upon the facts that $\bar{g}$ bisects the angle between g and g', the point sequence $(\bar{x})$ is similar to the point sequences (x) and (x'), the chord xx' projects onto $\bar{g}$, yielding segments of equal length, and finally that all of the chords will be perpendicular to $\bar{g}$, as long as *one* of them is (*). The last case actually occurs when the unscrewing reduces to a reversal. Secondly, if all chord midpoints coincide then there will be a whole bundle of lines through the chord midpoints. However, among them, there is only one pencil of axes of such unscrewings that make g coincide with g'; it will be defined by *the* lines of the bundle that cross g and g' at right angles. Among the associated unscrewings, there is only *one* reversal, in general, and only when the carriers of the point sequences g and g' coincide will all of those unscrewings be reversals, in particular.

The normal planes to the chords xx' *that belong to two congruent point sequences* g, *g either fill up pencil of planes* $\bar{\bar{h}}$ *or they coalesce into one and the same plane* (**).

(*) One finds a simple proof in **Schönflies**, *Geometrie der Bewegung* (Leipzig, 1886), pp. 81.

(**) **Möbius** gave a simple proof of this. (Ges. Werke, v. I, pp. 548 and 549).

In regard to the latter theorems, one can confer the remarks that will be made in § 5 (pp. 24, below).

In the first case, g can be made to coincide with g' by a certain rotation around the axis $\overline{\overline{h}}$, which can be finite or infinite; any plane of the pencil $\overline{\overline{h}}$ will be the normal plane to *one* chord xx'. (The pencil $\overline{\overline{h}}$ is projective to the point sequences g, g'.) If one of the chords xx' cuts the axis $\overline{\overline{h}}$ then all of them will; i.e., the rotation will then be a reversal. Secondly, if all normal planes coincide then there will be a whole ray-field of lines that each lie in all normal planes. However, among those lines, there is only a pencil of axes of rotations that make g coincide with g'; they run through the point of intersection of g and g'. In general, there is only one reversal among those rotations. It is only when the carriers of g and g' coincide such that the congruent point sequences lie involutorily on one and the same line that the rotation will be a reversal.

If we further consider two congruent point fields ω, ω' that lie in space in whatever way then we will have the corresponding theorem:

The midpoints $\bar{x}$ of the chords xx' that belong to two congruent point fields ω, ω' will either fill up a plane m, or a line n, or they will coalesce into one and the same point o.

We next consider the general case. The projections of ω, ω' onto m are then congruent. ω can be made to coincide with ω': First of all, by a reflection in the plane m and a prior or subsequent rotation around an axis s that is perpendicular to m, secondly, by a rotation around the line of intersection (ω, m) and a subsequent rotation around s, and thirdly, by the same rotation around s and a subsequent rotation around the line of intersection (m, ω').

In that, we already find that the point field $(\bar{x})$ is affinely-related to the point fields (x) and (x') in such a way its plane m defines the same angle with the planes ω and ω' as the lines (ω, m) and (m, ω') that correspond to ω and ω', resp., and finally that the points of intersection (ω, s) and (ω', s) correspond to each other. The rays of the pencil (m, s) in the plane m are distinguished by the fact that each of them is perpendicular to its associated chord.

We now turn to the second case:

If the chord midpoints $\bar{x}$ that belong to ω, ω' fill up a line n then that line will be the axis of an unscrewing, by which ω will be made to coincide with ω'.

In an arbitrary plane that goes through n, n will be the single axis of an unscrewing that makes one (and consequently, every) point sequence of ω coincide with the corresponding point sequence on ω'; the projection of corresponding figures on ω and ω' will be symmetrically equal to each other on such a plane. However, among the planes through the axis n, one finds a distinguished plane m, namely, the plane that is perpendicular to ω and ω'. In *that* plane, any line that is not parallel to n will be the axis of one-half an unscrewing that makes a certain point sequence g of ω coincide with the corresponding point sequence g of ω'. Corresponding figures of ω and ω' have projections in that plane that are not only symmetric, but also likewise congruent. Furthermore, m, together with ω and ω', will determine two symmetrically-equal spatial systems. The plane m enjoys properties that are entirely similar to those of the plane that is likewise distinguished in the general case. One must observe only that the axis *s* is now parallel to ω and ω' and that the rotations that belong to the lines (ω, *m*) and (*m*, ω') will be reversals.

Finally, in the third case, where ω and ω' go to each other by a *reflection through the point o*, we might denote any plane that is drawn through *o* by *m*: The projections of two corresponding figures of ω and ω' onto any such plane will now be congruent; any of them, together with ω and ω', will determine two symmetrically-equal spatial systems. Any line through *o* will be the axis of an unscrewing that makes a point sequence on ω coincide with the corresponding point sequence on ω'. ω can be made to coincide with ω' by a reflection in the plane *m* and a prior or subsequent reversal around the axis *s* that is perpendicular to *m* at the point *o*. One of the lines through the point *o* is distinguished by the fact that it is the line *n* that is perpendicular to ω and ω'. It is the axis of an unscrewing that takes ω to ω'. In a plane that is laid through it, the projections of the corresponding figures on ω and ω' will be not only congruent, but also symmetric.

The normal planes of the chords that belong to the two congruent point fields ω, ω' *either define a bundle of planes o, or a pencil of planes n, or they coalesce into one and the same plane m.*

In the general case, the point *o* is at first nothing but the point of intersection (*m*, *s*) that was discussed already. It has a position relative to ω and ω' that will determine two symmetrically-equal spatial systems: in one case, with ω, and in the other case, with ω'. Any plane through the point *o* will be the normal plane to a chord xx'; any line $\overline{\overline{h}}$ through the point *o* will be the axis of a rotation that makes a well-defined point sequence *g* on ω coincide with the corresponding point sequence *g* on ω'. The point fields ω, ω', and *m* are dualistically related in the bundle of planes.

If the normal planes that belong to the point-pairs of ω, ω' *define a pencil of planes then* ω *can be made to coincide with* ω' *by a rotation around the axis n.*

If one chooses an arbitrary point on *n* then *m* will be the single rotational axis that goes through that point and makes one (and consequently, any) point sequence *g* on ω coincide with the corresponding point sequence g' on ω'; the chosen points will determine two congruent spatial systems: in one case, together with ω, and in the other, with ω'. However, among the points of the axis *n*, one will find a distinguished point *o*, namely, the one at which *n* meets the planes ω, ω'. That point, together with corresponding points of ω and ω', will define figures that are not only congruent, but also symmetric. Any line through that point will be the axis of a rotation that makes a certain point sequence *g* on ω coincide with the corresponding point sequence *g* on ω'.

Finally, if thirdly the normal planes of the chords of ω and ω' coalesce into a single plane *m* (so ω and ω' go to each other by a reflection in the plane *m*) then any point that is chosen on *m* might be denoted by *o*. Any such point, together with ω and ω', will determine two symmetrically-equal spatial systems; any line of the plane *m* will be the axis of a rotation that makes a point sequence on ω coincide with the corresponding point sequence on ω'. One of the lines in the plane m is distinguished, namely, the line of intersection n of ω and ω'. Any of its points, together with corresponding points of ω and ω', define figures that are not only symmetric, but also congruent.

The theorems that were presented enable us to define the following table of concepts:

*If g and g' are congruent point sequences then we will call the axis of the unscrewing that makes g coincide with g' their **middle ray**.*	*If g and g' are congruent point sequences then we will call the axis of the rotation that makes g coincide with g' their **normal axis**.*

For such structures, one has the theorems:

Two congruent point sequences g, g' have only one middle ray, in general. However, if g and g' go to each other by a reflection through a point then there will be pencil of middle rays that belongs to that point.	*Two congruent point sequences have only one normal axis, in general. However, if g and g' go to each other by a reflection through a plane then there will be pencil of normal axes that belongs to that plane.*

The case in which a middle ray is, at the same time, a normal axis must be emphasized especially. g will then come to coincide with g' by a *reversal* around that axis. If the carriers of g and g' coincide, but the point sequences themselves have opposite directions, then the two aforementioned exceptional cases will occur simultaneously; g can then be taken to g' by ∞^1 reversals whose axes define a pencil of rays that is normal to g, g'.

If we consider corresponding point sequences of two congruent point fields then that will yield:

∞^2 middle rays always belong to the corresponding point sequences of two congruent point fields ω, ω'. *They lie in a plane, in general; however, if ω and ω' go to each other by a reflection through a point then they will define the bundle of rays that is determined by that point.*	*∞^2 normal axes always belong to the corresponding point sequences of two congruent point fields ω, ω'.* *They fill up a bundle of rays, in general; however, if ω and ω' go to each other by a reflection through a plane then they will define the field of rays that is determined by that plane.*

We would, moreover, also like to introduce some special terminology for the plane that was denoted by m above and the point that was denoted by o.

*We call the plane of any pencil of rays that is defined by the middle rays that are associated with the congruent point fields ω, ω' their **middle plane**.*	*We call the point of any pencil of rays that is defined by the normal axes that are associated with the congruent point fields ω, ω' their **central point**.*

Now, we can further say that:

Only a single middle plane belongs to two congruent point fields ω, ω', in general. It is only when ω and ω' go to each other by a reflection through a point that there will be infinitely many of them: Namely, any plane through that point will then be a middle plane of ω and ω'.

Only a single central point belongs to two congruent point fields ω, ω', in general. It is only when ω and ω' go to each other by a reflection through a plane that there will be infinitely many of them: Namely, any point of that plane will then be a central point of ω and ω'.

Since the two aforementioned exceptions cannot occur simultaneously, one will have *three* possibilities: Either there is one middle plane and one central point, or there is one middle plane and ∞^2 central points, or finally, there are ∞^2 middle planes and one central point. The following theorem is true for all three cases:

Any middle plane of the congruent point field ω, ω' is incident with any central point.

For the sake of later applications, we shall infer the following statements from the ones that we made about the middle plane *m* and the central point *o*:

"The point field ω can *always* be made to coincide with the congruent point field ω' by a reflection in the plane *m* and a prior or subsequent rotation around an axis *s* that is perpendicular to the plane *m* at the point *o*."

"If the central point *o* is finite then the point field ω can be made to coincide with the congruent point field ω' by a reflection in the point *o* and a prior or subsequent rotation around an axis that goes through the point *o*."

A peculiar *parallelism* emerges in the majority of the theorems that were given here, which we shall also encounter many times from now on. Midpoints and normals to a chord, unscrewing and rotation, reflection through a point and reflection through a plane, all seem to be interchangeable concepts. However, that parallelism is not complete, as the last two juxtaposed theorems show.

We cannot go further into the basis for the aforementioned remarkable phenomenon here. It goes back to the dualistic character of so-called non-Euclidian geometry and the nature of the passage to the limit by which one descends from that generalized geometry to ordinary geometry.

Along with the examination of the midpoints of the chords xx' that belong to two congruent point sequences g, g' or point fields ω, ω' that was carried out here, one can carry out a corresponding examination of the points at infinity on the chords. That subject shall be treated briefly, since it will not be necessary for us to come back to it from now on.

The points at infinity of the chords xx' that belong to two congruent point sequences g, g' either fill up a line at infinity $\bar{\bar{g}}$ or they coalesce into one and the same point.

The line $\bar{\bar{g}}$ is the line at infinity that is perpendicular to the aforementioned normal axis $\bar{\bar{h}}$. In the exceptional case for which the chords xx' are all parallel to each other, we will have two, or if one so desires, even three possibilities. One can make g coincide with g' either by a translation or a reflection in a plane, or finally, both kinds of transition together are possible.

The points at infinity that belong to the chords xx' to two congruent point fields ω, ω' will either fill up the entire plane at infinity, a line at infinity, or finally, they will coalesce into one and the same point.

The second case, in which the chords xx' are all parallel to a plane ε, can come about in two essentially different and generally equivalent ways (*).

a) The projections of ω and ω' onto the plane ε are congruent (i.e., congruent point fields).

ω can be made to coincide with ω' by a rotation around an axis that is perpendicular to ω, so ω and ω' will meet at a point o that is common to both point fields.

Moreover, ω can be made to coincide with ω' by a reflection in a certain plane through the point o, and a prior or subsequent rotation around the perpendicular that is erected to that plane at the point o.

b) The projections of ω and ω' onto the plane ε are symmetric (i.e., mirror-equal point fields).

ω can be made to coincide with ω' by a screwing motion whose axis is parallel to the line of intersection of ω and ω', and as a result, ω and ω' will meet in a point at infinity that is common to both point fields.

Moreover, ω can be made to coincide with ω' by a reflection through a plane m that is perpendicular to ε and a prior or subsequent translation in the direction of the line of intersection of m and ε.

The following case enjoys the properties of the two cases *a*) and *b*):

c) The planes ω and ω' are perpendicular to the plane ε.

Finally, if the chords xx' are all parallel to each other then one will have, in turn, two (three, resp.) possibilities. ω goes to ω' by either a translation, a reflection in a plane through the line of intersection (ω, ω'), or finally, one can make ω coincide with ω' by either of the two aforementioned processes.

One easily counts up the constants upon which two congruent point sequences or point fields depend when they are found in one of the special positions that were investigated in the present paragraph.

(*) **S. Möbius**, Ges. Werke, Bd. I, pp. 550, *et seq.*, in which the examination was not as thorough as it is here.

§ 5.

Motions in space.

Any motion in space is a screwing motion: It can be composed from a rotation around a certain axis n and a prior or subsequent translation in the direction of that axis. The angle of the rotation is already occasionally called the *screw angle*; it will be denoted by 2ϑ. We shall call the length of the segment that any point of the screw axis advances the *height of the screw;* we denote it by 2η. We consider both quantities, not as absolute numbers, but ones that are capable of taking on positive and negative values whose sign will be decided by the direction and sense of rotation of the axis n. The direction of a line n can be chosen arbitrarily. However, the positive sense of rotation shall be, perhaps, the one that is given by the hands of a clock for an observer that looks upon the point at infinity on the line in the positive direction.

Among the ∞^6 motions in space, we can distinguish the following ones:

The ∞^5 *unscrewings*, which are screwing motions with a screw angle of $2\vartheta = 2R$. These are the motions that will permute the points at infinity involutorily. If an unscrewing is repeated twice in succession then that will yield a translation in the direction of the screw axis through a segment that is equal to twice the height of the screw.

The ∞^5 *rotations,* which are motions for which the height of the screw is equal to zero.

Rotations and unscrewings have in common the fact that ∞^1 points and ∞^1 planes remain fixed by both of them: Rotations fix all of the points of the screw axis and all of the planes that are perpendicular to that line. Unscrewings fix all planes through the screw axis and the point at infinity that is perpendicular to them.

The ∞^4 *reversals*, which are motions that are, at the same time, rotations and unscrewings. They define the totality of all involutory motions; they can also be suitably called "reflections in the lines of space."

Finally, one has *the* ∞^3 *translations*, which are motions for which all of the points at infinity (i.e., all directions) are fixed individually. We consider a translation to be, as a rule, a limiting case of a rotation, and in that sense speak of a well-defined *axis* for the translation; we mean the line at infinity that is perpendicular to the direction of translation by that. However, a translation can also be regarded as a limiting case of a screwing motion whose screw angle vanishes; in that way of looking at things, the axis of the translation will refer to any line that is parallel to the direction of translation. The translations define a three-parameter group of commuting translation by themselves, which is an invariant subgroup of the group of all motions. The individual translation will be well-defined when one associates a given point in space with any other one.

The fundamental theorem that one finds at the summit of everything – namely, that every motion is a screwing motion – is obtained in the simplest way by the path that was taken by **H. Wiener** when one expresses a motion as two successive reversals. On the other hand, that very representation of a motion by reversals is a special case of several (four distinct ones, in total) remarkable, general ways of representing a motion by special motions. In connection with the developments of the previous paragraph, here we shall discuss a subset of the relevant theorems that define a closed totality in their own right, but for the sake of overview, we shall first given them without proof. The proofs (to the extent that such things are even required by the exposition in § 4 at all), together with several details whose immediate specification here would tend to perturb matters, shall then find their places in the next paragraph.

We first once more consider the midpoint $\bar{x}$ and normal plane $\bar{\bar{u}}$ of a chord xx', as well as the middle ray and normal axis of two corresponding point sequences g, g', and the middle plane and central point of two corresponding point fields ω, ω'. In order to avoid becoming too long-winded, we shall look at only those structures that do not coincide with the one that is associated with them by the motion S.

The midpoints of the chords will generally fill up all of point space. Only when the motion is a screw will they all lie along a line, namely, the axis of the screw.	The normal planes of the chords will generally fill up all of plane space. Only when the motion is a rotation will they all go through a line, namely, the axis of rotation.

The following theorems read precisely the same way:

The middle rays will generally fill up all of line space. Only when the motion is a screwing motion will they all meet a line, namely, the unscrewing axis.	The normal axes generally fill up all of line space. Only when the motion is a rotation will they all meet a line, namely, the rotational axis.

However, a third pair of theorems that relate to the middle plane and central point of two congruent point fields does not read analogously, and cannot be expressed as simply. We shall leave it to the reader to deduce a suitable formulation for them from what follows.

Along with the midpoints of the chords, we shall briefly consider the points at infinity of the chords and then the two planes that bisect the angle between two corresponding planes u, u'.

The points at infinity $\bar{\bar{x}}$ of the chords xx' fill up the entire plane at infinity. Only the rotations are excluded, and among them, the translations, once more. For a general rotation, the points $\bar{\bar{x}}$ will lie on the line at infinity that is perpendicular to the rotational axis, but for a translation they will all coalesce into the point at infinity of the direction of translation.

The two angle bisectors of a plane-pair u, u' exhibit a different sort of behavior.

The projections of the *angle bisectors of the first kind* $\bar{u}$, which lie in the planes u and u', onto mutually-related planar systems define planar systems that have equal areas and are affinely-coherent (*gleichstimmig-affine*), while the projections of the *angle bisectors of the second kind* $\bar{\bar{u}}$ will have equal areas, but will not be affinely-coherent. That is, if one projects, say, two corresponding circles in u and u' onto the plane $\bar{u}$ then one will get two ellipses that are affinely related to each other, have the same area, and the same sense of rotation, but when one projects them onto the plane $\bar{\bar{u}}$, the ellipses that result will have the opposite sense of rotation.

The angle bisector of the first kind generally runs through the entire plane-space when one allows the plane-pair u, u' to assume all possible positions. Only the unscrewings are excluded: The planes are all perpendicular to the screw axis for them. They then define only a pencil of (parallel) planes.

The angle bisector of the second kind is parallel to the screw axis. The totality of these planes then defines a bundle of planes, in general. However, if the motion is a rotation then the planes $\bar{\bar{u}}$ will define only a pencil; they will all go through the axis of rotation. If the motion is a translation, in particular, then all of the planes $\bar{\bar{u}}$ will coincide with the plane at infinity.

Now, a simple connection between the chord midpoints and the angle bisectors of the first kind, which is similar to the one in the geometry of the plane (§ 2), can be exhibited by the following theorem:

Two commuting affine transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$ are linked with any motion S that is not an unscrewing, and when they are performed one after the other, that will generate the motion S:

$$\mathfrak{T}_1\mathfrak{T}_2 = S = \mathfrak{T}_2\mathfrak{T}_1 .$$

Namely, if x, x′ are any pair of associated points, and u, u′ are any pair of associated planes such that:

$$x\{S\}x', \qquad u\{S\}\, u'$$

then the points x and x′ will correspond to the midpoint $\bar{x}$ of the chord xx′ under the transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2^{-1}$; likewise, the planes u and u′ will correspond to the angle bisector of the first kind $\bar{u}$ of u and u′ under the transformations $\mathfrak{T}_2$ and $\mathfrak{T}_1^{-1}$. In symbols (*):

$$x\{\mathfrak{T}_1\}\,\bar{x}\,\{\mathfrak{T}_2\}\, x', \qquad u\{\mathfrak{T}_2\}\,\bar{u}\,\{\mathfrak{T}_1\}\, u'.$$

However, when S goes to an unscrewing, the point transformations that are denoted by $\mathfrak{T}_1$ and $\mathfrak{T}_2^{-1}$ will degenerate. The transformation $\mathfrak{T}_2$ will then be undetermined, as the inverse of a degenerate transformation. Likewise, $\mathfrak{T}_2$ and $\mathfrak{T}_1^{-1}$ will degenerate, when regarded as plane transformations, and the plane transformation $\mathfrak{T}_1$ will be determined. The motion S can no longer be generated in stated way now.

A theorem that is entirely similar to the last one is connected with the theorems on the right above (pp. 18):

A dualistic transformation T is linked to any motion S that is not a rotation, and when it is applied twice in succession, it will generate the motion S:

$$S = T^2.$$

(*) The transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ probably first appeared in **Chasles** (Bull. de Férussac, Sect. I, t. 14, 1831), and then in **Rodrigues** (1840). However, both authors knew only the property of the formula that is presented on the left. **Chasles** first considered (1843) the transformation *T*, but without emphasizing the property that is expressed by the equation $S = T^2$, which is admittedly closely-related. We place special weight upon these theorems, since they directly mediate the connection with the theory of quadratic forms.

Namely, let x, x once more be any pair of corresponding points, so the point x will be associated with the normal plane $\bar{\bar{u}}$ of the chord xx', and that plane, in turn, will then be associated with the point x' under one and the same dualistic transformation T. In symbols [(*), prev. page]:

$$x \{T\} \bar{\bar{u}} \{T\} x'.$$

At the same time, the lines g and g' of any ray-pair naturally correspond to the associated normal axis, and the planes ω, ω' will correspond to the associated planes of the associated central point under the transformations T and T^{-1}.

If the motion S goes to a rotation then the transformations T and T^{-1} will degenerate, and the representation of the motion S that was stated in the theorem will be unusable.

In addition to the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T^{-1}, we have yet another remarkable dualistic transformation to record, which is an involutory transformation:

Any motion S that is either an unscrewing or a rotation is linked with the dualistic transformation $\mathfrak{W}$ of a null system whose principal axis is the axis of the screw S.

Under that transformation, one has a reciprocal correspondence between: Midpoints and normal planes to a chord xx', middle ray and normal axis of two associated points sequences g, g', and the middle plane and central point of two associated point fields ω, ω'.

An especially important role is played by the linear complex that is linked with our null system, which is then the locus of all lines that go through a chord midpoint $\bar{x}$ and lie in the associated normal plane $\bar{\bar{u}}$. For that reason, it shall bear a special name: We would like to call it the "middle complex" of the motion S. We might express its characteristic properties in two ways:

The middle complex is the locus of all middle rays that are perpendicular to their associated chords.

The middle complex is the locus of all normal axes that are cut by their associated chords.

The middle complex is the locus of all lines that are simultaneously middle rays and normal axes for the same pair of associated point sequences g, g'.

That definition does not depend upon whether the null system $\mathfrak{W}$ is or is not an invertible, dualistic transformation. If $\mathfrak{W}$ degenerates then the same thing will be true for the middle complex. In each of the two given cases, it will reduce to the totality of all lines that meet the screw axis of the motion S. Under a translation, in particular, the middle complex will consist of all lines that are perpendicular to the direction of translation. When no misunderstanding will arise, it might be permissible for us to represent the middle complex by the same sign $\mathfrak{W}$ as the null system that it is coupled with.

The cited theorems have a close connection with the representation of the motion S by certain special motions. By considering them, we shall arrive at not just a deeper conception of them, but also an essential extension of the knowledge that we have obtained up to now.

Any motion S can be generated in ∞^4 *ways by a rotation and a subsequent unscrewing. The rotational axis is associated with the unscrewing axis under the affine transformation* $\mathfrak{T}_1$.	*Any motion S can be generated in* ∞^4 *ways by an unscrewing and a subsequent rotation. The rotational axis is associated with the unscrewing axis under the affine transformation* $\mathfrak{T}_2^{-1}$.

It will suffice to give a thorough explanation for the theorem on the left.

If the motion *S* is not itself an unscrewing then one can choose either the rotational axis *g* or the unscrewing axis $\bar{g}$ (i.e., the middle ray of *g*, *g′*) at will; the rotation and unscrewing are then determined by that. Only the line at infinity can take the form of the axis of rotation or also naturally the axis of the unscrewing.

However, when *S* is an unscrewing, the following peculiarities will arise:

If one takes the rotational axis *g* to be a finite line that does not cross the screw axis *n* of *S* perpendicularly then the associated rotation will reduce to the identity transformation, $\bar{g}$ will coincide with *n*, and the associated unscrewing will coincide with *S*. However, if *g* is perpendicular to *n* then the associated rotation will be completely undetermined. The axis $\bar{g}$ of the rotation unscrewings that correspond to the rotation around *g* define a pencil of rays whose vertex lies on *n*, and whose plane is perpendicular to *g*. Furthermore, any line at infinity can now appear as the axis of rotation *g*. The associated translation will be undetermined. The unscrewing axes $\bar{g}$ that are associated with the individual translations define a pencil of parallel rays to which the axis *n* belongs, and whose plane is perpendicular to the plane that contains *g*. However, if *g* itself is perpendicular to *n*, in particular, then the axes $\bar{g}$ will all coincide with *n*.

Conversely, one can choose the axis $\bar{g}$ of the unscrewing arbitrarily only from among *the* lines that meet the axis *n*. The associated unscrewing will be undetermined in all cases. If *n* and $\bar{g}$ subtend a finite angle then all of the associated rotational axes *g* will define pencil of parallel rays that are perpendicular to *n* and $\bar{g}$. If *n* and $\bar{g}$ are parallel, but not identical, then the associated rotations will be translations whose directions belong to the plane ($\bar{g}$, *n*). Finally, if $\bar{g}$ coincides with *n* then either *g* will be the line at infinity that is perpendicular to *n*, and the associated rotation will be a translation in the direction of *n*, or *g* will be an arbitrary line in space and belong to the identity transformation.

The last double theorem stands alongside the following theorem:

Any motion S can be generated in ∞^4 *ways by two successive rotations. The axis of the first rotation corresponds to the axis of the second rotation under the dualistic transformation T.*

In general, one can choose either of the two rotational axes to be finite or at infinity arbitrarily. An exception to that is defined only by the case in which *S* itself is a rotation. Here, if one takes one of the rotational axes that we speak of to be a line that does not meet the axis n of S (when S is a translation, it will be a line that is not perpendicular to the direction of translation) then the associated rotation will reduce to the identity transformation, and the other rotation will coincide with the S itself. However, when the one rotational axis meets the line n, the associated rotation will be undetermined; the axis of the other rotation will then belong to a well-defined pencil of rays that also contains the line n itself. Finally, if the chosen rotational axis coincides with S then one will have to choose the associated rotation to either be a rotation that is different from S – in which case, the second rotational axis will likewise coincide with n – or the rotation that belongs to the given axis is S itself – in which case, the axis of the other rotation will be completely undetermined.

We will arrive at no-less-remarkable representations of a motion that have a special character from the representations that were discussed here when we demand that one of our special motions reduces to a reversal.

We might preface that with a remark that that relates to an entirely arbitrary group of such operations that can be ordered pair-wise as opposites. If S_1 and S_2 are any two operations of such a group then the products S_1S_2 and S_2S_1 will be on an equal status within the group, or when expressed otherwise:

If $S = S_a \cdot S_b$ then one can also set $S = S'_b \cdot S_a$, and $S = S_b \cdot S'_a$, where S_a and S'_a, and S_b and S'_b have an equal status inside the group.

We will now point out the special cases of this immediately enlightening general theorem when emphasize the following facts:

When an unscrewing, together with a prior (subsequent, resp.) rotation, generates the motion S, the same unscrewing will also generate the motion S, together with a subsequent (prior, resp.) rotation.

The angle of rotation will be the same in both cases.

When a rotation, together with a prior (subsequent, resp.) unscrewing generated the motion S, the same rotation will also generate the motion S, together will a subsequent (prior, resp.) unscrewing.

The height of the screw will be the same in both cases.

If a rotation, together with another prior (subsequent, resp.) rotation, generates the motion S then the same rotation will also generate the motion, together with a subsequent (prior, resp.) rotation that has the same angle of rotation as the first one.

These basically self-explanatory remarks contain the explanation for the fact that the structure in the following theorems that is denoted by $\mathfrak{W}_1$ can be described in various ways.

Any motion S can be generated in ∞^3 *ways by a rotation and a subsequent unscrewing.*

Any motion S can be generated in ∞^3 *ways by an unscrewing and a subsequent rotation.*

The locus of the axes of unscrewing is the middle complex $\mathfrak{W}$. *The locus of the rotational axes is, in general, likewise a linear complex* $\mathfrak{W}_1$ *whose principal axis is, in turn, the screw axis of S.*

We come to this theorem in two different ways: First, we start with the representation of a motion by a rotation and an unscrewing and demand that the unscrewing should reduce to a reversal. Secondly, we start with the representation of a motion by two rotations and demand that the one rotation must be a reversal. We will then have the following special cases to distinguish:

1) When the motion *S* is an unscrewing, but not, at the same time, a rotation, the middle complex $\mathfrak{W}$ will reduce to the screw axis *n*, as we said. However, the rays of the complex $\mathfrak{W}_1$ will be the lines in space that are perpendicular to *n*, and thus, the lines that meet the line at infinity that is perpendicular to *n*.

2) If the motion *S* is a rotation, but not, at the same, an unscrewing, the two complexes $\mathfrak{W}$, $\mathfrak{W}_1$ will consist of the lines that meet the rotational axis *n*. If *S* is a translation, in particular, then one will encounter the peculiarity that the any rotation that yields the motion *S*, together with an unscrewing, will be itself a reversal.

3) If the motion *S* is a reversal, so it is both an unscrewing and a rotation, then the last part of our theorem will suffer an *exception*. Namely, the rotation that is called left on the left, say, will either reduce to the identity transformation – so its axis will be a completely undetermined line in space – or it will be a proper rotation – so its axis will meet the unscrewing axis *n* at right angles. The rotations that belong to a finite angle of rotation will then no longer define a complex, now, but only a ray system of order one and class one.

When we go back to the question of expression a motion by a reversal and an unscrewing then we will raise our representation by one level of generality when we demand that the rotations that entered into the last theorem should reduce to reversals in any case. That will then imply:

Any motion S that is not a translation can be generated in ∞^2 *ways by two reversals in succession.*

The locus of reversal axes is a ray system of order one and class one that is the intersection of the complexes $\mathfrak{W}$, $\mathfrak{W}_1$, *namely, the normal system to the axis of the screwing motion S.*

Any two associated reversal axes will subtend one-half the screw angle ϑ *and will cut out one-half the height of the screw* η *from the screw axis n of S.*

However, when the motion *S* is a translation, as was remarked, the rotations in the last theorem will be reversals, in their own right; we thus get another theorem in regard to the translations:

Any translation can be represented in ∞^3 *ways by two reversals in succession.*

The locus of the reversal axes is the linear complex that is determined by the rotational axes at infinity.

Any two associated reversal axes run parallel to each other. Their separation distance is one-half the magnitude of translation in direction and absolute value.

The important construction of the composition of several motions that was found by **Halphen** and **Burnside** is founded upon the last theorems.

The path by which we arrived at this theory is neither as short nor as convenient as the one by which **H. Wiener** arrived at his own theory. Our way of looking at things would not be advisable then when one is dealing with just the composition of motions. Meanwhile, the general properties of a motion from which we started are not of lesser interest than its representation by reversals.

Moreover, as might be remarked incidentally, one can exhibit a connection between the two theories in another way. Namely, one can prove the theorems that were placed at the summit of § 4 and are the foundations of our further developments very easily with the help of reversals. If one ignores the chronology of events then one can also regard Chasles's theorem as a further development in the theorem of reversals.

In the theorems that were summarized here, it was, above all, the unscrewings and rotations, and secondly, the reversals and translations that defined the exceptional cases. It deserves to be said that those special transformations can assume a special position that emerges exquisitely in a comprehensive, analytical treatment of the subject. Without going deeper into the easily-explained geometric details, we would still like to emphasize the following:

The sequence of points $x, x', x'', x''', \ldots$ that is defined by the formula:

$$x\{S\}x'\{S\}\,x''\{S\}\,x''' \ldots$$

will not lie in a plane under a general motion S, and the sequence of planes u, u', u'', u''', … that is defined by the formula:

$$u\ \{S\}u'\{S\}u''\{S\}u''' \ldots$$

will go through one and the same point just as rarely.

The points x, x', x'', x''', …will lie in a plane only when S is a rotation or an unscrewing, and the planes u, u', u'', u''', …will all go through a point in only those cases.

The aforementioned points will all lie along a line only when S is a reversal or a translation, and all of the aforementioned planes will belong to the same pencil only in that case, as well.

Naturally, we have assumed a sufficiently general choice of the first point x and the first plane u.

§ 6.

Continuation: The general case of a motion in space.

In the previous paragraph, we summarized a series of the most important theorems on motions in space, while considering the exceptional cases completely. We would now like to explain the intrinsic connection between those theorems, along with various considerations that are connected with that. However, we would like to restrict ourselves in that to general case, in order to not drift too far afield. *We then expressly assume here that the motion S is either a rotation or an unscrewing.*

One immediately convinces oneself that with that assumption, the chord midpoints will, in fact, be all of point space, the normal planes of the chords will likewise fill up all of plane space, and that it will be precisely the unscrewings and rotations that define the exceptional cases.

