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SIMPLIFIED FOUNDATION FOR LIE'S SPHERE GEOMETRY I.

By. E. STUDY.
in Bonn

Translated by D. H. Delphenich

The present essay (which shall be followed by a second one) relates to the elements of sphere geometry (which is called "higher" by some). The algebraic facts that are the roots from which it grows shall be illuminated from a different side. The ideas that are expressed can be easily arranged in such a way that they can also serve as an introduction into that sphere of ideas. Whoever is already familiar with projective geometry will already be sufficiently prepared, and if he is not entirely foreign to quaternions then he will experience only minor difficulties. Only the nucleus of the facts shall be dealt with, and in the simplest way possible.

A prior investigation ([1]) embarked upon the geometry of circles, which includes most of space. This time, in essence, only the spheres will be considered ([2]).

As before, I have initially left all metric notions out of the discussion, which is similar to the usual presentation of projective geometry. That possibility not only exists, but it is also a requirement for a healthy methodology. The creators of the older designs, S. LIE and F. KLEIN, sought to introduce their concepts as "elementary" ones, namely, with the help of sphere radii and angles. However, their definitions were very sketchy, and if extended, they would lose their apparent simplicity and lead to a seemingly abstruse theory.

Metric notions and other details, among which are the special properties of real figures, are excluded completely from the present examination ([3]).

At the center of all consideration, one will find the orthogonal transformations of three and four variables, or rather, their systems of coefficients, together with the up-till-now scarcely-observed degeneracies in such systems. Their connection with sphere geometry is generally of a formal nature. It is only when one makes a special choice of coordinates that they come to light. The group property of orthogonal transformations does not come into play. Meanwhile, the overview and implementation of the formal apparatus will be eased considerably by the exploitation of such relationships. *Therefore, just about everything that will be used at the onset has long since existed in the theory of the simplest orthogonal transformations.* Entire systems of statements can then be

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([1]) Math. Annalen **89-91**(1922-1924).

([2]) The text in large print contains the actual train of thought and defines a connected whole by itself.

([3]) For them, see § 16 and § 17 of the cited treatise. [Math. Annalen, **91** (1924); in particular, propositions XXXIIIa, XXXV, XXXVI.]

phrased in fewer words. Likewise, the well-known parametric representation of the aforementioned transformations (which one obtains most conveniently from quaternions) also yields, with no further discussion, the connection between the projective geometry of a space R_3 and sphere geometry. The basic concepts that are called *point, line, plane,* and *facet* (element) will be assigned, with no gaps, to concepts that belong to "sphere space" M_3^2 as "images," namely – with my terminology that will be explained (repeatedly) later on – the concepts of *X-line*, *oriented sphere*, *Y-line*, and *leaf*, resp.

Things are even simpler in the geometry of circles. An M_2^2 (e.g., a sphere) enters in place of M_3^2. There, the *basic concepts* are "the oriented element" and "the oriented circle." The oriented elements contain the generators of the sphere, *which are also not "unions,"* while the point (even in a complex domain) belongs to the oriented circles. The images of the oriented circles are the lines in a linear complex, while the oriented elements have the pencils of such lines for their images.

It seems very profitable to me to express the basic ideas of sphere geometry that S. LIE sketched out more cleanly. It probably has a substantial interest that a complete, and likewise simple, duplicate of ordinary projective geometry is already present in the geometry of a quadratic point manifold M_3^2.

Modern geometry, especially higher geometry, knows many of these so-called conversion principles, very few of which have been sufficiently investigated. What follows here is also a term in a sequence that extends to infinity. However, the recursion does not have the same simplicity for more than three (real or complex) dimensions, and since it casts light upon the circumstances that belong to the best known of modern geometries, it can serve as one of the most instructive examples.

In the cited papers, whose contents have also served as the starting points for more recent authors, the point continua R_3 and M_3^2, and therefore, the open continuum of elementary geometry, were often lumped together. The enumerated counterparts to the concepts of projective geometry were either missing, or they entered by way of concepts that were only very imprecisely correlated with those statements. Correspondingly, the algebraic apparatus broke down, and they, or some arbitrarily-chosen fragment of the continua R_3 or M_3^2, would then appear.

While recognizing the value of the blueprint that originated with S. LIE, I have nonetheless regarded it as necessary to subject its implementation by him and others to a very disparaging criticism ([1]). As a result of an adaptation of KLEIN's lectures by W. BLASCHKE (1926), I must now once more underscore the statements that I made at the time. That mathematician (one of my former students!) should recognize that he has a description of the error before him, along with its correction.

The fact that his textbook, which is intended for students (!), dodges the concepts of real and imaginary, and of "space" and coordinate transformations defies description. Indeed, I believe that one can even give rise to ghosts with the help of such a methodology. Hence, the intersection figure of two spheres in § 29 is a (more or less) real circle, and in § 39, it consists of a circle and an imaginary specter that can certainly so-to-speak "materialize" as also real and "empty" (cf., pp. 50). Something must be said about taking care in one's statements, especially in the use of words such as "always," "all," "each," or "any." ("Mathematics *in négligée*") Since man indeed never says what he thinks, but often thinks what he says, a poorly (indeed, not at all) defined fragment of a putative "geometry in all space" can emerge (pp. 4). What is now being offered to us *for the third time* is so abortive that such a key notion as the equal status of points and planes under the line-sphere transformations does not come to light ([2]). As before, everything carries the stamp of the most hurried sort of writing.

Only at a single place (pp. 249) does it *almost* appear as if the author succumbed to a fit of pronounced conscientiousness. Namely, at that point, an attempt was made "incidentally" (!) to give a belated (!)

([1]) Jahresbericht der Deutschen Mathematikervereinigung **25** (1916). (Which was not cited by BLASCHKE, and with good reason.)

([2]) See propositions XXIII, XXVII, XXXI in the treatise that was cited above (and also be BLASCHKE).

precise version at least one of many to generally-held assertions. In general, that is also once more a nasty accident (!). However, the reader indeed learns when he can teach himself – namely, in the case where he (unlike the author of those lectures) should feel an irresistible urge to go deeper into the basic questions of sphere geometry…

That kind of mathematics – viz., *mathematics composed of half-truths, abuse of terminology, and incantations* – is truly irksome:

1. The mathematician who deals in approximations does not mean what he says, but something somewhat different.
2. The difference between those statements and their meanings does not come across for him very often.
3. He has said everything that actually matters. Period.

Honi soit qui mal y pense! (†)

Naturally, one cannot at all understand mathematicians who seek to evade every earnest criticism, or put even more simply, who can glide over them, while not remotely appreciating the ground rules of science.

The usual error that appears to be isolated is very harmless in comparison to that demanding (indeed, sometimes downright violent) kind of writing, which can be referred to, without exaggeration, as a *systematization of the error.*

In the future, these lectures will probably also find mostly relatively-uncritical readers. For that very reason, but also to spare the readers the necessity of looking things up, which is my wish for them, I have, moreover, once more analyzed some of the alleged facts upon which sphere geometry allegedly rests.

I.

The manifold M_4^2.

We consider a non-singular quadratic manifold M_4^2 in a domain of rank six; i.e., in a projective continuum R_5 of five *complex* dimensions, represent it by an equation of the special form:

$$\mathfrak{z}_0^2 - \mathfrak{z}_1^2 + \mathfrak{z}_2^2 - \mathfrak{z}_3^2 + \mathfrak{z}_4^2 - \mathfrak{z}_5^2 = 0. \tag{1}$$

We then understand $A_{00}, A_{11}, \dots, A_{33}$ and $B_{00}, B_{11}, \dots, B_{33}$ to mean the system of coefficients of two ternary, homogeneous, orthogonal transformations, so that each of the two systems of equations:

$$\begin{aligned} A_{00}\,\mathfrak{x}_1 &= A_{11}\,\mathfrak{x}_0 + A_{12}\,\mathfrak{x}_2 + A_{13}\,\mathfrak{x}_4 , \\ A_{00}\,\mathfrak{x}_3 &= A_{21}\,\mathfrak{x}_0 + A_{22}\,\mathfrak{x}_2 + A_{23}\,\mathfrak{x}_4 , \\ A_{00}\,\mathfrak{x}_5 &= A_{31}\,\mathfrak{x}_0 + A_{32}\,\mathfrak{x}_2 + A_{33}\,\mathfrak{x}_4 , \end{aligned} \tag{2.l}$$

$$\begin{aligned} -B_{00}\,\mathfrak{y}_1 &= B_{11}\,\mathfrak{y}_0 + B_{12}\,\mathfrak{y}_2 + B_{13}\,\mathfrak{h}_4 , \\ -B_{00}\,\mathfrak{y}_3 &= B_{21}\,\mathfrak{y}_0 + B_{22}\,\mathfrak{y}_2 + B_{23}\,\mathfrak{h}_4 , \\ -B_{00}\,\mathfrak{y}_5 &= B_{31}\,\mathfrak{y}_0 + B_{32}\,\mathfrak{y}_2 + B_{33}\,\mathfrak{h}_4 , \end{aligned} \tag{2.r}$$