If we now consider three points x, x', x'' that are related by:

(1) $$x \{S\} x' \{S\} x'',$$

as well as the midpoints $\bar{x}$ and $\bar{x}'$ of the chords xx' and $x'x''$, resp., and the normal planes $\bar{\bar{u}}$ and $\bar{\bar{u}}'$, resp. to those chords then we will have directly:

(2) $$\bar{x} \{S\} \bar{x}' \qquad \text{and} \qquad \bar{\bar{u}} \{S\} \bar{\bar{u}}' .$$

We can now define two affine transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$ by the formulas $x\{\mathfrak{T}_1\} \bar{x} \{\mathfrak{T}_2\}x'$, and a dualistic transformation T by the formula $x\{T\} \bar{\bar{u}}$. We then find directly that:

(3) $$x\{\mathfrak{T}_1\} \bar{x} \{\mathfrak{T}_2\}x'\{\mathfrak{T}_1\} \bar{x}' \{\mathfrak{T}_2\}x'',$$

(4) $$x\{T\} \bar{\bar{u}} \{T\}x'\{T\} \bar{\bar{u}}' \{T\}x'',$$

which imply the formulas $S = \mathfrak{T}_1\mathfrak{T}_2 = \mathfrak{T}_2\mathfrak{T}_1 = T^2$. Since $\bar{x}$ and $\bar{\bar{u}}$ are always incident, a null system $\mathfrak{W}$ will be defined by the assignment $\bar{x} \{\mathfrak{W}\} \bar{\bar{u}}$; one will then have:

(5) $$\bar{x} \{\mathfrak{W}\} \bar{\bar{u}} \{\mathfrak{W}\} \bar{x} ;$$

that immediately implies the remarkable theorem:

The two affine transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be composed from the dualistic transformations T and $\mathfrak{W}$:

(6) $$\mathfrak{T}_1 = T\,\mathfrak{W}, \qquad \mathfrak{T}_2 = \mathfrak{W}\,T.$$

When the dualistic transformations T and $\mathfrak{W}$ are applied repeatedly in succession, they will generate a group of infinitely many discrete collinear (in particular, affine) and dualistic transformations whose composition will be expressed by the two symbolic equations:

(7) $$\mathfrak{W}^2 = 1, \qquad T^2\mathfrak{W} = \mathfrak{W}T^2.$$

We now extend our figure by adding a pair of corresponding point sequences g, g' that shall contain the points x and x', resp. The middle ray $\bar{g}$ of g and g' then goes through the chord midpoint $\bar{x}$, and the normal axis $\bar{\bar{h}}$ lies in the normal plane $\bar{\bar{u}}$. That implies the further formulas:

$$g\{\mathfrak{T}_1\} \bar{g} \{\mathfrak{T}_2\}g', \qquad g\{T\} \bar{\bar{h}} \{T\}g',$$

(8)

$$\bar{g}\ \{\mathfrak{W}\}\ \bar{\bar{h}}\ \{\mathfrak{W}\}\ \bar{g}$$

From the theorems of § 4, the point sequence g can be made to coincide with g' by an unscrewing around the axis $\bar{g}$, as well as by a rotation around the axis $\bar{\bar{h}}$. The motion S can then be generated, firstly, by performing a certain rotation around the axis g and then an unscrewing around the axis $\bar{g}$, secondly, by the same unscrewing around $\bar{g}$ and a subsequent rotation around the axis g, thirdly, by a rotation around the axis g and subsequent rotation around $\bar{\bar{h}}$, and finally, by a rotation around $\bar{\bar{h}}$, followed by a rotation around g'.

Since any of the four lines g, g', $\bar{g}$, $\bar{\bar{h}}$ can be regarded as an entirely arbitrary line in space, and since each of the aforementioned decompositions of S is determined completely by that choice, all of those decompositions can be carried out in ∞^4 ways.

We will now obtain a decomposition of S into motions of a special character when demand that either the unscrewing around the axis $\bar{g}$ or the rotation around the axis $\bar{\bar{h}}$ reduces to a reversal. Both possibilities imply that $\bar{g}$ must coincide with $\bar{\bar{h}}$ in such a way that their common line must go through the point $\bar{x}$ and lie in the plane $\bar{\bar{u}}$ or that it must be a guiding line of the null system $\mathfrak{W}$, i.e., a ray of the middle complex. The locus of the associated rotational axes g and g' will then be naturally a linear complex $\mathfrak{W}_1$. We can then complete a theorem that was stated in the previous paragraph (pp. 22) as follows:

One and the same null system $\mathfrak{W}_1$ will emerge from the null system $\mathfrak{W}$ by the transformations $\mathfrak{T}_1^{-1}$, $\mathfrak{T}_2$, T^{-1}, T.

The guiding rays of $\mathfrak{W}_1$ are the axes of all rotations that generate the motion, in conjunction with a prior or subsequent reversal.

$$(9)\qquad \begin{cases} \mathfrak{W}_1 = \mathfrak{T}_1\mathfrak{W}\mathfrak{T}_1^{-1} = T\mathfrak{W}T^{-1}, \\ \mathfrak{W}_1 = \mathfrak{T}_2^{-1}\mathfrak{W}\mathfrak{T}_2 = T^{-1}\mathfrak{W}T, \end{cases} \qquad (\mathfrak{W}_1^2 = 1).$$

If we let v and v' denote the two planes through the points x and x', resp., that emerge from the normal $\bar{\bar{u}}$ to the chord xx' by the transformations $\mathfrak{T}_1^{-1}$ and $\mathfrak{T}_2$ (*) then those same planes will emerge from the chord midpoint $\bar{x}$ by the transformations T^{-1} and T. We can then now add the following formulas to (3) and (4):

$$(10)\qquad v\{\mathfrak{T}_1\}\,\bar{\bar{u}}\,\{\mathfrak{T}_2\}v'\{\mathfrak{T}_1\}\,\bar{\bar{u}}'\,\{\mathfrak{T}_2\}v'',$$

$$(11)\qquad v\ \{T\}\ \bar{x}\ \{T\}\ v'\{T\}\ \bar{x}'\ \{T\}\ v''.$$

(*) In § 5, we used the notation $\bar{\bar{u}}$ for the normal to the chord xx', and not the notation $\bar{v}$ that is more relevant to the present context, on grounds that will be explained later on.

The plane v' is then the normal to the chord $\bar{x}\,\bar{x}'$, while $\bar{\bar{u}}$ and $\bar{\bar{u}}'$ are perpendicular to the other two sides $\bar{x}\,x'$ and $x'\bar{\bar{x}}'$ of the isosceles triangle $\bar{x}\,x'\,\bar{x}'$ at the endpoints $\bar{x}$ and $\bar{x}'$. However, an important theorem that was pointed out in the previous paragraph will follow from this: The plane v' is one of the two planes that bisect the angles between $\bar{\bar{u}}$ and $\bar{\bar{u}}'$. [Cf., the addendum on pp. 63]

The points x, x', x'' are the null points of the planes v, v', v'', resp., in the null system $\mathfrak{W}_1$. We now extend our figure, in turn, when we add the null points p, p', p'', resp., of those planes in the null system $\mathfrak{W}$. One will then have that, e.g., p' is the foot of the perpendicular that is dropped from $\bar{x}$ or $\bar{x}'$ to v', and, at the same time, the midpoint of the chord $\bar{x}\,\bar{x}'$. We then have the formula:

$$(12)\qquad \begin{cases} x\{\mathfrak{W}_1\}v, \text{ etc.}, \quad p\{\mathfrak{W}\}v, \text{ etc.}, \\ p\{\mathfrak{T}_2\}\bar{x}\{\mathfrak{T}_1\}p'\{\mathfrak{T}_2\}\bar{x}'\{\mathfrak{T}_1\}p'', \\ p\{\mathfrak{T}_1\}\bar{p}\{\mathfrak{T}_2\}p'\{\mathfrak{T}_1\}\bar{p}'\{\mathfrak{T}_2\}p''. \end{cases}$$

One deduces the first half of the following theorem from this:

The null system $\mathfrak{W}$ will be taken to one and the same null system $\mathfrak{W}_2$ by both of the transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2^{-1}$, and its principal axis is, in turn, the axis of the screwing motion S.

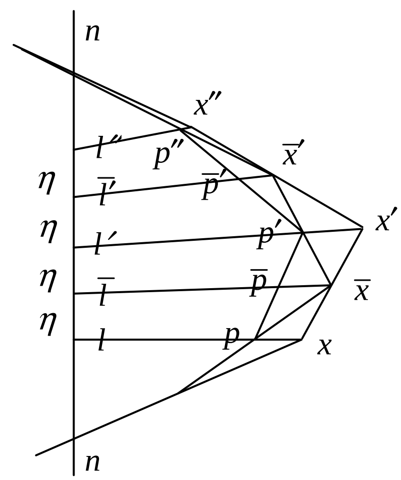
$$(13)\qquad \begin{cases} \mathfrak{W}_2 = \mathfrak{T}_1^{-1}\mathfrak{W}\mathfrak{T}_1 = \mathfrak{W}T\mathfrak{W}T^{-1}\mathfrak{W}, \\ \mathfrak{W}_2 = \mathfrak{T}_2\mathfrak{W}\mathfrak{T}_2^{-1} = \mathfrak{W}T^{-1}\mathfrak{W}T\mathfrak{W}, \end{cases} \qquad (\mathfrak{W}_2^2 = 1).$$

In fact, the plane $\bar{\bar{u}}$ corresponds to the point $\bar{p}$ in the null system $\mathfrak{W}_2$. When we go on to the second half of the theorem, whose validity is immediately self-evident, we remark:

The null systems $\mathfrak{W}_1$ and $\mathfrak{W}_2$ will be switched with each other by the dualistic transformation of the null system $\mathfrak{W}$. In symbols:

$$(14)\qquad \mathfrak{W}\mathfrak{W}_1 = \mathfrak{W}_2\mathfrak{W}, \qquad\qquad \mathfrak{W}_1\mathfrak{W} = \mathfrak{W}\mathfrak{W}_2.$$

If we now consider all rays of the pencil (x, v), and thus, all guiding rays of the complex $\mathfrak{W}_1$ that go through x or lie in v – in particular, *the* line l that is, at the same time, a guiding ray of the complex $\mathfrak{W}$ (i.e., the connecting line of the points x and p) – and we correspondingly call the connecting lines $\bar{x}\,\bar{p}$ and $x'p'$, $\bar{l}$ and l', resp., then we will come to the representation of S by two successive reversals: In the first place, S can be generated by the reversal around l and a subsequent reversal around $\bar{l}$, and secondly, by the reversal around $\bar{l}$, followed by a reversal around l'. It then follows from this that l, $\bar{l}$, l' will meet one and the same line n – viz., the screw axis of S – at right angles and separated by the same distance η, such that l' and $\bar{l}$, just like $\bar{l}$ and l', will subtend the same screw angle ϑ, etc. At the same time, one will get the theorem that the linear ray complexes $\mathfrak{W}$, $\mathfrak{W}_1$, $\mathfrak{W}_2$ intersect in one and the same ray system of order one and class one, namely, the normal system of the screw axis *n*, which will then be the common principal axis of the three null systems $\mathfrak{W}$, $\mathfrak{W}_1$, $\mathfrak{W}_2$. For now, we shall not go into the metric properties of the transformations *T*, $\mathfrak{W}$, etc., which are naturally linked with this, or the associated rotations and unscrewings. In the next paragraph, we will make use of only *one* known, simple theorem of that kind, which one might read about in **Reye**'s *Geometrie der Lage* (Section 2, Lecture 10).



§ 7.

Groups of collinear and dualistic transformations that are linked with a motion in space.

We will arrive at a deeper understanding of the theory that was developed in §§ 5 and 6 when we examine the totality of all motions that have the same screw axis, instead of the individual motion *S* that we have directed our attention to, up to now. Thus, certain more general transformations that are likewise determined by the given screw axis will be worthy of consideration, in their own right.

We first restrict ourselves to the general case, and thus assume that the screw axis is finite.

All screwing motions around a finite axis n define a two-parameter, continuous group G_2 *of commuting transformations.*

Just as illuminating is the theorem:

A one-parameter, continuous group G_1 of affine transformations is determined by the axis n, namely, the group of all affine transformation that fix the normal system to the axis n.

One will obtain the general transformation of G_1 when one associates a point x in space with another one x^* in such a way that the connecting line xx^* meets the screw axis n at a point o, and that the ratio $ox : ox^*$ has a positive or negative value λ that is independent of the position of x.

Aside from the identity transformations, the groups G_1 and G_2 intersect in only the reversal $\mathfrak{U}$ around the axis n.

The transformations of G_1 and G_3 collectively generate a three-parameter, simple-transitive group G_3 of commuting affine transformations that transform the normal system to the axis n like the screwing motions of the group G_2 .

One can associate any finite point in space that does not lie on the axis n with another point with the same property arbitrarily; the most general transformation $\mathfrak{t}$ of G_3 is determined in that way, and in fact uniquely. It can obviously be put into the form:

$$\mathfrak{t} = B\,\mathfrak{r} = \mathfrak{r}\,B, \tag{1}$$

as long as B means a transformation of G_2, and $\mathfrak{r}$ means a transformation of G_1, and in fact that will be possible in two different ways. If the one decomposition of $\mathfrak{t}$ into a transformation of G_2 and a transformation of G_1 is represented by (1) then the other decomposition will be $\mathfrak{t} = (B\,\mathfrak{U})(\mathfrak{U}\;\mathfrak{r})$.

A family H_1 of ∞^1 null systems $\mathfrak{w}$ is determined by the line n: viz., the totality of all null systems that have the axis n for their principal axis.

These ∞^1 dualistic transformations $\mathfrak{w}$ define a group, together with the transformations of G_1 .

The first part of the statement follows from the metric properties of the null system $\mathfrak{W}$ that were mentioned in the previous paragraph. In order to prove the second half of the theorem, we again let $\mathfrak{r}$ be any transformation of G_1 . $\mathfrak{W}\mathfrak{r}$ will then be a dualistic transformation that associates any point with a plane that goes through it, and thus, another null system; as one sees directly, it will be a null system of the family H_1:

$$\mathfrak{w} = \mathfrak{W}\mathfrak{r} = \mathfrak{r}^{-1}\,\mathfrak{W}. \tag{2}$$

At the same time, that implies a remarkable property of our group G_1, H_1: Any transformation of G_1 will commute with the inverse of any transformation of H_1 ; in symbols:

$$\mathfrak{r}\;\mathfrak{w} = \mathfrak{w}\mathfrak{r}^{-1}, \qquad \mathfrak{w}\;\mathfrak{r} = \mathfrak{r}^{-1}\mathfrak{w}. \tag{3}$$

The special case of $\mathfrak{r} = \mathfrak{U}$ deserves special attention: The null system $\mathfrak{w}$ will be associated with the family H_1 in pairs by the reversal $\mathfrak{U}$, such that two associated null systems will always be so-called conjugates (i.e., null systems in involutory position); two such null systems, together with $\mathfrak{U}$, will determine a group of three commuting involutory transformations whenever two of those transformations yields the third one when they are performed in succession.

Since the null system $\mathfrak{W}$ and the transformations $\mathfrak{r}$ will go to themselves under all screws with the axis *n*, it will follow that:

If one composes any transformation $\mathfrak{w}$ of the family H_1 with all of the transformations of the group G_2 then that will produce a family H_2 of ∞^2 dualistic transformations. The families G_2 and H_2, in turn, define a group, and in fact a group of commuting transformations.

In fact, if we let *S* and *B* denote any two screws around the axis *n* then it will follow directly that:

$$S \cdot B\,\mathfrak{W}\mathfrak{r} = SB \cdot \mathfrak{W}\mathfrak{r} = B\,\mathfrak{W}\mathfrak{r} \cdot S,$$
$$S\mathfrak{W}\mathfrak{r} \cdot B\,\mathfrak{W}\mathfrak{r} = SB = B\,\mathfrak{W}\mathfrak{r} \cdot S\,\mathfrak{W}\mathfrak{r}\,.$$

There are ∞^1 families H_2 , corresponding to the ∞^1 null systems $\mathfrak{w} = \mathfrak{W}\mathfrak{r}$. If we combine all of these families together then we will find:

If one composes all transformations of the family H_1 with all transformations of the group G_2, or any transformation $\mathfrak{w}$ of H_1 with all transformations of the group G_3 then that will produce a family H_3 of ∞^3 dualistic transformations t. They will define a group, together with the transformations of G_3 .

The general transformation of the family H_3 reads:

$$t = B\,\mathfrak{w} = \mathfrak{w}\,B. \tag{4}$$

We now see how the transformations of the group G_3, H_3 behave under the transformations of that group itself. We might next complete a theorem that was already pointed out in part, by applying the terminology that was used by **S. Lie**:

The group G_2 of the screws around the axis n is a distinguished subgroup of the group G_3, H_3 . In addition, the group G_1, the group G_1, H_1, and naturally also the group G_3 itself, are invariant subgroups.

If we then subject the transformations of the family H_3 to the transformations of G_2 then they will be fixed. If we subject them to the transformations of G_1 then they will be permuted with each other by a one-parameter group, and they will be permuted with each other in exactly the same way when we subject them to the transformations of G_3 .

Moreover, the transformations of H_3 will be permuted with each other under a one-parameter (discontinuous) group when we subject them to the transformations of the extended group G_1, H_1, and they will be permuted amongst themselves in precisely the same way by the transformations of G_3, H_3 .

The transformations of the family H_3 do not generally commute; only the transformations of one family H_2 will commute with themselves. The family H_2 will be permuted amongst itself by the transformations of G_1 and likewise the transformations of G_3 ; moreover, it will be permuted amongst itself under the transformations of G_1, H_1, and likewise the transformations of G_3, H_3 .

The ∞^1 groups G_2, H_2 are then subgroups of the group G_3, H_3 that have the same status.

The following theorem is now important in our present context:

The transformations of the group G_3 can be associated with pairs $\mathfrak{t}_1$, $\mathfrak{t}_2$ in a well-defined way.

Any two elements of an associated pair will be permuted with each other under all transformations of the family H_3:

$$\mathfrak{t}_1\, t = t\, \mathfrak{t}_2\,, \qquad t\, \mathfrak{t}_1 = \mathfrak{t}_2\, t\,. \tag{5}$$

Moreover, $\mathfrak{t}_1^{-1}\mathfrak{t}_2$ and $\mathfrak{t}_1\,\mathfrak{t}_2^{-1}$ are (inverse) transformations of the group G_1; finally, when $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are performed in succession that will yield a transformation of G_2, namely, a motion.

Conversely, one can decompose any screwing motion around the axis n into two associated transformations of G_2 in ∞^1 ways.

In fact, if one sets t equal to any well-defined transformation of H_3 and takes, perhaps, the first of formulas (5) to be the defining equations of the associated transformations $\mathfrak{t}_1$, $\mathfrak{t}_2$ then one will find directly that:

$$\mathfrak{t}_1 = S^{1/2}\, \mathfrak{r}, \qquad \mathfrak{t}_2 = S^{1/2}\, \mathfrak{r}^{-1}, \tag{6}$$

as long as $S^{1/2}$ and $\mathfrak{r}$ mean transformations of G_2 and G_1 , resp.; one then easily infers the remaining relations that were given in the theorem from this:

$$\mathfrak{t}_1^{-1}\mathfrak{t}_2 = \mathfrak{r}^{-2},\ \mathfrak{t}_1\,\mathfrak{t}_2^{-1} = \mathfrak{r}^2, \qquad \mathfrak{t}_1\mathfrak{t}_2 = \mathfrak{t}_2\mathfrak{t}_1 = S.$$

If S is given then $S^{1/2}$ will naturally be only doubly determined; if $S^{1/2}$ is one value then $S^{1/2}\, \mathfrak{U}$ will be the other one. One easily recognizes the geometric meaning of the transformations $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Let xx' be any chord, $\bar{x}$, its midpoint, and let $\bar{l}$ be the perpendicular that is dropped from $\bar{x}$ to the axis n. Furthermore, let $r = r'$ be the distance between the points x and x' on the axis n, and let ρ be the distance from a point x^* on the line l to the axis n. One will then have $x\{\mathfrak{t}_1\}x^*\{\mathfrak{t}_2\}x'$, $\mathfrak{t}_1\mathfrak{t}_2 = S$. The ratio $r : \rho$ will then have a value λ that is independent of the position of the chord xx'. If one takes $\lambda = 1 / \cos \vartheta$ or $\lambda = -1 / \cos \vartheta$ according to the choice of motion that $S^{1/2}$ refers to then the first part of the theorem will follow:

The transformations $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *that were treated in* §§ 5, 6 *are a pair of associated transformations of the group* G_3 .

Moreover, as was said, any two inverse transformation of G_1 *belong together; however, any transformation of* G_2 *– i.e., any screwing motion around the axis n – will be associated with itself.*

Finally, we deduce the following theorem from formula (4):

Any transformations of the family H_3 *will produce a transformation of* G_2 *– namely, a motion – when performed twice in succession.*

Conversely, any screwing motion around the axis n can be represented in ∞^1 *ways by repeating a dualistic transformation that belongs to the family* H_3 *that is determined by n.*

One will find all transformations t of H_3 that satisfy the equation $t^2 = S$ when one associates any normal l to the axis n with the normal $\overline{l}$ that corresponds to l under the two transformations $S^{1/2}$, so a point of l will now be assigned to a plane of $\overline{l}$; t will be determined completely in that way.

Among the transformations of the family H_3 *that generate a given screwing motion S, one finds the transformation T that was considered in* §§ 5, 6, *in particular.*

In fact, if , as we have done up to now, we let $\mathfrak{W}$ denote the null system that belongs to the middle complex of S, and we let $\mathfrak{R}$ denote the transformation of G_1 that satisfies the two equations:

(7) $$\mathfrak{T}_1 = S^{1/2}\,\mathfrak{R}, \qquad \mathfrak{T}_2 = S^{1/2}\,\mathfrak{R}^{-1}$$

then we will find directly that:

(8) $$T = S^{1/2}\,\mathfrak{R}\mathfrak{W} = \mathfrak{W}\,S^{1/2}\,\mathfrak{R}^{-1};$$

at the same time, that will imply anew the theorem that was already proved in a different context:

Not only will the null system $\mathfrak{W}$ *belong to the family* H_1 *that is determined by a screwing motion S of the group* G_2*, but also the null systems* $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *that were discussed in* §§ 5, 6.

On the basis of formulas (9) and (13) of § 6:

(9) $$\mathfrak{W}_1 = \mathfrak{R}^2\mathfrak{W} = \mathfrak{W}\mathfrak{R}^{-2},$$

(10) $$\mathfrak{W}_2 = \mathfrak{W}\,\mathfrak{R}^2 = \mathfrak{R}^{-2}\,\mathfrak{W}\,.$$

If we no longer think of the axis n as being given, as we have up to now, but a motion S of the group G_2 then we will come to the same results, in general. The axis n, and thus, the entire system of groups G_1, G_2, G_3, G_1H_1, G_2H_2, G_3H_3, will then be determined by S. However, special behavior will occur when S goes to a *translation*, since the axis n will then be undetermined. In that case, our construction will yield ∞^1 different decompositions of S into two associated affine transformations $\mathfrak{t}_1$, $\mathfrak{t}_2$, and likewise ∞^3 dualistic transformations t that will generate S when they are performed twice in succession. Among the ∞^2 groups G_2 that contain a given translation, one will find ∞^1 groups, in particular, that consist of nothing by translations. In that case, the group G_3 will be the group of all translations, but the transformations of the family H_3 will degenerate.

We might extend the considerations that were just discussed in yet another direction, by asking when the group G_2 consists of *all* motions that commute with a certain transformation S of G_2.

It is easy to decide the conditions under which two motions S and B will commute. Let S first not be a translation, so the axis of the screw B must go to itself under S; i.e., the axis of B will either be identical with the screw axis of S, or it will be the line at infinity that is perpendicular to it, or S will be an unscrewing, and the axis of S will meet the axis of B. If that happens at a point at infinity then B will be a translation, and S and B will not commute; if it happens at a finite point then S and B will be reversals whose axes intersect perpendicularly. If S is a translation then B will likewise either be a translation or one will come back to the case that was treated already by switching S and B. One will then have the theorem:

If two motions S and B commute then either they will be translations, or they will both be screwing motions around the same axis n, or finally, S and B will be reversals around two axes that intersect at right angles.

In the last case, the composed transformation will be a new reversal around a third axis that is perpendicular to both of them.

The last theorems now imply the answer to the question that was posed before.

If S is either a reversal or a translation then there will be a family of ∞^2 motions B that commute with S; they define the continuous group G_2. However, if S is a reversal then a second family of ∞^2 motions B (reversals) will come about that will once more define a group, together with G_2. Finally, if S is a translation then there will be ∞^4 motions B that commute with S, which define a continuous group, namely, the group of all screwing motions with the same axis direction [Cf, § 9, pp. 39, 2)].

§ 8.

The chord complex. Infinitely-small motions.

We now return to the considerations of § 6, in order to specialize in another direction the decomposition that was made there of a motion *S* into an unscrewing and a rotation or two rotations.

Here, as in § 6, we assume that *S* is either a rotation or an unscrewing.

One immediately convinces oneself that the totality of chords xx' defines a manifold of ∞^3 lines, namely, a (special tetrahedral) quadratic ray complex that can just as well be defined to be the locus of the planes that correspond to lines of intersection, or the locus of lines that are cut by the lines that are associated with them by S^{-1} or *S*. We would like to call that complex the *chord complex* of the motion *S*. Without going further into the otherwise remarkable properties of that complex (*), we would like to pose the question of what sort of peculiarities that the rotations and unscrewings that were treated in § 6 will take on when one of the lines that were denoted by g, g', $\bar{g}$, $\bar{\bar{h}}$ is a ray of the complex.

If we let, say, the line $\bar{g}$ that goes through the point $\bar{x}$ coincide with the chord xx' then the lines g, g', and $\bar{\bar{h}}$ will, at the same time, be rays of the chord complex. That is obvious for g and g'. However, $\bar{\bar{h}}$ belongs to the chord complex as the middle ray of the intersecting corresponding point sequences $(v, \bar{\bar{u}})$ and $(\bar{\bar{u}}, v')$. *S* can be generated: Firstly, by an unscrewing around $\bar{g}$ and a prior rotation around g or a subsequent rotation around g'. Secondly, by an unscrewing around $\bar{\bar{h}}$ and a prior rotation around $(v, \bar{\bar{u}})$ or a subsequent rotation around $(\bar{\bar{u}}, v')$. Thirdly, by a rotation around $\bar{\bar{h}}$ and a prior rotation around g or a subsequent rotation around g'. Finally, by a rotation around $\bar{g}$ and a prior rotation around (v, u) or a subsequent rotation around $(\bar{u}, v')$. (Cf., also § 4, pp. 12) Thus, $\bar{g}$ will be perpendicular to $(v, \bar{\bar{u}})$, $\bar{\bar{h}}$, $(\bar{\bar{u}}, v')$, and likewise, $\bar{\bar{h}}$ will be perpendicular to g, $\bar{g}$, g'. We then come to the following theorem, which includes an extension of the considerations that were presented in § 5, 6.

The motion S can be represented in ∞^3 ways by an unscrewing and a prior or subsequent rotation in such a way that the unscrewing axis and rotational axis intersect.

Moreover, the motion S can be decomposed in ∞^3 ways into two successive rotations whose axes cross at right angles.

The locus of rotational axes, like that of the unscrewing axes, is the chord complex in either of those cases.

That ray complex will then go to itself under each of the affine transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$ in such a way that any two associated rays will intersect, and it will go to itself under

(*) Cf., **Schönflies**, *Geometrie der Bewegung*, chap. 3, § 7. (pp. 109).

each of the dualistic transformations T, $\mathfrak{W}$ in such a way that any two associated rays cross at right angles.

The rays of our complex that lie in the plane $\bar{\bar{u}}$ – namely, the connecting chords of corresponding points of the lines (v, $\bar{\bar{u}}$), ($\bar{\bar{u}}$, v') – envelope a parabola whose vertex tangent is the middle ray $\bar{\bar{h}}$ of the point sequences (v, $\bar{\bar{u}}$) and ($\bar{\bar{u}}$, v'), and whose focal point is the point x. The principal axis of the parabola is the line $\bar{l}$ that was considered in a § 6, which was an angle bisector of (v, $\bar{\bar{u}}$) and ($\bar{\bar{u}}$, v').

We shall not go further into the special circumstances at this time. It might suffice to stress that the chord complex will decompose into two special linear complexes under the transition to an unscrewing or a rotation – namely, the secant system of the screw axis and the secant system of the line at infinity that is perpendicular to it – and that one must regard as chords in the narrow sense, in the first case, only the secants of the unscrewing axis, and in the second case, the lines that are perpendicular to the rotational axis. A special position is therefore assumed once more by the reversals and the translations.

Let a finite axis n be given and any non-degenerate null system $\mathfrak{W}$ that has the line n for its principal axis. Let $\bar{x}$ be any point that does not lie on n, and let $\bar{\bar{u}}$ be its null plane relative to $\mathfrak{W}$; finally, let s be the perpendicular to the plane $\bar{\bar{u}}$ that is erected at the point $\bar{x}$. If we now choose any point-pair x, x' on s whose midpoint is $\bar{x}$ then we can define a screwing motion with the axis n by the formula $x\{S\}x'$ whose middle complex is the given complex $\mathfrak{W}$, and whose chord complex contains the ray s. If we choose another point x^* on s then the formula $x\{\mathfrak{t}\}x^*$ will define a transformation $\mathfrak{t}$ of the group G_3 that was discussed in the previous paragraph. If we now subject the point-pair x, x' to all transformations of G_3 then we will obtain all point-pairs y, y' that are associated with each other by the motion S (viz., $y\{S\}y'$); all rays of the chord complex will then emerge from s by the transformations of G_3 . If we further subject the point-pair x, x^* to all transformations of G_3 then we will likewise obtain all point-pairs y, y^* that satisfy the condition $y\{\mathfrak{t}\}y^*$.

We can consider the ray s that is employed in this construction to be an arbitrary line in space. Namely, if s is given then one will find the point $\bar{x}$ directly as the foot of the shortest connecting line $\bar{l}$ of s and n; $\bar{\bar{u}}$ will then be the plane of the axis $\bar{l}$, which is perpendicular to s; however, the null system will be also known with that.

We can then formulate the following theorems:

Let n and s be any two finite lines in space that either intersect or cross at right angles. If one then subjects the line s to all transformations of the group G_3 that is determined by the axis n then all rays of a certain quadratic complex will arise from s. It will be the chord complex for a continuous family of ∞^1 motions that all have the axis n for their screw axis, and which belong to one and the same middle complex.

The quadratic ray complex is, moreover, the locus of the connecting lines of corresponding points and the lines of intersection of corresponding plane for a continuous family of ∞^3 transformations of the group G_3 (to which, those ∞^1 motions belong).

In addition, when we partially invert our theorem and complete it, we will find that:

A given linear complex is the middle complex of ∞^1 *motions that all have the same chord complex.*

Those motions define a continuous family (but not a group), in general. Only when the given complex consists of lines that cut a finite axis n will the family of associated motions, namely, into the rotations and unscrewings around the axis.

The unscrewings will drop out when one shifts the axis n to infinity; one will once more obtain a continuous family of motions that do not define a group.

As is known, the special case of that theorem that is most meaningful is one that was discovered by **Möbius**, which we might formulate as:

Any linear complex is linked to an infinitely-small motion whose middle complex is that complex, and conversely.

It is not our intention to devote an in-depth presentation of the sufficiently-known theory of infinitely-small motions; we refer to the aforementioned treatise of Chasles and to the interesting work of **Mannheim, Ball, Schönflies**, and others that is connected with it. In order to ease the transition, and likewise to prepare for our own further considerations, the following might be mentioned:

If we go from a finite motion S to an infinitely-small one (perhaps by establishing the middle complex) then the dualistic transformation T will approach the null system $\mathfrak{W}$. In the limit, the points that we denoted by x, $\bar{x}$, x' will coincide, and likewise, the lines g, $\bar{g}$, g', the lines (v, $\bar{\bar{u}}$), ($\bar{\bar{u}}$, v'), and its middle ray, and finally, the planes v, $\bar{\bar{u}}$, v'. If one then represents the infinitely-small motion by two successive rotation then their axes will correspond reciprocally in the null system $\mathfrak{W}$; the associated rotations are themselves infinitely-small, and therefore commute (when one looks at only first-order quantities).

One can think of any infinitely-small rotation as being coupled with a *force*, or a *line segment*, in the **Grassmann** sense, when one measures out a segment along the rotational axis that is proportional to the angle of rotation in the positive direction, as determined by the sense of rotation. If one represents any small rotation as a sequence of two unscrewings then one will see that rotations whose axes intersect will be combined when one adds the associated forces or line segments geometrically. That will yield the representation of an infinitely-small motion by the geometric sum of two forces or line segments, and the very important theorem that **Möbius** discovered:

Infinitely-small motions will be combined according to the same rules as systems of forces.

Any infinitely-small motion will generate a *one-parameter group* of motions by "infinite repetition" whose paths are helices, circles, or straight lines. The ratio of the height of the screw 2η to the screw angle 2ϑ is constant for all transformation of such a group.

One can easily find all infinitely-small motions that generate a given finite motion *S* or all one-parameter groups to which *S* belongs.

If *S* is not a rotation then one will obtain infinitely-many discrete groups that all belong to the screw axis of *S*. If *S* is a rotation, but not a translation, then there will be only one infinitely-small motion that generates *S*. Finally, if *S* is a translation then one will get infinitely-many discrete families of ∞^2 one-parameter groups that each contain *S*. All of those families will intersect in the one-parameter group of translation that *S* belongs to.

§ 9.

Continuous groups of motions.

It is easy to find all groups of motions that are generated by infinitesimal transformations, not only by the methods of **S. Lie**, but also with the tools of elementary geometry. Since we will have to speak of those groups occasionally in the next section, we will carry out the solution of the stated problem here briefly.

One addresses all possible ways of combining the one-parameter groups of motions that were discussed in the previous paragraph into continuous families such that every family again defines a group.

We first determine the subgroups of the group of *motions in the plane*, while remarking that the presence of two infinitely-small motions will not imply the presence of all motions in a group of motions when the midpoint of just one of them also lies at a finite point. There are then only three types of subgroups:

1) The group of all rotations around a finite point (∞^2 groups with the same status).

2) The group of all translations in a given direction (limiting case of the previous one, ∞^2 groups).

3) The group of all translations (a two-parameter invariant subgroup of commuting transformations).

Secondly, if we consider the group of all *rotations around a fixed point in space* then we will likewise find that there is only one type of (real) subgroup, namely, the group of all rotations around a fixed axis through that point. There will again be ∞^2 such groups with the same status.

Now, in order to find the subgroups of the *motions in space*, we observe that an isomorphic group G' will emerge from any such group G when one replaces any transformation *S* of the group with a rotation S' around a finite fixed point whose axis is parallel to the axis of *S* and whose angle is equal to the angle of rotation of *S*. From what was just said about the rotations around a fixed point, we can thus distinguish three main cases:

A) The directions (i.e., the points at infinity) are transformed by zero (i.e., not at all).

B) They are transformed by a one-parameter family.

C) They are transformed by a two-parameter family.

A) leads to the directly-specifiable group of translations.

B) The group G in question consists of screwing motions whose axes are all parallel to each other. We consider that a plane ε that is perpendicular to the screw axis will be transformed by the permutation of its normals through the transformations of G, by way of an isomorphic group G'' that contains at least one proper rotation. From what was previously said about the motions in the plane, the aforementioned isomorphic group G'' will then be either one-parameter or two-parameter. If it is one-parameter then G itself will be either one-parameter or two-parameter, since every transformation of G can be composed of a transformation of G'' and a translation that is perpendicular to the plane ε. We then obtain two groups: A one-parameter group of screwing motions around an axis n, and the two-parameter group of all screwing motions around the axis n.

If G'' is three-parameter then G will be three-parameter or four-parameter. We first assume that G is three-parameter, and let G_1 denote any one-parameter subgroup of G that corresponds to a group G_1'' of proper rotations in G''. We will then already obtain ∞^3 transformations of G when we subject G_1 to all of the ∞^2 transformations of G itself that correspond to translations in G''. However, since every screwing motion goes to itself under translations in the direction of its axis, one will come to the same ∞^3 transformations when one subjects the group G_1 to all translations that are parallel to the plane ε. It follows from this that G consist of all screwing motions whose axes are perpendicular to the plane ε, and whose screw heights 2η have a given ratio with the screw angle 2ϑ. – If we assume that G is four-parameter then G will consist of nothing but all screwing motions whose axes are perpendicular to the plane ε.

C) Any prescribed direction will be associated with at least one one-parameter group of G whose axis has that direction. We next assert that G cannot contain any infinitely-small translation. Namely, the appearance of one would imply the presence of all remaining translations, since the ∞^2 directions (i.e., their points at infinity) would go to each other under the transformations of G in the most general way. However one would then directly obtain a rotation for any prescribed direction of axis, and thus, all motions in space, moreover, with aid of the translations.

We now consider two one-parameter groups G_1 and G_1' of G whose axes n and n' cross or intersect at right angles. If we subject G_1' to an unscrewing that is contained in G_1 then a new group G_1'' will arise whose axis n'' is parallel to n'. If the line n'' were different from n' then one could derive an infinitely-small translation from G_1' and G_1''; n'' would then coincide with n'. However, that would be possible only if n and n' intersected, and if G_1 were the group of all rotations around the axis n. Naturally, G_1' would also be a group of rotations then, and G itself would be the group of all rotations around the point of intersection of n and n'.

With that, we have found all (real) groups of motions that are generated by infinitesimal transformations. We summarize them briefly. [Cf., pp. 63].

Six-parameter groups.

1) The group of all motions.

Four-parameter groups.

2) The group of all screwing motions with parallel axes. ∞^2 groups with the same status.

Three-parameter groups.

3) The group of all rotations around a finite point. ∞^3 groups with the same status.

4) The group of a screwing motions with parallel axes, for which the ratio $\eta : \vartheta$ of the screw height to the screw angle has a constant value k. ∞^3 groups, of which, ∞^2 have the same status, correspond to the different values of k. Any of them will be invariant under a group of type 2). For $k = 0$, one will get:

4*b*) The group of all rotations with parallel axes or the group of all rotations around a point at infinity, which is a degeneration of type 3).

5) The group of all translations, which is then a group of commuting transformations. It is invariant in 1) and 2); it is likewise a limiting case of the groups of type 4) that corresponds to the value $k = \infty$.

Two-parameter groups.

6) The group of all screwing motions around a finite axis n. ∞^3 groups of commuting transformations that have the same status.

7) The group of all translations that are perpendicular to a given direction, which is a degeneration of type 6). ∞^2 groups with the same status. Each of them is invariant under a group of type 2) and under all of its three-parameter subgroups.

One-parameter groups.

8) The group of all screwing motions around a finite axis n, for which the ratio $\eta : \vartheta$ has a constant value k. ∞^5 groups, ∞^4 of which have the same status. For $k = 0$, one gets:

8*b*) The group of all rotations around an axis n.

9) The group of all translations in a given direction, which is a degeneration of 8) or 8*b*). ∞^2 groups with the same status. Any of them will be distinguished in a group of type 2).

It might be left to the reader to find the subgroups of these individual groups, make it clear how they are permuted with each other, how the points of space behave under them, etc.

§ 10.

Transfers in space.

Every transfer can be represented by a reflection in a plane and a prior or subsequent rotation around an axis that is perpendicular to that plane. In general, that will happen in only one way, but in special cases, there will be ∞^2 ways. [cf., pp. 563].

We call the plane of the reflection the *middle plane*, the axis of the rotation, the *rotational axis*, and its point of intersection with the middle plane, the *midpoint* of the transfer. As the theorem says, these structures are all present just once, in general. For points of the middle plane, the transfer will reduce to a motion (viz., a rotation). If one performs a transfer twice in succession then one will not obtain a general screwing motion as a result, but only a rotation.

Corresponding to the cases in which the transfer goes to a translation, the identity transformation, or a reversal for the points of its middle plane, we will now have the following special types of transfers in space to point out:

The ∞^5 transfers with midpoints at infinity. These are characterized by the fact that they transform the points of the plane at infinity involutorily. In general, no finite point will be fixed by a transfer with a midpoint at infinity; however, those transformations include:

The ∞^3 reflections in the planes in space. All points of the middle plane are individually fixed by such a reflection. Any plane through the midpoint is a middle plane, and any line through the midpoint is a rotational axis; the associated rotation is a reversal.

The reflections in the points, together with the translations, define a *group* that is an invariant subgroup of the group of all motions and transfers in space.

The reflections in the planes and the reflections in the points collectively make up the totality of all involutory transfers in space; as we already remarked, both of them have similar properties. Along with the theorem that was formulated to begin with, one poses yet another one that reads similarly, but generally does not have the same domain of validity:

A transfer can, in general, be represented by a reflection through a point and a prior or subsequent rotation around an axis that goes through that point. In fact, that will happen in either one or ∞^2 ways.

The only exceptions are the transfers that have no finite midpoint; they cannot be represented in that way.

In fact, for a general transfer S, the point of reflection will be the midpoint of the transfer, but the axis of the applied rotation will again be the aforementioned rotational axis of the transfer. Let 2ϑ be the rotational angle, but let $2\vartheta'$ be the angle of the rotation that one must apply when one employs the reflection in the middle plane, instead of the reflection in the midpoint, so the relation $\vartheta - \vartheta' \equiv R$ will exist between the two angles. If one shifts the midpoint of the transfer to infinity then the present representation will become impossible, except when S goes to a reflection in a plane, in particular; in that case, one will have ∞^2 decompositions of S, for each of which the applied rotation will be a reversal ($\vartheta = R$). Finally, if S itself is a reflection in a point then the rotational axis will indeed by undetermined, but the rotation itself will reduce to the identity transformation.

As we did before in similar cases, we now once more consider the midpoint and point at infinity of the chord xx', as well as the angle bisector of the planes that belong to the pair u, u'.

Under a transfer S, the locus of the chord midpoints, the middle rays of any corresponding point sequences, and finally, the middle planes of any two corresponding point fields will generally be a plane, namely, the middle plane of the transfer.

Only when S is a reflection in a point will the locus of all these structures be a point, namely, the point of reflection.

Under a transfer S, the locus of normal planes to the chords, the normal axis of any two corresponding point sequences, and finally, the central points of any two corresponding point-fields will generally be a point, namely, the midpoint of the transfer.

Only when S is a reflection in a plane will the locus of all these structures be a plane, namely, the plane of reflection.

The points at infinity of the chords generally fill up the entire plane at infinity. However, they will fill up a line when the transfer is the composition of a reflection in a plane followed by a translation. Finally, they coincide in a single point when the transfer goes to a reflection in a plane.

The two angle bisectors of a plane-pair u, u' once more differ by the properties of the projections of the congruent point-fields that lie in u and u': Those projections behave the same for the *angle bisectors of the first kind*, while for the *angle bisectors of the second kind* they behave differently under affine transformations; naturally, in both cases, they will both have the same area. (Cf., § 5, pp. 18)

The angle bisectors of the first kind generally fill up the bundle of planes that belongs to the midpoint of the transfer.

Only when that point is undetermined – i.e., when the transfer goes to a reflection in a plane – will an exception occur: All of the aforementioned planes will coincide with the plane of reflection.

The angle bisectors of the second kind generally fill up the bundle of planes that belongs to the point at infinity on the rotational axis.

Only when that point is undetermined – i.e., when the transfer is a reflection in a point – will an exception occur: All of the aforementioned planes will coincide with the plane at infinity.

The consideration of chord midpoints and normal planes to the chords, as well as the angle bisectors of associated planes, are also connected with the theory of certain collinear and dualistic transformations, which are, admittedly, defined, not for all of space, but only for the rays and planes of the midpoints of the transfers (the points at infinity on the rotational axis, resp.), or, what generally amounts to the same thing, for the points and lines of the plane at infinity (the middle plane of the transfer, resp.). Of the various garments that one can dress that theory in, we shall choose one that seems most convenient for our further exposition.

Two affine transformation $\mathfrak{T}_1'$ and $\mathfrak{T}_2'$ are linked (from § 2) with the points and rays of the middle plane *m* of any transfer *S* that is not a reflection.

We choose the points *x*, $\bar{x}$, *x′* and lines *h*, $\bar{h}$, *h′* of the plane *m*, corresponding to the conditions:

$$x\{\mathfrak{T}_1'\}\bar{x}\{\mathfrak{T}_2'\}x', \qquad h\{\mathfrak{T}_2'\}\bar{h}\{\mathfrak{T}_1'\}h',$$

resp., and let *g*, *g′* denote the lines that are perpendicular to the plane *m* at *x*, *x′*, resp., and let $\bar{u}$ denote the plane in $\bar{h}$ that is perpendicular to *m*: The lines *g* and *g′* will then be the respective loci of the endpoints of all chords that have their midpoints at $\bar{x}$, and the lines *h*, *h′* will be the loci of all pairs of associated planes *u*, *u′*, resp., whose angle bisector of the first kind is the plane $\bar{u}$.

The following corollary to these theorems is of great significance:

A dualistic transformation T is linked with the rays and planes through the midpoint o of any transfer S that is not a reflection in a plane. That transformation will associate any line g through o with the common normal plane to all chords that belong to g, g′, and that plane itself will again be associated with the ray g that corresponds to g under the transfer S.

The rays and planes through the points o will be permuted by the transformation T^2 *that arises from performing T twice as they are by the transfer S.*

If *S* goes to a reflection in the midpoint *o* then *g* and *g′* will coincide, and the ray *g* = *g′* will be perpendicular to the plane that is associated with it. *T* is then an involutory transformation, namely, the association of poles and polars relative to the so-called absolute cone through the point *o*. If *S* goes to a reflection in a plane then *T* will degenerate.

Along a train of thought that is entirely similar to the one that was followed previously in the theory of motions, we now come to the transformations $\mathfrak{T}_1'$, $\mathfrak{T}_2'$ of the plane *m*, and starting with the transformation *T* of the bundle *o*, to the decomposition of the transfer *S* into special motions and transfers in a way that is more general than what was formulated in the theorems that were formulated at the beginning of the paragraph.

Any transfer that is not a reflection in a plane can be generated in ∞^2 *ways (but* ∞^3 *ways for any reflection in a plane) by a rotation and a subsequent reflection in a plane.*

The rotational axis and the plane of reflection go through the midpoint o of the transfer. The first one will correspond to the second one under the dualistic transformation T of the bundle o.

Any transfer that is not a reflection in a plane can be generated in ∞^2 *ways (but* ∞^3 *ways for any reflection in a plane) by a reflection in a plane and a subsequent rotation.*

The plane of reflection and the rotational axis go through the midpoint o of the transfer. The first one will correspond to the second one under the dualistic transformation T^{-1} *of the bundle o.*

If the plane of reflection of both decompositions is the same then the rotational axis *g* in the theorem on the left and the rotational axis *g′* in the theorem on the right will correspond under the transfer S; the associated rotational angles will be equal and opposite to each other. If S is itself a reflection in a plane then the rotation around g, g' will again either reduce to the identity transformation (in which case, g will be completely undetermined) or g and g' will coalesce into a line in the plane of reflection; they will correspond to an undetermined angle of rotation. If S is a reflection in a point then g and g' will likewise coincide; the associated motions will both be reversals.

Any transfer that is not a reflection in a point can be generated in ∞^2 *ways (but* ∞^3 *ways for any reflection in a point) by a rotation and a subsequent reflection in a point.*	*Any transfer that is not a reflection in a point can be generated in* ∞^2 *ways (but* ∞^3 *ways for any reflection in a point) by a reflection in a point and a subsequent rotation.*

The point of reflection lies in the middle plane, and the rotational axis is perpendicular to that plane.

The point of intersection of the middle plane and the rotational axis corresponds to the point of reflection under any affine transformation $\mathfrak{T}_1'$ *of the middle plane that is determined by the given transfer.*	*The point of intersection of the middle plane and the rotational axis corresponds to the point of reflection under any affine transformation* $\mathfrak{T}_2'^{-1}$ *of the middle plane that is determined by the given transfer.*

The angle of the rotation that corresponds to the theorem on the left and that of the rotation that corresponds to the theorem on the right are both equal to the angle 2ϑ that was defined above; they will then be independent of the special type of decomposition that is carried out.

If the transfer S is a reflection in a plane then the rotational axis will always go through the point of reflection; the associated rotation will be a reversal. If S is itself a reflection through a point then each of the applied rotations will be a translation; the associated reflection point will be an arbitrarily-prescribed point in space.

Any transfer that is not a reflection in a point can be generated in ∞^2 *ways (but* ∞^3 *ways, for a reflection in a plane) by a screwing motion and a subsequent reflection in a plane.*	*Any transfer that is not a reflection in a point can be generated in* ∞^2 *ways (but* ∞^3 *ways, for a reflection in a plane) by a reflection in a plane and a subsequent screwing motion.*

An unscrewing axis lies in the middle plane and the plane of reflection will be perpendicular to that plane.

The line of intersection of the middle plane and the plane of reflection correspond to the unscrewing axis under any affine transformation $\mathfrak{T}_2'^{-1}$ *of the middle plane that is determined by the*	*The line of intersection of the middle plane and the plane of reflection correspond to the unscrewing axis under any affine transformation* $\mathfrak{T}_1'$ *of the middle plane that is determined by the given*

given transfer.

transfer.

The unscrewing axis and the plane of reflection then subtend a constant angle, namely, the angle ϑ' ($-\vartheta'$, resp.). If the plane of reflection is the same for both kinds of decomposition then the associated unscrewing axes that correspond to each other under the transfer S will go to each other under the reflection. (One should compare the conclusion of § 3.)

If the transfer S is itself a reflection in a plane then the unscrewing to be applied will reduce to a reversal in both cases. The plane of reflection will go through the reversal axis. A representation of the reflection S will then arise that is also included in the expression for S in terms of a rotation and a reflection.

If S is a reflection through a point then the unscrewing axis will go through that point. The plane of reflection will be any plane that is perpendicular to the unscrewing axis.

Along with the three double theorems that were presented, one can finally pose yet a fourth one that deals with the decomposition of a transfer S into a reflection through a point and a prior or subsequent unscrewing. However, that theorem has a special character. Namely, such a decomposition is obviously possible only when S has a midpoint at infinity, when the points of the plane at infinity are transformed involutorily by the transfer S, but then the decomposition can be effected in ∞^3 ways in *all* cases. The point of reflection is an arbitrary point in space. The unscrewing axis is perpendicular to the middle plane. Its distance to the point of reflection is equal and opposite (equal, resp.) to one-half the magnitude of the translation that coincides with the transfer S for the points of the middle plane. The height of the unscrewing is equal to twice the distance from the middle plane to the point of reflection.

We now consider some special cases that are no less interesting than the general theorems.

If we represent the transfer S, in one case, by a rotation and a reflection in a plane, and in the other case, by a unscrewing and a reflection in a plane, and demand that the rotation or the unscrewing is a reversal, in particular, then we will come to the following theorem in both cases consistently:

Any non-involutory transfer can be represented in ∞^1 ways by a reversal and a subsequent reflection in a plane.

Any non-involutory transfer can be represented in ∞^1 ways by a reflection in a plane and a subsequent reversal.

The reversal axis and the plane of reflection go through the midpoint of the transfer; the reversal axis lies in the middle plane and the plane of reflection is perpendicular to it.

The reversal axis and the plane of the reflection subtend a constant angle of ϑ', viz., one-half the angle of the rotation that generates the transfer, along with the reflection in the middle plane.

The plane of the reflection and the reversal axis subtend a constant angle of ϑ', viz., one-half the angle of the rotation that generates the transfer, along with the reflection in the middle plane.

The theorem reads somewhat differently for the involutory reversals (cf., pp. 40, above).

Every reflection in a plane can be represented in ∞^2 ways by a reversal and a prior or subsequent reflection in a plane.

The plane of latter the reflection is perpendicular to the plane of the given reflection; both planes intersect in the axis of the reversal.

Every reflection in a plane can be represented in ∞^2 ways by a reversal and a prior or subsequent reflection in a plane.

The axis of reversal is perpendicular to the plane of the reflection; they intersect at the midpoint of the given reflection.

The decomposition of a transfer S into a reversal and a subsequent reflection in a point, or into a reflection in a point and subsequent reversal is once more possible only when S has a midpoint at infinity, but always in ∞^2 ways then. (Cf., pp. 44, above.) If S is involutory, in addition (viz., a reflection in a plane) then each of the ∞^2 reversals will commute with the associated reflection.

Finally, we can specialize our general theorems in yet another direction by demanding that the theorem on the left must agree with the theorem on the right in every case; i.e., that the two transformations into which S decomposes must *commute.*

If we start with the representation of S by a rotation and a reflection in a plane then we will come to the first theorems that were already expressed at the beginning of this paragraph, and if we start with the representation by a rotation a reflection in a point then we will come to the second one. We will once more obtain a special result when we link the representation of S with an unscrewing and a reflection in a plane:

"Any non-involutory transfer with a midpoint at infinity can be represented in ∞^2 ways by a reflection in a plane an unscrewing that commutes with that reflection. The plane of reflection is perpendicular to the middle plane of the given transfer, as well as the direction of the midpoint at infinity; it intersects the middle plane in the unscrewing axis. The height of the unscrewing is identical to the magnitude of the translation to which the transfer reduces for the points of its middle plane."

By contrast, if a transfer is involutory – i.e., if it goes to a reflection in a plane – then, as we already saw before, there will be ∞^2 representations of the kind that were given in the theorem. The associated unscrewings will reduce to reversals.

Finally, a transfer is representable by a reflection in a point and an unscrewing that commutes with it only when it is itself a reflection in a plane:

"Any reflection in a plane can be represented in ∞^2 ways by a reflection in a point and a reversal that commutes with it. The reversal axis is perpendicular to the plane of the given reflection and intersects it in the associated midpoint of the reflection."

This result was also stated before in another context.

The theorems that were developed in the present paragraph make it possible for one to compose two or more motions and transfers. We shall leave it to the reader to illustrate these constructions, which can be accomplished in various ways.

§ 11.

Continuation. – Addenda to the theory of motions.

Part of the content of the theorems that were developed in § 10 can be put into another form that is more expedient for many purposes; one once more arrives that the composition of rotations and unscrewing into two reflections in their own right. We then come to solution of the problem:

Decompose a given transfer into three successive reflections in all possible ways.

First of all, the following theorem emerges from the first of the double theorems that were pointed out above (pp. 42):

Any transfer can be decomposed into three successive reflections in planes in space in ∞^3 *ways, or if it is a reflection in a plane, in* ∞^4 *ways.*

Conversely, if the transfer S is given by a succession of three reflections in planes then the intersection of those planes will be the midpoint of the transfer. The middle plane, the angles ϑ, ϑ', etc., can also be found directly by a construction that corresponds precisely to the construction of the succession of three reflections in the plane that was given in § 3.

If the three planes of reflection intersect perpendicularly then S will be a reflection in a point; if they belong to the same pencil then S will be a reflection in a plane.

Any transfer can be decomposed into three reflections in ∞^3 *ways, or* ∞^4*, or if it is a reflection in a point, of which, either the first, the second, or third one will be a reflection in a point, while the other two will be reflections in planes.*

If we let S_x, S_y, S_z be reflections in the points x, y¸ z, resp., and if we likewise denote the reflections in the planes u, v, w by S_u, S_v, S_w, resp., then we will have three representations for the transfer S of the following form:

$$S = S_x S_{v_1} S_w, \quad S = S_u S_y S_w, \quad S = S_u S_{v_2} S_z .$$

These formulas give us the link to the double theorems on pp. 42 and 43; namely, if we combine two successive reflections in the following way:

$$S = S_x (S_{v_1} S_w), \qquad S = (S_u S_{v_2}) S_z$$

then we will come to the representation of a transfer by a reflection in a point and a rotation; however, if we compose two reflections, first, according to the formulas:

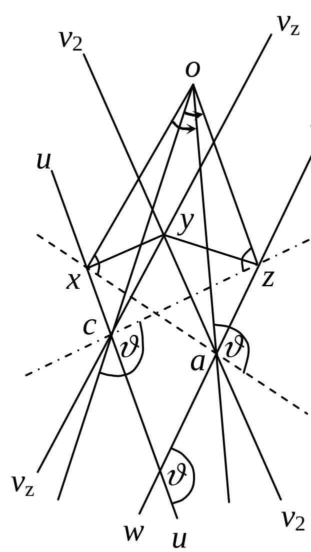
$$S = (S_x S_{v_1}) S_w \qquad S = (S_u S_y) S_w ,$$

and then according to the formulas:

$$S = S_u(S_yS_w), \qquad S = S_u(S_{v_2}S_z),$$

then we will come to the representation of S by a reflection in a plane and an unscrewing.

Conversely, we will easily find the midpoint, the middle plane, etc. of the transfer S when S is defined in the given way by a succession of three reflections.



Let, e.g., u, y, and w be given such that $S = S_u\,S_y\,S_w$; we then let x and z, in particular, denote the feet of the altitudes that are dropped from y to u and w, resp., and let v_1 and v_2 denote the planes through the point y that are parallel to u and w, resp. One will then have, at the same time:

$$S = S_xS_{v_1}S_w, \qquad S = S_uS_{v_2}S_z.$$

The plane xyz is the middle plane of the transfer, but the midpoint o will be found as follows: One connects the point x in the middle plane with the trace a of the line (v_1, w) and the point z with the trace c of the line (u, v_2). One then erects perpendiculars to both of those connecting lines at x and z, and in addition draws the lines ax and cz to a and c, resp., at the angle ϑ (the planes u and v at an angle of $-\vartheta$, resp.), as is shown in the accompanying figure. The four lines thus-constructed will go through the midpoint o of the transfer. At the same time, $\angle\, xoa = \angle\, coz$ is one-half the angle ϑ of the rotation that generates the transfer S, along with the reflection in the plane xyz.

If the two planes of reflection are parallel in any of the three decompositions then S will be a reflection in a point. If both planes go through the point of reflection, and if they are perpendicular to each other, in addition, then S will be a reflection in a plane.

One will again obtain theorems of a special character when one requires a transfer to decompose into three successive reflections, at least two of which are reflections in points:

"Any transfer with a midpoint at infinity can be decomposed in ∞^4 ways into three successive reflections such that either the first, the second, or the third one is a reflection in a plane, while the other two are reflections in points."

In all cases, the plane of reflection is parallel to the middle plane of the transfer *S*. The connecting line of the point of reflection will be perpendicular to the middle plane, when *S* is a reflection in that plane. One can also replace the two reflections in points with a translation, etc.

Finally, the following theorem has a more specialized character:

"Any reflection in a point can be decomposed in ∞^4 ways into three successive reflections in points of space."

The decomposition of a transfer into motions and transfers of a special kind that was treated in the previous paragraph is a corollary to the theory of the decomposition of a motion into two special motions that was developed in § 5. However, one can find yet another analogy between the theory of motions and the theory of transfers when one decomposes a given motion into two transfers in all possible ways and then specializes those transfers in a suitable way.

We especially need to highlight some theorems that will shed a new light upon the properties of the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, and *T* that we know from § 5.

Any motion S can be decomposed in ∞^3 ways into two transfers, the second of which is a reflection in a plane.

The plane of reflection thus corresponds to the midpoint and the middle plane of the first transfer under the transformations T^{-1} and $\mathfrak{T}_2^{-1}$, resp.

Any motion S can be decomposed in ∞^3 ways into two transfers, the first of which is a reflection in a plane.

The plane of reflection thus corresponds to the midpoint and the middle plane of the second transfer under the transformations T and $\mathfrak{T}_1$, resp.

The midpoint and middle plane of the first (second, resp.) transfer are associated with each other reciprocally in the null system $\mathfrak{W}$ that belongs to the middle complex of the motion *S*. However, the rotational axis is a ray of the chord complex of *S*.

One can choose the plane of the reflection arbitrarily in any situation. The decomposition of the motion *S* will then be determined completely by that.

If *S* is a *screwing motion,* but not, at the same time, a rotation, then the aforementioned transfer (perhaps in the theorem on the left) will have a completely determined midpoint. Moreover, it will also have a completely determined middle plane when the plane of the reflection is not perpendicular to the unscrewing axis. In fact, the middle plane is the plane of the unscrewing axis *n* that is perpendicular to the plane of reflection. However, when the plane of reflection meets the axis *n* at a right angle, the transfer will become a reflection in a point of the axis *n*; it will then have a well-defined middle plane.

If S is a *rotation*, but not at the same time an unscrewing, then the transfer that we spoke of will have a well-defined middle plane. If the plane of reflection does not contain the rotational axis n then the transfer will also have a well-defined midpoint: viz., the point of intersection of the plane of reflection with the axis n. However, when the axis n goes through the plane of reflection, the transfer itself will become a reflection in a plane; it will then have infinitely many midpoints.

All of this is also true, in particular, when the axis n lies at infinity, so when S goes to a translation.

Finally, if S is a *reversal* – i.e., both an unscrewing and a rotation together – then there will be three cases to distinguish: If the plane of reflection goes through either the reversal axis or the line at infinity that is perpendicular to it then the transfer that we spoke of will have a well-defined midpoint (viz., the point of intersection of the reversal axis n and the plane of reflection), and furthermore, a well-defined middle plane (viz., the plane through the axis n that is perpendicular to the plane of reflection). If the plane of the reflection is perpendicular to the axis n then the middle plane will be undetermined. Finally, when the plane of reflection includes the axis n, the midpoint of the transfer will be undetermined.

Any motion S can be decomposed in ∞^3 *ways into two transfers, the second of which is a reflection in a point.*

The midpoint of the first transfer will then correspond to the midpoint of the reflection under the affine transformation $\mathfrak{T}_1$.

Any motion S can be decomposed in ∞^3 *ways into two transfers, the first of which is a reflection in a point.*

The midpoint of the second transfer will then correspond to the midpoint of the reflection under the affine transformation $\mathfrak{T}_2^{-1}$.

Moreover, the middle planes of the first transfer (e.g., in the theorem on the left) correspond to the reflection point under a dualistic transformation, but in a degenerate way: The middle plane of the transfer is *the* plane through its middle point that is perpendicular to the screw axis n of the motion S.

Only the case in which S is an unscrewing occupies a special place in regard to that theorem. If one chooses the midpoint of the reflection along the unscrewing axis n then the other transfer will become a reflection in a plane that perpendicular to the axis n; in that case, it will then have infinitely many midpoints.

The given theorems can be brought into an even closer connection with the theory that was developed in § 5 when one decomposes the motion S into four involutory transfers in all possible ways.

Therefore, the decomposition of S into two rotations, as well as the decomposition of S into two transfers, one of which is a reflection in a plane will be contained in the theorem:

Any motion can be replaced with four successive reflections in planes of space in ∞^6 *ways.*

A corresponding statement is true for the theorem:

Any motion can be represented in ∞^6 *ways by four involutory transfers, one of which is a reflection in a point, while the other three are reflections in planes.*

From this point onward, one comes to the decomposition of S into an unscrewing and a rotation, in one case, and then to the decomposition of S into a reflection in a point or a plane and another transfer. However, the second of these decompositions and third one, in part, can also be connected with the following theorem:

Any motion that is not a translation can be decomposed in ∞^6 *ways into four involutory transfers and any translation in* ∞^7 *ways, two of which are reflections in points, and two of which are reflections in planes.*

At the same time, that will imply the decomposition of S into a rotation and a translation, which we did not emphasize especially, and the decomposition of S into two transfers, which we intentionally passed over.

However, two further theorems of a kind that we shall follow through with, for the sake of completeness, have a more specific character than the aforementioned three:

"Any unscrewing can be decomposed into four involutory transfers in ∞^7 ways, one of which is a reflection in a plane, while the other three will be reflection in points."

"Any translation can be decomposed into four successive reflections in points in ∞^7 ways."

All of these theorems, which are significant for an insight into the internal connections of the theory, can be easily linked with our developments up to this point; one can give an account of the positions of the planes and points of reflection.

However, one can also invert the train of thought by putting the involutory transformations at the center of attention. They will now be the reversals and reflections, since we are treating a combined development of the theories of motions *and* transfers. As the author has convinced himself, one will also come to the properties of the transformations that were denoted by $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, etc., in that way with no great effort. Naturally, the entire theory will take on a different character under such a change of viewpoint. The reflections, which depend upon three constants, seem to be simpler than the reversals, which have four constants, and the transfers, which are representable by three reflections, seem to be simpler than the motions, for whose expression one will need four reflections.

Perhaps it is convenient to put the following problem at the pinnacle of the entire theory: *Decompose the identity transformation into four, six, … reflections in all possible ways.*

One can then couple that problem, which is not as specialized as it might seem upon first glimpse, with numerous further developments quite easily.

We would prefer to content ourselves with these suggestions than to promise to go into **H. Wiener**'s further investigation into those questions. We make only one relevant remark, since it will once again bring to our attention the distinguished position of the unscrewings and rotations collectively assume in the theory of motions:

The unscrewings and rotations are the only motions that are already representable by ***two*** *reflections.*

One convinces oneself directly that among those transformations, in turn, the *reversals* and the *translations* exhibit a special behavior.

§ 12.

Motion in a ray bundle.

One recognizes generalizations of the theory that was developed in § 3 in several theorems of §§ 10 and 11. In order to arrive at a clear insight into this situation, which is important for an understanding of the study of motions and transfers, we consider the motions and transfers under which a finite point *o* in space will be fixed, when we regard these transformations as associations between the rays or planes through the point *o*.

The transfers that fix the point *o* arise from the rotations around axes through the point *o* quite simply when one performs a reflection through the point *o* before or after a rotation. That involutory transformation will then commute with all of our motions and transfers; it will be a distinguished transformation of the group in question (viz., the only one besides the identity transformation). Since all rays and planes through the point *o* will be fixed, it will follow that no difference will exist between motions and transfers for the rays and planes of the bundle *o*: If one introduces the rays or planes through *o* as spatial elements then one will obtain an irreducible family, namely, a continuous group of ∞^3 transformations. It might be referred to as the *motions in the ray bundle.*

In the study of that group, we must observe, above all, that its transformations will fix the dualistic transformation $\mathfrak{P}$ of the absolute polar system that belongs to the bundle *o*. The relationship between a ray g and the plane γ through the point *o* that is perpendicular to it (and thus, between poles and polars relative to the so-called imaginary absolute cone of the point *o*) is invariant under all motions of the bundle *o*. They will define the totality of *all* collinear transformations of the rays of our bundle that commute with the dualistic transformation $\mathfrak{P}$.

The following things correspond to each other reciprocally under the transformation $\mathfrak{P}$,:

Two rays g, g'	Their polar planes γ, γ'.
The two angle bisectors (rays) $\bar{g}$ and $\bar{\bar{g}}$ of g and g'.	The angle bisectors (planes) $\bar{\gamma}$ and $\bar{\bar{\gamma}}$ of γ and γ'.
The two *normal planes* $\bar{\gamma}$ and $\bar{\bar{\gamma}}$ of the rays-pair g, g' that are the polar planes of $\bar{g}$ and $\bar{\bar{g}}$, and thus, planes that go through $\bar{g}$ and $\bar{\bar{g}}$, resp., and are perpendicular to the planes of g and g'.	The two *normal rays* $\bar{g}$ and $\bar{\bar{g}}$ of the plane-pair γ, γ', which are the poles of $\bar{\gamma}$ and $\bar{\bar{\gamma}}$, and thus rays that lie in $\bar{\bar{\gamma}}$ and $\bar{\gamma}$, resp., and are perpendicular to the line of intersection of γ and γ'.

We now let g, g' mean a pair of rays that correspond under a motion S of the ray bundle, and consider the motion s or the transfer σ in space that permutes the rays of our bundle in the prescribed way. In the first case, (s) will then be, perhaps, the angle bisector $\bar{g}$ of the locus of the chord midpoints that belongs to the points of g, g', and the normal plane $\bar{\bar{\gamma}}$ (viz., the polar to the other angle bisector $\bar{\bar{g}}$) will be the common normal plane to the aforementioned chords; however, in the case of the transfer (s), $\bar{\bar{g}}$ will be the locus of the chard midpoints, and $\bar{\gamma}$ will be the normal plane to all chords that belong to g, g'. We must then only reorganize the information that we obtained already in order to arrive at the theorem:

The two angle bisectors and normals to any two corresponding rays or planes will exhibit different behavior under a motion S of the ray bundle.

The angle bisectors if the first kind $\bar{g}$ of all pairs g, g' of associated rays will either fill up the entire ray bundle or they will coalesce into one and the same ray (viz., the rotational axis *of S*).

The angle bisectors of the second kind $\bar{\bar{g}}$, by contrast, will lie in a well-defined plane under all circumstances, namely, the polar plane of the rotational axis; viz., the "middle plane" of the motion S.

Moreover, the normal planes of the first kind $\bar{\gamma}$ will either fill up the entire bundle of planes or they will coalesce into a single plane (viz., the middle plane).

By contrast, the normal planes of the second kind $\bar{\bar{\gamma}}$ will all go through a well-defined ray under all circumstances, namely, the rotational axis of the motion.

The angle bisectors if the first kind $\bar{\gamma}$ of all pairs γ, γ' of associated planes will either fill up the entire plane bundle or they will coalesce into one and the same plane (viz., the middle plane *of S, which is likewise defined).*

The angle bisectors of the second kind $\bar{\bar{\gamma}}$, by contrast, will go through a well-defined ray under all circumstances, namely, the rotational axis of the motion S (*).