(†) Translator's note: "Shame upon those who think evil!" (This is also the motto of the British Order of the Garter.)

will yield one of the $2 \cdot \infty^{2 \cdot 3}$ planes that lie on M_4^2, and indeed, in the first case, we will obtain what we would like to call *a left plane* E_ξ, and the second, a *right plane* E_ϕ. The *Grassmann coordinates* of those planes, whose specification is essential for all that follows, can be calculated with no further assumptions. In the first case, they will be pair-wise equal to each other, and in the second, pair-wise equal and opposite ([1]), and indeed we will obtain the *coordinates of* E_ξ as the ratios ([2]):

$$
\boxed{\begin{array}{ccc}
 & \mathfrak{X}_{024} = A_{00} = \mathfrak{X}_{135}, & \\
\mathfrak{X}_{124} = A_{11} = \mathfrak{X}_{035}, & \mathfrak{X}_{014} = A_{12} = \mathfrak{X}_{235}, & \mathfrak{X}_{021} = A_{13} = \mathfrak{X}_{435}, \\
\mathfrak{X}_{324} = A_{21} = \mathfrak{X}_{105}, & \mathfrak{X}_{034} = A_{22} = \mathfrak{X}_{125}, & \mathfrak{X}_{023} = A_{23} = \mathfrak{X}_{145}, \\
\mathfrak{X}_{524} = A_{31} = \mathfrak{X}_{130}, & \mathfrak{X}_{054} = A_{32} = \mathfrak{X}_{132}, & \mathfrak{X}_{025} = A_{33} = \mathfrak{X}_{134}.
\end{array}} \tag{3.l}
$$

When one temporarily sets $\bar{\mathfrak{Y}}_{ikl}$ in place of $-\mathfrak{Y}_{ikl}$, for the sake of better clarity, one will likewise find the system of *coordinates of* E_ϕ:

$$
\boxed{\begin{array}{ccc}
 & \mathfrak{Y}_{024} = B_{00} = \bar{\mathfrak{Y}}_{135}, & \\
\bar{\mathfrak{Y}}_{124} = B_{11} = \mathfrak{Y}_{035}, & \bar{\mathfrak{Y}}_{014} = B_{12} = \mathfrak{Y}_{235}, & \bar{\mathfrak{Y}}_{021} = B_{13} = \mathfrak{Y}_{435}, \\
\bar{\mathfrak{Y}}_{324} = B_{21} = \mathfrak{Y}_{105}, & \bar{\mathfrak{Y}}_{034} = B_{22} = \mathfrak{Y}_{125}, & \bar{\mathfrak{Y}}_{023} = B_{23} = \mathfrak{Y}_{145}, \\
\bar{\mathfrak{Y}}_{524} = B_{31} = \mathfrak{Y}_{130}, & \bar{\mathfrak{Y}}_{054} = B_{32} = \mathfrak{Y}_{132}, & \bar{\mathfrak{Y}}_{025} = B_{33} = \mathfrak{Y}_{134}.
\end{array}} \tag{3.r}
$$

We will then reduce the twenty *GRASSMANN* coordinates of a plane E_ξ or E_ϕ to half that number, when we employ *either* the symbols that appear in the left version of (3) *or* the ones that appear in the right version, and we then have the theorem that in the case of the left (right, resp.) planes E_ξ (E_ϕ, resp.), those coordinates will be equal to the coefficients of a proper (improper, resp.) homogeneous, ternary, orthogonal transformation, in succession ([3]).

The non-vanishing of the quantities A_{00}, B_{00} is assumed here, which has the consequence that the system of equations (2) will define a non-closed continuum of planes E_ξ and E_ϕ on M_4^2 that does not exhaust the totality of those planes then. However, the equations (3) that one derives from (2) are not subject to that restriction. Rather, one has the theorem:

([1]) This theorem can obviously be extended to an indeterminate even number of variables, and we shall make some occasional applications of that fact.

([2]) For the laws of constructing those formulas, see the treatises on the KUMMER configuration and the PASCAL hexangle, Leipziger Berichte, 1892 and 1895.

([3]) This theorem, and even some others, can be extended to an indeterminate number of variables *mutatis mutandis*; i.e., to the theory of a non-singular M_{2m}^2. For more details, see Math. Annalen **91** (1924), pp. 102 (remark). Moreover, it is naturally irrelevant which family of planes on M_4^2 one would wish that the proper orthogonal transformations should correspond to.

The same equations (viz., twenty linearly-independent quadratic equations) exist between the GRASSMANN coordinates of a left (right, resp.) plane on the quadratic manifold M_4^2 that exist between the coefficients of a homogeneous, ternary, proper (improper, resp.) orthogonal transformation, and no further dependencies of any sort.

Along with the proof of this statement, we shall point out the richer content in it. Indeed, with the help of EULER's known formulas, we can express the quantities A_{00}, A_{ik}, and B_{00}, B_{ik}, and in essentially only one way, in terms of two systems of homogeneous parameters:

(4) $$\xi_0 : \xi_1 : \xi_2 : \xi_3\,, \qquad \phi_0 : \phi_1 : \phi_2 : \phi_3\,,$$

by way of:

(5.*l*)
$$\begin{gathered} A_{00} = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2\,, \\ A_{11} = \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ A_{23} = 2(\xi_2\,\xi_3 + \xi_0\,\xi_1)\,, \qquad A_{32} = 2(\xi_2\,\xi_3 - \xi_0\,\xi_1)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{gathered}$$

(5.*r*)
$$\begin{gathered} B_{00} = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2\,, \\ B_{11} = \phi_0^2 + \phi_1^2 - \phi_2^2 - \phi_3^2\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ B_{23} = 2(\phi_2\,\phi_3 + \phi_0\,\phi_1)\,, \qquad B_{32} = 2(\phi_2\,\phi_3 - \phi_0\,\phi_1)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{gathered}$$

We now consider the parameters ξ_i , ϕ_k to be the homogeneous coordinates of a point and a plane, resp., in a projective continuum R_3 that shall be called the *image space* from now on. We then calculate the (so-called PLÜCKER) coordinates of the connecting lines and intersection lines of two points ξ, ξ' and two planes ϕ, ϕ', resp.:

$$\begin{aligned} \Xi_{01} &= \xi_0\,\xi_1' - \xi_1\,\xi_0' = \Phi_{23}, & \Xi_{23} &= \xi_2\,\xi_3' - \xi_3\,\xi_2' = \Phi_{01}\,, \\ \mathrm{H}_{01} &= \phi_2\,\phi_3' - \phi_3\,\phi_2' = \Psi_{23}, & \mathrm{H}_{23} &= \phi_0\,\phi_1' - \phi_1\,\phi_0' = \Psi_{01}\,. \end{aligned}$$

We then bring the (PLÜCKER) equation that exists between such line coordinates:

$$\mathrm{Z}_{01}\,\mathrm{Z}_{23} + \mathrm{Z}_{02}\,\mathrm{Z}_{31} + \mathrm{Z}_{03}\,\mathrm{Z}_{12} = 0$$

into the form (1) by means of the substitutions:

(6)
$$\boxed{\begin{aligned} \mathfrak{z}_0 &= \mathrm{Z}_{01} + \mathrm{Z}_{23}, & \mathfrak{z}_2 &= \mathrm{Z}_{02} + \mathrm{Z}_{31}, & \mathfrak{z}_4 &= \mathrm{Z}_{03} + \mathrm{Z}_{12}, \\ \mathfrak{z}_1 &= \mathrm{Z}_{01} - \mathrm{Z}_{23}, & \mathfrak{z}_3 &= \mathrm{Z}_{02} - \mathrm{Z}_{31}, & \mathfrak{z}_5 &= \mathrm{Z}_{03} - \mathrm{Z}_{12}. \end{aligned}}$$

The bundle of lines through a point ξ will then *always* correspond to a plane E_ξ on M_4^2, and likewise the line field in a plane ϕ will correspond to a plane E_ϕ. In order to find the

GRASSMANN coordinates of, say, the plane E_ξ, one needs only to connect the point ξ to the three corners of the coordinate tetrahedron that is being used with lines, and after referring to formulas (6), looking for the corresponding points $\mathfrak{z}'$, $\mathfrak{z}''$, $\mathfrak{z}'''$ on the manifold M_4^2. If the corners employed were, say, (0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 1 : 0) then one would obtain just the expressions (5.*l*) with the factor ξ_0 as the coordinates of the connecting plane E_ξ of $\mathfrak{z}'$, $\mathfrak{z}''$, $\mathfrak{z}'''$. One will obtain the same expressions, with the factors ξ_1 or ξ_2 or ξ_3, when one employs one of the other triples of corners of the coordinate tetrahedron. However, since the quantities ξ_0, ξ_1, ξ_2, ξ_3 cannot all simultaneously have the value zero as homogeneous coordinates, the general validity of formulas (3.*l*) is then proved, and one likewise proves the validity of formulas (3.*r*).