Moreover, the normal rays of the first kind $\bar{g}$ will either fill up the entire bundle of rays or they will coalesce into a single ray (viz., the rotational axis).

By contrast, the normal rays of the second kind $\bar{\bar{\gamma}}$ will all lie in a well-defined plane under all circumstances, namely, the middle plane of the motion.

The special cases that were mentioned in the theorem will occur when the motion is involutory S; thus, when the motion s of space that is bound with the motion S of the ray bundle goes to a reversal around an axis of points o, and at the same time, the associated transfer σ goes to a reflection in a plane (viz., the polar plane of that reversal axis).

One can now connect the statements above with further theorems in a natural way:

Any non-involutory motion S of the ray bundle is linked with two commuting, collinear transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$ of the bundle that will generate the motion when they are performed in sequence:

(*) One can distinguish between the two angle bisectors as follows: If one projects the corresponding pencils of rays of γ and γ' (orthogonally) onto the angle bisector of the first kind $\bar{\gamma}$ then two projective pencils of rays with the same sense of rotation will arise; if one projects them onto the angle bisector of the second kind $\bar{\bar{\gamma}}$ then pencils of rays with opposite senses of rotation will arise. Naturally, corresponding differences will also exist for the other structures that were treated in the text.

(1) $$\mathfrak{T}_1\mathfrak{T}_2 = S = \mathfrak{T}_2\mathfrak{T}_1 .$$

The transformation $\mathfrak{T}_1$ assigns each ray g of the bundle to the angle bisector $\bar{g}$ of the first kind of the ray-pair g, g'.

The transformation $\mathfrak{T}_2$ assigns any plane γ of the bundle to the angle bisector $\bar{\gamma}$ of the first kind of the plane-pair γ, γ'.

The transformation $\mathfrak{T}_2^{-1}$ assigns each ray g' of the bundle to the angle bisector $\bar{g}$ of the first kind of the ray-pair g, g'.

The transformation $\mathfrak{T}_1^{-1}$ assigns any plane γ' of the bundle to the angle bisector $\bar{\gamma}$ of the first kind of the plane-pair γ, γ'.

In symbols:

(2) $$g\,\{\mathfrak{T}_1\}\,\bar{g}\,\{\mathfrak{T}_2\}g', \qquad \gamma\{\mathfrak{T}_2\}\,\bar{\gamma}\,\{\mathfrak{T}_1\}\gamma'.$$

Moreover:

Any non-involutory motion S of the ray bundle will be coupled with two dualistic transformations T, T' of the bundle, each of which will generate the motion S when it is performed twice:

(3) $$S = T^2, \qquad S = T'^2.$$

The transformation T assigns any ray g to the normal plane $\bar{\gamma}$ of the first kind of the ray-pair g, g', and that plane, in turn, to the ray g'.

The transformation T' assigns any plane γ to the normal ray $\bar{g}$ of the first kind of the plane-pair γ, γ', and that ray, in turn, to the plane γ'.

In symbols:

(4) $$g\,\{T\}\,\bar{\gamma}\,\{T\}g', \qquad \gamma\{T'\}\,\bar{g}\,\{T'\}\gamma'.$$

The following simple relations exist between the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T':

The transformations T and T' will be permuted under the dualistic transformation $\mathfrak{P}$ of the absolute polar system:

(5) $$T' = \mathfrak{P}T\,\mathfrak{P}, \qquad T = \mathfrak{P}\,T'\mathfrak{P}.$$

The collinear transformations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be composed of the dualistic transformation T and $\mathfrak{P}$, as well as T' and $\mathfrak{P}$:

(6) $$\begin{cases} \mathfrak{T}_1 = T\mathfrak{P}, & \mathfrak{T}_2 = \mathfrak{P}T, \\ \mathfrak{T}_1 = \mathfrak{P}T', & \mathfrak{T}_2 = T'\mathfrak{P}, \end{cases}$$

$\mathfrak{T}_1$ and $\mathfrak{T}_2$ will then be switched with each other by each of the dualistic transformations T, $\mathfrak{P}$, T'; in symbols:

$$(7)\qquad \begin{cases} \mathfrak{T}_1 T = T\mathfrak{T}_2, & \mathfrak{T}_1\mathfrak{P} = \mathfrak{P}\mathfrak{T}_2, & \mathfrak{T}_1 T' = T'\mathfrak{T}_2, \\ T\mathfrak{T}_1 = \mathfrak{T}_2 T, & \mathfrak{P}\mathfrak{T}_1 = \mathfrak{T}_2\mathfrak{P}, & T'\mathfrak{T}_1 = \mathfrak{T}_2 T'. \end{cases}$$

The meaning of the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T' will be further clarified by the following remarks:

One can decompose any motion S in the ray bundle into two motions, the second of which is involutory, in ∞^2 *ways.*	*One can decompose any motion S in the ray bundle into two motions, the first of which is involutory, in* ∞^2 *ways.*
The rotational axis of the first motion corresponds to the rotational axis and the middle plane of the second one under the transformations $\mathfrak{T}_1$ *and T, resp.*	*The rotational axis of the second motion corresponds to the rotational axis and the middle plane of the first one under the transformations* $\mathfrak{T}_2^{-1}$ *and* T^{-1}*, resp.*
The middle plane of the first motion corresponds to the middle plane and rotational axis of the second one under the transformation $\mathfrak{T}_2$ *and T′, resp.*	*The middle plane of the second motion corresponds to the middle plane and rotational axis of the first one under the transformation* $\mathfrak{T}_1^{-1}$ *and* T'^{-1}*, resp.*

If we demand that the decomposition on the left must agree with the one on the right then we will come to the theorem that we already put forth in a different context (pp. 44), which might be repeated here for the sake of clarity:

Any motion S in a ray bundle can be decomposed in ∞^1 *ways into two involutory motions.*

The associated rotational axes lie in the middle plane of S and subtend one-half the angle of rotation.	*The associated middle planes go through the rotational axis of S and subtend one-half the angle of rotation.*

The composition of motions in the ray bundle is based upon this theorem in a well-known way.

The transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T' under examination can be regarded as special cases of more general transformations that exhibit some of their properties. The theory of those transformations is entirely similar to the theory of the transformations $\mathfrak{t}_1$, $\mathfrak{t}_2$, t that was treated in § 7, which belongs to a motion of three-fold extended space.

A one-parameter group Γ_1 *of motions of the ray bundle is determined by an axis n through the point o or the polar plane v, namely, the group of all rotation around the axis n.*

Furthermore, a one-parameter group G_1 *of perspective transformations of the ray bundle is determined by the axis n, namely, the group of all perspective transformations whose perspectivity axis is the line n and whose perspectivity plane if the plane v.*

Let g and g^* be two rays through the point o that lie in a plane with n, so the general transformation $\mathfrak{r}$ of G_1 will be determined by the association $g\{\mathfrak{r}\}g^*$, as long as one assumes that the ratio tan (g, n) : tan (g^*, n) should have a value λ that is independent of the position of the ray g^*. One gets the first part of the theorem for $\lambda = -1$:

The groups G_1 and Γ_1 intersect in the involutory motion that is determined by the axis n, in addition to the identity transformation.

A two-parameter continuous group G_2 of commuting transformations arises by composing the transformations of G_1 and Γ_1, namely, the group of linear transformations $\mathfrak{t}$ of the bundle that permute the planes of the axis n and the rays of the plane v with each other like the motions of the group Γ_1.

If we denote the polar planes of the rays g and g^* in question by γ and γ^*, and the traces of those planes in the plane through g and n by g_1 and g_1^*, resp., then we will have cot (γ, n) : cot $(\gamma^*, n) = \lambda$ or tan (g_1, n) : tan $(g_1^*, n) = 1 : \lambda$. However, it follows from this that the transformation $\mathfrak{r}$ of G_1 is exchanged with its inverse transformation by the dualistic transformation $\mathfrak{P}$ of the absolute polar system:

(8) $$\mathfrak{P}\,\mathfrak{r} = \mathfrak{r}^{-1}\,\mathfrak{P}, \qquad \mathfrak{r}\,\mathfrak{P} = \mathfrak{P}\mathfrak{r}^{-1}.$$

We infer the following conclusion from this:

If one composes all transformations of G_1 with the dualistic transformation $\mathfrak{P}$ of the absolute polar system then a continuous family H_1 of ∞^1 polar systems will arise, namely, the totality of all polar systems $\mathfrak{p}$ that belong to the cone of rotation whose axis is n.

The family H_1 of transformations $\mathfrak{p}$ defines a group along with the transformations of G_1.

The order surface of the polar system:

$$\mathfrak{p} = \mathfrak{P}\,\mathfrak{r} = \mathfrak{r}^{-1}\,\mathfrak{P}$$

has imaginary or real generators according to whether the value of λ that belongs to $\mathfrak{r}$ is positive or negative, resp.

If one composes all transformations of Γ_1 with any of the dualistic transformations $\mathfrak{p}$ of H_1 then a family H_1 of ∞^1 dualistic transformations will arise, each of which will take the cone of rotation that belongs to $\mathfrak{p}$ to itself.

There are ∞^1 such families H_1 and groups Γ_1, H_1, including one of them that emerges from the absolute polar system $\mathfrak{P}$, and which therefore fixes the absolute cone of the point o.

In fact, if S and B are transformations of Γ_1 will then we have commuting transformations before us in the form of S, B, and $\mathfrak{p} = \mathfrak{P}\mathfrak{r}$; the theorem will follow from that directly. If $\mathfrak{r}$ goes to the identity transformation then the family H_1 will arise, which belongs to the absolute polar system and contains the transformation $\mathfrak{P}$. The cones of rotation with the axis n will be permuted pair-wise by the transformations of that family (as they are by all of the remaining families H_1).

If one composes all transformations of the group G_2 with any of the transformations of H_1 or all transformations of the group G_1 with all transformations of any of the families H_1, or finally, composes all transformations of Γ_1 with all transformations of H_1 then a continuous family H_2 of ∞^2 dualistic transformations will arise, each of which will take the totality of all cones of rotation with the axis n into itself.

The family H_2, together with the group G_2, again define a group (of non-commuting transformations).

The general transformation of H_2 reads:

$$t = S\,\mathfrak{P}\mathfrak{r} = S\,\mathfrak{r}^{-1}\,\mathfrak{P}. \tag{9}$$

One easily proves that two such transformations will commute only when they belong to the same family H_1 .

The following theorem will provide information about the relationship between the six groups:

$$G_1;\ \Gamma_1\ ;\ G_2\ ;\ -\ G_1,\ H_1\ ;\ \Gamma_1,\ \mathsf{H}_1\ ;\ G_2,\ H_2\ .$$

The groups G_1 ; G_1 , H_1 (and naturally, G_2, as well) are invariant subgroups of the group G_2, H_2, but the group Γ_1 is a distinguished subgroup, moreover. The ∞^1 subgroups all have the same status.

That will be explained partially by the following remarks:

The transformations of the group G_2 can be ordered into pairs $\mathfrak{t}_1$, $\mathfrak{t}_2$ in a certain way.

Any two transformations that are thus paired will be permuted by all transformations of the family H_2 :

$$\mathfrak{t}_1\, t = t\, \mathfrak{t}_2, \qquad t\, \mathfrak{t}_1 = \mathfrak{t}_2\, t. \tag{10}$$

Furthermore, $\mathfrak{t}_1^{-1}\mathfrak{t}_2$ and $\mathfrak{t}_1\mathfrak{t}_2^{-1}$ are (inverse) transformations of the group G_1; finally, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ will yield a transformation of Γ_1 when they are performed in succession, namely, a motion.

Conversely, any rotation around the axis n can be decomposed into two associated transformation of G_2 in ∞^1 ways.

Let $S^{1/2}$ be one of the two rotations that generate the rotation S when they are performed twice in succession, so:

$$\mathfrak{t}_1 = S^{1/2}\,\mathfrak{r}, \qquad \mathfrak{t}_2 = S^{1/2}\,\mathfrak{r}^{-1} \tag{11}$$

will be the general expressions for a pair of associated transformations of G_2 .

Let g, g^*, g' be three rays through o that satisfy the condition:

$$g\{\mathfrak{t}_1\}g^*\{\mathfrak{t}_2\}\,g',$$

and let γ, γ^*, γ' be their polar planes, one will then have, at the same time:

$$\gamma\{\mathfrak{t}_2\}\gamma^*\{\mathfrak{t}_1\}\gamma'.$$

g^* is a ray of the normal plane of the second kind $\overline{\overline{\gamma}}$ of g and g', and γ^* is a plane through the normal ray of the second kind $\overline{\overline{g}}$ of γ and γ'.

Any two inverse transformations of the group G_1 define a pair of associated transformations of the group G_2; however, any transformation of the group Γ_1 is associated with itself.

We can deduce the following theorem from formula (9):

Any transformation t of the family H_2 will generate a transformation of Γ_1 when it is performed twice in succession, namely, a motion:

$$S = t^2. \tag{12}$$

Conversely, any rotation around the axis n can be generated in ∞^1 ways by repeating a transformation of the family H_2 .

Any of these transformations t associates any ray g with a plane that goes through the angle bisector $\overline{\overline{g}}$ of the second kind of the ray-pair g, g', and that plane, in turn, will be associated with the ray g'. Moreover, it associates any plane γ with a ray that lies in the angle bisector of the second kind $\overline{\overline{\gamma}}$ of the plane-pair γ, γ', and that ray will, in turn, be associated with the plane γ'.

One will remark that in the theory of the group G_2, H_2, all of the polar systems, or the concentric cones of rotation that they are associated with, will have the same status. We now return to our previous train of thought, in which we single out the absolute polar system $\mathfrak{P}$:

Any decomposition of the rotation S into associated transformations of G_2 by means of the absolute polar system:

$$S = \mathfrak{t}_1\mathfrak{t}_2 = \mathfrak{t}_2\mathfrak{t}_1$$

is associated with two well-defined ways of representing the same motion by transformations of H_2 :

$$S = t^2 \quad \text{and} \quad S = t'^2.$$

The relationship between t *and* t' *and* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *is expressed by the formulas:*

$$(13) \qquad \begin{cases} t = \mathfrak{t}_1\mathfrak{P} = \mathfrak{P}\mathfrak{t}_2, \\ t' = \mathfrak{P}\mathfrak{t}_1 = \mathfrak{t}_2\mathfrak{P}. \end{cases}$$

If we use the symbols g, g^*, g', γ, γ^*, γ' with the meanings that they had before (pp. 55) then we can perhaps best clarify the mutual relationships between the transformations $\mathfrak{P}$, $\mathfrak{t}_1$, $\mathfrak{t}_2$, t, t', and S by a schematic figure:

$$\begin{tikzcd}
g \arrow[r, "\mathfrak{t}_1"] \arrow[d, leftrightarrow, "\mathfrak{P}"'] \arrow[dr, "t"] & g^* \arrow[r, "\mathfrak{t}_2"] \arrow[d, leftrightarrow, "\mathfrak{P}"] \arrow[dr, "t'"'] & g' \arrow[d, leftrightarrow, "\mathfrak{P}"] \\
\gamma \arrow[r, "\mathfrak{t}_2"'] \arrow[ur, "t'"'] & \gamma^* \arrow[r, "\mathfrak{t}_1"'] \arrow[ur, "t"] & \gamma'
\end{tikzcd}$$

As we said, g^* lies in the plane $\bar{\bar{\gamma}}$ in this, and γ^* goes through the ray $\bar{\bar{g}}$.

The relationship between the transformations $\mathfrak{t}_1$, $\mathfrak{t}_2$, t, and t' that is represented coincides with the connection between the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T' that is expressed by the formulas (2) and (4). One then also has the theorem:

The transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T' *define a quadruple of associated transformations* $\mathfrak{t}_1$, $\mathfrak{t}_2$, t, t' *of the group* G_2, H_2 .

In fact, one can set:

$$\mathfrak{T}_1 = S^{1/2}\,\mathfrak{R}, \qquad \mathfrak{T}_2 = S^{1/2}\,\mathfrak{R}^{-1}, \quad T = S^{1/2}\,\mathfrak{R}\mathfrak{P}, \quad T' = S^{1/2}\,\mathfrak{R}^{-1}\mathfrak{P},$$

as long as $\mathfrak{R}$ means a transformation of the group G_1 whose parameter λ has the value $1 / \cos\vartheta$ or $-1 / \cos\vartheta$ according to one's choice of $S^{1/2}$. As always, 2ϑ is the angle of the rotation S. g^* and γ^* now go to $\bar{g}$ and $\bar{\gamma}$ for all positions of the line g.

§ 13.

Continuation. – Return to motions and transfers in the plane.

The fact that most of the considerations that we applied to the last theorems agree (at least, the sound of the words) with those of § 7 deserves a more precise explanation. It can be found in the following argument:

The group G_3, H_3 *that was defined in* § 7 *contains an invariant subgroup* G_2', H_2' *that is holohedrally isomorphic to the group* G_2, H_2 *that was defined in* § 12.

One will see that easily when one constructs the theorem that was presented in § 7 somewhat differently from what was done there. If one connects the transformations of the group G_1 (§ 7), not with the group G_2 of all screwing motions around the axis n, but only with the group G_1' of rotations around that axis, then a group G_2' of affine transformations of space will arise that will permute the rays through any point o that is chosen on n in precisely the same way as the transformations of the group G_2 of § 12. If one now adds the null system $\mathfrak{W}$ to the transformations of G_2' and the polar system $\mathfrak{P}$ to the transformations of G_2 then the group G_2, H_2 of § 12 will arise in the second case, and a certain group G_2', H_2' in the first one. Both groups are holohedrally isomorphic in the restricted sense of the term that is used in Galois theory. If one relates the transformations of G_2 and G_2' with each other in the given way then their transformations will be associated with each other in a *single-valued* and invertible way, and when one then also associates the dualistic transformations $\mathfrak{W}$ and $\mathfrak{P}$ with each other.

The translations in the direction of the axis of n commute with the transformations of G_2', H_2'; if one then combines them with the transformations of G_2', H_2' then the group G_3, H_3 will once more arise. We can then also say:

One can relate the group G_3, H_3 *of* § 7 *to the group of* G_2, H_2 *in* § 12 *isomorphically in such a way that the translations that are parallel to the axis n will be associated with identity transformation.*

One might remark in passing that the developments of § 12 can themselves be expressed on the basis of an intuition in a manner that is entirely similar to the usual one; namely, one will obviously have the following two (probably mutually-implicit) theorems:

One can relate the group of all motions and transfers in space to the group of all rotations and transfers around a fixed point isomorphically in such a way that one associates the translations with the identity transformation.

One can relate the group of motions and transfers in space to the group of motions in a ray bundle isomorphically in such a way that one associates the translations and the reflections in points with the identity transformations.

That is an immediate consequence of the fact that the points of the plane at infinity will be permuted with each other by the most general motions and transfers in exactly the same way as the rotations around a fixed point.

No less important than the relationship between the motions in a ray bundle and the motions and transfers in space is their relationship with the motions and transfers in the plane, which has a totally different nature.

One can relate the geometry of the ray bundle o with plane geometry by a dualistic transformation of space as by a central projection. We choose the second one, which is an intuitively-approachable process. We thus now choose any plane π that does not contain the point o, and replace every ray g through the point o with its trace x in the plane π and any plane γ through o with trace line u. We can then carry over the theory that was developed in § 12 to the plane π with no further discussion. *A geometric theory of collinear and dualistic transformations of a conic section in the plane* (which is an imaginary circle, at first) *then arises* that can naturally also be given a basis immediately. We must return to this situation repeatedly on some later occasions; for now, we restrict ourselves to the consideration of a limiting case that leads us back to the investigations in § 2 and § 3.

If the point o is shifted to infinity in a certain direction then the continuous family of motions in the ray bundle o will decompose into two separate families. When one replaces the rays and planes through the point o with their traces in a plane π that is perpendicular to the direction of o, they will go to the motions and transfers in the plane π.

The theory of motions and transfers in the Euclidian plane can then be regarded as a limiting case of the theory of motions in a ray bundle (*).

One can easily clarify the details of this.

Before carrying out our passage to the limit, we let x, $\bar{x}$, x', $\bar{\bar{x}}$ denote the traces of the rays g, $\bar{g}$, g', $\bar{\bar{g}}$, resp., in the plane p and let u, $\bar{u}$, u', $\bar{\bar{u}}$ denote the traces of those rays on the polar planes γ, $\bar{\gamma}$, γ', $\bar{\bar{\gamma}}$, resp. (Cf., § 12, pp. 51) Therefore, perhaps, the points x and x', which are associated with each other by the motion S of the ray bundle, might lie at infinity; they shall be fixed when o goes to infinity in a direction that is perpendicular to π.

Now, in the limiting case, one of the lines $\bar{u}$, $\bar{\bar{u}}$ must obviously go to the line at infinity in the plane π.

(*) **F. Klein**, "Ueber die sogennante Nicht-Euclidische Geometrie," Math. Ann., Bd. 4.
The theorem that was posed on pp. 602 there that the real linear transformations of (general, belonging to a polar system) conic section decompose into two disjoint families is correct only for conic sections with *real* points. The division of linear transformations of a conic section into "proper" and "improper" that **Lindemann** made in a book that will appear shortly, and is excellent in many respects (*Vorlesungen über Geometrie*, v. II, Leipzig, 1891, pp. 382-385), is completely incomprehensible to me.

We first assume that the line $\bar{u}$ is shifted to infinity: We then obtain a motion in the plane π as a limiting case of the motion S in the ray bundle. The midpoint of the motion is the limiting position of the point of intersection of the rotational axis n with the plane π; however, the middle plane v meets π in the line at infinity. $\bar{x}$ will be the midpoint of the chord xx', and $\bar{\bar{x}}$ will be its point at infinity. $\bar{\bar{u}}$ will be the normal to the chord xx', while u and u' will, in turn, go to the line at infinity.

Secondly, the line $\bar{\bar{u}}$ might be shifted to infinity: A transfer in the plane π will then arise. The middle line of the transfer is the line of intersection of π with the limiting position of the middle plane v of the motion of the ray bundle; however, the limiting position of the trace of the rotational axis will be point at infinity that is perpendicular to the middle line. $\bar{\bar{x}}$ will be the middle of the chord xx', and $\bar{x}$ is its point at infinity. $\bar{u}$ will be the normal to the chord xx', while u and u' will coincide with the line at infinity, as before.

Here, we have carried out the passage to the limit in such a way that we fixed two points x, x' that lay in the plane π. However, one can just as well also fix two associated lines u, u'. The points x, $\bar{x}$, x', $\bar{\bar{x}}$ will all lie at infinity then. If we go to a motion then $\bar{\bar{u}}$ will be the angle bisector of u and u', which contains the center of rotation; $\bar{u}$ will be the other angle bisector. If we go to a transfer then $\bar{u}$ will be parallel to the middle line of the transfer, and $\bar{\bar{u}}$ will be perpendicular to it.

We can summarize all of these facts in a simple way:

If one goes from a motion of the ray bundle to a motion of the plane then the collinear transformations $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *of the bundle will go to the similarly-denoted affine transformations of the plane; on the other hand, the dualistic transformations* T *and* T' *will degenerate.*

However, if one goes from a motion in the ray bundle to a transfer in the plane then only the dualistic transformation that is denoted by T *will remain; by contrast, the transformations* $\mathfrak{T}_1$, $\mathfrak{T}_2$, *and* T' *will degenerate.*

Some of the properties of a motion in a ray bundle will then be found for the motions in the plane, and others for the transfers; however, other properties will be lost in the limiting case. Naturally, one can also pursue all of the other theorems of § 12 to the limit in a similar way. Following up on that thought might be left to the reader.

In order to be able to connect up with these things later on, we must at least mention that we can implement the transition from the motions and transfers around a fixed point o in space to the motions and transfers in a plane in yet another way. Namely, if we enclose the point o with a sphere, and we then consider its points to be spatial elements then (as is known) we will again have a three-parameter group in the motions and transfers of that surface that goes to the group of motions and transfers of a plane in the limit. As one sees, an essential difference exists between both kinds of generalization of elementary geometry. In the geometry of the ray bundle, a *continuous group* appears in place of the motions and transfers of the plane, while in the geometry of the spherical surface, we have before us a group that consist of two *disjoint families* of transformations. Those families will not split under the passage to the plane, as the motions of the ray bundle do. Moreover, the motions of the sphere go to the motions of the plane, and the transfers of the sphere go to the transfers of the plane.

The relationship between the continuous group of rotations in a ray bundle and the group of rotations and transfers of the spherical surface, which consists of two families, is mediated by the reflection in the center of the sphere. That involutory transformation is missing from the first group, but it is contained as a distinguished element in the second. It is associated with just the identity transformation under the isomorphic relationship of the two groups. It vanishes under passage to the limit: In that way, it will become possible for both groups to have one and the same decomposable group as their limiting positions when one is continuous and the other decomposes.

We regard it as useful to discuss this simple relationship more thoroughly, since we have not, in fact, made it sufficiently clear up to now. We will then be able to summarize it more quickly on later occasions.

From what was said, the theory of motions and transfers in the plane will be fused with the theory of motions of a ray bundle at a higher level, to some extent; however, equality would not be true if one chose the other kind of generalization.

Can one also regard the motions and transfers in three-fold extended space to be a limiting case of a six-parameter, continuous group? As is known, the answer proves to be no: No matter how one might perform the generalization, one will always obtain two disjoint families of transformations. There is simply no radical analogy between the metric geometry of the plane and that of space: Moreover, one can put only the even-dimensional manifolds and the odd-dimensional manifolds into a series whose terms offer a close analogy; this is a fact that is frequently remarked and emphasized in the theory of quadratic forms.

With those remarks, we have already exceeded the bounds of this study.

Addenda to section I.

Added to § 6 (pp. 27). One can also pose the theorem on pp. 19 above in a simpler way: One lets w, $\bar{w}$, w' denote the three planes that are associated with x, $\bar{x}$, x' in the null system $\mathfrak{W}$. w and w' will then intersect in a line of $\bar{w}$. However, that plane is identical with the plane that we have usually denoted by $\bar{\bar{u}}$, which is the normal plane to the chord xx'. $\bar{w}$ is then one of the two angle bisectors of w and w'. One now has, by construction:

$$w\{\mathfrak{W}\}x\{\mathfrak{T}_1\}\,\bar{x}\,\{\mathfrak{W}\}\,\bar{w}\,\{\mathfrak{T}_2\}x'\{\mathfrak{W}\}w',$$

so, from formula (6) and (7), pp. 25:

$$w\,\{\mathfrak{T}_2\}\,\bar{w}\,\{\mathfrak{T}_1\}\,w'.$$

It is easy to see that $\bar{w}$ is the angle bisector of the *first* kind of w and w'. Cf., the theorem on pp. 48 below.

Added to § 9 (pp. 37). I was unaware that **C. Jordan** had dealt with the problem of the groups of motion in the most general way in his well-known treatise (Annali di Matematica, ser. III, t. II). Nonetheless, perhaps many readers will welcome the brief exposition in § 9, even if it does not contain anything new.

Added to § 10. The fundamental theorem at the center of everything is well-known. From a communication of **Schönflies**, it goes back to the investigations that **Hessel** that were devoted to crystal structure.

II.

On the parametric representation of motions and transfers.

In the present section, we will now address the question of finding the simplest-possible analytical representation of the motions in three-fold extended space. It shall be shown that one can associate those transformations with eight homogeneous parameters that are coupled by a quadratic relation in a single-valued, invertible manner, and that those parameters have entirely similar properties to **Euler**'s parameters for the rotations around a fixed point. The parametric representation thus-found will then be extended to transfers. It will also be shown how one can derive **Euler**'s corresponding formulas for motions and transformations in the plane from the formulas.

The considerations are, on the one hand, based upon certain ideas of **Cayley** and **Clifford** on the algebra of quaternions, and also upon some recent investigations into systems of complex numbers, in their own right, and in another direction, upon the well-known work of **Hermite**, **Cayley**, **Frobenius**, and others into the parametric representation of the linear transformations of a quadratic form. For the author, the starting point was in the theory of complex numbers; that is also the foundation for the following presentation. The relationships to the other cited work (whose main facts are easy to exhibit, moreover) will be first set down rigorously in a later section that will deal with the application of the methods of the theory of invariants to our situation.

The following analytical developments are, on the whole, independent of the geometric theory that was set down in the first section. It is only when brevity permits that we have carried over a number of the theorems from the first section, without deriving them anew with the tools of analysis.

Both sections extend each other reciprocally. They define a connected whole when they are considered together. First of all, in a methodological context, we have strived to give a presentation that is as elementary as possible here by employing rectangular parallel coordinates exclusively. Secondly, from a practical standpoint, we prefer to take advantage here of the linear and dualistic transformations that were denoted by $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, etc. before. Those transformations exhibit a further property that is now important: Their coefficients will become the simplest possible *linear* functions of seven independent ratios in a suitable notation. One can then treat three of the theorems that were summarized in § 7. *One* of them has already been known for fifty years by **Rodrigues**, and whose meaning will appreciated completely. In the meantime, that discovery, which defined the foundations of his original theory, has remained unnoticed.

In order to avoid repeating one and the same argument, we will now pursue a different train of thought from the one that was followed in the first section. We begin immediately with the motions in three-fold extended space and follow that with the associated transfers, in order to eventually ascend to the motions and transfers of the plane.

§ 1.

The transformation coefficients.

As we said, in the present study, we shall appeal to rectangular Cartesian coordinates. We then express a *motion* by a system of equations of the form:

$$(1)\qquad \begin{cases} a_{00}z_1' = a_{10} + a_{11}z_1 + a_{12}z_2 + a_{13}z_3, \\ a_{00}z_2' = a_{20} + a_{21}z_1 + a_{22}z_2 + a_{23}z_3, \\ a_{00}z_3' = a_{30} + a_{31}z_1 + a_{32}z_2 + a_{33}z_3. \end{cases}$$

Should these formulas actually imply a motion, then some known condition equations would have to exist between the coefficients a_{ik}. We briefly summarize those relations, in order to refer to them from now on. We will introduce several brief notations into their left-hand sides, for the sake of later considerations. However, for the sake of simplicity, whenever three formulas follow from each other by cyclic permutation of the indices 1, 2, 3, only one of them will be exhibited. The numeral in square brackets will give the number of relations that are equivalent to the stated one.

We put the expression for the determinant of our transformation at the center of all attention, which is a relation that has degree three in the coefficients $a_{i\kappa}$:

$$(2)\qquad \Delta = |\,a_{11}\ a_{22}\ a_{33}\,| = a_{00}^3\ .$$

We will not employ other relations of degree three or higher; however, it is probably important to understand the relations of degree two in the coefficients a_{ik} as precisely as possible.

They are the following ones:

$$(3)\qquad R_{i\kappa} = a_{00}\,a_{i\kappa} - \frac{\partial\Delta}{\partial a_{i\kappa}} = 0 \qquad [9],$$

or, more thoroughly:

$$(3b)\qquad \begin{cases} R_{11} = a_{00}a_{11} - a_{22}a_{33} + a_{23}a_{32} = 0, & [3], \\ R_{23} = a_{00}a_{23} - a_{12}a_{31} + a_{11}a_{32} = 0, & [3], \\ R_{32} = a_{00}a_{32} - a_{21}a_{13} + a_{11}a_{23} = 0, & [3], \end{cases}$$

and furthermore:

$$(4)\qquad \begin{cases} P_1 = a_{00}^2 - a_{11}^2 - a_{12}^2 - a_{13}^2 = 0, & [3], \\ P_1' = a_{00}^2 - a_{11}^2 - a_{21}^2 - a_{31}^2 = 0, & [3], \end{cases}$$

$$(5)\qquad \begin{cases} Q_1 = a_{21}a_{31} + a_{21}a_{31} + a_{21}a_{31} = 0, & [3], \\ Q_1' = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, & [3]. \end{cases}$$

As is known, the six equations $P_i = 0$, $Q_i = 0$ are independent of each other; it follows from them that:

$$\Delta = \pm a_{00}^3 .$$

If one chooses the upper sign in this then that will yield the remaining relations. Three of the nine coefficient ratios $a_{11} : a_{00}$, …, $a_{33} : a_{00}$ will then remain arbitrary, as they must.

The second-degree relations (3), …, (5) that were written down (there are twenty-one of them, in all) are obviously not linearly-independent of each other; namely, if the quantities $a_{i\kappa}$ have arbitrary values then one will also have the identity:

$$P_1 + P_2 + P_3 = D = P_1' + P_2' + P_3' . \tag{6}$$

Therefore, any one of the six equations (4) will be superfluous; it can be dropped without affecting the symmetry.

The remaining twenty relations are linearly-independent. They represent the complete system of all second-degree relations that exist for the coefficients $a_{i\kappa}$.

That is, if one has any second-degree expression $\Omega(a_{i\kappa})$ that vanishes for the coefficients $a_{i\kappa}$ of an orthogonal substitution then Ω can be represented linearly with constant coefficients in terms of the expressions $R_{i\kappa}$, Q_i, Q_i', and the five expressions P_i, P_i'.

We will carry out the proof of this (as it seems) hitherto-unnoticed theorem later (in § 5) in a very simple way when we put the system of equations (3), …, (6) into another form.

Equations (1), whose coefficients are coupled with each other by the conditions (2), …, (5), include six linearly-independent infinitesimal transformations, namely, infinitely-small motions whose expressions were given already by Euler. We write them in the following way, by applying the symbolism that was introduced by **S. Lie**:

$$\left\{ \begin{aligned} X_1 f &= 2\left(z_3 \frac{\partial f}{\partial z_2} - z_2 \frac{\partial f}{\partial z_3} \right), & Y_1 f &= -2\frac{\partial f}{\partial z_1}, \\ X_2 f &= 2\left(z_1 \frac{\partial f}{\partial z_3} - z_3 \frac{\partial f}{\partial z_1} \right), & Y_2 f &= -2\frac{\partial f}{\partial z_2}, \\ X_3 f &= 2\left(z_2 \frac{\partial f}{\partial z_1} - z_1 \frac{\partial f}{\partial z_2} \right), & Y_3 f &= -2\frac{\partial f}{\partial z_3}. \end{aligned} \right. \tag{7}$$

As one easily sees, $X_1 f$, …, $X_3 f$ represent infinitely-small rotations around the coordinate axes, while $Y_1 f$, …, $Y_3 f$ are translations along those axes.

With the choice of infinitesimal transformations that we made, the *compositions* in the group of motions will be given by the formulas:

$$
(8) \qquad \begin{cases}
(X_2X_3) = 2X_1,\ (X_3X_1) = 2X_2,\ (X_1X_2) = 2X_3, \\
(X_2Y_3) = 2Y_1 = (Y_2X_3), \\
(X_3Y_1) = 2Y_2 = (Y_2X_1), \\
(X_1Y_2) = 2Y_3 = (Y_1X_2), \\
(Y_i\,Y_i) = 0,\ (X_iY_\kappa) = 0 \\
(i, \kappa = 1, 2, 3).
\end{cases}
$$

For the sake of a later application in connection with this, we shall determine the infinitesimal transformations of the group of motions that fix a given point z_1^0, z_2^0, z_3^0. That group contains three independent infinitesimal transformations, namely, rotations around axes that run parallel to the coordinate axes:

$$
(9) \qquad \begin{cases}
Z_1 f = 2(z_3 - z_3^0)\dfrac{\partial f}{\partial z_2} - 2(z_2 - z_2^0)\dfrac{\partial f}{\partial z_3}, \\
\quad = X_1 f - z_2^0\, Y_3 f + z_3^0\, Y_2 f, \\
Z_2 f = 2(z_1 - z_1^0)\dfrac{\partial f}{\partial z_3} - 2(z_3 - z_3^0)\dfrac{\partial f}{\partial z_1}, \\
\quad = X_2 f - z_3^0\, Y_1 f + z_1^0\, Y_3 f, \\
Z_3 f = 2(z_2 - z_2^0)\dfrac{\partial f}{\partial z_1} - 2(z_1 - z_1^0)\dfrac{\partial f}{\partial z_2}, \\
\quad = X_3 f - z_1^0\, Y_2 f + z_2^0\, Y_1 f.
\end{cases}
$$

In our later considerations, we shall treat the problem of expressing the transformation coefficients $a_{i\kappa}$ in the simplest-possible way in terms of a smaller number of quantities.

We think of the transformations S_1, S_2, … of an r-parameter continuous group as being represented in terms of $r + s + 1$ homogeneous parameters, between which, s mutually-independent relations exist. We would then like to say that *bilinear combinations* of those parameters exist when the parameters of the composed transformation S_1S_2 are homogeneous, linear functions of the parameters of S_1, as well as also the parameters of S_2. This kind of composition of the parameters, which is distinguished by its particular simplicity, will be present, for example, in the general projective group:

$$x_i' = c_{i0}\, x_0 + c_{i1}\, x_1 + c_{i2}\, x_2 + c_{i3}\, x_3 \qquad (i = 0, 1, 2, 3),$$

when one takes the sixteen coefficients $c_{i\kappa}$ themselves to be the parameters. One likewise finds that bilinear combinations exist for our group (1) of motions when one regards the thirteen coefficients $a_{i\kappa}$ as the parameters. However, there is an essential difference between both cases: Whereas the number of parameters $c_{i\kappa}$ has the smallest conceivable value (viz., sixteen) for the general projective group, we have thirteen homogeneous parameters for the six-parameter group of motions, and therefore, no less than *six* more parameters than would be ultimately required for the representation of the motions by homogeneous parameters. One remarks that the ease of the analytical, as well as geometric, treatment of a group depends precisely upon the ease of composing several transformations of the group with each other, which itself raises the question of whether one can express the motions in terms of less than thirteen homogeneous parameters while preserving the requirement of bilinear combination. We then pose the problem:

Represent the motions in space by the smallest-possible number of homogeneous parameters with bilinear combination.

The solution of that problem cannot be determined completely; it still contains arbitrary elements. If one has found a system of parameters with the stated property then one will directly obtain infinitely-many parametric representations of the same kind by a linear transformation. Furthermore, the expression for the coefficients a_{ik} in terms of the parameters in question will necessarily be undetermined when one shows that the smallest number of parameters with bilinear combination is greater than seven: In that case, one can alter the form of the coefficients a_{ik} in many ways with the help of the relations that then exist between them. However, if one regards all such parameter systems that can be derived from a single one as identical then it will not be clear from the outset that no other ones are possible that one could not arrive in that way.

One can pose a problem for entirely arbitrary continuous groups that is similar to the one that was just formulated; from the investigations of **S. Lie**, there are projective groups of any arbitrary composition. Meanwhile, two essentially different parametric representations with bilinear combination already exist for one-parameter groups, one of which is a limiting case of the other one, and for three-parameter groups, there also already exists the case in which several essentially different parametric representations of equal generality are possible.

The fact that the group of motions exhibits a different, and indeed, simpler, character will be seen in the next paragraph.

§ 2.

Biquaternions.

The problem that was formulated in § 1 can now be reduced to the following one, by an exposition that the author gave on another occasion:

"Find all types of systems of complex numbers with $7 + s$ principal units for a smallest-possible value of the number s, when those systems have the property that the associated $(7 + s - 1)$-parameter, simply-transitive group $G^{(1)}$ contains a subgroup that has the same composition as the group of motions (*)."

(*) Cf., the treatise: "Ueber Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen," Wiener Monatshefte 1890 (I. Jahrg.), 8-10 Heft. G_1 is written there, instead of the notation $G^{(1)}$ of the present text.

Now, it emerges from the investigations of **Scheffers** (*) that the smallest value of the number s is $s = 1$, and that only *one* system of complex numbers belongs to that value whose group $G^{(1)}$ contains one (and only one) subgroup with the desired composition. One can conclude from this that:

A representation of the motions by seven homogeneous parameters with bilinear combinations is impossible.

By contrast, there is one, and in fact, essentially only one, such representation by eight homogeneous parameters, between which one relation exists (**).

In essence, that means: except for linear transformations of the parameters, and naturally, except for transformations by virtue of the identity that shall be assumed to exist between the parameters.

The system of complex numbers in question is one that was discovered by **Clifford** and is the number system that is referred to as *biquaternions*. When one splits the eight principal units into two groups of four $e_0, \ldots, e_3, \varepsilon_0, \ldots, \varepsilon_3$, its multiplication table will read (***):

(*) **G. Scheffers**, "Zurückführung complexer Zahlensysteme auf typische Formen," Math. Ann. **39** (1891), pp. 293, § 10, *et seq.*

(**) It might perhaps be of interest to some readers for us to expressly prove that the theorem in the text does not so much pertain to the theory of Scheffers, as much as it expresses the non-existence of a second, simpler, or even just-as-simple, parametric representation of the motions. Moreover, (according to the nature of the situation) we will not have to make use of not-entirely-simple considerations, and in particular, of the important consequence that was just pointed out, either.

(***) **Clifford**, who died too soon, considered three kinds of "biquaternions" in several partially-incomplete treatises, which one will find published in his "Mathematical Papers" (London, 1882), and he related them to the so-called elliptic, parabolic (Euclidian), and hyperbolic geometries. He arrived at them by the operation that Scheffers called "multiplication" when he coupled Hamilton's quaternions with the three binary number systems ($i_0 = 1$, $i_1^2 = i_0$), ($i_0 = 1$, $i_1^2 = 0$), ($i_0 = 1$, $i_1^2 = -i_0$), resp. He operated with them in roughly the same way that Hamilton and his followers operated with ordinary quaternions. Clifford did not seem to arrive at a parametric representation and composition of the motions, any more than **Buchheim**, who sought to give a summary presentation and extension of Clifford's ideas. ["A memoir on Biquaternions," Am. J. Math. **8** (1885)]

With the author's terminology, the biquaternions that belong to the elliptic and hyperbolic geometries are different *forms* of the same *type*, but the one that belongs to parabolic geometry corresponds to a new type that is a degenerate case of the first one. In the following sections of this investigation, we shall speak no more of these and two other, similarly-constructed number systems.

One will find the system that is employed here in **Scheffers**, pp. 380. There, it is denoted by Q_3 .

(1)

	e_0	e_1	e_2	e_3	ε_0	ε_1	ε_2	ε_3
e_0	e_0	e_1	e_2	e_3	ε_0	ε_1	ε_2	ε_3
e_1	e_1	$-e_0$	e_3	$-e_2$	ε_1	$-\varepsilon_0$	ε_3	$-\varepsilon_2$
e_2	e_2	$-e_3$	$-e_0$	e_1	ε_2	$-\varepsilon_3$	$-\varepsilon_0$	ε_1
e_3	e_3	e_2	$-e_1$	$-e_0$	ε_3	ε_2	$-\varepsilon_1$	$-\varepsilon_0$
ε_0	ε_0	ε_1	ε_2	ε_3	0	0	0	0
ε_1	ε_1	$-\varepsilon_0$	ε_3	$-\varepsilon_2$	0	0	0	0
ε_2	ε_2	$-\varepsilon_3$	$-\varepsilon_0$	ε_1	0	0	0	0
ε_3	ε_3	ε_2	$-\varepsilon_1$	ε_0	0	0	0	0

If we write $(e_i\, e_k)$ as an abbreviation for $e_i\, e_k - e_i\, e_k$ then from this table, we will have the relations:

$$(2)\qquad \begin{cases} (e_2e_3) = 2e_1,\ (e_3e_1) = 2e_2,\ (e_1e_2) = 2e_3, \\ (e_2\varepsilon_3) = 2\varepsilon_1 = (\varepsilon_2 e_3), \\ (e_3\varepsilon_1) = 2\varepsilon_2 = (\varepsilon_3 e_1), \\ (e_1\varepsilon_2) = 2\varepsilon_3 = (\varepsilon_1 e_2), \\ (e_i\varepsilon_i) = 0,\ (i = 1, 2, 3),\ (\varepsilon_i\varepsilon_k) = 0,\ (i, k = 1, 2, 3), \end{cases}$$

$$(3)\qquad (e_i\ \varepsilon_0) = 0, \qquad (\varepsilon_i\ \varepsilon_0) = 0, \qquad (i = 1, 2, 3).$$

If we introduce the further abbreviations:

$$a = \sum_{i=0}^{3}(\alpha_i e_i + \beta_i \varepsilon_i), \qquad a' = \sum_{i=0}^{3}(\alpha_i' e_i + \beta_i' \varepsilon_i), \text{ etc.,}$$

$$x = \sum_{i=0}^{3}(\xi_i e_i + \eta_i \varepsilon_i), \qquad x' = \sum_{i=0}^{3}(\xi_i' e_i + \eta_i' \varepsilon_i), \text{ etc.,}$$

then we can write the seven-parameter group $G^{(1)}$ (which we can write more briefly as G_7) that is linked with the biquaternions (1) as:

$$(4)\qquad x' = x\, a.$$

Formulas (8) of § 1 and formulas (2), (3) of the present paragraph then imply (cf., Wiener, Ber. *loc. cit.*, § 9) that the group G_7, in fact, contains a single subgroup G_6, and in fact, an *invariant* one, that has the same composition as the group of motions, which is a group whose general infinitesimal transformation reads:

$$(5)\qquad x' = x\,[e_0 + (\mathfrak{a}_1\, e_1 + \mathfrak{a}_2\, e_2 + \mathfrak{a}_3\, e_3 + \mathfrak{b}_1\, \varepsilon_1 + \mathfrak{b}_2\, \varepsilon_2 + \mathfrak{b}_3\, \varepsilon_3)\ \delta t].$$

One further infers that a (single) *distinguished* subgroup G_1 is present, whose infinitesimal transformation is:

(6) $$x' = x\,[e_0 + \varepsilon_0\,\delta t].$$

The group G_6 is related isomorphically to the group of motions in space when one writes the general infinitesimal motion in the form:

(7) $$\mathfrak{a}_1\,X_1 f + \mathfrak{a}_2\,X_2 f + \mathfrak{a}_3\,X_3 f + \mathfrak{b}_1\,Y_1 f + \mathfrak{b}_2\,Y_2 f + \mathfrak{b}_3\,Y_1 f,$$

and assigns the same values to the constants $\mathfrak{a}_i$, $\mathfrak{b}_i$ that they had in formula (5).

On top of that, we will now treat, above all, the exhibition of the *finite* equations of the group G_6 that is generated by the infinitesimal transformations (5), i.e., finding the condition that the eight parameters α_i, β_i must satisfy if the transformation (4) of G_7 is to belong to G_6 . However, before we do that, we can point out some important consequences of the theorems that were derived already. Namely, we can now give the formula for the composition of the parameters (α, β).

Let S, S' be any two motions, and let S'' be their product (= SS'), and furthermore, let $x' = xa$, $x' = xa'$, $x' = xa''$ be three transformations of G_6 that correspond to the motions S, S', S'', so one will have $a'' = aa'$; i.e., the formulas for the composition of the parameters (α, β) will be provided by the so-called *multiplication theorem* of our number system (1).

If we abandon the algorithm of the complex numbers then we can now formulate the theorem:

If one has expressed the motions in space in terms of any system of eight homogeneous parameters with bilinear combination then the formulas for the composition of parameters can always be assigned to a linear transformation of the parameters of the following form:

(8) $$\begin{cases} \alpha_0'' = \alpha_0\alpha_0' - \alpha_1\alpha_1' - \alpha_2\alpha_2' - \alpha_3\alpha_3', \\ \alpha_1'' = \alpha_0\alpha_1' + \alpha_1\alpha_0' + \alpha_2\alpha_3' - \alpha_3\alpha_2', \\ \alpha_2'' = \alpha_0\alpha_2' + \alpha_2\alpha_0' + \alpha_3\alpha_1' - \alpha_1\alpha_3', \\ \alpha_3'' = \alpha_0\alpha_3' + \alpha_3\alpha_0' + \alpha_1\alpha_2' - \alpha_2\alpha_1', \end{cases}$$

$$
(8)\ [sic]\qquad \left\{\begin{aligned}
\beta_0'' &= \alpha_0\beta_0' - \alpha_1\beta_1' - \alpha_2\beta_2' - \alpha_3\beta_3' \\
&\quad + \beta_0\alpha_0' - \beta_1\alpha_1' - \beta_2\alpha_2' - \beta_3\alpha_3', \\
\beta_1'' &= \alpha_0\beta_1' + \alpha_1\beta_0' + \alpha_2\beta_3' - \alpha_3\beta_2' \\
&\quad + \beta_0\alpha_1' + \beta_1\alpha_0' + \beta_2\alpha_3' - \beta_3\alpha_2', \\
\beta_2'' &= \alpha_0\beta_2' + \alpha_2\beta_0' + \alpha_3\beta_1' - \alpha_1\beta_3' \\
&\quad + \beta_0\alpha_2' + \beta_2\alpha_0' + \beta_3\alpha_1' - \beta_1\alpha_3', \\
\beta_3'' &= \alpha_0\beta_3' + \alpha_3\beta_0' + \alpha_1\beta_2' - \alpha_2\beta_1' \\
&\quad + \beta_0\alpha_3' + \beta_3\alpha_0' + \beta_1\alpha_2' - \beta_2\alpha_1'.
\end{aligned}\right.
$$

The first four of these equations represent a well-known system of formulas – viz., the multiplication of quaternions – by whose extension the number system (1) will arise.

If we consider the determinant of the system of equations (8), after we have ordered them relative to the quantities α_i', β_i', then we will get an expression $N^4(\alpha, \beta)$, in which we have set:

$$
(9)\qquad N(\alpha, \beta) = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2,
$$

to abbreviate. If we then order it relative to the quantities α_i, β_i, and once more take the determinant then that will yield the value $N^4(\alpha', \beta')$. The expression N, which is completely independent of the quantities β_i, will then play the role of a *discriminant* for our system of equations. The values of the quantities (α, β) for which N vanishes define the degenerate transformations of the group G_7. We will always have to consider non-degenerate transformations; we can then drop the factor N from any equation that contains that factor.

The formula:

$$
(10)\qquad N(\alpha) \cdot N(\alpha') = N(\alpha''),
$$

which is likewise well-known from the theory of quaternions, follows from the first four of equations (8).

We arrived at these theorems without knowing the condition equation between the parameters α_i, β_i. Later, we will see that the expressions for the motions themselves are independent of that relation, to a certain extent. (Cf., § 6)

§ 3.

The group G_6 and its adjoint group.

We now set $x = \sum \xi_i\, e_i + \eta_i\, \varepsilon_i$, $a = \sum \alpha_i\, e_i + \beta_i\, \varepsilon_i$, as in the previous paragraph, and interpret the quantities $\xi_i : \xi_0$, $\eta_i : \xi_0$ as, say, Cartesian coordinates in a seven-dimensional space R_7. Since the group $x' = xa$, which is our group G_7, is simply-transitive, the space R_7 will decompose into a family of ∞^1 six-fold extended spaces under the transformations of G_6, each of which will be transformed transitively. The family of manifolds can now be given easily: It will be represented by the equation:

$$\begin{aligned}(1)\qquad f &= \lambda(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2) \\ &\quad + \mu(\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3) = 0\end{aligned}$$

when one assigns all possible values to the parameter $\lambda : \mu$. In fact, one easily convinces oneself that the increase in the function f will vanish for any infinitesimal transformation of the group G_6. [Cf., formula (5), § 2] (*)

The condition equation between the parameters α_i, β_i that is characteristic of the transformations of G_6 must now take the form:

$$f = \lambda(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \mu(\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) = 0.$$

Namely, let $x' = xa$ and $x'' = x'a'$ be any two transformations of G_6, and let $x'' = xa''$ be the transformation of the two combined, so one has $a'' = aa'$; however, that equation will once more have the form $x' = xa$. Now, since the same relation must exist between the parameters α_i'', β_i'' that exists between the parameters α_i, β_i and α_i', β_i', it will follow that the desired relation will emerge from one of equations (1) when one writes α_i for ξ_i and β_i for η_i. We then only have to specialize the parameters $\lambda : \mu$ in a suitable way. Since the infinitesimal transformations of G_6 are already known, one will find that $\lambda = 0$, with no further analysis. We then have the theorem:

With the choice of eight homogeneous parameters for a motion that we made, the relation that exists between them will read:

$$(2)\qquad L(\alpha, \beta) = \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.$$

Equation (2) is linear in each of the eight parameters α_i, β_i. When interpreted in a seven-dimensional space, it represents a general quadratic manifold that can be described by two real families of linear three-fold extended spaces.

One can confirm by a generally laborious calculation that the equation $L(\alpha, \beta) = 0$ actually determines the transformations of a subgroup of G_7. Namely, due to equations (8), § 2, one will have the identity:

$$(3)\qquad L(\alpha'', \beta'') = N(\alpha', \beta')\cdot L(\alpha, \beta) + N(\alpha, \beta)\cdot L(\alpha', \beta'),$$

(*) The increase in f takes the value:

$$\mu(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2)\,\delta t,$$

under an infinitesimal transformation of G_1 (cf., (6), § 2), which is once more an expression of the form f. The manifolds $f = 0$ will then be permuted with each other under the group G_1. Since that situation is already clear from equations (2) and (3) of § 2, one can find the invariant $\lambda : \mu$ of the group G_6 immediately from a theorem of **S. Lie**. Meanwhile, the application of the general theory of integration requires laborious calculations; the suggested verification might then suffice.

from which, the theorem will follow. (*)

We can make an important application of the relation $L(\alpha, \beta) = 0$ directly. Namely, if we compute the inverse $x' = xa^{-1}$ of a transformation $x' = xa$ of G_7, with the help of formulas (8), § 2, then we will generally find expressions of degree *three* in α_i, β_i for its parameters. However, if we have a transformation of G_6 before us, in particular, then we can already represent the parameters of its inverse transformation by expressions of degree *one* in α_i, β_i. In fact, if we set:

$$\bar{a} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3 + \beta_0 \varepsilon_0 - \beta_1 \varepsilon_1 - \beta_2 \varepsilon_2 - \beta_3 \varepsilon_3 , \tag{4}$$

to abbreviate, then we will have the identity:

$$a\{N \cdot \bar{a} - 2L \cdot (\alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3)\} = N^2 \cdot e_0 , \tag{5}$$

from which, one can deduce the value of a^{-1}, as well as the second-degree identity that follows from it:

$$a\bar{a} = \bar{a}a = N(\alpha, \beta) \cdot e_0 + 2L(\alpha, \beta) \cdot \varepsilon_0 . \tag{6}$$

If one sets L equal to zero then that will yield the theorem:

If the parameters of a motion S are:

$$\alpha_0 : \quad \alpha_1 : \quad \alpha_2 : \quad \alpha_3 : \quad \beta_0 : \quad \beta_1 : \quad \beta_2 : \quad \beta_3 ,$$

then

$$\alpha_0 : \ -\alpha_1 : \ -\alpha_2 : \ -\alpha_3 : \quad \beta_0 : \ -\beta_1 : \ -\beta_2 : \ -\beta_3$$

will be the parameters of the inverse motion S^{-1}.

On the basis of that remark, we will arrive at an expression for the finite equations of the *adjoint* of the group G_6 (or, what amounts to the same thing, the group of motions): The adjoint group will be expressed simply by the equation:

$$x' = \bar{a}xa \tag{7}$$

as long as one assigns the values:

$$\xi_0 = 1, \qquad \eta_0 = 1, \qquad \xi_i = \mathfrak{a}_i, \qquad \eta_i = \mathfrak{b}_i \qquad (i = 1, 2, 3)$$

to the coefficients ξ_i, η_i, and corresponding values to the coefficients ξ_i', η_i' (**).

We also write these equations down in explicit form, since we will have to employ them later on. They read:

(*) One will find a brief derivation of formula (3) in § 6 (pp. 87).
(**) Wiener Ber., *loc. cit.*, § 7 and § 10.

$$(8)\qquad \begin{cases} N\mathfrak{a}'_i = a_{i1}\mathfrak{a}_1 + a_{i2}\mathfrak{a}_2 + a_{i3}\mathfrak{a}_3, \\ N\mathfrak{b}'_i = b_{i1}\mathfrak{a}_1 + b_{i2}\mathfrak{a}_2 + b_{i3}\mathfrak{a}_3 \qquad\qquad (i = 1, 2, 3), \\ \qquad + a_{i1}\mathfrak{b}_1 + a_{i2}\mathfrak{b}_2 + a_{i3}\mathfrak{b}_3, \end{cases}$$

in which we have set:

$$(9)\qquad \begin{cases} a_{23} = 2(\alpha_2\alpha_3 + \alpha_0\alpha_1),\ a_{32} = 2(\alpha_2\alpha_3 - \alpha_0\alpha_1), \\ a_{11} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2, \\ b_{11} = 2(\alpha_0\beta_0 + \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3), \\ b_{23} = 2(\alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_0\beta_1 + \alpha_1\beta_0), \\ b_{32} = 2(\alpha_2\beta_3 + \alpha_3\beta_2 - \alpha_0\beta_1 - \alpha_1\beta_0), \quad \text{etc.}, \end{cases}$$

to abbreviate.

Hopefully, no confusion will arise from the fact that we have already employed the notations a_{ik} in § 1with a different meaning: In both cases, the expressions for the coefficients a_{ik} in terms of the parameters (α, β) are the same, as we see straightaway.

§ 4.

The parametric representation of motions.

We easily arrive at the parametric representation of the motions themselves from the expression that was found in the previous paragraph for the adjoint group.

Equations (8) and (9) of § 3 show how the infinitesimal transformations of the group G_6 commute with each other under the finite transformations of G_6. However, since the group $G6$ is related isomorphically to the group of motions in space by formulas (5) and (7) of § 2, we will also know then how the infinitesimal motions commute with each other by the finite motions that are expressed in terms of the parameters (α, β). We will then be in a position to calculate, for example, the infinitesimal transformations of the three-parameter group G'_3 that emerges from the group G_3 of all rotation around the fixed point z_1^0, z_2^0, z_3^0 by the motion $S(\alpha, \beta)$. If we then determine the midpoint z'_1, z'_2, z'_3 of the new group G'_3 then we will already have the desired expression of the motions with that. The coordinates z'_i will obviously be linear functions for the coordinates z_i^0 and *rational* functions of the parameters α_i, β_i.

We have already given the infinitesimal transformations of the group G_3 [in § 1, (9)]. For, e.g., Z_1f, we have $\mathfrak{a}_1 = 1$, $\mathfrak{a}_2 = 0$, $\mathfrak{a}_3 = 0$, $\mathfrak{b}_1 = 0$, $\mathfrak{b}_2 = z_3^0$, $\mathfrak{b}_3 = -z_3^0$. The expression for the transformed infinitesimal transformation $Z'_1 f$ then follows from that with the help of formulas (8) and (9), § 3:

$$\begin{aligned} N \cdot Z'_1 f &= a_{11}\, X_1 f + a_{21}\, X_2 f + a_{31}\, X_3 f \\ &\quad + (b_{11} + a_{12}\, z_3^0 - a_{13}\, z_2^0)\, Y_1 f, \\ &\quad + (b_{21} + a_{22}\, z_3^0 - a_{23}\, z_2^0)\, Y_2 f, \\ &\quad + (b_{31} + a_{32}\, z_3^0 - a_{33}\, z_2^0)\, Y_3 f\, ; \end{aligned}$$

corresponding expressions for $Z_2' f$ and $Z_3' f$ arise by cyclic permutation of the indices 1, 2, 3.

In order to find the point z_1', z_2', z_3' that is fixed by the transformation $Z_1' f$, $Z_2' f$, $Z_3' f$, we would do best to introduce three new transformations $\bar{Z}_1 f$, $\bar{Z}_2 f$, $\bar{Z}_3 f$, in place of $Z_1' f$, $Z_2' f$, $Z_3' f$, that again have the form $Z_1 f$, $Z_2 f$, $Z_3 f$. We then obtain the expressions for each coordinate z_1', z_2', z_3' a second time, and thus have, at the same time, a check for the validity of the calculations. One finds, with no effort:

$$\begin{aligned}\bar{Z}_1 f &= X_1 f - z_2' Y_3 f + z_3' Y_2 f \\ &= N^{-1} \cdot (a_{11} Z_1' f + a_{12} Z_2' f + a_{13} Z_3' f), \qquad \text{etc.}\end{aligned}$$

Not just $X_2 f$ and $X_3 f$ drop out of the right-hand side of this equation, but also $Y_1 f$, as a result of the existence of the relation $L(\alpha, \beta) = 0$. A comparison of the coefficients of $X_1 f$, $Y_3 f$, and $Y_2 f$ then gives the desired transformation formulas, namely, the expressions for the coefficients a_{ik} of § 1 in terms of the parameters (α, β), as long as we again write z_1^0, z_2^0, z_3^0 in place of z_1, z_2, z_3.

Furthermore, for the ultimate formulation of our results, we would not to like to appeal to the ordinary rectangular coordinates z_1, z_2, z_3, as we have up to now, but *homogeneous coordinates* $x_0 : x_1 : x_2 : x_3$, which are coupled by the equations:

$$z_1 = \frac{x_1}{x_0}, \qquad z_2 = \frac{x_2}{x_0}, \qquad z_3 = \frac{x_3}{x_0}.$$

We let $u_0 : u_1 : u_2 : u_3$ denote the associated plane coordinates, and let $p_{ik} = - p_{ki}$ denote the associated complex or line coordinates, such that, one will get, for example:

$$\begin{aligned} p_{01} &= x_0 y_1 - x_1 y_0, & p_{23} &= x_2 y_3 - x_3 y_2, \\ p_{02} &= x_0 y_2 - x_2 y_0, & p_{31} &= x_3 y_1 - x_1 y_3, \\ p_{03} &= x_0 y_3 - x_3 y_0, & p_{12} &= x_1 y_2 - x_2 y_1, \end{aligned}$$

for the coordinates of the connecting line of the point x and y.

With that, we can now express the motions in space in point, line, and plane coordinates:

$$(1) \qquad \left\{ \begin{aligned} x_0' &= a_{00} x_0, \\ x_1' &= a_{10} x_0 + a_{11} x_1 + a_{12} x_2 + a_{13} x_3, \\ x_2' &= a_{20} x_0 + a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \\ x_3' &= a_{30} x_0 + a_{31} x_1 + a_{32} x_2 + a_{33} x_3; \end{aligned} \right.$$

$$(2)\quad \begin{cases} p'_{01} = a_{11}p_{01} + a_{12}p_{02} + a_{13}p_{03}, & [3] \\ p'_{23} = b_{11}p_{01} + b_{12}p_{02} + b_{13}p_{03} & \\ \qquad + a_{11}p_{23} + a_{12}p_{31} + a_{13}p_{12}, & [3]; \end{cases}$$

$$(3)\quad \begin{cases} u'_0 = a_{00}u_0 \ + a_{01}u_1 + a_{02}u_2 + a_{03}u_3, \\ u'_1 = \quad \cdot \quad + a_{11}u_1 + a_{12}u_2 + a_{13}u_3, \\ u'_2 = \quad \cdot \quad + a_{21}u_1 + a_{22}u_2 + a_{23}u_3, \\ u'_3 = \quad \cdot \quad + a_{31}u_1 + a_{32}u_2 + a_{33}u_3. \end{cases}$$

The formulas represent the general motion in space, as long as one expresses the coefficients a_{ik} , b_{ik} in terms of the parameters (α, β) as follows:

$$(4)\quad \begin{cases} a_{00} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\ a_{11} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2, \\ a_{22} = \alpha_0^2 + \alpha_2^2 - \alpha_3^2 - \alpha_1^2, \\ a_{33} = \alpha_0^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2, \\ a_{23} = 2(\alpha_2\alpha_3 + \alpha_0\alpha_1), \qquad a_{32} = 2(\alpha_2\alpha_3 - \alpha_0\alpha_1), \\ a_{31} = 2(\alpha_3\alpha_1 + \alpha_0\alpha_2), \qquad a_{13} = 2(\alpha_3\alpha_1 - \alpha_0\alpha_2), \\ a_{12} = 2(\alpha_1\alpha_2 + \alpha_0\alpha_3), \qquad a_{21} = 2(\alpha_1\alpha_2 - \alpha_0\alpha_3), \end{cases}$$

$$(5)\quad \begin{cases} a_{10} = 2(\alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_0\beta_1 + \alpha_1\beta_0), \\ a_{20} = 2(\alpha_3\beta_1 - \alpha_1\beta_3 - \alpha_0\beta_2 + \alpha_2\beta_0), \\ a_{30} = 2(\alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_0\beta_3 - \alpha_3\beta_0), \\ a_{01} = 2(\alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_0\beta_1 - \alpha_1\beta_0), \\ a_{02} = 2(\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_0\beta_2 - \alpha_2\beta_0), \\ a_{03} = 2(\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_0\beta_3 - \alpha_3\beta_0), \end{cases}$$

and finally:

$$(6)\quad \begin{cases} b_{11} = 2(\alpha_0\beta_0 + \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3), & [3], \\ b_{23} = 2(\alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_0\beta_1 + \alpha_1\beta_0), & [3], \\ b_{32} = 2(\alpha_2\beta_3 + \alpha_3\beta_2 - \alpha_0\beta_1 - \alpha_1\beta_0), & [3]. \end{cases}$$

The connection between the quantities b_{ik} and the quantities a_{ik} will be given by the formulas:

$$(7)\quad \begin{cases} a_{20}a_{31} - a_{30}a_{21} = a_{00}b_{11}, & [3], \\ a_{30}a_{13} - a_{10}a_{33} = a_{00}b_{23}, & [3], \\ a_{10}a_{22} - a_{20}a_{12} = a_{00}b_{32}, & [3], \end{cases}$$

which are equations that one can solve for a_{10}, a_{20}, a_{30} in many ways [cf., § 5, pp. 84, (14)]. Finally, the connection between the quantities a_{i0} and the quantities a_{0i} will be represented by the formulas:

$$(8) \qquad \begin{cases} a_{00}a_{0i} + a_{10}a_{1i} + a_{20}a_{2i} + a_{30}a_{3i} = 0, & (i = 1, 2, 3), \\ a_{00}a_{i0} + a_{01}a_{i1} + a_{02}a_{i2} + a_{03}a_{i3} = 0, & (i = 1, 2, 3). \end{cases}$$

We observe that formulas (2) will agree precisely with formulas (8) of the previous paragraph as soon as one adds the factor a_{00} to their left-hand sides, and we can conclude:

One will get the adjoint of the group of motions when one interprets the coordinates p_{ik} of the linear complex that the motions are subjected to as the Cartesian coordinates in a six-fold extended space.

That should not be surprising: The theorem is only the analytical expression for the fact that is already familiar to us that an infinitesimal motion is linked with every linear complex, and conversely.

If one sets the quantities β_i all equal to zero then formulas (1) and (4) will go to the well-known **Euler** formulas for rotations around a fixed point, namely, the origin of the coordinates z_1, z_2, z_3 .

We have already pointed out that one can alter the expressions for the coefficients a_{ik} in terms of the parameters (α, β) in many ways. One might perhaps feel that there is something wrong with the fact that for the case $\alpha_0 = 1$, $\alpha_i = 0$, $\beta_0 = 0$, our formulas will yield negative values for the quantities a_{10}, a_{20}, a_{30} for positive values of the quantities β_1, β_2, β_3, and thus translations that are opposite to the positive directions of the coordinate axes. That will be easily remedied by the introduction of new parameters $\beta_k = -\beta_k$, or the parameters $\bar{\alpha}_0 = \alpha_0$, $\bar{\alpha}_i = -\alpha_i$, $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_i = -\beta_i$, in place of the parameters (α, β). The first change is not recommended, due to the changes that one will then make, as in § 7. However, as for the second one (against which, there is nothing to recall), the relationship between our parametric representation and Euler's formulas, as well as the theory of quaternions, is no longer quite as immediate as it once was. Namely, in place of the multiplication table for biquaternions, one now finds the so-called reciprocal table, which emerges from the latter by switching the rows with the columns. We have thus preferred to keep the expressions for the coefficients a_{i0} in the form (5), and to take the aforementioned minor flaw into account.

§ 5.

The transformation coefficients and the parameters (α, β).