Now, it is clear that a point ξ and a plane ϕ that are not united will correspond to planes E_ξ and E_ϕ, resp., on M_4^2, that have no common point, and that when ξ and ϕ are united, the planes E_ξ and E_ϕ will intersect in a line. A necessary and sufficient condition for this "united position" of E_ξ and E_ϕ is then the existence of the equation:

$$\boxed{(\xi\,\phi) = \xi_0\,\phi_0 + \xi_1\,\phi_1 + \xi_2\,\phi_2 + \xi_3\,\phi_3 = 0.} \tag{7}$$

One likewise explains how any two left planes E_ξ and E_η will intersect in the point that corresponds to the connecting line of ξ and η, and that correspondence will also be true for two right planes E_ϕ and E_χ.

The point ξ and the plane ϕ in united position define a figure with five complex constants that BIANCHI called a *facet* (and LIE called it a "surface element," or more briefly, an "element"), and they collectively determine a pencil of lines that will map to a line $\mathfrak{Z}$ on M_4^2. Conversely, every line $\mathfrak{Z}$ on M_4^2 naturally corresponds to a facet that is the image of the figure that consists of two planes E_ξ and E_ϕ in united position, and thus $\mathfrak{Z}$ is also their line of intersection. We desire *to calculate the GRASSMANN coordinates of the line $\mathfrak{Z}$ when the facet (ξ, ϕ) is given, and conversely, to find the coordinates of the facet (ξ, ϕ) when the coordinates of $\mathfrak{Z}$ are given.*

This problem is similar to the one that was treated before, from which one obtained the dependencies $\xi \leftrightarrow E_\xi$, $\phi \leftrightarrow E_\phi$, and it is likewise elementary. However, the necessary calculation is more tedious in the present case.

In order to be able to grasp it briefly, I shall next introduce the following system of symbols:

$$\begin{aligned}
C_{00} &= \quad \xi_0\,\phi_0 + \xi_1\,\phi_1 + \xi_2\,\phi_2 + \xi_3\,\phi_3 = \Omega_{00}\,,\\[1ex]
C_{01} &= -\,\xi_0\,\phi_1 + \xi_1\,\phi_0 - \xi_2\,\phi_3 + \xi_3\,\phi_2 = \Omega_{24}\,,\\
C_{02} &= -\,\xi_0\,\phi_2 + \xi_1\,\phi_3 + \xi_2\,\phi_0 - \xi_3\,\phi_1 = \Omega_{40}\,,\\
C_{03} &= -\,\xi_0\,\phi_3 - \xi_1\,\phi_2 + \xi_2\,\phi_1 + \xi_3\,\phi_0 = \Omega_{02}\,,\\[1ex]
C_{10} &= \quad \xi_0\,\phi_1 - \xi_1\,\phi_0 - \xi_2\,\phi_3 + \xi_3\,\phi_2 = \Omega_{35}\,,\\
C_{20} &= \quad \xi_0\,\phi_2 + \xi_1\,\phi_3 - \xi_2\,\phi_0 - \xi_3\,\phi_1 = \Omega_{51}\,,\\
C_{30} &= \quad \xi_0\,\phi_3 - \xi_1\,\phi_2 + \xi_2\,\phi_1 - \xi_3\,\phi_0 = \Omega_{13}\,,
\end{aligned}$$

(8)

$$\begin{aligned}
C_{11} &= \quad \xi_0\,\phi_0 + \xi_1\,\phi_1 - \xi_2\,\phi_2 - \xi_3\,\phi_3 = \Omega_{01}\,, \\
C_{12} &= \quad \xi_0\,\phi_2 + \xi_1\,\phi_1 - \xi_2\,\phi_2 - \xi_3\,\phi_3 = \Omega_{01}\,, \\
C_{13} &= -\,\xi_0\,\phi_3 + \xi_1\,\phi_2 + \xi_2\,\phi_1 + \xi_3\,\phi_0 = \Omega_{21}\,, \\[1em]
C_{21} &= -\,\xi_0\,\phi_3 + \xi_1\,\phi_1 + \xi_2\,\phi_1 - \xi_3\,\phi_0 = \Omega_{01}\,, \\
C_{22} &= \quad \xi_0\,\phi_0 - \xi_1\,\phi_1 + \xi_2\,\phi_2 - \xi_3\,\phi_3 = \Omega_{01}\,, \\
C_{23} &= \quad \xi_0\,\phi_1 + \xi_1\,\phi_0 + \xi_2\,\phi_3 + \xi_3\,\phi_2 = \Omega_{21}\,, \\[1em]
C_{31} &= \quad \xi_0\,\phi_2 + \xi_1\,\phi_3 + \xi_2\,\phi_0 + \xi_3\,\phi_1 = \Omega_{03}\,, \\
C_{32} &= -\,\xi_0\,\phi_1 - \xi_1\,\phi_0 + \xi_2\,\phi_3 + \xi_3\,\phi_2 = \Omega_{25}\,, \\
C_{33} &= \quad \xi_0\,\phi_0 - \xi_1\,\phi_1 - \xi_2\,\phi_2 + \xi_3\,\phi_0 = \Omega_{45}\,.
\end{aligned}$$

In regard to that, one should immediately note that these equations can be solved for the products $\xi_i\,\phi_k$. For example, one will have:

$$C_{00} + C_{11} + C_{22} + C_{33} = 2\xi_0\,\phi_0\,.$$

The dependencies that exist between the sixteen quantities C_{ik} (viz., 36 linearly-independent quadratic equations) are then linear transformations of the ones that exist between the products $\xi_i\,\phi_k$. They will then differ from the latter only in form. If they are fulfilled then one can always calculate the homogeneous coordinates of a point ξ and plane ϕ from them, and when the equation $C_{00} = 0$ exists, moreover, those figures will collectively define a facet.

However, the property of the quantities C_{ik} that is important here can be expressed in the theorem:

When the square-root quantity:

$$\sqrt{N\xi}\cdot\sqrt{N\phi} = \sqrt{\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2}\sqrt{\phi_0^2+\phi_1^2+\phi_2^2+\phi_3^2}$$

is non-zero, the quantities C_{ik} , together with one or the other of the values of $\sqrt{N\xi}\cdot\sqrt{N\phi}$, will define the system of coefficients of any quaternary, proper, homogeneous orthogonal transformation. However, in general, the dependencies between the coefficients C_{ik} will be exhausted by the homogeneous equations that exist between the coefficients of such a transformation.

In fact, the quaternion formula:

$$\sqrt{N\xi}\cdot\sqrt{N\phi}\cdot\omega^* = \tilde{\xi}\;\omega\,\phi,$$

by which, one can indeed express every proper orthogonal transformation in terms of four variables ω_0, ω_1, ω_2, ω_3, delivers precisely that system of coefficients ([1]). If $N\xi \cdot N\phi = 0$ then one will obtain a system of quantities C_{ik} that will also be useful from now on, but which no longer belong to a transformation. Whether the one or the other occurs is immaterial here.

Note that when $N\xi \neq 0$, $N\xi \neq 0$, the ratios A_{00}, A_{ik} and B_{00}, B_{ik} – i.e., the system of coefficients of the ternary transformations:

$$N\xi \cdot \lambda^* = \bar{\xi}\,\lambda\,\xi, \qquad N\phi \cdot \rho^* = \tilde{\phi}\,\rho\,\phi,$$

that are invariantly-coupled to the quaternary transformations – can also always be calculated without having to go the way of the products $\xi_i\,\phi_k$. Namely, any two complementary two-rowed determinants from the matrix of quantities C_{ik} are equal to each other. If one now chooses the nine possible types of two pairs of such determinants in such a way that all sixteen quantities C_{ik} are employed in that way then sums and differences of them will be equal to the quantities $A_{11}, \ldots, A_{33}$, $B_{11}, \ldots, B_{33}$, multiplied by the factors $N\phi$ or $N\xi$, resp. For example, one has:

$$N\phi \cdot A_{11} = B_{00} \cdot A_{11} = \begin{vmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{vmatrix} + \begin{vmatrix} C_{02} & C_{03} \\ C_{11} & C_{13} \end{vmatrix} = \begin{vmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{vmatrix} + \begin{vmatrix} C_{20} & C_{21} \\ C_{30} & C_{31} \end{vmatrix},$$

$$N\xi \cdot B_{11} = A_{00} \cdot B_{11} = \begin{vmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{vmatrix} - \begin{vmatrix} C_{02} & C_{03} \\ C_{11} & C_{13} \end{vmatrix} = \begin{vmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{vmatrix} - \begin{vmatrix} C_{20} & C_{21} \\ C_{30} & C_{31} \end{vmatrix}.$$

Moreover, we will not employ this proposition and everything that is connected with it here ([2]). Another property of the quantities C_{ik} can also be mentioned only as an appendix:

The equations $C_{ik} = 0$ or $\Omega_{\alpha\beta} = 0$, when considered in isolation and interpreted in the projective continuum R_3, represent the group of sixteen two-sided collineations that is linked with any KUMMER configuration, and as a result, with $\infty^{2\cdot 3}$ of them ([3]).