Several important consequences are connected with the remark that one can solve Euler's equations (4), § 4 for the squares and products of the parameters α_i . One finds immediately that:

$$(1)\qquad \left\{\begin{array}{c} a_{00}+a_{11}+a_{22}+a_{33}=4\alpha_0^2, \\ a_{00}+a_{11}-a_{22}-a_{33}=4\alpha_1^2, \\ a_{00}+a_{22}-a_{33}-a_{11}=4\alpha_2^2, \\ a_{00}+a_{33}-a_{11}-a_{22}=4\alpha_3^2, \\ a_{23}-a_{32}=4\alpha_0\alpha_1, \qquad a_{23}+a_{32}=4\alpha_2\alpha_3, \\ a_{31}-a_{13}=4\alpha_0\alpha_2, \qquad a_{31}+a_{13}=4\alpha_3\alpha_1, \\ a_{12}-a_{21}=4\alpha_0\alpha_3, \qquad a_{12}+a_{21}=4\alpha_1\alpha_2. \end{array}\right.$$

That next yields the proof of the theorem that was discussed in § 1, namely, that the relations $P_i = 0$, $P_i' = 0$, $Q_i = 0$, $Q_i' = 0$, $R_{ik} = 0$ that were presented there include the complete system of all linearly-independent identities of degree two between the coefficients of an orthogonal substitution. One can then write down the identities between the squares and products of four independent quantities directly. There are six of type $\alpha_i^2 \cdot \alpha_k^2 - \alpha_i\,\alpha_k \cdot \alpha_i\,\alpha_k = 0$, twelve of type $\alpha_i^2 \cdot \alpha_k\,\alpha_i - \alpha_i\,\alpha_k \cdot \alpha_i\,\alpha_i = 0$, and two independent ones of type $\alpha_i\,\alpha_k \cdot \alpha_i\,\alpha_m - \alpha_i\,\alpha_i \cdot \alpha_k\,\alpha_m = 0$, and thus, twenty, in all. We then have a *complete* system of second-degree relations between the coefficients a_{ik} in the following formulas:

$$(2)\qquad \left\{\begin{array}{ll} A_{01}=(a_{00}+a_{11})^2-(a_{22}+a_{33})^2-(a_{23}-a_{32})^2=0, & [3], \\ A_{23}=(a_{00}-a_{11})^2-(a_{22}-a_{33})^2-(a_{23}+a_{32})^2=0, & [3], \\ B_{01}=(a_{00}+a_{11}+a_{22}+a_{33})(a_{23}+a_{32})-(a_{31}-a_{13})(a_{12}+a_{21})=0, & [3], \\ B_{10}=(a_{00}+a_{11}-a_{22}-a_{33})(a_{23}+a_{32})-(a_{12}+a_{21})(a_{31}+a_{13})=0, & [3], \\ B_{23}=(a_{00}-a_{11}+a_{22}-a_{33})(a_{23}-a_{32})-(a_{31}-a_{13})(a_{12}+a_{21})=0, & [3], \\ B_{32}=(a_{00}-a_{11}+a_{33}-a_{22})(a_{23}-a_{32})-(a_{31}+a_{13})(a_{12}-a_{21})=0, & [3], \\ C_1=(a_{23}-a_{32})(a_{23}+a_{32})-(a_{31}-a_{13})(a_{31}+a_{13})=0, & \\ C_2=(a_{12}-a_{21})(a_{12}+a_{21})-(a_{23}-a_{32})(a_{23}+a_{32})=0, & \\ C_3=(a_{23}-a_{32})(a_{23}+a_{32})-(a_{31}-a_{13})(a_{31}+a_{13})=0. & \end{array}\right.$$

One must consider in this that the last three relations are not linearly-independent; namely, one will have:

$$(3)\qquad C_1 + C_2 + C_3 = 0$$

identically, such that one can write them more briefly as:

$$(4)\qquad a_{23}^2-a_{32}^2 = a_{31}^2-a_{13}^2 = a_{12}^2-a_{21}^2 .$$

The left-hand sides of relations (2) on the one hand, and the left-hand side of relations (3), .., (5) in § 1, on the other, can now be expressed in terms of each other reciprocally, as follows:

$$(5)\qquad \begin{cases} 4P_1 = A_{02} + A_{03} + A_{12} + A_{13} + 2C_1, \\ 4P_1' = A_{02} + A_{03} + A_{12} + A_{13} - 2C_1, \\ 4R_{11} = A_{01} - A_{23}, \\ 4Q_1 = B_{01} - B_{10} - B_{23} + B_{32}, \\ 4Q_1' = B_{01} - B_{10} + B_{23} + B_{32}, \\ 4R_{23} = B_{01} + B_{10} + B_{23} + B_{32}, \\ 4R_{32} = B_{01} + B_{10} - B_{23} - B_{32}, \end{cases}$$

$$(6)\qquad \begin{cases} C_1 = P_1 - P_1', \\ A_{01} = -P_1 - P_1' + D + 2R_{11}, \\ A_{23} = -P_1 - P_1' + D - 2R_{11}, \\ B_{01} = R_{23} + R_{32} + Q_1 + Q_1', \\ B_{10} = R_{23} + R_{32} - Q_1 - Q_1', \\ B_{23} = R_{23} - R_{32} - Q_1 + Q_1', \\ B_{32} = R_{23} - R_{32} + Q_1 - Q_1'. \end{cases}$$

Any of these identities can represent *three* mutually-equivalent formulas.

With that, we have proved the theorem in question. At the same time, we have brought the system of relations between the coefficients into another form, namely, the form (2), independently of the theory of the parameters (α, β), and which is a form for these relations that is to be preferred in some cases.

In order to give an example, we consider the system of relations:

$$\begin{aligned} a_{11}^2 - a_{23}^2 &= a_{23}^2 - a_{31}^2 = a_{33}^2 - a_{12}^2, \\ a_{11}^2 - a_{32}^2 &= a_{23}^2 - a_{13}^2 = a_{33}^2 - a_{21}^2, \end{aligned}$$

which (we might remark, in passing) has a simple connection with the so-called *theorem of sines* in spherical trigonometry. One finds, in the given way, e.g.:

$$\begin{aligned} &(a_{23}^2 - a_{31}^2) - (a_{33}^2 - a_{12}^2) \\ &= \tfrac{1}{4}\{A_{02} - A_{03} + A_{31} - A_{12} - 2C_1\} \\ &= \tfrac{1}{2}\{-P_1 + P_1' - P_2 - P_2' + P_3 + P_3'\} = P_3 - P_2 . \end{aligned}$$

One can easily extend the argument that was just presented to relations between the coefficients a_{ik} of arbitrarily high degree:

There exist:

$$\binom{n+9}{9} - \binom{2n+3}{3}$$

linearly-independent relations of degree n ($n \geq 2$) *between the coefficients* $a_{00}, a_{11}, \ldots, a_{33}$ *of an orthogonal substitution of three variables. They can all be written in the form:*

$$\sum_{i=1}^{20} F_i\,\Phi_i = 0,$$

as long as F_i *are homogeneous functions of degree* $n - 2$ *in the coefficients* $a_{00}, a_{11}, \ldots, a_{33}$ *, and* Φ_i *mean the left-hand sides of the twenty relations of degree two that exist between them* (*).

Any expression F of degree n in the quantities a_{ik} can, in fact, be written as an expression of degree $2n$ in the parameters $\alpha_0, \ldots, \alpha_3$, and conversely. However, there are only $\binom{2n+3}{3}$ such expressions; the remaining expressions F must then be equal to zero identically. If F vanishes identically then, once one has expressed the coefficients a_{ik} in terms of the parameters $\alpha_0, \ldots, \alpha_3$, the various terms of the form:

$$A_\nu \alpha_0^{k_0} \alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \qquad (k_0 + k_1 + k_2 + k_3 = 2n)$$

must affect each other reciprocally. However, those individual terms differ by only the numerical values of the coefficients A_ν (whose sum is equal to zero), and by the way that the parameters α_i, α_k are combined pair-wise in them. If one then lets Φ denote any expressions of the form $\alpha_i\alpha_k \cdot \alpha_j\alpha_s - \alpha_j\alpha_k \cdot \alpha_i\alpha_s$ then one can put F into the form $\sum F_i\,\Phi_i$, without decomposing the products $\alpha_i\alpha_k$ into their factors, as the theorem asserts. Naturally, there are many forms that the expressions for the individual relation $F = 0$ in the form $\sum F_i\,\Phi_i$ can take. For example, one will get the mutually-equivalent representations for the third-degree relations (2) in § 1, *et al.*:

$$(7) \qquad \left\{ \begin{aligned} & a_{00}^3 - |\,a_{11}a_{22}a_{33}\,| \\ &= a_{i1}R_{i1} + a_{i2}R_{i2} + a_{i3}R_{i3} + a_{00}P_i \\ &= a_{1i}R_{1i} + a_{2i}R_{2i} + a_{3i}R_{3i} + a_{00}P_i' \end{aligned} \right. \qquad (i = 1, 2, 3).$$

Like the second-degree relations between the coefficients $a_{00}, a_{11}, \ldots, a_{33}$, formulas (1) also easily lead to *bilinear relations* between those coefficients and the parameters $\alpha_0, \ldots, \alpha_3$ – i.e., the expressions that are linear and homogeneous in the quantities α_i , as well as in the quantities a_{ik} , and which will vanish identically when one expresses the coefficients a_{ik} in terms of the parameters α_i .

We shall communicate those relations here in the form and arrangement by which they can be usefully employed in many calculations.

One first has the three formulas:

(*) As will be shown in another place, the methods of the theory of invariants lead to a deeper insight into the structure of these systems of relations.

$$
(8) \qquad \begin{cases} \alpha_1 a_{11} + \alpha_2 a_{21} + \alpha_3 a_{31} = a_{00} \cdot \alpha_1 , \\ \alpha_1 a_{12} + \alpha_2 a_{22} + \alpha_3 a_{32} = a_{00} \cdot \alpha_2 , \\ \alpha_1 a_{13} + \alpha_2 a_{23} + \alpha_3 a_{33} = a_{00} \cdot \alpha_3 , \end{cases}
$$

which go to each other by cyclic permutation of the 1, 2, 3. Moreover, there are nine more similarly-constructed formulas whose representatives might be the following three:

$$
(9) \qquad \begin{cases} \alpha_0 a_{11} - \alpha_3 a_{21} + \alpha_2 a_{31} = a_{00} \cdot \alpha_0 , & [3], \\ \alpha_0 a_{12} - \alpha_3 a_{22} + \alpha_2 a_{32} = a_{00} \cdot \alpha_3 , & [3], \\ \alpha_0 a_{13} - \alpha_3 a_{23} + \alpha_2 a_{33} = a_{00} \cdot \alpha_2 , & [3]. \end{cases}
$$

These twelve bilinear relations yield twelve more when one switches a_{ik} with a_{ki} and simultaneously replaces $\alpha_1, \alpha_2, \alpha_3$ with $-\alpha_1, -\alpha_2, -\alpha_3$.

We then know 24 bilinear relations. However, they are not linearly-independent, but they are coupled by four mutually-independent identities, namely, these:

$$
(10) \qquad \begin{cases} \quad [\alpha_0 (a_{11} - a_{00}) - \alpha_3 a_{21} + \alpha_2 a_{31}] \\ -[\alpha_0 (a_{11} - a_{00}) + \alpha_3 a_{12} - \alpha_2 a_{13}] \\ +[\alpha_0 (a_{22} - a_{00}) - \alpha_1 a_{32} + \alpha_3 a_{12}] \\ -[\alpha_0 (a_{22} - a_{00}) + \alpha_1 a_{31} - \alpha_3 a_{12}] \\ +[\alpha_0 (a_{33} - a_{00}) - \alpha_2 a_{13} + \alpha_1 a_{23}] \\ -[\alpha_0 (a_{33} - a_{00}) + \alpha_2 a_{31} - \alpha_1 a_{32}] = 0, \\ \\ \quad [\alpha_0 a_{32} - \alpha_2 a_{12} + \alpha_1 (a_{22} + a_{00})] \\ +[\alpha_0 a_{23} + \alpha_2 a_{21} - \alpha_1 (a_{22} + a_{00})] \\ -[\alpha_0 a_{23} - \alpha_1 (a_{33} + a_{00}) + \alpha_3 a_{13}] \\ -[\alpha_0 a_{32} + \alpha_1 (a_{33} + a_{00}) - \alpha_3 a_{31}] \\ -[\alpha_1 (a_{11} - a_{00}) + \alpha_2 a_{21} + \alpha_3 a_{31}] \\ +[\alpha_1 (a_{11} - a_{00}) + \alpha_2 a_{12} + \alpha_3 a_{13}] = 0 \quad [3]. \end{cases}
$$

Four of our relations are superfluous in this. *The remaining twenty, however, are linearly-independent. The represent the totality of all linearly-independent bilinear relations between the parameters $\alpha_0, \dots, \alpha_3$ and the transformation coefficients $a_{00}, a_{11}, \dots, a_{33}$.*

The proof is entirely similar to the argument that was carried out above (pp. 78).

We would now like to apply some of the formulas that were summarized here by investigating how the parameters (α, β) can be expressed in terms of the transformation coefficients.

Equations (1) yield four different expressions for the ratios of the quantities α_i that can be represented as *linear* functions of the transformation coefficients a_{ik} :

$$
(11) \qquad \begin{cases} \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 \\ = (a_{00}+a_{11}+a_{22}+a_{33}) : (a_{23}-a_{32}) : (a_{31}-a_{13}) : (a_{12}-a_{21}) \\ = (a_{23}-a_{32}) : (a_{00}+a_{11}-a_{22}-a_{33}) : (a_{12}+a_{21}) : (a_{31}+a_{13}) \\ = (a_{31}-a_{13}) : (a_{12}+a_{21}) : (a_{00}+a_{22}-a_{33}-a_{33}) : (a_{23}+a_{32}) \\ = (a_{12}-a_{21}) : (a_{31}+a_{13}) : (a_{23}+a_{32}) : (a_{00}+a_{33}-a_{11}-a_{22}). \end{cases}
$$

These equations represent a somewhat clearer form of relations (2), at their basis. One then sees that:

If any of the sixteen terms on the right-hand side of (11) *vanish then, at the same time, either all terms that are in a column with them or all of them that are contained in the same row as them will vanish.*

In the first case, the corresponding parameters α_i will have the value zero. Likewise, due to the symmetry of the table (11) in its diagonal, one of its rows will also vanish, and one of its four proportions will take on the undetermined form 0 : 0 : 0 : 0. Since that obviously cannot be the case for all four proportions, one will get a uniquely-determined, rational expression for the ratios of the quantities α_i under *all* circumstances.

One might use, perhaps, the first of our proportions, as long as $a_{00} + a_{11} + a_{22} + a_{33} \neq 0$. However, when that expression vanishes, the coefficient matrix will be symmetric ($a_{ik} = a_{ki}$), and one will get the ratios of the quantities α_i from one of the proportions:

$$
(11b) \qquad \begin{cases} \alpha_0 : \quad \alpha_1 \quad : \quad \alpha_2 \quad : \quad \alpha_3 \\ = 0 : (a_{00}+a_{11}) : \quad a_{12} \quad : \quad a_{31} \\ = 0 : \quad a_{12} \quad : (a_{00}+a_{22}) : \quad a_{23} \\ = 0 : \quad a_{31} \quad : \quad a_{23} \quad : (a_{00}+a_{33}). \end{cases}
$$

Of these, the first one is useful only when $a_{00} + a_{11} \neq 0$. If $a_{00} + a_{11}$ vanishes then one will also have $a_{12} = a_{31} = 0$, and one will have two proportions:

$$
(11c) \qquad \begin{cases} \alpha_0 : \alpha_1 : \quad \alpha_2 \quad : \quad \alpha_3 \\ = 0 : 0 : (a_{00}+a_{22}) : \quad a_{23} \\ = 0 : 0 : \quad a_{23} \quad : (a_{00}-a_{22}), \end{cases}
$$

of which, once more, perhaps the first one is useful as long as $a_{00} + a_{22}$ does not vanish. If that case also occurs then one will ultimately have:

$$(11d)\qquad \begin{cases} \alpha_0 : \alpha_1 : \alpha_2 : \quad \alpha_3 \\ = 0 : 0 : 0 : (a_{00} + a_{33}). \end{cases}$$

Once we have found the ratios of the parameters α_i from this, we will get the parameters β_i by solving the linear equations:

$$(12)\qquad \begin{cases} \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0, \\ \alpha_1\beta_0 - \alpha_0\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 = \frac{1}{2}a_{10} \cdot \dfrac{\sum \alpha_i^2}{a_{00}}, \\ \alpha_2\beta_0 + \alpha_3\beta_1 - \alpha_0\beta_2 - \alpha_1\beta_3 = \frac{1}{2}a_{20} \cdot \dfrac{\sum \alpha_i^2}{a_{00}}, \\ \alpha_3\beta_0 - \alpha_2\beta_1 + \alpha_1\beta_2 - \alpha_0\beta_3 = \frac{1}{2}a_{30} \cdot \dfrac{\sum \alpha_i^2}{a_{00}}. \end{cases}$$

In order to find β_i from this, one multiplies the left-hand and right-hand side of each of these equations by a factor α_k that will make the coefficient of β_i in the sum of products on the left equal $\sum \alpha_i^2$. One will then come to the formulas:

$$(13)\qquad \begin{cases} \alpha_1 a_{10} + \alpha_2 a_{20} + \alpha_3 a_{30} = \ \ 2a_{00} \cdot \beta_0, \\ \alpha_0 a_{10} - \alpha_3 a_{20} + \alpha_2 a_{30} = -2a_{00} \cdot \beta_1, \\ \alpha_2 a_{10} + \alpha_0 a_{20} - \alpha_1 a_{30} = -2a_{00} \cdot \beta_2, \\ -\alpha_2 a_{10} + \alpha_1 a_{20} + \alpha_0 a_{30} = -2a_{00} \cdot \beta_3. \end{cases}$$

Not only does a well-defined motion belong to any system of parameters (α, β), *but conversely, any motion belongs to a system of parameters.*

Here, it is assumed that the determinant of the system of equations (12) does not vanish, so the motion does not degenerate. Naturally, in that case, which we will exclude, the theorem will no longer be true.

The considerations that were communicated in this paragraph can be generalized in many directions. In regard to them, we remark only that one can solve equations (5) and (6) of § 4 for the products $\alpha_i\ \beta_k$, and that one can consequently give the mutually-independent second-degree relations that exist between the 16 quantities a_{ik} and the nine quantities b_{ik} with no further assumptions. Due to the large number of those relations (there are no less than 208 of them), we shall, however, avoid treating them exhaustively. It might suffice to exhibit, in addition to the relations (8), § 4 that we gave already, some of them that emerge from equations (7), § 4 with the help of the formulas that were developed already:

$$(14) \qquad \begin{cases} a_{21}b_{31} + a_{22}b_{32} + a_{23}b_{33} = a_{00}a_{10} & [3], \\ a_{31}b_{21} + a_{32}b_{22} + a_{33}b_{23} = -a_{00}a_{10} & [3], \\ a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} = \qquad 0 & [3]. \end{cases}$$

When the equations of motion are given in line coordinates, they will yield the expressions for the coefficients a_{i0} .

A second system of relations of this kind will arise when one exchanges the quantities a_{ik}, b_{ik} in (14) with the quantities a_{ki}, b_{ki} .

§ 6.

Another derivation and generalization of the parameters (α, β).

One can arrive at the parameters (α, β) by which we have represented the motions in space in several other ways. Here, we would like to communicate a derivation that has the advantage of being very short, and assumes nothing beyond the known connection between **Euler**'s formulas and the theory of quaternions, moreover.

Let e_0, e_1, e_2, e_3 be the quaternion units, such that:

$$e_1^2 = - e_0 \qquad [3], \qquad e_2\, e_3 = - e_3\, e_2 = e_1 \qquad [3],$$

and let α, β, x, … be quaternions – for example, ones for which:

$$(1) \qquad \alpha = \alpha_0\, e_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3\,,$$

and finally, let:

$$(2) \qquad \bar{\alpha} = \alpha_0\, e_0 - \alpha_1\, e_1 - \alpha_2\, e_2 - \alpha_3\, e_3$$

be the so-called conjugate quaternion to the quaternion α; one will then have:

$$(3) \qquad \alpha\bar{\alpha} = \bar{\alpha}\alpha = \sum \alpha_i^2 = N(\alpha).$$

One will further have:

$$(4) \qquad x' = \bar{\alpha} x \alpha$$

for the abbreviated expression for Euler's representation of the rotations around a fixed point, which will be the origin of the coordinates, as long as one interprets the quantities $z_i = x_i / x_0$ as rectangular coordinates in space; The formulas for the connection between the parameters likewise coalesce into a single equation:

$$(5) \qquad \alpha'' = \alpha\, \alpha'.$$

We will now have an expression for a general motion before us if we write the following equation in place of (4):

(6) $$x' = \bar{\alpha} x \alpha - 2\bar{\alpha}\beta \cdot x_0 ,$$

and as long as we choose the quaternion β such that the second term on the right-hand side of (6) makes no contribution to the value of x_0'. That will be facilitated by the demand that one should have:

(7) $$\alpha\bar{\beta} + \beta\bar{\alpha} = \bar{\alpha}\beta + \bar{\beta}\alpha$$
$$= 2\,(\alpha_0\,\beta_0 + \alpha_1\,\beta_1 + \alpha_2\,\beta_2 + \alpha_3\,\beta_3) = 0.$$

If we calculate the expression on the right-hand side of (6) under that assumption then we will come to the system of equations (1), (4), (5), of § 4. One then shows, as in § 5, that any system of coefficients a_{ik} belongs to just one system of parameters (α, β).

One now asks how the parameters (α, β) that are found in that way are combined.

If we perform the transformation:

$$x'' = \bar{\alpha}' x' \alpha' - 2\bar{\alpha}'\beta' x_0'$$

after the transformation (6) then we will find directly that:

$$x'' = \bar{\alpha}'\,\bar{\alpha}\, x' \alpha\, \alpha' - 2(\bar{\alpha}'\,\bar{\alpha}\,\beta' + \bar{\alpha}'\,\bar{\alpha}\,\alpha\,\beta')\, x_0' ;$$

should this equation once more have the form (6), that is, should one have:

$$x'' = \bar{\alpha}'' x\, \alpha'' - 2\bar{\alpha}''\beta'' x_0 ,$$

then it would have to follow that:

(8) $$\alpha'' = \alpha\alpha', \qquad \bar{\alpha}'' = \bar{\alpha}'\,\bar{\alpha}, \qquad \beta'' = \alpha\beta' + \beta\alpha'.$$

The second of these equations is only another form of the first one; however, if one calculates the first and third ones then formulas (8) of § 2 will arise.

With that, we have proved the most important of the theorems that we have derived up to now in a new way, while generally ignoring the knowledge that we obtained in § 1 that our parametric representation is the only one of its kind. However, at the same time, we arrive at a new result.

Namely, instead of formula (6), we can make a somewhat more general Ansatz by employing eight parameters α_i, β_i that are not coupled by any relation. The formula:

(9) $$x' = \bar{\alpha} x\, \alpha - 2(\bar{\alpha}\beta - \bar{\beta}\alpha) x_0$$

can be employed to represent the general motion in space. If we once more ask about the composition of parameters then we will get the following formula in place of (8):

(10) $$\begin{cases} \alpha'' = \alpha\alpha', & \beta'' = \alpha\beta' + \beta\alpha', \\ \bar{\alpha}'' = \bar{\alpha}'\alpha, & \bar{\beta}'' = \bar{\alpha}'\bar{\beta} + \bar{\beta}'\bar{\alpha}. \end{cases}$$

The last two equations are again only another form of the first two. However, if one expands them then one will obtain formulas (8) of § 2 anew. If one expands the right-hand side of (9) then one will, in turn, come to the transformation formulas (1), (4), (5) of § 4. We then have the theorem:

If one considers the parameters in our expressions for the coefficients a_{ik} in terms of the parameters (α, β), *not, as before, as being dependent upon each other, but as completely arbitrary quantities then the same formulas will be true for the representation of motions that are true for the composition of parameters.*

Finally, in connection with what was stated, we might prove formula (3) of § 3. With our present notation, it reads:

$$\alpha''\bar{\beta}'' + \beta''\bar{\alpha}'' = \alpha'\bar{\alpha}'(\alpha\bar{\beta} + \beta\bar{\alpha}) + \alpha\,\bar{\alpha}(\alpha'\bar{\beta}' + \beta'\bar{\alpha}') . \tag{11}$$

However, it follows immediately from formulas (10) that:

$$\begin{aligned}\alpha''\bar{\beta}'' + \beta''\bar{\alpha}'' &= \alpha\,\alpha'(\bar{\alpha}'\bar{\beta} + \bar{\beta}'\bar{\alpha}) + (\alpha\beta' + \beta\alpha')\,\bar{\alpha}'\,\bar{\alpha} \\ &= \alpha'\,\bar{\alpha}'(\alpha\bar{\beta} + \beta\bar{\alpha}) + \alpha(\alpha'\bar{\beta}' + \beta'\bar{\alpha}')\,\bar{\alpha} \\ &= \alpha'\,\bar{\alpha}'(\alpha\bar{\beta} + \beta\bar{\alpha}) + \alpha\,\bar{\alpha}(\alpha'\bar{\beta}' + \beta'\bar{\alpha}') .\end{aligned}$$

The fact that the theorem that was just pointed out must be true could have been predicted; it can just as easily be linked to the theorem about the group G_7 that was treated in § 2 and § 3.

As we saw there, the group G_7 contains a one-parameter distinguished subgroup G_1 whose general finite transformation reads, in the notation of § 2:

$$x' = x\,(e_0 + h\,\varepsilon_0), \tag{12}$$

as long as h means a numerical parameter. [Cf., § 2, (6)]. Now, for the moment, let $x' = xa^*$ denote any transformation of G_7 , so one can always determine the quantity h, and in only one way, such that the second factor on the right-hand side of the equation:

$$a^* = (e_0 + h\,\varepsilon_0)\,a \tag{13}$$

satisfies the condition $L(\alpha, \beta) = 0$, so:

$$x' = xa^* \quad \text{and} \quad x' = xa \tag{14}$$

will be corresponding transformations of the isomorphically-related groups G_7 and G_6 .

One finds directly that:

(15) $$h = \frac{L(\alpha^*, \beta^*)}{N(\alpha^*, \beta^*)},$$

(16) $$\alpha_i^* = \alpha_i\,, \qquad \beta_i^* = \beta_i + h\,\alpha_i\,.$$

If one now introduces the parameters (α^*, β^*), which do not satisfy the condition $L(\alpha^*, \beta^*) = 0$, into formulas (4) and (5) of § 4, in place of the parameters (α, β), which satisfy the condition $L(\alpha, \beta) = 0$, by means of the substitutions (16) then those formulas will not change in form. That is the theorem that was proved above in a new form. We likewise recognize wherein the meaning of that theorem lies:

The theorem that was just pointed out will no longer be true when one changes the expressions for the coefficients a_{ik} in terms of the parameters (α, β) in such a way that one adds multiples of the vanishing expression $L(\alpha, \beta)$ to their right-hand sides.

We can then say that, in a certain sense, the expressions that we chose are the *simplest* of all the forms of representation that one would like to consider to be equivalent. The isomorphism of the group of motions with the group G_7 that belongs to the biquaternions finds its purest expression in them.

Naturally, it is, as a rule, much more convenient to represent the motions by the parameters (α, β), which are present in only one way, than to employ the parameters (α^*, β^*), which contain the superfluous parameter h; we will then make no use of the parameters (α^*, β^*) in this study.

The latter considerations relate to only our expression of the motion by *point* or *plane coordinates*. However, a corresponding theorem is also true for *line coordinates* when one write the unabbreviated equations in place of equations (2), § 4, which are implied by equations (1), § 4:

(17) $$\begin{cases} p'_{01} = a_{00}(a_{11}p_{01} + a_{12}p_{02} + a_{13}p_{03}) & [3], \\ p'_{23} = (a_{00}b_{11} - 2La_{11})\,p_{01} + a_{00}a_{11}p_{23} & \\ \qquad + (a_{00}b_{12} - 2La_{12})\,p_{02} + a_{00}a_{12}p_{31} & \\ \qquad + (a_{00}b_{13} - 2La_{13})\,p_{03} + a_{00}a_{13}p_{12} & [3]. \end{cases}$$

If one adds the factor $a_{00}^2 = N^2$ to the left-hand side of this and replaces the quantities p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12} with $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ then one will obtain the expression for the adjoint of the group G_7 in terms of the parameters (α, β) of that group.

We would then arrive at a knowledge of the theorem that was already stated on pp. 87 if we did not abbreviate the calculations there from the outset by employing the relation $L(\alpha, \beta) = 0$.

§ 7.

Geometric interpretation of the parameters (α, β).

Our arguments up to now have been based upon the single assumption that the motions being examined did not degenerate, so a_{00} was non-zero; everything that was presented is just as true for real values of the coefficients a_{ik} and parameters (α, β), as it is for complex values. However, from now on, for the sake of brevity, we would like to restrict the consideration to real values of the stated quantities, and thus to the investigation of *real motions.*

In what follows, we shall ascertain the geometric meaning that the parameters (α, β) – or rather, their ratios – take on relative to the coordinate system (z_1, z_2, z_3). The simple calculations that relate to that might be left to the reader; we remark only that one can appeal to the relations that were developed in § 5 in performing them to one's advantage.

In order to find the middle complex and screw axis of the motion $S(\alpha, \beta)$, one can perhaps proceed in such a way that one will discover all (real) linear complexes that are fixed by the motion by starting with equations (2) in § 4. One finds that, in general, their coordinates p_{ik} will be represented by the equations:

$$(1)\qquad \left\{ \begin{array}{lll} p_{01} = \mu\alpha_1, & p_{02} = \mu\alpha_2, & p_{03} = \mu\alpha_3, \\ p_{23} = \mu\beta_1 - \nu\alpha_1, & p_{31} = \mu\beta_2 - \nu\alpha_2, & p_{12} = \mu\beta_3 - \nu\alpha_3, \end{array} \right.$$

and further such complexes will appear only when either α_0 or $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ vanishes. The middle complex, as well as the *screw axis* of the motion S is contained among the complexes (1), among others. One finds that the first of the parameter values corresponds to $\nu : \mu = 0$, while the second one corresponds to the parameter value:

$$(2)\qquad \frac{\nu}{\mu} = \frac{\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3}{\alpha_1\alpha_1 + \alpha_2\alpha_2 + \alpha_3\alpha_3} = -\frac{\alpha_0\beta_0}{a_{00} - \alpha_0^2}.$$

The first result of this needs to be emphasized especially:

The middle complex of the motion $S(\alpha, \beta)$ has the coordinates:

$$(3)\qquad \left\{ \begin{array}{l} p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} \\ = \alpha_1 : \alpha_2 : \alpha_3 : \beta_1 : \beta_2 : \beta_3. \end{array} \right.$$

The null system $\mathfrak{W}$ – viz., the association of a chord midpoint $\bar{x}$ with the normal plane $\bar{\bar{u}}$ of the associated chord – will then be represented by the equations:

$$(4)\qquad \begin{cases} \bar{\bar{u}}_0 = \cdot & -\beta_1\bar{x}_1 & -\beta_2\bar{x}_2 & -\beta_3\bar{x}_3, \\ \bar{\bar{u}}_1 = \beta_1\bar{x}_0 & \cdot & -\alpha_3\bar{x}_2 & +\alpha_2\bar{x}_3, \\ \bar{\bar{u}}_2 = \beta_2\bar{x}_0 & +\alpha_3\bar{x}_1 & \cdot & -\alpha_2\bar{x}_3, \\ \bar{\bar{u}}_3 = \beta_3\bar{x}_0 & -\alpha_2\bar{x}_1 & +\alpha_1\bar{x}_2 & \cdot\ , \end{cases}$$

$$(5)\qquad \begin{cases} \bar{x}_0 = \cdot & -\alpha_1\bar{\bar{u}}_1 & -\alpha_2\bar{\bar{u}}_2 & -\alpha_3\bar{\bar{u}}_3, \\ \bar{x}_1 = \alpha_1\bar{\bar{u}}_0 & \cdot & -\beta_3\bar{\bar{u}}_2 & +\beta_2\bar{\bar{u}}_3, \\ \bar{x}_2 = \alpha_2\bar{\bar{u}}_0 & +\beta_3\bar{\bar{u}}_1 & \cdot & -\beta_2\bar{\bar{u}}_3, \\ \bar{x}_3 = \alpha_3\bar{\bar{u}}_0 & -\beta_2\bar{\bar{u}}_1 & +\beta_1\bar{\bar{u}}_2 & \cdot\ . \end{cases}$$

Each of these two systems of equations is the solution of the other one; their determinants have the common value:

$$(6)\qquad (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)^2 = \alpha_0^2\beta_0^2.$$

The two cases in which the null system $\mathfrak{W}$ degenerates then correspond to the two cases in which either α_0 or β_0 vanishes.

We will obtain a more precise breakdown of the meaning of the equations $\alpha_0 = 0$, $\beta_0 = 0$ when we calculate the screw height and the screw angle of the screwing motion $S(\alpha, \beta)$. Both expressions are summarized in the following theorem:

Let 2ϑ be the screw angle, and let 2η be the screw height of the screwing motion $S(\alpha, \beta)$; finally, let λ_1, λ_2, λ_3 be the angles that the screw axis makes with the coordinate axes so one has the proportions:

$$(7)\qquad \begin{cases} \cot\vartheta : \cos\lambda_1 : \cos\lambda_2 : \cos\lambda_3 : \eta \\ = \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0. \end{cases}$$

One infers the special consequences from this:

$$(8)\qquad \begin{cases} \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3 : \\ = 0 : p_{01} : p_{02} : p_{03} : 0 : p_{23} : p_{31} : p_{31}. \end{cases}$$

It follows from (7) that:

$$(9)\qquad \cot\vartheta = \frac{\alpha_0}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}},$$

$$(10)\qquad \eta = \frac{\beta_0}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}},$$

in which one must determine the sign of the square root to have the same sense both times. If the denominator vanishes then S will be a translation. The expression for η will then take on the undetermined form 0 : 0; one finds the true value of one-half the translation magnitude to be:

(10*b*) $$\eta = \frac{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}{\alpha_0}.$$

We will also be led to some very remarkable results when we express the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, and T that are linked with the motion S in terms of the parameters (α, β).