In the present investigation, the quantities C_{ik} will now be considered to be ratios, and the fifteen GRASSMANN coordinates of any line on M_4^2 will be used for their representation. C_{00} is then to be set equal to zero, which will have the effect that the quantities C_{11}, C_{22}, C_{33} can be written a little more easily; e.g.:

$$C_{11} = 2\,(\xi_0\,\phi_0 + \xi_1\,\phi_1) = -\,2\,(\xi_2\,\phi_2 + \xi_3\,\phi_3).$$

One will get the desired coordinates from elementary calculations that are similar to the ones above in the following compilation:

([1]) The appearance of quaternary, orthogonal transformations in this context can be excluded from the outset. However, I shall omit an explanation of that fact, which would seem to necessitate a good number of words that are not required here.

([2]) Cf., Am. J. Math. **29** (1906) or (1907).

([3]) See the previously-cited paper in Leipziger Berichte, 1892. Further groups of similar structure (which can all be represented in a similar way) were determined exhaustively in the Göttinger Nachrichten from the year 1912.

(10)

		(0)	(2)	(4)
	0	$\mathfrak{Z}_{24} = C_{01} = -\mathfrak{U}_{24}$	$\mathfrak{Z}_{40} = C_{02} = -\mathfrak{U}_{40}$	$\mathfrak{Z}_{24} = C_{01} = -\mathfrak{U}_{24}$
(1)	$-\mathfrak{Z}_{25} = C_{10} = \mathfrak{U}_{35}$	$\mathfrak{Z}_{01} = C_{11} = \mathfrak{U}_{01}$	$\mathfrak{Z}_{21} = C_{12} = \mathfrak{U}_{21}$	$\mathfrak{Z}_{41} = C_{13} = \mathfrak{U}_{41}$
(3)	$-\mathfrak{Z}_{51} = C_{20} = \mathfrak{U}_{51}$	$\mathfrak{Z}_{03} = C_{21} = \mathfrak{U}_{03}$	$\mathfrak{Z}_{23} = C_{22} = \mathfrak{U}_{23}$	$\mathfrak{Z}_{43} = C_{23} = \mathfrak{U}_{43}$
(5)	$-\mathfrak{Z}_{13} = C_{30} = \mathfrak{U}_{13}$	$\mathfrak{Z}_{05} = C_{31} = \mathfrak{U}_{05}$	$\mathfrak{Z}_{25} = C_{32} = \mathfrak{U}_{25}$	$\mathfrak{Z}_{45} = C_{33} = \mathfrak{U}_{45}$

The symbols $\mathfrak{U}_{ik}$ that are appended, which can naturally be replaced with the symbols $-\mathfrak{U}_{ik}$, mean the GRASSMANN coordinates of the polar figure to the line $\mathfrak{Z}$ in relation to the manifold M_4^2, and therefore the coordinates of a certain line space $\mathfrak{R}_3$. For the sake of simplicity, we shall write them with two indices, instead of four. (e.g., $\mathfrak{U}_{\alpha\beta\gamma\delta} = \mathfrak{U}_{\varepsilon\zeta}$, if $\alpha\beta\gamma\delta\varepsilon\zeta$ is an even permutation of 012345) Of the quantities $\mathfrak{Z}_{ik}$ (and likewise, of the quantities $\mathfrak{U}_{ik}$), fifteen times six of them – namely, the ones that contain four indices – will satisfy the PLÜCKER equation, and in addition, as in the case of $C_{00} \neq 0$, twenty-one quadratic equations will exist between them that are linearly-independent of them and each other.

That proves:

The fifteen ratios $\mathfrak{Z}_{ik}$ (= $\pm\,\Omega_{\alpha\beta}$) are the coordinates of the intersecting line of the planes E_ξ and E_ϕ, and the fifteen ratios are the coordinates of the polar of $\mathfrak{Z}$, and therefore of the linear space $\mathfrak{R}_3$ that connects the planes E_ξ and E_ϕ.

The association of the figures E_ξ and E_ϕ, and $\mathfrak{Z}$ or $\mathfrak{U}$ that lie in the space R_5 with the figures ξ, ϕ, and *(ξ, ϕ) in the image space R_3* is ***birational and completely free of singularities.***

This last fact is obviously very essential. The simplicity of the following developments rests upon it.

If one performs a collineation or a correlation on the image space then that will induce a proper or improper collineation, resp., of the manifold M_4^2, and the converse is likewise true. The groups Γ_{15}, H_{15}, and $\mathfrak{G}_{15}$, $\mathfrak{H}_{15}$ that come under consideration are holomorphic. Since the collineations and correlations in R_3 are capable of two essentially-different representations by matrices with sixteen quantities, the same thing will be true for the automorphic collineations of M_4^2. However, our investigation has produced not only that fact (which is sufficiently well-known), but a further-qualified formal apparatus that we will need henceforth ([1]).

([1]) The representation of automorphic collineations of our M_4^2 that is borrowed from projective geometry has still not attained its formally-simplest expression, moreover. See Journal für Mathematik **157** (1926), pp. 58, 59.

For the anti-collineations and anti-correlations that appear in that context, see Math. Annalen **91** (1924).

2.

The manifold M_3^2.

We can now derive a further group G_{15}, H_{15}, and indeed *a group of non-projective transformations*, from the last group that was considered $\mathfrak{G}_{15}$, $\mathfrak{H}_{15}$, simply by choosing any linear space R_4 in the space R_5 that does not contact M_4^2, and then intersecting the planes E_ξ and E_ϕ with that R_4. The figures of intersection, or *traces,* of E_ξ and E_ϕ will be lines $\mathfrak{X}$ and $\mathfrak{Y}$ that lie in the intersection of R_4 and M_4^2, and thus on a non-singular M_3^2, and the planes E_ξ and E_ϕ are associated with each other in a one-to-one invertible and everywhere continuous way. If the planes E_ξ and E_ϕ were then exchanged with each other by a transformation of $\mathfrak{G}_{15}$, $\mathfrak{H}_{15}$ then the lines $\mathfrak{X}$ and $\mathfrak{Y}$ would be exchanged correspondingly, and in that way, the group G_{15}, H_{15} would arise as the one whose (simplest) space-element would seem to be the $2 \cdot \infty^{2\cdot 3}$ lines $\mathfrak{X}$ and $\mathfrak{Y}$. *The manifold M_3^2, as the locus of the lines that lie in it, will be doubly-covered (with two sheets) when one each of its lines can either take on the role of a $\mathfrak{X}$-line or also the role of a $\mathfrak{Y}$-line* ([1]). The $\mathfrak{X}$-lines and $\mathfrak{Y}$-lines will be permuted amongst themselves by the transformations of G_{15}, each sheet by itself, but the two sheets will transform according to a rule of a different sort – viz., *contragrediently,* as we would like to say. Namely, as one will see with no further discussion, each transformation of G_{15} and the permutation of the $\mathfrak{X}$ and $\mathfrak{Y}$ lines that it implies will correspond to a collineation in the image space R_3 that permutes the points ξ and planes ϕ correspondingly. Similarly, a transformation of the family $\mathfrak{H}_{15}$ on M_3^2 will act as a transformation of the $\mathfrak{X}$ and $\mathfrak{Y}$-lines that permutes the two sheets of lines with each other (viz., $\mathfrak{X} \leftrightarrow \mathfrak{Y}^*$, $\mathfrak{Y} \leftrightarrow \mathfrak{X}^*$), and that transformation will have a correlation (viz., $\xi \to \phi^*$, $\phi \to \xi^*$) as its image in R_3. Naturally, the group G_{15}, H_{15} will also be mapped holomorphically to the groups Γ_{15}, H_{15} in that way.

We shall now explain the concept of *united position of an $\mathfrak{X}$-line and a $\mathfrak{Y}$-line* by way of the united position of the associated planes E_ξ and E_ϕ. Since E_ξ and E_ϕ intersect each other in a line $\mathfrak{Z}$ then, two position relationships for the united lines $\mathfrak{X}$ and $\mathfrak{Y}$ will be possible: They can *either* have a uniquely-determined point $\mathfrak{z}$ in common *or* they can overlap each other. That latter possibility will occur when $\mathfrak{Z}$ lies in R_4, and therefore coincides with $\mathfrak{X}$ and $\mathfrak{Y}$. With no further discussion, it is clear that: *The united position of an $\mathfrak{X}$-line and a $\mathfrak{Y}$-line is a property that is invariant under transformations of* G_{15}, H_{15}, *while the overlapping of two such lines (on different sheets) is not.* Furthermore, a subgroup G_{10}, H_{10} of G_{15}, H_{15} will be defined by the further demand that the overlapping of two lines $\mathfrak{X}$ and $\mathfrak{Y}$ should also be invariant, and that subgroup will reduce to the group

([1]) A similar situation necessarily exists in radially-projective geometry. [*Geometrie der Dynamen*, 1903, Third Section.]

$\mathfrak{G}_{10}$ of automorphic collineations of M_3^2, to which it is meromorphic, due to the indistinguishability of the two sheets.

It further follows immediately that:

Two lines $\mathfrak{X}$ and $\mathfrak{Y}$ in united position have a well-defined facet (ξ, ϕ) as their image in the projective continuum R_3, and in particular, two lines $\mathfrak{X}$ and $\mathfrak{Y}$ that overlap each other will have an image that is a facet whose point ξ and plane ϕ belong together as the zero-point and zero-plane relative to a certain linear complex or the null systems that is linked with it.

Naturally, by means of this association or mapping, the group G_{10}, H_{10} will correspond to the group Γ_{10}, H_{10} of automorphic collineations and correlations of the complex or null-system, and in particular, the exchange of the two sheets ($\mathfrak{X} \leftrightarrow \mathfrak{Y}$) will correspond to the correlative reflection ($\xi \leftrightarrow \phi$) in the complex, and the pairing of ξ and ϕ is by the associated null-system. We imagine that the linear complex has been definitely chosen and call it the *principal complex* of the image space R_3 .

The problem now arises of carrying out what was just now developed conceptually in terms of algebraic calculations. However, that is an easy thing to do, on the basis of the presentation in § 1.

We assume that the R_4 that is employed in our construction is given by the equation:

$$\boxed{\mathfrak{z}_0 = \mathrm{Z}_{01} + \mathrm{Z}_{23} = 0,} \tag{11}$$

which merely means a specialization of the coordinate system. We then have the equation of our M_3^2:

$$-\mathfrak{z}_1^2 + \mathfrak{z}_2^2 - \mathfrak{z}_3^2 + \mathfrak{z}_4^2 - \mathfrak{z}_5^2 = 0, \tag{12}$$

and likewise the associated principal complex, as well as the null-system that is coupled with it:

$$\boxed{\xi_0 : \xi_1 : \xi_2 : \xi_3 = \phi_1 : -\phi_0 : \phi_3 : -\phi_2 .} \tag{13a}$$

For the sake of what will follow, we shall also express that association – hence, the *correlative reflection in the principal complex* – in line coordinates:

$$\Xi_{01} : \Xi_{02} : \Xi_{03} : \Xi_{23} : \Xi_{31} : \Xi_{12} = \mathrm{H}_{23} : -\mathrm{H}_{02} : -\mathrm{H}_{03} : \mathrm{H}_{01} : -\mathrm{H}_{31} : -\mathrm{H}_{12} . \tag{13b}$$

Only the basic point (1 : 0 : 0 : 0 : 0 : 0) of the coordinate hexatope in the space R_5 does not belong to the space R_4 . When one suppresses the index 0, the ratios $\mathfrak{X}_{0\alpha\beta}$ and $\mathfrak{Y}_{0\alpha\beta}$ will immediately yield the *coordinates of the traces $\mathfrak{X}$ and $\mathfrak{Y}$ of E_ξ and E_ϕ*:

(14.l)
$$\boxed{\begin{array}{ccc} & \mathfrak{X}_{24} = A_{00}, & \\ \mathfrak{X}_{35} = A_{11}, & \mathfrak{X}_{14} = A_{12}, & \mathfrak{X}_{21} = A_{13}, \\ \mathfrak{X}_{51} = A_{21}, & \mathfrak{X}_{34} = A_{22}, & \mathfrak{X}_{23} = A_{23}, \\ \mathfrak{X}_{13} = A_{31}, & \mathfrak{X}_{54} = A_{32}, & \mathfrak{X}_{25} = A_{33}, \end{array}}$$

(14.r)
$$\boxed{\begin{array}{ccc} & \mathfrak{Y}_{24} = B_{00}, & \\ \mathfrak{Y}_{35} = B_{11}, & \mathfrak{Y}_{14} = -B_{12}, & \mathfrak{Y}_{21} = -B_{13}, \\ \mathfrak{Y}_{51} = B_{21}, & \mathfrak{Y}_{34} = -B_{22}, & \mathfrak{Y}_{23} = -B_{23}, \\ \mathfrak{Y}_{13} = B_{31}, & \mathfrak{Y}_{54} = -B_{32}, & \mathfrak{Y}_{25} = -B_{33}. \end{array}}$$

When A_{00} and B_{00} do not vanish, we will then be dealing with two proper orthogonal transformations.

The lines $\mathfrak{X}$ and $\mathfrak{Y}$ will lie united when $(\xi\,\phi) = 0$ in (7), and when the proportion (13a) does not exist, they will have a well-defined point of intersection $\mathfrak{z}$ whose coordinates one extracts from equations (10):

(15)

0	*	$\mathfrak{z}_1 = -C_{02}$	$\mathfrak{z}_2 = C_{03}$
*	$\mathfrak{z}_1 = C_{11}$	*	*
*	$\mathfrak{z}_3 = C_{21}$	*	*
*	$\mathfrak{z}_5 = C_{31}$	*	*

The connecting plane of the lines $\mathfrak{X}$ and $\mathfrak{Y}$ also remains well-defined in the case that was excluded here. If the lines $\mathfrak{X}$ and $\mathfrak{Y}$ overlap each other then it will be the contact plane of M_3^2 along $\mathfrak{X}$ or $\mathfrak{Y}$, and thus, the polar of the two overlapping lines relative to M_3^2. It is the intersecting plane – *or trace – of the linear space R_3, or "flat (Flach)"* $\mathfrak{U}$ *of formulas* (10). Its coordinates (which are denoted with two indices, instead of three) are then to be extracted from the same Table (10):

(16)

0	$\mathfrak{U}_{24} = -C_{01}$	*	*
$\mathfrak{U}_{35} = C_{10}$	*	$\mathfrak{U}_{21} = C_{21}$	$\mathfrak{U}_{41} = C_{13}$
$\mathfrak{U}_{51} = C_{20}$	*	$\mathfrak{U}_{23} = C_{22}$	$\mathfrak{U}_{43} = C_{23}$
$\mathfrak{U}_{23} = C_{30}$	*	$\mathfrak{U}_{25} = C_{32}$	$\mathfrak{U}_{45} = C_{33}$

The sixteen quantities:

(17)

0	$-\mathfrak{U}_{24}$	$-\mathfrak{z}_4$	$\mathfrak{z}_2$
$\mathfrak{U}_{35}$	$\mathfrak{z}_1$	$\mathfrak{U}_{21}$	$\mathfrak{U}_{41}$
$\mathfrak{U}_{51}$	$\mathfrak{z}_3$	$\mathfrak{U}_{23}$	$\mathfrak{U}_{43}$
$\mathfrak{U}_{23}$	$\mathfrak{z}_5$	$\mathfrak{U}_{25}$	$\mathfrak{U}_{45}$

then satisfy the same thirty-six *homogeneous* quadratic equations that exist between the coefficients C_{ik} of a quaternary, proper orthogonal transformation that is specialized by the assumption $C_{00} = 0$. Of them, we need only the ones that are homogeneous in $\mathfrak{z}$, as well as in $\mathfrak{U}$. They are initially equations in just the $\mathfrak{U}$: viz., the five PLÜCKER equations, and a further one that says that the plane $\mathfrak{U}$ contacts the manifold M_3^2

(18) $$\mathfrak{U}_{24}^2 + \mathfrak{U}_{35}^2 + \mathfrak{U}_{51}^2 + \mathfrak{U}_{13}^2 - \mathfrak{U}_{21}^2 - \mathfrak{U}_{23}^2 - \mathfrak{U}_{25}^2 - \mathfrak{U}_{41}^2 - \mathfrak{U}_{43}^2 - \mathfrak{U}_{45}^2 = 0.$$

Furthermore, equation (12) is contained in it. Of the remaining equations, five of them are bilinear in $\mathfrak{z}$ and $\mathfrak{U}$ and have four terms; those equations say that $\mathfrak{z}$ lies on $\mathfrak{U}$. Ten more equations are bilinear on $\mathfrak{z}$ and $\mathfrak{U}$ and have three terms; as a result of them, $\mathfrak{z}$ is the contact point, or in the limiting case, which can be included here, one of the contact points of $\mathfrak{U}$ with M_3^2.

If a plane $\mathfrak{U}$ in the space R_3 is given by the prescription that is included in that then *the pair* of lines $\mathfrak{X}$ and $\mathfrak{Y}$ will always be determined by it, as well, by the lines at which the plane cuts the manifold M_3^2. However, those lines, as a rule, cannot be rationally separated. The question then arises of how one can find them individually.

We shall also provide some information about our formulas in regard to that.