As in § 5 of the first section, we consider the figures that are composed of points and planes whose elements are related to each other by:

$$x\{\mathfrak{T}_1\}\,\bar{x}\,\{\mathfrak{T}_2\}x', \qquad u\{\mathfrak{T}_2\}\,\bar{u}\,\{\mathfrak{T}_1\}u', \qquad x\{T\}\,\bar{\bar{u}}\,\{T\}x'.$$

If we recall the meanings of these formulas then we will come to the theorems:

I. *The dependency of the endpoints x, x' of a chord xx' upon the midpoint of the chord $\bar{x}$ will be expressed by the formulas* (*):

(11*a*) $$\mathfrak{T}_1^{-1}: \quad \begin{cases} x_0 = \alpha_0\bar{x}_0 \\ x_1 = \beta_1\bar{x}_0 + \alpha_0\bar{x}_1 - \alpha_3\bar{x}_2 + \alpha_2\bar{x}_3, \\ x_2 = \beta_2\bar{x}_0 + \alpha_3\bar{x}_1 + \alpha_0\bar{x}_2 - \alpha_1\bar{x}_3, \\ x_3 = \beta_3\bar{x}_0 - \alpha_2\bar{x}_1 + \alpha_1\bar{x}_2 + \alpha_0\bar{x}_3; \end{cases}$$

(11*b*) $$\mathfrak{T}_2: \quad \begin{cases} x'_0 = \quad \alpha_0\bar{x}_0 \\ x'_1 = -\beta_1\bar{x}_0 + \alpha_0\bar{x}_1 + \alpha_3\bar{x}_2 - \alpha_2\bar{x}_3, \\ x'_2 = -\beta_2\bar{x}_0 - \alpha_3\bar{x}_1 + \alpha_0\bar{x}_2 + \alpha_1\bar{x}_3, \\ x'_3 = -\beta_3\bar{x}_0 + \alpha_2\bar{x}_1 - \alpha_1\bar{x}_2 + \alpha_0\bar{x}_3; \end{cases}$$

(*) These formulas (11), or ones that do not differ from them very much define the nucleus of the treatise of **Rodrigues**: "Des lois géométriques qui régissent les déplacéments d'un système solide dans léspace," Liouville J. **5** (1840), which is still worthy of attention to this day. They seemed so remarkable to the author that he did not consider it to be trivial to prove them in four different ways. He erred only insofar as he believed that he had shown that his formulas were valid without exception.

The parameters that were employed by Rodrigues are connected with the ones that are used here very simply. One has:

$$\mathrm{A} = -2\frac{\beta_1}{\alpha_0}, \qquad \mathrm{B} = -2\frac{\beta_2}{\alpha_0}, \qquad \Gamma = -2\frac{\beta_3}{\alpha_0},$$

$$m = 2\frac{\alpha_1}{\alpha_0}, \qquad n = 2\frac{\alpha_2}{\alpha_0}, \qquad p = 2\frac{\alpha_3}{\alpha_0}.$$

The only exceptions are the unscrewings (viz., the motions whose parameter α_0 *vanishes).*

II. *The dependency of the two mutually-associated planes u, u′ on their angle bisector of the first kind* $\bar{u}$ *will be expressed by the formulas:*

$$(12a)\qquad \mathfrak{T}_2^{-1}: \quad \begin{cases} u_0 = \alpha_0\bar{u}_0 - \beta_1\bar{u}_1 - \beta_2\bar{u}_2 - \beta_3\bar{u}_3, \\ u_1 = \quad \cdot \quad + \alpha_0\bar{u}_1 - \alpha_3\bar{u}_2 + \alpha_2\bar{u}_3, \\ u_2 = \quad \cdot \quad + \alpha_3\bar{u}_1 + \alpha_0\bar{u}_2 - \alpha_1\bar{u}_3, \\ u_3 = \quad \cdot \quad - \alpha_2\bar{u}_1 + \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3, \end{cases}$$

$$(12b)\qquad \mathfrak{T}_1: \quad \begin{cases} u'_0 = \alpha_0\bar{u}_0 + \beta_1\bar{u}_1 + \beta_2\bar{u}_2 + \beta_3\bar{u}_3, \\ u'_1 = \quad \cdot \quad + \alpha_0\bar{u}_1 + \alpha_3\bar{u}_2 - \alpha_2\bar{u}_3, \\ u'_2 = \quad \cdot \quad - \alpha_3\bar{u}_1 + \alpha_0\bar{u}_2 + \alpha_1\bar{u}_3, \\ u'_3 = \quad \cdot \quad + \alpha_2\bar{u}_1 - \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3. \end{cases}$$

Once again, the unscrewings are excluded, however.

III. *The dependency of the endpoints x, x′ of a chord on its normal plane* $\bar{\bar{u}}$ *will be expressed by the formulas* (*):

$$(13a)\qquad T^{-1}: \quad \begin{cases} x_0 = \quad \cdot \quad - \alpha_1\bar{\bar{u}}_1 - \alpha_2\bar{\bar{u}}_2 - \alpha_3\bar{\bar{u}}_3, \\ x_1 = \alpha_1\bar{\bar{u}}_0 + \beta_0\bar{\bar{u}}_1 - \beta_3\bar{\bar{u}}_2 + \beta_2\bar{\bar{u}}_3, \\ x_2 = \alpha_2\bar{\bar{u}}_0 + \beta_3\bar{\bar{u}}_1 + \beta_0\bar{\bar{u}}_2 - \beta_1\bar{\bar{u}}_3, \\ x_3 = \alpha_3\bar{\bar{u}}_0 - \beta_2\bar{\bar{u}}_1 + \beta_1\bar{\bar{u}}_2 + \beta_0\bar{\bar{u}}_3; \end{cases}$$

$$(13b)\qquad T: \quad \begin{cases} x'_0 = \quad \cdot \quad + \alpha_1\bar{\bar{u}}_1 + \alpha_2\bar{\bar{u}}_2 + \alpha_3\bar{\bar{u}}_3, \\ x'_1 = -\alpha_1\bar{\bar{u}}_0 + \beta_0\bar{\bar{u}}_1 + \beta_3\bar{\bar{u}}_2 - \beta_2\bar{\bar{u}}_3, \\ x'_2 = -\alpha_2\bar{\bar{u}}_0 - \beta_3\bar{\bar{u}}_1 + \beta_0\bar{\bar{u}}_2 + \beta_1\bar{\bar{u}}_3, \\ x'_3 = -\alpha_3\bar{\bar{u}}_0 + \beta_2\bar{\bar{u}}_1 - \beta_1\bar{\bar{u}}_2 + \beta_0\bar{\bar{u}}_3; \end{cases}$$

Here, the rotations (viz., motion whose parameter β_0 *vanishes) are excluded.*

If one then eliminates the coordinates $\bar{x}$ *of the midpoint of the chord from equations* (11) *or the coordinates* $\bar{u}_i$ *of the angle bisector from equations* (12), *or finally, the coordinates* $\bar{\bar{u}}_i$ *of the normal plane from equations* (13), *then the equations of the general motion in space will come about.*

(*) Formulas (11), …, (13) were given by the author in a tentative publication on the topic that is treated in this paper. (Sächs. Ber. Oct. 1890, pps. 347 and 348) At the same time, **Lindemann** also made some suggestions on the parametric representation of motions that one finds in formulas (12). [*Vorlesungen über Geometrie*, Bd. II (1891), pp. 373, below]. Meanwhile, he did not follow through on his ideas. He also (if I understand correctly) denied the presence of formal pairs of type (11) or (13).

Meanwhile, the unscrewings are excluded in the first two cases and the rotations in the last one: These special motions cannot be obtained in that way.

In order to obtain a clear presentation of how the exceptional cases come about, we imagine that we have added a proportionality factor ρ to the let-hand sides of the systems of equations (11), (12), and (13), strictly speaking. If we take that factor to be equal to α_0 in the first two cases and β_0 in the last one then our equations will be completely general. Equations (11), e.g., then say that the chord midpoint lies along the screw axis in the case $\alpha_0 = 0$. Naturally, one can now no longer express the ratios of the quantities x_i or x'_i in terms of the ratios of the quantities $\bar{x}'_i$. Corresponding statements are true in the other cases.

The determinants of equations (11a), (11b), (12a), or (12b) will have the value $a_{00} \cdot \alpha_0^2$, while the determinants of equations (13a) or (13b) will have the value:

$$(14) \qquad (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\,\beta_0^2 + (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)^2 = a_{00} \cdot \beta_0^2.$$

One then sees that equations (11) and (12) can be solved for the quantities $\bar{x}_i$ ($\bar{u}_i$, resp.) when $\alpha_0 \neq 0$, and equations (13) for the quantities $\bar{\bar{u}}_i$ when $\beta_0 \neq 0$. In this, it is assumed that β_0 must vanish at the same time as $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$. However, even if one would like to regard the parameters that enter into formulas (13) as entirely independent quantities, they would still not be useful in the representation of rotations: In that case, one would obtain only the identity transformation by eliminating the quantities $\bar{\bar{u}}_i$. [One compares formulas (4) and (5), pp. 89.] If one now actually solves the stated equations then one will find that:

$$(11c) \qquad \mathfrak{T}_1: \quad \begin{cases} \bar{x}_0 = 2a_{00}x_0 \\ \bar{x}_1 = a_{10}x_0 + (a_{11} + a_{00})x_1 + a_{12}x_2 + a_{13}x_3, \\ \bar{x}_2 = a_{20}x_0 + a_{21}x_1 + (a_{22} + a_{00})x_2 + a_{23}x_3, \\ \bar{x}_3 = a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + (a_{33} + a_{00})x_3, \end{cases}$$

(after moving a factor of $2a_{00} \cdot \alpha_0$ to the left-hand side), and furthermore:

$$(12c) \qquad \mathfrak{T}_2: \quad \begin{cases} \bar{u}_0 = 2a_{00}u_0 + a_{01}u_1 + a_{02}u_2 + a_{03}u_3, \\ \bar{u}_1 = \cdot \qquad (a_{11} + a_{00})u_1 + a_{12}u_2 + a_{13}u_3, \\ \bar{u}_2 = \cdot \qquad a_{21}u_1 + (a_{22} + a_{00})u_2 + a_{23}u_3, \\ \bar{u}_3 = \cdot \qquad a_{31}u_1 + a_{32}u_2 + (a_{33} + a_{00})u_3, \end{cases}$$

(again, after moving a factor of $2a_{00} \cdot \alpha_0$ to the left-hand side). Finally, one will have:

$$(13c) \qquad T: \quad \begin{cases} \bar{u}_0 = 2d_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3, \\ \bar{u}_1 = a_{10}x_0 + (a_{11} - a_{00})x_1 + a_{12}x_2 + a_{13}x_3, \\ \bar{u}_2 = a_{20}x_0 + a_{21}x_1 + (a_{22} - a_{00})x_2 + a_{23}x_3, \\ \bar{u}_3 = a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + (a_{33} - a_{00})x_3, \end{cases}$$

after moving a factor of $-2a_{00} \cdot \beta_0$ to the left-hand side and writing:

$$(15) \qquad d_{00} = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2,$$

to abbreviate.

If one connects formulas (11*b*) and (11*c*), (12*b*) and (12*c*), (13*b*) and (13*c*) then, as stated, one will come back to the transformation formulas of § 4 in all cases; however, as one can anticipate, one will not obtain every formula in the form that was given there immediately, but only when one has dropped the factors $2\alpha_0$, $2\alpha_0$, $2\beta_0$, resp. from their right-hand sides.

The way that the parameters (α, β) enter into formulas (4), (5), (11*a*), (11*b*), (12*a*), (12*b*), (13*a*), 13*b*) is remarkable in several respects. They enter into all of them only linearly. Furthermore, the parameters α_0, β_0 are missing from (4) and (5), while the parameter β_0 is missing from (11) and (12), and α_0 is missing from (13). In emphasizing this last fact, we might remark that the transformations $\mathfrak{T}_1^{-1}$ and $\mathfrak{T}_2$, $\mathfrak{T}_1$ and $\mathfrak{T}_2^{-1}$, T and T^{-1} invert their roles when one switches the motion S with the inverse motion S^{-1}. It then suffices to say:

The motions in space that are linked with the transformations $\mathfrak{T}_2$ ($\mathfrak{T}_1$, *resp.*) *define a six-fold extended* ***linear*** *manifold when regarded as transformations between points (planes, resp.).*	*The motions in space that are linked with the transformations T define a six-fold extended* ***linear*** *manifold when regarded as transformations between points and planes.*

If one interprets the ratios $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_1 : \beta_2 : \beta_3$ of the coefficients of, e.g., $\mathfrak{T}_2$, as homogeneous coordinates in a six-fold extended space R_6 then one will obtain a generally single-valued, invertible map of the manifold of motions to the points of that space. Only the *unscrewings* exhibit singular behavior under that map, besides the degenerate motions: All unscrewings with the same axis will correspond to one and the same point of the space R_6 . An entirely similar map will arise when none starts with the theorem on the right. However, the *rotations* will now be the singular elements. All rotations around the same axis will correspond to the same point; however, the identity transformation will correspond to a completely indeterminate point.

Naturally one can also establish theorems I, II, III immediately with no knowledge of the theory that was developed in §§ 1, …, 6, which is, in fact, what **Rodrigues** succeeded in doing in the case of equations (11). If one then introduces the missing superfluous parameters with the help of the relation $L(\alpha, \beta) = 0$ in every case then one will have a third way of arriving at our parametric representation of motions. We content ourselves with the suggestion that a deeper insight into the essence of the last theorems can be deduced only from the general theory of quadratic forms, which would then require tools (for a proper presentation) whose application we will deny ourselves here.

§ 8.

The canonical parameters of motions. Continuous groups of motions.

If we specialize our parameters by the introduction of an infinitely small quantity δt whose square can be neglected, in the following way:

(1) $$\alpha_0 = 1, \quad \beta_0 = 0, \quad \alpha_i = \mathfrak{a}_i\,\delta t, \quad \beta_i = \mathfrak{b}_i\,\delta t, \qquad (i = 1, 2, 3)$$

then we will once more get the formulas:

(2) $$\begin{cases} z_1' = z_1 - 2\{\mathfrak{b}_1 - \mathfrak{a}_3 z_2 + \mathfrak{a}_2 z_3\}\delta t, \\ z_2' = z_2 - 2\{\mathfrak{b}_2 - \mathfrak{a}_1 z_3 + \mathfrak{a}_3 z_1\}\delta t, \\ z_3' = z_3 - 2\{\mathfrak{b}_3 - \mathfrak{a}_2 z_1 + \mathfrak{a}_1 z_2\}\delta t, \end{cases}$$

which represent the general infinitely-small motion; we have already introduced that infinitesimal transformation, whose symbol is:

(3) $$Xf = \sum_{i=1}^{3} (\mathfrak{a}_i X_i f + \mathfrak{b}_i Y_i f),$$

in §§ 1 and 2.

That infinitesimal motion generates the general motion "by infinite repetition," and, following **S. Lie**, its equation can be written in this way:

(4) $$z_i' = z_i + X(z_i) + \frac{1}{2!} X\big(X(z_i)\big) + \frac{1}{3!} X\Big(X\big(X(z_i)\big)\Big) + \dots$$

The parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$ that appear in this are called the *canonical parameters* of the motion. One asks how they might be coupled with the parameters (α, β).

In order to ascertain that connection, we will proceed most simply and intuitively in such a way that we now express the screw axis n of the motion S, its screw height 2η, and the screw angle 2ϑ, which we have all represented in terms of the parameters (α, β), in terms of the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$, as well. One then obtain the screw axis, as well as the quantities η and ϑ for the infinitesimal motion Xf from (1). If one considers that the screw axis will be the same for all motions in the one-parameter group that is generated by Xf, and that η and ϑ change in proportion to the parameter t of the group then one will get the following expression for the parameters (α, β) of our general transformation (t) of our one-parameter group:

$$(5)\qquad \begin{cases} \rho\alpha_0 = \sqrt{\mathfrak{m}}\cot\sqrt{\mathfrak{m}}\,t, & \rho\beta_0 = \mathfrak{n}t, \\ \rho\alpha_1 = \mathfrak{a}_1, & \rho\beta_1 = \mathfrak{b}_1 + \dfrac{\mathfrak{n}}{\mathfrak{m}}(\sqrt{\mathfrak{m}}\,t\cdot\cot\sqrt{\mathfrak{m}}\,t - 1)\mathfrak{a}_1, \\ \rho\alpha_2 = \mathfrak{a}_2, & \rho\beta_2 = \mathfrak{b}_2 + \dfrac{\mathfrak{n}}{\mathfrak{m}}(\sqrt{\mathfrak{m}}\,t\cdot\cot\sqrt{\mathfrak{m}}\,t - 1)\mathfrak{a}_2, \\ \rho\alpha_3 = \mathfrak{a}_3, & \rho\beta_3 = \mathfrak{b}_3 + \dfrac{\mathfrak{n}}{\mathfrak{m}}(\sqrt{\mathfrak{m}}\,t\cdot\cot\sqrt{\mathfrak{m}}\,t - 1)\mathfrak{a}_3, \end{cases}$$

in which, we have set:

$$(6)\qquad \mathfrak{m} = \mathfrak{a}_1\mathfrak{a}_1 + \mathfrak{a}_2\mathfrak{a}_2 + \mathfrak{a}_3\mathfrak{a}_3, \qquad \mathfrak{n} = \mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{a}_2\mathfrak{b}_2 + \mathfrak{a}_3\mathfrak{b}_3,$$

to abbreviate.

One easily convinces oneself by actual calculation that equations (5), in fact, represent the group that is generated by the infinitesimal transformation (2) as t varies. If one sets the proportionality factor ρ equal to unity, for the sake of simplicity, and one lets α_i^*, β_i^* denote the values of the parameters (α, β) that emerge from the values α_i'', β_i'' in (8), § 2 when one replaces α_i, β_i and α_i', β_i' with the values of $\alpha_i(t)$, $\beta_i(t)$ and $\alpha_i(t')$, $\beta_i(t')$ then the quantities α_i^*, β_i^* will be proportional to the values of $\alpha_i(t + t')$, $\beta_i(t + t')$ that emerge from (5):

$$(7)\qquad \begin{cases} \alpha_i^* = \sqrt{\mathfrak{m}}(\cot\sqrt{\mathfrak{m}}\,t + \cot\sqrt{\mathfrak{m}}\,t')\cdot\alpha_i(t+t'), \\ \beta_i^* = \sqrt{\mathfrak{m}}(\cot\sqrt{\mathfrak{m}}\,t + \cot\sqrt{\mathfrak{m}}\,t')\cdot\beta_i(t+t'); \end{cases}$$

the relation $S(t)\cdot S(t') = S(t + t')$ then exists between the three motions $S(t)$, $S(t')$, $S(t + t')$.

If $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$ do not have the value zero simultaneously then for a finite value of t, formulas (5) generally give a well-defined infinite value for the ratios of the parameters (α, β); these formulas only appear to contain the irrationality $\sqrt{\mathfrak{m}}$. However, if t is a whole number multiple of $\pi/\sqrt{\mathfrak{m}}$ – say, $\kappa\pi/\sqrt{\mathfrak{m}}$ – then the proportionality factor ρ must increase to infinity in order to still get finite values for the quantities α_i, β_i. The associated motions will be translations:

$$(5b)\qquad \begin{cases} \rho\alpha_0 = \dfrac{\mathfrak{m}\sqrt{\mathfrak{m}}}{\pi}, & \rho\beta_i = \mathfrak{n}\cdot\kappa\cdot\mathfrak{a}_i, \\ \beta_0 = 0, & \alpha_i = 0, \end{cases} \qquad (i = 1, 2, 3).$$

Finally, translations will also arise when $\mathfrak{m}$ vanishes:

$$(5c)\qquad \rho\,\alpha_0 = 1, \qquad \beta_0 = 1, \qquad \alpha_i = 0, \qquad \rho\beta_i = \mathfrak{a}_i\,t \qquad (i = 1, 2, 3).$$

The formulas that were developed will give the connection between the parameters (α, β) of a motion and the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$ of the same motion completely, as soon as one sets $t = 1$ in them. If we solve equations (5) under that assumption then we will find:

$$
(8) \qquad \left\{ \begin{aligned}
\mathfrak{a}_1 &= \cos\lambda_1 \cdot \vartheta, & \mathfrak{b}_1 &= \cos\lambda_1 \cdot \eta \cdot (\vartheta \cot\vartheta - 1) + \mu_1 \cdot \vartheta, \\
\mathfrak{a}_2 &= \cos\lambda_2 \cdot \vartheta, & \mathfrak{b}_2 &= \cos\lambda_2 \cdot \eta \cdot (\vartheta \cot\vartheta - 1) + \mu_2 \cdot \vartheta, \\
\mathfrak{a}_3 &= \cos\lambda_3 \cdot \vartheta, & \mathfrak{b}_3 &= \cos\lambda_3 \cdot \eta \cdot (\vartheta \cot\vartheta - 1) + \mu_3 \cdot \vartheta,
\end{aligned} \right.
$$

in which the quantities $\cot \vartheta$, η, $\cos \lambda_i$, μ_i have the following values:

$$
(9) \qquad \begin{aligned}
\cot \vartheta &= \frac{\alpha_0}{\sqrt{a_{00} - \alpha_0^2}}, & \eta &= \frac{\beta_0}{\sqrt{a_{00} - \alpha_0^2}}, \\
\cos \lambda_i &= \frac{\alpha_i}{\sqrt{a_{00} - \alpha_0^2}}, & \mu_i &= \frac{\beta_i}{\sqrt{a_{00} - \alpha_0^2}}.
\end{aligned}
$$

Equations (8) and (9) provide the canonical parameters for any motion that is not a translation. One will get infinitely many values for the quantities $\mathfrak{a}_i$, $\mathfrak{b}_i$ for any motion $S(\alpha, \beta)$, which correspond to just as many infinitely-small motions that generate the motion $S(\alpha, \beta)$. However, the case in which S is a rotation ($\eta = 0$) is excluded. In that case, the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$ have uniquely-determined ratios; S will be generated by only *one* infinitely-small motion, although it will be repeated infinitely often.

The translations exhibit an essentially different behavior. The canonical parameters of a translation will, in fact, be indeterminate. If one considers that the relations:

$$
(10) \qquad \left\{ \begin{aligned}
\mathfrak{m} &= \mathfrak{a}_1\mathfrak{a}_1 + \mathfrak{a}_2\mathfrak{a}_2 + \mathfrak{a}_3\mathfrak{a}_3 = \vartheta^2, \\
\mathfrak{n} &= \mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{a}_2\mathfrak{b}_2 + \mathfrak{a}_3\mathfrak{b}_3 = -\eta\vartheta
\end{aligned} \right.
$$

follow from equations (8) and (9), and that the quantities on the right-hand side also take on finite values for the translations then, with the use of the abbreviations:

$$
(11) \qquad \cos \lambda_i = \frac{-\beta_i}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}, \qquad \eta = \frac{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}{\alpha_0},
$$

that will imply that one can make two different Ansätze for a translation with the parameters $\alpha_0, 0, 0, 0, 0, \beta_1, \beta_2, \beta_3$. Either one sets:

$$
(8b) \qquad \mathfrak{a}_1 = \cos \lambda_1 \cdot \kappa\pi, \qquad \mathfrak{a}_2 = \cos \lambda_2 \cdot \kappa\pi, \qquad \mathfrak{a}_3 = \cos \lambda_3 \cdot \kappa\pi,
$$

in which one understands κ to mean a non-zero whole number, and chooses $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ according to the condition:

$$\mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{a}_2\mathfrak{b}_2 + \mathfrak{a}_3\mathfrak{b}_3 = -\eta \cdot \kappa\pi, \tag{10b}$$

but otherwise arbitrary, or one sets:

$$\left\{ \begin{array}{lll} \mathfrak{a}_1 = 0, & \mathfrak{a}_2 = 0, & \mathfrak{a}_3 = 0, \\ \mathfrak{b}_1 = -\eta\cos\lambda_1, & \mathfrak{b}_2 = -\eta\cos\lambda_2, & \mathfrak{b}_3 = -\eta\cos\lambda_3. \end{array} \right. \tag{8c}$$

Equations (8*b*) and (10*b*) collectively contain the general solution to equations (5*b*). Corresponding to any value of κ, one will obtain ∞^2 values of the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$; they generate the translation $S(\alpha, \beta)$ by infinitely-small screwing motions. However, equations (8*c*) correspond to equations (5*c*). They give the parameters of the infinitely-small screw that generates the translation $S(\alpha, \beta)$.

Finally, the canonical parameters of the identity transformation (whose easily-derived expressions might be suppressed) are undetermined to an even higher degree than the canonical parameters of a general translation.

It is remarkable that not only the quantities η, ϑ, λ_i, which were discussed already, but also the quantities μ_i on the right-hand sides of equations (8), have a simple geometric meaning. Namely, the middle complex of the motion $S(\alpha, \beta)$ is linked in a known way with an extension of rank two or a force system that one can represent by:

$$\left\{ \begin{array}{l} \cos\lambda_1[e_0e_1] + \cos\lambda_2[e_0e_2] + \cos\lambda_3[e_0e_3] \\ \quad +\mu_1[e_2e_3] + \quad \mu_2[e_3e_1] + \quad \mu_3[e_1e_2], \end{array} \right. \tag{12}$$

if one applies **Grassmann**'s notation. In this, e_0 means the origin of the coordinates, e_1, e_2, e_3 are the segments of length one that are parallel to the coordinate axes, and $[e_i\, e_\kappa]$ are the exterior products of rank two of those quantities whose exterior product of rank four $[e_0\, e_1\, e_2\, e_3]$ is set equal to one. If one represents the extensions (12) as sums of two line segments then (as is known) they will describe a tetrahedron of constant volume J; $6J$ will be equal to the exterior square of the expressions (12); i.e., one will have the equation:

$$J = -\tfrac{1}{3}\,\eta\cot\vartheta. \tag{13}$$

The quantities μ_1, μ_2, μ_3 are then represented directly in the simplest way as the six-fold sums of two tetrahedra, or as the moments of the force system relative to the coordinate axes.

One can introduce the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$ into any expression that contains the transformation coefficients $a_{i\kappa}$ or the parameters (α, β) of a motion with the help of the formulas that were developed. One can derive the finite equations of the canonical parameter group, e.g., from equations (8), § 2. The construction of those equations will then become so complicated that there is no point in writing them down. In most cases, one will appeal to the parameters (α, β) to much greater advantage then the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$, whose introduction would require the solution of transcendental equations.

An important peculiarity of the canonical parameters $\mathfrak{a}_i$, $\mathfrak{b}_i$ consists of the fact that they can be represented by *linear* equations for all continuous groups of motions. The parameters (α, β) do not have that property. However, it is noteworthy that, at the very least, *the* groups of motions whose translations can be characterized by *algebraic* equations in the coefficients $a_{i\kappa}$ [namely, the subgroups 2), 3), 4*b*), 5), 6), 7), 8*b*), 9) of the classification that was given in the first section of § 9] with the use of the parameters (α, β) can likewise be defined by *linear* equations in those parameters.

One will arrive at the group of *rotations around a given point* $y_0 : y_1 : y_2 : y_3$ when one chooses the parameters (α, β) according to the conditions:

$$(14) \qquad \left\{ \begin{aligned} \beta_0 &= 0, \\ y_0\beta_1 &= \alpha_3 y_2 - \alpha_2 y_3, \\ y_0\beta_2 &= \alpha_1 y_3 - \alpha_3 y_1, \\ y_0\beta_3 &= \alpha_2 y_1 - \alpha_1 y_2. \end{aligned} \right.$$

There is no difficulty associated with expressing all continuous groups of motion with the help of the parameters (α, β) and examining them. Those parameters will probably also be of use in the representation of non-continuous groups of motions (as well as the groups of motions and transfers; cf., § 9). In general, I have still not made any investigations of that vast subject.

§ 9.

Expressing transfers in space in terms of the parameters (α, β).

In order to go from the analytical expression for the general motion in space to the expression of the general transfer, one needs only to perform a reflection in the origin of the coordinates before or (as we would like to do) after the motion. The following formulas will then emerge from formulas (1), (2), (3), of § 4:

$$(1) \qquad \left\{ \begin{aligned} x_0' &= -a_{00}x_0 \\ x_1' &= \quad a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= \quad a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= \quad a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3; \end{aligned} \right.$$

$$(2) \qquad \left\{ \begin{aligned} p_{01}' &= -a_{11}p_{01} - a_{12}p_{02} - a_{13}p_{03}, \qquad [3] \\ p_{23}' &= \quad b_{11}p_{01} + b_{12}p_{02} + b_{13}p_{03} \\ &\qquad a_{11}p_{01} + a_{12}p_{02} + a_{13}p_{03}; \qquad [3] \end{aligned} \right.$$

$$(3) \qquad \begin{cases} u_0' = -a_{00}u_0 - a_{01}u_1 - a_{02}u_2 - a_{03}u_3, \\ u_1' = \qquad\qquad a_{11}u_1 + a_{12}u_2 + a_{13}u_3, \\ u_2' = \qquad\qquad a_{21}u_1 + a_{22}u_2 + a_{23}u_3, \\ u_3' = \qquad\qquad a_{31}u_1 + a_{32}u_2 + a_{33}u_3. \end{cases}$$

These equations then represent the general transfer in space, as long as one expresses the coefficients $a_{i\kappa}$, $b_{i\kappa}$ in terms of the parameters (α, β) *with the prescription of* § 4. Every system of these parameters that satisfies the conditions $N(\alpha, \beta) \neq 0$, $L(\alpha, \beta) = 0$ corresponds to a transfer in space, and conversely.

Since the transfers define a group in conjunction with the motions, that raises the question of whether one can also summarize the transformations of that extended group with the help of the parameters (α, β) in a simple way. That is, in fact, the case: One also finds that there is a *bilinear combination* of the parameters (α, β) for the enveloping group. Namely, it consists in the following easily-proved theorems, the first of which is, in principle, only a slight extension of the main theorem that was proved in § 2:

If $S(\alpha, \beta)$ is a motion and $S'(\alpha', \beta')$ is a motion or transfer in space then the parameters (α'', β'') of the composed transformation $S'' = SS'$, which is a motion in the former case and a transfer in the latter one, are both obtained from formulas (8), § 2 (pp. 71).

By contrast, if $S(\alpha, \beta)$ is a transfer and $S'(\alpha', \beta')$ is a motion or a transfer then the parameters (α'', β'') of the composed transformations $S'' = SS'$, which is a transfer (motion, resp.), will be deduced from the formulas:

$$(4) \qquad \begin{cases} \alpha_0'' = \quad \alpha_0\alpha_0' - \alpha_1\alpha_1' - \alpha_2\alpha_2' - \alpha_3\alpha_3', \\ \alpha_1'' = \quad \alpha_0\alpha_1' + \alpha_1\alpha_0' + \alpha_2\alpha_3' - \alpha_3\alpha_2', \text{ etc., [3]} \\ \beta_0'' = -\alpha_0\beta_0' + \alpha_1\beta_1' + \alpha_2\beta_2' + \alpha_3\beta_3' \\ \qquad + \beta_0\alpha_0' - \beta_1\alpha_1' - \beta_2\alpha_2' - \beta_3\alpha_3', \\ \beta_1'' = -\alpha_0\beta_1' - \alpha_1\beta_0' - \alpha_2\beta_3' + \alpha_3\beta_2' \\ \qquad + \beta_0\alpha_1' + \beta_1\alpha_0' + \beta_2\alpha_3' - \beta_3\alpha_3', \text{ etc., [3]} \end{cases}$$

Formulas (4) differ from formulas (8) of § 8 simply by the fact that the quantities β_i' are replaced with $-\beta_i'$ on the right-hand side. An especially important special case of the last theorem is the following one:

The inverse of a transfer with the parameters:

$$\alpha_0 : \quad \alpha_1 : \quad \alpha_2 : \quad \alpha_3 : \quad \beta_0 : \beta_1 : \beta_2 : \beta_3$$

has the parameters:

$$\alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 : -\beta_0 : \beta_1 : \beta_2 : \beta_3 .$$

Here, as everywhere, as well, we assume that the parameters are chosen to satisfy the condition $L(\alpha, \beta) = 0$.

The geometric structures that are linked with a transfer can be represented analytically by the parameters (α, β) relatively simply, correspond to what we have seen in the theory of motions.

The middle plane of a transfer $S(\alpha, \beta)$ has the coordinates:

$$u_0 : u_1 : u_2 : u_3 = \beta_0 : \alpha_1 : \alpha_2 : \alpha_3 \tag{5}$$

The middle point of a transfer $S(\alpha, \beta)$ has the coordinates:

$$x_0 : x_1 : x_2 : x_3 = \alpha_0 : \beta_1 : \beta_2 : \beta_3. \tag{6}$$

The rotational axis of a transfer $S(\alpha, \beta)$ has the coordinates:

$$\begin{aligned} & p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} \\ & = \alpha_0\alpha_1 : \alpha_0\alpha_2 : \alpha_0\alpha_3 : -(\alpha_2\beta_3 - \alpha_3\beta_2) : -(\alpha_3\beta_1 - \alpha_1\beta_3) : -(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned} \tag{7}$$

The special cases that emerge in the theory of transfers are characterized as follows:

The parameter α_0 vanishes for transfers with an infinitely-distant middle point.

The parameters:

$$\alpha_0, \beta_1, \beta_2, \beta_3$$

vanish for reflections in the planes in space.

The parameters:

$$\beta_0, \alpha_1, \alpha_2, \alpha_3$$

vanish for reflections in the points of space.

We then have the following principal types of transfers to distinguish, analytically, as well as geometrically, in their classification:

I. *The general case:* $\alpha_0 \neq 0, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0.$

Only one finite point will be fixed – namely, the middle point of the transfer – and likewise one point at infinity, whose coordinates are:

$$x_0 : x_1 : x_2 : x_3 = 0 : \alpha_1 : \alpha_2 : \alpha_3 .$$

Moreover, in addition to the plane at infinity, yet another plane will be fixed, namely, the middle plane.