In might happen, to begin with, that $\mathfrak{U}$ is a singular contact plane of M_3^2, so its lines of intersection $\mathfrak{X}$ and $\mathfrak{Y}$ will overlap. $\mathfrak{X}$ and $\mathfrak{Y}$ will then be identical with the polar of $\mathfrak{U}$ relative to M_3^2, if one overlooks their distribution on the two sheets. One infers their coordinates (naturally, except for the proportionality factors, which they do not depend upon) from the following table, in which $\mathfrak{X}$ and $\mathfrak{Y}$ have been replaced with $\mathfrak{Z}$, which coincides with them:

(19)

0	$\mathfrak{Z}_{24} = -\mathfrak{U}_{24}$	*	*
$\mathfrak{Z}_{35} = -\mathfrak{U}_{35}$	*	$\mathfrak{Z}_{21} = \mathfrak{U}_{21}$	$\mathfrak{Z}_{41} = \mathfrak{U}_{41}$
$\mathfrak{Z}_{51} = -\mathfrak{U}_{51}$	*	$\mathfrak{Z}_{23} = \mathfrak{U}_{22}$	$\mathfrak{Z}_{43} = \mathfrak{U}_{43}$
$\mathfrak{Z}_{12} = -\mathfrak{U}_{12}$	*	$\mathfrak{Z}_{25} = \mathfrak{U}_{32}$	$\mathfrak{Z}_{45} = \mathfrak{U}_{45}$

In fact, when one or the other of the two reciprocal substitutions:

$$\phi_0 \leftrightarrow -\xi_1\,, \qquad \phi_1 \leftrightarrow -\xi_0\,, \qquad \phi_2 \leftrightarrow -\xi_3\,, \qquad \phi_3 \leftrightarrow -\xi_2$$

have been performed, one will obtain precisely the corresponding ratios $\mathfrak{X}_{ik} = \mathfrak{Y}_{ik}$ (14) from the quantities $\mathfrak{U}_{ik}$ or $-\mathfrak{U}_{ik}$ that are enumerated in (16). At the same time, the remaining coordinates of the linear $\mathfrak{R}_3$ in the space R_5 that belong to $\mathfrak{U}$ [whose locations in (19) are denoted by asterisks] all take the value zero.

(20) $$\boxed{\mathfrak{U}_{01} = \mathfrak{U}_{02} = \mathfrak{U}_{03} = \mathfrak{U}_{04} = \mathfrak{U}_{05} = 0,}$$

and conversely, the case that is now being treated will occur when equations (20) are verified. The elimination of the quantities $\mathfrak{U}_{0k}$ will yield:

The necessary and sufficient condition for a prescribed plane $\mathfrak{U}$ in the space R_4 to be a singular contact plane of M_3^2 consists of saying that its coordinates must fulfill the equations:

$$\boxed{\begin{aligned}\mathfrak{U}_{24}^2 &= \mathfrak{U}_{35}^2 + \mathfrak{U}_{51}^2 + \mathfrak{U}_{23}^2 = \mathfrak{U}_{21}^2 + \mathfrak{U}_{23}^2 + \mathfrak{U}_{25}^2 = \mathfrak{U}_{41}^2 + \mathfrak{U}_{43}^2 + \mathfrak{U}_{45}^2,\\ \mathfrak{U}_{24}^2 &= \mathfrak{U}_{35}^2 + \mathfrak{U}_{21}^2 + \mathfrak{U}_{43}^2 = \mathfrak{U}_{51}^2 + \mathfrak{U}_{23}^2 + \mathfrak{U}_{43}^2 = \mathfrak{U}_{13}^2 + \mathfrak{U}_{25}^2 + \mathfrak{U}_{45}^2.\end{aligned}} \tag{21}$$

The plane $\mathfrak{U}$ will then be cut out by an $\mathfrak{R}_3$ in the space R_5 that contacts the manifold M_4^2 along a line $\mathfrak{Z}$ in the space R_4 .

In general, however, the two lines of intersection of a contact plane $\mathfrak{U}$ of M_3^2 will be different from each other. $\mathfrak{U}$ is then the trace of two linear $\mathfrak{R}_3'$, $\mathfrak{R}_3''$, each of which contacts the manifold at all points of a line ($\mathfrak{Z}'$: $\mathfrak{Z}''$) and goes through it along a pair of planes $(E'_\xi, E'_\phi, E''_\xi, E''_\phi)$. (In the aforementioned limiting case, the two pairs of planes merge together, so $E'_\xi = E''_\xi$, $E' = E''_\phi$.) If the two lines of intersection are known and distributed on the two sheets of lines on M_3^2, and therefore denoted by $\mathfrak{X}$ and $\mathfrak{Y}$ individually, then the one will appear to be the trace of a well-defined plane E_ξ , and the other one, the trace of a well-defined plane E_ϕ . The splitting of the linear spaces $\mathfrak{R}_3'$, $\mathfrak{R}_3''$, and thus, the decision for a well-defined singular contact $\mathfrak{R}_3$ of M_4^2, will however be effected in such a way that one seeks to fill up the gaps (viz., the asterisks) in the coordinate table (19), which are not zero, as a rule, by conferring the table (10). Since one must have:

$$\begin{aligned}\mathfrak{U}_{35}^2 + \mathfrak{U}_{51}^2 + \mathfrak{U}_{13}^2 - \mathfrak{U}_{21}^2 - \mathfrak{U}_{23}^2 - \mathfrak{U}_{25}^2 &= \mathfrak{U}_{40}^2,\\ \mathfrak{U}_{35}^2 + \mathfrak{U}_{51}^2 + \mathfrak{U}_{13}^2 - \mathfrak{U}_{41}^2 - \mathfrak{U}_{42}^2 - \mathfrak{U}_{45}^2 &= \mathfrak{U}_{02}^2,\\ \mathfrak{U}_{51}^2 + \mathfrak{U}_{13}^2 - \mathfrak{U}_{21}^2 - \mathfrak{U}_{41}^2 &= \mathfrak{U}_{01}^2,\\ \mathfrak{U}_{13}^2 + \mathfrak{U}_{35}^2 - \mathfrak{U}_{23}^2 - \mathfrak{U}_{43}^2 &= \mathfrak{U}_{03}^2,\\ \mathfrak{U}_{35}^2 + \mathfrak{U}_{51}^2 - \mathfrak{U}_{25}^2 - \mathfrak{U}_{45}^2 &= \mathfrak{U}_{05}^2,\end{aligned} \tag{22}$$

the squares of the five quantities $\mathfrak{U}_{0k}$ will be known already. If they are all equal to zero then one will be addressing the special case that was spoken of above. However, if at least one if the $\mathfrak{U}_{0k}^2$ are non-zero then all of the quantities $\mathfrak{U}_{0k}$ will be determined *uniquely* by means of the bilinear equations that exist between the $\mathfrak{U}_{ik}$, after deciding upon a single value for the root. With that, a particular space $\mathfrak{R}_3$ will be chosen from the two linear space $\mathfrak{R}_3'$, $\mathfrak{R}_3''$.

If all fifteen quantities $\mathfrak{U}_{ik}$ are known then, as we have seen, so are the products $\xi_i\,\phi_k$, and with them, the dual ratios ξ_i , ϕ_k , as well as the ratios A_{00}, A_{ik} and B_{00}, B_{ik} , and therefore, ultimately the planes E_ξ and E_ϕ whose traces are the lines $\mathfrak{X}$ and $\mathfrak{Y}$. A change of sign in all of the quantities $\mathfrak{U}_{0k}$ obviously means the projective reflection in the plane R_4 and its pole (1 : 0 : 0 : 0 : 0 : 0), and thus, the exchange of $E_\xi = E'_\xi$ and $E_\phi = E'_\phi$ with E''_ξ and E''_ϕ, and thus the exchange of $\mathfrak{X}$ and $\mathfrak{Y}$.

Finally, Table (15) also provides us with the traces of the line $\mathfrak{Z}$, and therefore the contact point of the plane $\mathfrak{U}$ and the point of intersection of $\mathfrak{X}$ and $\mathfrak{Y}$, when it is defined:

$$\boxed{\mathfrak{z}_1 : \mathfrak{z}_2 : \mathfrak{z}_3 : \mathfrak{z}_4 : \mathfrak{z}_5 = \mathfrak{U}_{01} : -\mathfrak{U}_{02} : \mathfrak{U}_{03} - \mathfrak{U}_{04} : \mathfrak{U}_{05}.} \tag{23}$$

In fact, when that point is defined, it will then have as its image the likewise-well-defined line:

$$\begin{aligned} \Xi_{01} &= \xi_0\,\phi_0 + \xi_1\,\phi_1, & \Xi_{02} &= -\xi_0\,\phi_3 + \xi_2\,\phi_1, & \Xi_{03} &= \xi_0\,\phi_2 + \xi_3\,\phi_1, \\ \Xi_{23} &= \xi_2\,\phi_2 + \xi_3\,\phi_3, & \Xi_{31} &= \xi_3\,\phi_0 - \xi_1\,\phi_2, & \Xi_{12} &= -\xi_1\,\phi_3 - \xi_2\,\phi_0, \end{aligned}$$

which belongs to the facet (ξ, ϕ) and lies in the principal complex $\Xi_{01} + \Xi_{23} = 0$.

If, from now on, we say *leaf* (Blatt) to mean the figure of the plane $\mathfrak{U}$ and its lines of intersection with M_3^2, which are referred to as left and right, according to the schema – hence, $\mathfrak{X}$ and $\mathfrak{Y}$ – then we can summarize what we have proved as follows:

Any plane $\mathfrak{U}$ that contacts the manifold M_3^2 will belong to two leaves that are not distinct from each other only in the case of a singular plane $\mathfrak{U}$ (which contacts M_3^2 in a line). The singular contact planes then define the branching figure for the double covering of the contact planes of M_3^2 with the leaves.

Any leaf corresponds to a well-defined facet in image space R_3, and conversely ([1]).

Overlapping leaves correspond to facets that are paired with each other by the null system of the principal complex in image space, and conversely.

If the plane $\mathfrak{U}$ is given by the ratios of its ten coordinates $\mathfrak{U}_{ik}$ [*that must satisfy the equation* (18)] *then the separation of the two associated leaves will result when one*

([1]) Math. Ann. (1925), pp. 107, Proposition XXVII.

S. LIE phrases this as: "every" element (i.e., every facet) in the image space corresponds to "one" element in the sphere space, and conversely. In KLEIN-BLASCHKE, in the sphere space, in place of an element (although it does not quite follow), one will find an oriented (i.e., "directed") element, whose wording must also refer to the "total space," and in turn, give a theorem that is false. No less than three so-called proofs of this are presented, in which some very remarkable things come about (such as the line-sphere transformation as a "surface transformation").

Let me say that I have produced a "closer examination" of those "relationships." That implies a misleading of the reader, who naturally cannot guess that I find myself to be in contradiction to BLASCHKE, and indeed, it seems to me, at some very essential points.

determines the five more quantities $\mathfrak{U}_{0k}$ *that can therefore serve as* ***coordinates of a leaf****, along the ten given quantities.*

Each individual quantity $\mathfrak{U}_{0k}$ *is the root of a purely-quadratic equation. Rational dependencies exist between those five roots in such a way that each of the quantities* $\mathfrak{U}_{0k}$ *that is non-zero will be determined uniquely from the remaining ones. If all of the quantities* $\mathfrak{U}_{0k}$ *are equal to zero then only one leaf (hence, a singular one) will belong to the plane* $\mathfrak{U}$.

Furthermore, it emerges from what was said that *when determining the leaves that belong to a plane* $\mathfrak{U}$, *it is impossible to always rely upon the same aforementioned roots, or also to replace all of them with a single square root.*

Henceforth, we shall assign the symbol $\underline{\mathfrak{U}}$ to a leaf, in order to distinguish it from its plane $\mathfrak{U}$, and call the transition from a leaf to the one that lies above it (and is different from it, as a rule), the *inversion* of the leaf, and it will have the coordinates $\underline{\mathfrak{U}}_{ik}$, $-\underline{\mathfrak{U}}_{0k}$, or ones that are proportional to them. *Inversion corresponds to the correlative reflection in the principal complex.*

One can also represent the lines $\mathfrak{X}$, $\mathfrak{Y}$ by systems of linear equations, instead of coordinates. One then has only to express the idea that in image space, a line Z of the principal complex $Z_{01} + Z_{23} = 0$ lies united with a point or a plane, and then introduce the point-coordinates of the space R_5 : $\mathfrak{z}_0$ (= 0), $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\mathfrak{z}_3$, $\mathfrak{z}_4$, $\mathfrak{z}_5$. Equations of the form:

$$(24.l)\qquad \boxed{\begin{array}{rrrrl} * & z_4\,\xi_1 & +z_1\,\xi_2 & -z_3\,\xi_3 & =0, \\ -z_4\,\xi_0 & * & -z_2\,\xi_2 & +z_0\,\xi_3 & =0, \\ -z_1\,\xi_0 & +z_2\,\xi_1 & * & -z_4\,\xi_3 & =0, \\ z_3\,\xi_0 & -z_0\,\xi_2 & +z_4\,\xi_2 & * & =0, \end{array}} \qquad (z = x)$$

$$(24.r)\qquad \boxed{\begin{array}{rrrrl} * & z_4\,\phi_1 & +z_1\,\phi_2 & +z_3\,\phi_3 & =0, \\ -z_4\,\phi_0 & * & -z_2\,\phi_2 & +z_0\,\phi_3 & =0, \\ -z_1\,\phi_0 & -z_3\,\phi_1 & * & -z_4\,\phi_3 & =0, \\ z_3\,\phi_0 & -z_1\,\phi_1 & +z_4\,\phi_2 & * & =0, \end{array}} \qquad (z = y)$$

will then arise in which (for a suitable determination of a proportionality factor) the coordinates z_k (or x_k , y_k) will be connected with the coordinates $\mathfrak{x}_k$ (or $\mathfrak{x}_k$, $\mathfrak{y}_k$) by the equations:

$$(25)\qquad \boxed{\mathfrak{z}_1 = 2z_4, \quad \mathfrak{z}_2 = z_0 - z_1, \quad \mathfrak{z}_3 = z_0 + z_1, \quad \mathfrak{z}_4 = z_2 + z_3 \quad \mathfrak{z}_5 = z_2 - z_3.}$$

If the figures ξ and ϕ are given then equations (24.*l*) and (25.*r*) will yield the associated lines $\mathfrak{X}$ and $\mathfrak{Y}$ (or X and Y, here) as the loci of the points $\mathfrak{z} = \mathfrak{x}$ and $\mathfrak{z} = \mathfrak{y}$ (or $z = x$ and $z = y$), in which *one must observe that despite the surplus in these equations, none of them are dispensable* ([1]). Equations (25) also already contain the corresponding dependencies between the line coordinates X_{ik} and $\mathfrak{X}_{ik}$, Y_{ik} and $\mathfrak{Y}_{ik}$ that belong to the two coordinate systems $\{z\}$ and $\{\mathfrak{z}\}$, and therefore they again deliver formula (14) for us. Moreover, if a point z (or $\mathfrak{z}$) of the space R_4 is given then equations (24) will imply nothing contradictory only when their common determinant or its square root, namely:

$$z_0\, z_1 - z_2\, z_3 + z_4^2 = \tfrac{1}{4}\{\mathfrak{z}_1^2 - \mathfrak{z}_2^2 + \mathfrak{z}_3^2 - \mathfrak{z}_4^2 + \mathfrak{z}_5^2\}, \tag{26}$$

has the value zero, and thus, if the point $\mathfrak{z}$ (or z) lies on M_3^2. Both systems of equations will then yield the image of the given point, namely, a line in the principal complex, in one case, as the locus of its points, and as the locus of its planes, in the other.

Equations (12) yield plane coordinates $U_{\alpha\beta}$ that are naturally likewise connected with the coordinates $\mathfrak{U}_{ik}$ that used here by a linear transformation ([2]).

3.

Spheres and oriented spheres

We saw before what kind of figures in the "sphere space" M_3^2 have points and planes in the projective continuum R_3 for their images. Here, we are dealing with something even simpler that does, however, require a more careful treatment (like all mathematics).

The manifolds M_4^2 and M_3^2 differ from real spherical manifolds in a (Euclidian or non-Euclidian) space R_5 or R_4 only by their unusual choice of coordinates, and thus basically not at all. Therefore, e.g., the intersections of M_3^2 with planes can properly be called *circles* (e.g., regular, simple, and doubly-singular circles). The intersection of M_3^2 with lines $\mathfrak{R}_3$ or *surfaces* in the space R_4 will be called *spheres.* A sphere $\mathfrak{u}$ is then defined by a linear equation in point coordinates $\mathfrak{x}_1, \ldots, \mathfrak{x}_5$:

$$\mathfrak{u}_1\, \mathfrak{x}_1 + \mathfrak{u}_2\, \mathfrak{x}_2 + \mathfrak{u}_3\, \mathfrak{x}_3 + \mathfrak{u}_4\, \mathfrak{x}_4 + \mathfrak{u}_5\, \mathfrak{x}_5 = 0.$$

Since the relationship of pole and polar exists between points and surfaces in R_4, one can also put the last equation into the following notation (which is probably a bit more suitable):

([1]) For S. LIE and some more recent authors, only a kind of grotesque mutation of the system of equations (24.*l*) appears:

$$(X + iY) + xZ + z = 0, \qquad z\,(X - iY) - Z - y = 0.$$

([2]) Cf., Math. Ann. **91** (1924), pp. 113.