The transfer can be generated by a reflection in the middle plane (5), and a prior or subsequent rotation around the axis (7). (Cf., I, § 10, pp. 40)

The reflection has the parameters:

$$0 : : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : 0 : 0 : 0, \tag{8a}$$

the rotation has the parameters:

$$(8b)\qquad -(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) : \alpha_0\alpha_1 : \alpha_0\alpha_2 : \alpha_0\alpha_3 : 0 : -(\alpha_2\beta_3 - \alpha_3\beta_2) : -(\alpha_3\beta_1 - \alpha_1\beta_3) : -(\alpha_1\beta_2 - \alpha_2\beta_1)\,;$$

the associated angle $2\vartheta'$ is then given by the equation:

$$(8c)\qquad \cot\vartheta' = -\frac{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}{\alpha_0}\,.$$

The transfer can, moreover, be generated by a reflection in the midpoint:

$$x_0' = -\alpha_0\,x_0\,,\quad x_i' = -2\beta_i\,x_0 + \alpha_0\,x_i \qquad (i = 1, 2, 3);$$

i.e., by a transfer with the parameters:

$$(9a)\qquad \alpha_0 : 0 : 0 : 0 : 0 : \beta_1 : \beta_2 : \beta_3\,,$$

and a prior or subsequent rotation around the axis (7) whose parameters are:

$$(9b)\qquad \alpha_0^2 : \alpha_0\alpha_1 : \alpha_0\alpha_2 : \alpha_0\alpha_3 : 0 : -(\alpha_2\beta_3 - \alpha_3\beta_2) : -(\alpha_3\beta_1 - \alpha_1\beta_3) : -(\alpha_1\beta_2 - \alpha_2\beta_1)$$

and an angle of rotation 2ϑ for which:

$$(9c)\qquad \cot\vartheta = \frac{\alpha_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,.$$

[Cf., formula (8*c*), as well as formula (9) in § 7, pp. 90]

In fact, if one performs the sequence of transformations (8*a*) and (8*b*) on pp. 101 or the transformations (9*a*) and (9*b*) that was given above, one after the other, then one will come back to formulas (1), after one has removed the factor $-(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$ in the first case (and α_0^2, in the second) from the parameters of the composed transformation.

II. *Transfers with midpoints at infinity.*

$$\alpha_0 = 0, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0, \qquad \beta_1^2 + \beta_2^2 + \beta_3^2 \neq 0.$$

As before, the point at infinity $0 : \alpha_1 : \alpha_2 : \alpha_3$ that is perpendicular to the middle plane will now be fixed, but also every point at infinity of the middle plane, and thus, every point of the line:

$$0 : 0 : 0 : \alpha_1 : \alpha_2 : \alpha_3\,.$$

Moreover, in addition to the middle plane, any plane of a pencil of planes that are parallel and perpendicular to the middle plane will be fixed, namely, any plane through the line:

$$0 : 0 : 0 : \beta_1 : \beta_2 : \beta_3 .$$

III. *Reflection in the plane of space:*

$$\alpha_0 = 0, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0, \qquad \beta_1 = \beta_2 = \beta_3 = 0.$$

The middle point is an undetermined point x of the middle plane:

$$\beta_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0.$$

Among the planes, in addition to the middle plane (plane of reflection), all planes that are perpendicular to will be fixed – i.e., all planes whose coordinates satisfy the equation:

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0.$$

IV. *Reflections in the points of space.*

$$\alpha_0 \neq 0, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \quad \beta_0 = 0.$$

Among the points, in addition to the midpoint, all points of the plane at infinity will be fixed.

The following theorems that are based in formulas (7*a*) and (9*a*) might be pointed out, in particular (*):

The reflection in a given plane $v_0 : v_1 : v_2 : v_3$ *has the parameters:*

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3 = 0 : v_1 : v_2 : v_3 : v_0 : 0 : 0 : 0.$$

The reflection in a given point $y_0 : y_1 : y_2 : y_3$ *has the parameters:*

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3 = y_0 : 0 : 0 : 0 : 0 : y_1 : y_2 : y_3.$$

(*) **Buchheim** has remarked already that it is useful to relate biquaternions of the special form:

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \beta_0 \varepsilon_0$$

with planes, and biquaternions of the form:

$$\alpha_0 e_0 + \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 + \beta_3 \varepsilon_3$$

with points. He was led to that convention, not by considering reflections, but by considerations of convenience of a different kind. (Am. J., v. VII, *loc. cit.*, no. 8). The relationship between biquaternions of the special kind:

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 + \beta_3 \varepsilon_3$$

and linear complexes was found already by Clifford. Naturally, it is only an expression of a fact that was known in a different form for a long time. (Cf., the rem. on pp. 69)

§ 10.

Motions in a ray bundle.

If one sets all of the parameters β_i equal to zero then the analytical representation for *the* motions and transfers that fix the coordinate origin o will emerge from our formulas.

As in § 12, we now consider the ray g and the plane γ through the point o that is perpendicular to it to be spatial elements. If we are given a line that includes the origin and has the coordinates:

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} = g_1 : g_2 : g_3 : 0 : 0 : 0,$$

and a plane that goes through the origin and has the coordinates:

$$u_0 : u_1 : u_2 : u_3 = 0 : \gamma_1 : \gamma_2 : \gamma_3$$

then the dualistic transformation $\mathfrak{P}$ of the absolute polar system will be represented by the equations:

(1) $$g_1 : g_2 : g_3 = \gamma_1 : \gamma_2 : \gamma_3 .$$

In place of the motions and transfers, we now obtain a continuous group:

(2) $$\begin{cases} g_i' = a_{i1} g_1 + a_{i2} g_2 + a_{i3} g_3 , \\ \gamma_i' = a_{i1} \gamma_1 + a_{i2} \gamma_2 + a_{i3} \gamma_3 , \end{cases} \qquad (i = 1, 2, 3),$$

in which the expressions for the coefficients $a_{i\kappa}$ are once more inferred from formulas (4) of § 4. (Cf., pp. 77)

As before, the formulas for the composition of parameters will be provided by the multiplication theorem for quaternions, whose equations we would now like to arrange somewhat differently:

(3) $$\begin{cases} \alpha_0'' = \alpha_0 \alpha_0' - \alpha_1 \alpha_1' - \alpha_2 \alpha_2' - \alpha_3 \alpha_3', \\ \alpha_1'' = \alpha_0 \alpha_1' + \alpha_1 \alpha_0' + \alpha_2 \alpha_3' - \alpha_3 \alpha_2', \\ \alpha_2'' = \alpha_0 \alpha_2' - \alpha_1 \alpha_3' + \alpha_2 \alpha_0' + \alpha_3 \alpha_1', \\ \alpha_3'' = \alpha_0 \alpha_3' + \alpha_1 \alpha_2' - \alpha_2 \alpha_1' + \alpha_3 \alpha_0'. \end{cases}$$

We can now write down the expressions for the transformations $\mathfrak{T}_1$, $\mathfrak{T}_2$, T, T' that were introduced in § 12 of the Part I with no further discussion. They are:

(4) $$\begin{cases} \mathfrak{T}_1^{-1} : g = \Phi(\alpha, \bar{g}), & \mathfrak{T}_2 : g' = \Phi(\bar{\alpha}, \bar{g}), \\ \mathfrak{T}_2^{-1} : \gamma = \Phi(\alpha, \bar{\gamma}), & \mathfrak{T}_1 : \gamma' = \Phi(\bar{\alpha}, \bar{\gamma}), \\ T^{-1} : g = \Phi(\alpha, \bar{\gamma}), & T : g' = \Phi(\bar{\alpha}, \bar{\gamma}), \\ T'^{-1} : \gamma = \Phi(\alpha, \bar{g}), & T' : \gamma' = \Phi(\bar{\alpha}, \bar{g}), \end{cases}$$

when one writes, for example, the equation $g = \Phi(\alpha, \bar{g})$ as an abbreviation for the system:

$$(4b)\qquad \begin{cases} g_1 = \alpha_0\bar{g}_1 + \alpha_3\bar{g}_2 + \alpha_2\bar{g}_3, \\ g_2 = \alpha_3\bar{g}_1 + \alpha_0\bar{g}_2 - \alpha_1\bar{g}_3, \\ g_3 = -\alpha_2\bar{g}_1 + \alpha_1\bar{g}_2 - \alpha_0\bar{g}_3, \end{cases}$$

and the equation $g' = \Phi(\bar{\alpha}, \bar{g})$ for the system of formulas (*):

$$(4c)\qquad \begin{cases} g'_1 = \alpha_0\bar{g}_1 + \alpha_3\bar{g}_2 - \alpha_2\bar{g}_3, \\ g'_2 = -\alpha_3\bar{g}_1 + \alpha_0\bar{g}_2 + \alpha_1\bar{g}_3, \\ g'_3 = \alpha_2\bar{g}_1 - \alpha_1\bar{g}_2 + \alpha_0\bar{g}_3. \end{cases}$$

In order to get back to formulas (2) from formulas (4*b*) and (4*c*), one must solve the system of equations (4*b*) for the quantities $\bar{g}_i$ and then introduce the values thus-found into equations (4*c*). One will then obtain the following equations:

$$\alpha_0 \cdot a_{00} \cdot g'_i = \alpha_0\,(a_{i1}\,g_1 + a_{i2}\,g_2 + a_{i3}\,g_3),$$

in place of equations (2).

One then sees that formulas (4) will become useless in the case for which $\alpha_0 = 0$, without formulas (2) themselves ceasing to apply, in turn (**).

The program whose foundation was the purpose of this entire study – namely, the construction of a theory of invariants of motions that is analogous to the usual theory of invariants – can be realized most easily in the case that was discussed here. In the year 1886, on the basis of formulas that are not essentially different from those of **Euler**, the author had already developed a complete theory of invariants of the group of a conic section in plane, which can be regarded as distinguished by the great simplicity of the formulas and the fact that it is an intermediary between the theory of binary forms and that of ternary forms. He hopes to be able to present those investigations soon in order to then extend the argument to the complicated behavior of the Euclidian plane and non-Euclidian and Euclidian space.

It is easy to devise a passage to the limit by which one can go from the formulas that were just developed to similar expressions for motions and transfers in the plane.

We first set:

$$(5)\qquad \begin{cases} g_1 = x_1, \quad g_2 = \lambda x_2, \quad g_3 = \lambda x_3, \\ \gamma_1 = \lambda u_1, \quad \gamma_2 = u_2, \quad \gamma_3 = u_3. \end{cases}$$

(*) As we already said, the originator of these formulas was **Rodrigues**. Later, **Hermite** and **Cayley** extended the argument to quadratic forms that are not represented as sums of squares.

(**) Cf., **Bachmann**, Crelle's J. Bd. 76 and **Hermite**, *Ibidem*, Bd. 78.

If we then pass to the limit $\lambda = 0$, and interpret the ratios:

$$y = \frac{x_2}{x_1} \qquad \text{and} \qquad z = \frac{x_3}{x_1}$$

as the rectangular Cartesian coordinates in the plane then the absolute cone of the point *o*:

$$g_1^2 + g_2^2 + g_3^2 = 0, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0$$

will become the pair of so-called circle points at infinity:

$$x_1 = 0, \qquad u_2^2 + u_3^2 = 0.$$

At the same time, the motions of the ray bundle will go to motions and transfers in the plane; indeed, one will obtain the *motions* when one introduces new parameters by the formulas:

(6) $$\alpha_0 = \alpha_0^*, \qquad \alpha_1 = \alpha_1^*, \qquad \alpha_2 = \lambda\alpha_2^*, \qquad \alpha_3 = \lambda\alpha_3^*,$$

simultaneously with the substitutions (5), and the *transfers*, when one appeals to the substitutions:

(7) $$\alpha_0 = \lambda\alpha_0^*, \qquad \alpha_1 = \lambda\alpha_1^*, \qquad \alpha_2 = \alpha_2^*, \qquad \alpha_3 = \alpha_3^*,$$

instead of (6).

The formulas that will be summarized in the next paragraph will be obtained in that way. As one will see, they confirm, in every respect, results that were derived in § 13 of Part I by geometric considerations.

§ 11.

Motions and transfers in the plane.

Just as for the motions in a ray bundle, so can one also assign four homogeneous parameters to the motions in a plane in essentially only one way such that bilinear combination exists for those parameters (*).

At the same time, a parametric representation of transfers is determined (up to linear transformations of the parameters) by this representation of motions such that bilinear combinations likewise exist for the motions and transfers collectively.

The formulas for *motions* read, in rectangular parallel coordinates *y*, *z*:

(1*a*) $$\begin{cases} (\alpha_0^2 + \alpha_1^2)y' = (\alpha_0^2 - \alpha_1^2)y + 2\alpha_0\alpha_1 \cdot z + 2(\alpha_1\alpha_2 - \alpha_0\alpha_3), \\ (\alpha_0^2 + \alpha_1^2)z' = -2\alpha_0\alpha_1 \cdot y + (\alpha_0^2 - \alpha_1^2)z + 2(\alpha_3\alpha_1 + \alpha_0\alpha_2), \end{cases}$$

(*) Wiener Monatsh., *loc. cit.*, §§ 15 and 16.

or, in homogeneous point coordinates $x_2 / x_1 = y$, $x_3 / x_1 = z$:

$$\text{(1)} \qquad \left\{ \begin{aligned} x_1' &= (\alpha_0^2 + \alpha_1^2) x_1 , \\ x_2' &= 2(\alpha_1\alpha_2 - \alpha_0\alpha_3) x_1 + (\alpha_0^2 + \alpha_1^2) x_2 + 2\alpha_0\alpha_1 \cdot x_3 , \\ x_3' &= 2(\alpha_3\alpha_1 + \alpha_0\alpha_2) x_1 - 2\alpha_0\alpha_1 \cdot x_2 + (\alpha_0^2 - \alpha_1^2) x_3 , \end{aligned} \right.$$

or finally in the associated line coordinates:

$$\text{(2)} \qquad \left\{ \begin{aligned} u_1' &= (\alpha_0^2 + \alpha_1^2) u_1 + 2(\alpha_1\alpha_2 + \alpha_0\alpha_3) u_2 + 2(\alpha_3\alpha_1 - \alpha_0\alpha_2) u_3 , \\ u_2' &= \quad \cdot \quad + (\alpha_0^2 - \alpha_1^2) u_2 + 2\alpha_0\alpha_1 \cdot u_3 , \\ u_3' &= \quad \cdot \quad - 2\alpha_0\alpha_1 \cdot u_2 + (\alpha_0^2 - \alpha_1^2) u_3 . \end{aligned} \right.$$

However, the corresponding formulas for the *transfers* are these:

$$\text{(3}a\text{)} \qquad \left\{ \begin{aligned} -(\alpha_2^2 + \alpha_3^2) y' &= (\alpha_2^2 - \alpha_3^2) y + 2\alpha_2\alpha_3 \cdot z + 2(\alpha_1\alpha_2 - \alpha_0\alpha_3), \\ -(\alpha_2^2 + \alpha_3^2) z' &= 2\alpha_2\alpha_3 \cdot y - (\alpha_2^2 - \alpha_3^2) z + 2(\alpha_3\alpha_1 + \alpha_0\alpha_2), \end{aligned} \right.$$

or

$$\text{(3)} \qquad \left\{ \begin{aligned} x_1' &= -(\alpha_2^2 + \alpha_3^2) x_1 , \\ x_2' &= 2(\alpha_1\alpha_2 - \alpha_0\alpha_3) x_1 + (\alpha_2^2 - \alpha_3^2) x_2 + 2\alpha_2\alpha_3 \cdot x_3 , \\ x_3' &= 2(\alpha_3\alpha_1 + \alpha_0\alpha_2) x_1 + 2\alpha_2\alpha_3 \cdot x_2 - (\alpha_2^2 - \alpha_3^2) x_3 , \end{aligned} \right.$$

or finally:

$$\text{(4)} \qquad \left\{ \begin{aligned} u_1' &= -(\alpha_2^2 + \alpha_3^2) u_1 + 2(\alpha_1\alpha_2 + \alpha_0\alpha_3) u_2 + 2(\alpha_3\alpha_1 - \alpha_0\alpha_2) u_3 , \\ u_2' &= \quad \cdot \quad + (\alpha_2^2 - \alpha_3^2) u_2 + 2\alpha_2\alpha_3 \cdot u_3 , \\ u_3' &= \quad \cdot \quad + 2\alpha_2\alpha_3 \cdot u_2 - (\alpha_2^2 - \alpha_3^2) u_3 . \end{aligned} \right.$$

The product $S'' = SS'$ of two transformations $S(a)$ and $S'(a')$ of our group is:

When S and S' are motions, a motion whose parameters are:

$$\text{(5)} \qquad \left\{ \begin{aligned} \alpha_0'' &= \alpha_0\alpha_0' - \alpha_1\alpha_1' , \\ \alpha_1'' &= \alpha_0\alpha_1' + \alpha_1\alpha_0' , \\ \alpha_2'' &= \alpha_0\alpha_2' - \alpha_1\alpha_3' + \alpha_2\alpha_0' + \alpha_3\alpha_1' , \\ \alpha_3'' &= \alpha_0\alpha_3' + \alpha_1\alpha_2' - \alpha_2\alpha_1' + \alpha_3\alpha_0' . \end{aligned} \right.$$

When S and S' are transfers, a motion whose parameters are:

$$(6)\quad \begin{cases} \alpha_0'' = & -\alpha_2\alpha_2' - \alpha_3\alpha_3', \\ \alpha_1'' = & +\alpha_2\alpha_3' - \alpha_3\alpha_2', \\ \alpha_2'' = \alpha_0\alpha_2' - \alpha_1\alpha_3' + \alpha_2\alpha_0' + \alpha_3\alpha_1', \\ \alpha_3'' = \alpha_0\alpha_3' + \alpha_1\alpha_2' - \alpha_2\alpha_1' + \alpha_3\alpha_0'. \end{cases}$$

When S is a motion and S′ is a transfer, a motion whose parameters are:

$$(7)\quad \begin{cases} \alpha_0'' = \alpha_0\alpha_2' - \alpha_1\alpha_1' - \alpha_2\alpha_2' - \alpha_3\alpha_3', \\ \alpha_1'' = \alpha_0\alpha_1' + \alpha_1\alpha_0' + \alpha_2\alpha_3' - \alpha_3\alpha_2', \\ \alpha_2'' = \alpha_0\alpha_2' - \alpha_1\alpha_3' \qquad , \\ \alpha_3'' = \alpha_0\alpha_3' + \alpha_1\alpha_2' \qquad . \end{cases}$$

When S is a transfer and S′ is a motion, a transfer whose parameters are:

$$(8)\quad \begin{cases} \alpha_0'' = \alpha_0\alpha_2' - \alpha_1\alpha_1' - \alpha_2\alpha_2' - \alpha_3\alpha_3', \\ \alpha_1'' = \alpha_0\alpha_1' + \alpha_1\alpha_0' + \alpha_2\alpha_3' - \alpha_3\alpha_2', \\ \alpha_2'' = & +\alpha_2\alpha_0' + \alpha_3\alpha_1', \\ \alpha_3'' = & -\alpha_2\alpha_1' + \alpha_3\alpha_0'. \end{cases}$$

Formulas (5) define the multiplication theorem of a system of complex numbers that is a degeneration of the quaternions:

$$(9)\quad \begin{array}{c|cccc} & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_1 & e_2 & e_3 \\ e_1 & e_1 & -e_0 & e_3 & -e_2 \\ e_2 & e_2 & -e_3 & 0 & 0 \\ e_3 & e_3 & e_2 & 0 & 0 \end{array}$$

If one appeals to the quaternions themselves instead of this system (9) then one can summarize formulas (1), …, (4) and formulas (5), …, (8) with the help of a further symbol ε very simply. If α_i are the parameters of a motion then we set:

$$(10a)\qquad \alpha = \alpha_0\, e_0 + \alpha_1\, e_1 + \varepsilon\, \alpha_2\, e_2 + \varepsilon\, \alpha_3\, e_3\,,$$

to abbreviate, but if α_i are the parameters of a transfer then we write, correspondingly:

$$(10b)\qquad \alpha = \varepsilon\, \alpha_0\, e_0 + \varepsilon\, \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3\,.$$

In addition, we set:

$$(11)\quad \begin{cases} x = \quad x_1e_1 + \varepsilon\, x_2e_2 + \varepsilon\, x_3e_3, \\ u = \varepsilon\, u_1e_1 + \quad u_2e_2 + \quad u_3e_3. \end{cases}$$

If we now assume that e_2 is equal to zero and that the quantity ε commutes with the quaternion units then we can draw formulas (1), …, (4) together into two formulas:

(12) $$x' = \bar{\alpha} x \alpha, \qquad u' = \bar{\alpha} u \alpha,$$

and formulas (5), …, (8) into a single one:

(13) $$\alpha'' = \alpha\alpha'.$$

In that way, we have brought the defining law for these formulas into its simplest expression.

One easily ascertains the geometric meaning that the formulas (5), …, (8) take on when one interprets that quantities α_i as homogeneous coordinates in a triply-extended space. (Cf., Wiener Monatsh., *loc. cit.*, § 16) Naturally, the relationship between formulas (1) and (3) and the geometric defining data of a motion or transfer can also be given with no further analysis.

In the case of a *motion*:

(14) $$x_1 : x_2 : x_3 = \alpha_1 : \alpha_2 : \alpha_3$$

are the coordinates of the center of rotation; the angle of rotation 2ϑ is given by the equation:

(15) $$\tan\vartheta = \mp\frac{\alpha_1}{\alpha_0},$$

such that the motion (1) will coincide with the rotation:

(16) $$\begin{cases} \left(y' - \dfrac{\alpha_2}{\alpha_1}\right) = \left(y - \dfrac{\alpha_2}{\alpha_1}\right)\cos 2\vartheta \mp \left(z - \dfrac{\alpha_3}{\alpha_1}\right)\sin 2\vartheta, \\ \left(y' - \dfrac{\alpha_3}{\alpha_1}\right) = \pm\left(y - \dfrac{\alpha_2}{\alpha_1}\right)\sin 2\vartheta + \left(z - \dfrac{\alpha_3}{\alpha_1}\right)\cos 2\vartheta. \end{cases}$$

The appearance of the double sign in formulas (15) and (16) goes back to the fact that we have still made no convention as to the sense in which we are measuring the angle in the (y, z) plane. If we then decide upon the positive sense of rotation as being the one that takes the positive y-axis to the positive z-axis then only the upper sign will be valid, while only the lower sign will be true for the other case.

From what was said, the motion will be a transfer when α_0 vanishes, and it will be a translation when α_1 vanishes.

If $S(a)$ is a *transfer* then:

(17) $$u_1 : u_2 : u_3 = \alpha_1 : \alpha_2 : \alpha_3$$

will be the line coordinates of the *middle line;* one-half the magnitude of the translation η will have the components:

$$\frac{\alpha_0\alpha_3}{\alpha_2^2+\alpha_3^2}, \qquad -\frac{\alpha_0\alpha_2}{\alpha_2^2+\alpha_3^2}$$

and thus, the length:

(18) $$\eta = \frac{\alpha_0}{\sqrt{\alpha_2^2+\alpha_3^2}}.$$

The transfer will therefore be a reflection when α_0 vanishes.

The expression $\alpha_0^2 + \alpha_1^2$ ($\alpha_2^2 + \alpha_3^2$, resp.) plays the role of a *discriminant* for the motions (transfers, resp.); it can always be assumed to be non-zero. If one denotes it by N then one will have the following formula for the composition of motions, as well as transfers:

(19) $$N(\alpha)\,N(\alpha') = N(\alpha''),$$

just as in the theory of quaternions. [Cf., eq. (10), § 2] The following theorem can likewise be carried over with no further discussion:

The inverse of a motion or transfer whose parameters are:

$$\alpha_0 : \quad \alpha_1 : \quad \alpha_2 : \quad \alpha_3$$

will have the parameters:

$$\alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 .$$

If $S(a)$ is a non-involutory *motion* then the endpoints of a chord xx' will depend upon its midpoint $\bar{x}$ in this way:

(20) $$\mathfrak{T}_1^{-1}: \quad \begin{cases} x_1 = \alpha_0\bar{x}_1 , \\ x_2 = \alpha_3\bar{x}_1 + \alpha_0\bar{x}_2 - \alpha_1\bar{x}_3, \\ x_3 = -\alpha_2\bar{x}_1 + \alpha_1\bar{x}_2 + \alpha_0\bar{x}_3, \end{cases}$$

(20*b*) $$\mathfrak{T}_2: \quad \begin{cases} x_1' = \alpha_0\bar{x}_1 , \\ x_2' = -\alpha_3\bar{x}_1 + \alpha_0\bar{x}_2 + \alpha_1\bar{x}_3, \\ x_3' = \alpha_2\bar{x}_1 - \alpha_1\bar{x}_2 + \alpha_0\bar{x}_3, \end{cases}$$

and similarly, the connection between any pair of corresponding lines u, u' and their angle bisector $\bar{u}$ of the first kind will be:

(21) $$\mathfrak{T}_2^{-1}: \quad \begin{cases} u_1 = \alpha_0\bar{u}_1 - \alpha_3\bar{u}_2 + \alpha_2\bar{u}_3, \\ u_2 = \quad + \alpha_0\bar{u}_2 - \alpha_1\bar{u}_3, \\ u_3 = \quad + \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3, \end{cases}$$

$$(21b) \qquad \mathfrak{T}_2: \quad \begin{cases} u_1' = & \alpha_0\bar{u}_1 + \alpha_3\bar{u}_2 - \alpha_2\bar{u}_3, \\ u_2' = & \qquad + \alpha_0\bar{u}_2 + \alpha_1\bar{u}_3, \\ u_3' = & \qquad - \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3. \end{cases}$$

By contrast, if $S(\alpha)$ is a non-involutory *transfer* then one can represent the endpoints of the chord xx' in a simple way in terms of the normal $\bar{u}$:

$$(22) \qquad T^{-1}: \quad \begin{cases} x_1 = & \qquad - \alpha_3\bar{u}_2 + \alpha_2\bar{u}_3, \\ x_2 = & \alpha_3\bar{u}_2 + \alpha_0\bar{u}_2 - \alpha_1\bar{u}_3, \\ x_3 = & -\alpha_2\bar{u}_1 + \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3; \end{cases}$$

$$(22b) \qquad T: \quad \begin{cases} x_1' = & \qquad + \alpha_3\bar{u}_2 - \alpha_2\bar{u}_3, \\ x_2' = & -\alpha_3\bar{u}_2 + \alpha_0\bar{u}_2 + \alpha_1\bar{u}_3, \\ x_3' = & \alpha_2\bar{u}_1 - \alpha_1\bar{u}_2 + \alpha_0\bar{u}_3. \end{cases}$$

We have derived the parametric representation of motions and transfers in the plane from the formulas for motions in a ray bundle by a passage to the limit in order to obtain them in a convenient form directly. We might remark, in passing, that one can also manage without this passage to the limit. The motions in the plane, as well as the transfers, can be regarded as motions or transfers of a three-fold extended space that contains the plane, if desired.

If we start with, say, the *motions* in space then we will need only to introduce the assumption that:

$$(23a) \qquad \alpha_2 = \alpha_3 = \beta_0 = \beta_1 = 0$$

into the formulas of § 4 in order to obtain the motions in the plane $x_1 = 0$. Likewise, the transfers in that plane will arise when we make the assumption:

$$(23b) \qquad \alpha_0 = \alpha_1 = \beta_2 = \beta_3 = 0 .$$

Of the formulas that emerge from the formulas (1) of § 4, we drop the equation that gives the expression for x_1' in terms of x_1, and then once more write x_1, instead of x_0 . Finally, if we introduce some new parameters, and in fact with the assumption (23*a*), by the substitutions:

$$(24a) \qquad \alpha_0 = \alpha_0^*, \qquad \alpha_1 = \alpha_1^*, \qquad \beta_2 = \alpha_3^*, \qquad \beta_3 = -\alpha_2^*,$$

and with the assumption (23*b*), by the substitutions:

$$(24b) \qquad \beta_0 = \alpha_0^*, \qquad \beta_1 = \alpha_1^*, \qquad \alpha_2 = \alpha_3^*, \qquad \alpha_3 = -\alpha_2^*,$$

then we will, in fact, arrive at the formulas that were just presented all over again.

At this point, it deserves to be remarked that one can, admittedly, simplify the (developed) form of formulas (1), …, (8) with the help of an imaginary transformation (*).

Namely, there is a natural system of parallel coordinates in the plane that consist of the two pencils of straight lines that run through the so-called circle points at infinity. It will be introduced by the imaginary transformation:

$$y + i\,z = x, \qquad y - i\,z = \bar{x},$$

or if we set:

$$x = \frac{\xi_2}{\xi_1}, \qquad \bar{x} = \frac{\xi_3}{\xi_1},$$

in order to make things homogeneous, by the transformation:

$$\text{(25)} \qquad \begin{cases} x_1 = \xi_1, \quad 2x_2 = \xi_2 + \xi_3, \quad 2ix_3 = \xi_2 - \xi_3, \\ u_1 = \omega_1, \quad u_2 = \omega_2 + \omega_3, \quad u_3 = (\omega_2 - \omega_3)i. \end{cases}$$

If we convert, e.g., formulas (1*a*) accordingly then that will itself suggest new parameters, which we would like to denote by:

$$\text{(26)} \qquad \begin{array}{ll} \gamma_{11} = \alpha_0 + i\alpha_1, & \gamma_{12} = -\alpha_3 - i\alpha_2, \\ \gamma_{21} = \alpha_3 - i\alpha_2, & \gamma_{22} = \ \ \alpha_0 - i\alpha_1. \end{array}$$

We now give the formulas that emerge from our formulas (1), …, (8) by these substitutions, and labeled by the same equations numbers:

$$(1a)^* \qquad \begin{cases} \gamma_{11} \cdot x' = \gamma_{22} \cdot x - 2\gamma_{21}, \\ \gamma_{22} \cdot \bar{x}' = \gamma_{11} \cdot \bar{x} + 2\gamma_{12}. \end{cases}$$

$$(1)^* \qquad \begin{cases} \xi_1' = \gamma_{11}\gamma_{22} \cdot \xi_1, \\ \xi_2' = \gamma_{22}(-2\gamma_{21} \cdot \xi_1 + \gamma_{22} \cdot \xi_2), \\ \xi_3' = \gamma_{11}(\ \ 2\gamma_{12} \cdot \xi_1 + \gamma_{11} \cdot \xi_3). \end{cases}$$

$$(2)^* \qquad \begin{cases} \omega_1' = \gamma_{11}\gamma_{22} \cdot \omega_1 + 2\gamma_{11}\gamma_{21} \cdot \omega_2 - 2\gamma_{12}\gamma_{22} \cdot \omega_3, \\ \omega_1' = \gamma_{11}^2 \cdot \omega_2, \qquad \omega_3' = \gamma_{22}^2 \cdot \omega_3. \end{cases}$$

$$(3a)^* \qquad \begin{cases} \gamma_{12} \cdot x' = -\gamma_{21} x - 2\gamma_{22}, \\ \gamma_{21} \cdot \bar{x}' = -\gamma_{12}\bar{x} + 2\gamma_{11}. \end{cases}$$

(*) A corresponding simplification is also possible for the formulas of § 10; however, there, it will come about through a nonlinear transformation.

$$(3)^* \quad \begin{cases} \xi_1' = \gamma_{12}\gamma_{21} \cdot \xi_1, \\ \xi_2' = -\gamma_{21}(2\gamma_{22} \cdot \xi_1 + \gamma_{21} \cdot \xi_3), \\ \xi_3' = \quad \gamma_{12}(2\gamma_{11} \cdot \xi_1 - \gamma_{12} \cdot \xi_2). \end{cases}$$

$$(4)^* \quad \begin{cases} \omega_1' = -\gamma_{11}\gamma_{22} \cdot \omega_1 - 2\gamma_{11}\gamma_{21} \cdot \omega_2 + 2\gamma_{12}\gamma_{22} \cdot \omega_3, \\ \omega_1' = \gamma_{13}^2 \cdot \omega_3, \qquad \omega_3' = \gamma_{21}^2 \cdot \omega_2. \end{cases}$$

$$(5)^* \quad \begin{cases} \gamma_{11}'' = \gamma_{11}\gamma_{11}' \quad \cdot \quad, & \gamma_{12}'' = \gamma_{12}\gamma_{11}' + \gamma_{22}\gamma_{12}', \\ \gamma_{21}'' = \gamma_{11}\gamma_{21}' + \gamma_{21}\gamma_{22}', & \gamma_{22}'' = \quad \cdot \quad + \gamma_{22}\gamma_{22}', \end{cases}$$

$$(6)^* \quad \begin{cases} \gamma_{11}'' = \quad \cdot \quad + \gamma_{21}\gamma_{12}', & \gamma_{12}'' = \gamma_{12}\gamma_{11}' + \gamma_{22}\gamma_{12}', \\ \gamma_{21}'' = \gamma_{11}\gamma_{21}' + \gamma_{21}\gamma_{22}', & \gamma_{22}'' = \gamma_{22}\gamma_{22}' \quad \cdot \quad, \end{cases}$$

$$(7)^* \quad \begin{cases} \gamma_{11}'' = \gamma_{11}\gamma_{11}' + \gamma_{21}\gamma_{12}', & \gamma_{12}'' = \quad \cdot \quad + \gamma_{22}\gamma_{12}', \\ \gamma_{21}'' = \gamma_{11}\gamma_{21}' \quad \cdot \quad, & \gamma_{22}'' = \gamma_{12}\gamma_{21}' + \gamma_{22}\gamma_{22}', \end{cases}$$

$$(8)^* \quad \begin{cases} \gamma_{11}'' = \gamma_{11}\gamma_{11}' + \gamma_{21}\gamma_{12}', & \gamma_{12}'' = \gamma_{22}\gamma_{12}' \quad \cdot \quad, \\ \gamma_{21}'' = \quad \cdot \quad + \gamma_{21}\gamma_{22}', & \gamma_{22}'' = \gamma_{12}\gamma_{21}' + \gamma_{22}\gamma_{22}'. \end{cases}$$

These formulas, which we will have to speak about on a later occasion, show clearly the behavior of the two families of curves that are invariant under the motions, as well as the decomposition of the degenerate motions and transfers into two disjoint families, which is important in the theory of invariants of our group. It is in just that way that it becomes possible for the coefficients $\gamma_{\iota\kappa}$ to enter into formulas $(1a)^*$ and $(3a)^*$ only linearly.

Since one can easily derive formulas $(1a)^*$, …, $(8)^*$ from the usual formulas for the transformation of rectangular coordinates, one will then have another path into the theory that was founded in the present paragraph.

Marburg, 26 July 1891.