(27) $$* - \mathfrak{z}_1 \mathfrak{x}_1 + \mathfrak{z}_2 \mathfrak{x}_2 - \mathfrak{z}_3 \mathfrak{x}_3 + \mathfrak{z}_4 \mathfrak{x}_4 - \mathfrak{z}_5 \mathfrak{x}_5 = 0 \qquad \{\mathfrak{z}_1 = - \mathfrak{u}_1,\ \mathfrak{z}_2 = - \mathfrak{u}_2, \text{ etc.}\}$$

Like all non-singular second-order surfaces, our "spheres" also carry two families of $\infty^{2\cdot 1}$ lines, and they can be distributed on the sheets of the $\mathfrak{X}$ and $\mathfrak{Y}$-lines in two kinds of ways. Along with the concept of sphere, one can mention yet another concept that shall be referred to as an "oriented sphere." Here, I would like to use the term *union*, which shall be explained later, since it is less biased, at least, for certain manifolds of leaves. Let me explain:

If each of $\infty^{2\cdot 1}$ $\mathfrak{X}$-lines lie united with each of $\infty^{2\cdot 1}$ $\mathfrak{Y}$-lines ([1]) *then the $\infty^{2\cdot 2}$ leaves* ($\mathfrak{X}$, $\mathfrak{Y}$) *that one derives from them will define a union that shall be called an "oriented sphere."*

It follows from this that, *first of all*, any oriented sphere will again give rise to another one under each transformation of the group G_{15}, H_{15} . *Secondly*, one has that there are two kinds of oriented spheres. The one of them, which shall be called *regular*, doubly-cover a unoriented sphere. The other kind, which we shall called *singular*, are nothing but the points of M_3^2 when they are likewise considered to be loci (unions) of $\infty^{2\cdot 2}$ leaves. Namely, one can associate each point z to all of the $\infty^{2\cdot 2}$ leaves that have their proper point at z when they are regular and one of their points at z when they are singular. *Thirdly,* one sees that the unions of the second kind are limiting case of those of the first kind, and collectively define the *branching manifold* of the double covering of the regular spheres. *Fourthly,* not only are manifolds of $\infty^{2\cdot 2}$ points linked with the regular oriented spheres, which are just the associated unoriented spheres, but also with the singular ones: viz., *the null spheres.* Any singular leaf on a singular sphere has, in fact, $\infty^{2\cdot 2}$ points, and they collectively define a cone, namely, the associated *null cone.*

Under these circumstances, it is advisable to introduce the term *point sphere,* along with the word "null sphere." The word "point sphere" will then mean precisely the same thing as the word *point* (on M_3^2), insofar as only the latter word (which is proper here) will be employed to refer to the associated unions of $\infty^{2\cdot 2}$ leaves. The term "point sphere" will then express the idea that the point, as a union, means a special case of an oriented sphere. *Hence, from now on, the "null sphere" will refer (only) to the unions of $\infty^{2\cdot 1}$ singular leaves that have points at the point z.*

The *point sphere* then has $\infty^{2\cdot 2}$ leaves, like a regular sphere, and those leaves also define a single analytic continuum in which, however, along with the leaf ($\mathfrak{X}$, $\mathfrak{Y}$), the inverted leaf ($\mathfrak{X}' = \mathfrak{Y}$, $\mathfrak{Y}' = \mathfrak{X}$) will also appear; they will then differ from the regular, oriented spheres in that way. The $\infty^{2\cdot 1}$ singular leaves of the associated *null sphere* will then function as the branching manifold in this figure. The point-sphere is also distinguished from the regular, oriented sphere by the fact that it contains singular leaves. The condition for the presence of a point-sphere is obviously:

([1]) Naturally, that means the *analytic* continua of $\mathfrak{X}$-lines and $\mathfrak{Y}$-lines.

(28) $$\mathfrak{z}_1^2 - \mathfrak{z}_2^2 + \mathfrak{z}_3^2 - \mathfrak{z}_4^2 + \mathfrak{z}_5^2 = 0.$$

Obviously, one likewise has the theorem:

An oriented sphere has a line in the space R_3 for its image, and conversely, each of those lines will correspond to an oriented sphere.

The point-spheres or points correspond to the lines in the principal complex.

Here, we then once more arrive at the association that we already expressed by equations (24) and (25).

It still remains for us to assign coordinates to the regular, oriented spheres. We get them from the remark that the leaf (27) will be cut by two spaces $\mathfrak{N}'_4$, $\mathfrak{N}''_4$ that contact M_4^2. We obtain the contact point with no further discussion by extracting a square root:

(29) $$\boxed{\mathfrak{z}_0 = \sqrt{\mathfrak{z}_1^2 - \mathfrak{z}_2^2 + \mathfrak{z}_3^2 - \mathfrak{z}_4^2 + \mathfrak{z}_5^2}\,.}$$

In general, we will consider the ratios:

(30) $$\mathfrak{z}_0 : \mathfrak{z}_1 : \mathfrak{z}_2 : \mathfrak{z}_3 : \mathfrak{z}_4 : \mathfrak{z}_5$$

that are explained by that fact to be the coordinates of an oriented sphere.

In fact, with their help, we can distinguish the two overlapping oriented spheres. We already know the image of the point $\mathfrak{z}$ in space R_5, namely, on the basis of equations (6), or then on the basis of the equations:

(31) $$\boxed{\begin{array}{lll} 2Z_{01} = \mathfrak{z}_0 + \mathfrak{z}_1, & 2Z_{02} = \mathfrak{z}_0 + \mathfrak{z}_3, & 2Z_{03} = \mathfrak{z}_4 + \mathfrak{z}_5, \\ 2Z_{23} = \mathfrak{z}_0 - \mathfrak{z}_1, & 2Z_{31} = \mathfrak{z}_2 - \mathfrak{z}_3, & 2Z_{12} = \mathfrak{z}_4 - \mathfrak{z}_5. \end{array}}$$

Moreover, we know the conditions for a point ξ or a plane ϕ to lie united with the line Z. Thus, we also have, however, the conditions for a given line on M_3^2 to belong to the oriented sphere $\mathfrak{z}$ as an $\mathfrak{X}$-line or a $\mathfrak{Y}$-line: *We have separated the two families of lines on an initially unoriented sphere by determining the root $\mathfrak{z}_0$.*

We shall now say that *two oriented spheres contact each other* when they have a leaf in common with each other. If we correspondingly say that *two lines in image space contact each other* when they – as unions of facets (ξ, ϕ) – have a facet in common, and thus, when they cut each other, it will then follow that:

Oriented spheres that contact each other have lines that contact (cut) each other for their images.

We already know the condition for this positional relationship between oriented spheres $\mathfrak{z}'$, $\mathfrak{z}''$. Namely, it consists of the fact that the connecting line for the two points $\mathfrak{z}'$, $\mathfrak{z}''$ is a line $\mathfrak{Z}$ that lives in M_4^2:

The condition for the positional relationship in the last theorem is:

$$(32)\quad \begin{aligned}&\mathfrak{z}_0'\,\mathfrak{z}_0'' - \mathfrak{z}_1'\,\mathfrak{z}_1'' + \mathfrak{z}_2'\,\mathfrak{z}_2'' - \mathfrak{z}_3'\,\mathfrak{z}_3'' + \mathfrak{z}_4'\,\mathfrak{z}_4'' - \mathfrak{z}_5'\,\mathfrak{z}_5''\\ &= 2\,\{Z_{01}'\,Z_{23}'' + Z_{02}'\,Z_{31}'' + Z_{03}'\,Z_{12}'' + Z_{23}'\,Z_{01}'' + Z_{31}'\,Z_{02}'' + Z_{12}'\,Z_{03}''\} = 0.\end{aligned}$$

Any two distinct contacting spheres then contact each other in a single leaf, and they will be contained in a *pencil of oriented spheres* that contact pair-wise that the leaf determines. $\infty^{2\cdot 5}$ exemplars of that pencil exist, like the plane pencils of lines in R_3 that are their images. However, like the associated $\infty^{2\cdot 5}$ leaves, they will overlap each other in pairs, by which, a *branching manifold* will once more appear that corresponds to the $\infty^{2\cdot 3}$ facets in the principal complex. Furthermore, (as is self-explanatory) one will get:

In order for two (or more) oriented spheres to contact each other, it is already sufficient that they should have an $\mathfrak{X}$-line ***or*** *a $\mathfrak{Y}$-line in common with each other.*

With that, I believe that I have established the elementary facts upon which any further development in sphere geometry must rest. We shall speak on that subject, and especially on the general concept of a *union of leaves*, and on the representation of the line-sphere transformation as a type of *contact transformation* ([1]) in a second article.

However, it would first be good for us to clarify the great distinction that exists between the contact of oriented spheres and the concept of the contact of spheres, *per se*.

If we have two unoriented spheres – i.e., the intersection of M_3^2 with linear spaces $\mathfrak{R}_3'$, $\mathfrak{R}_3''$ – then we will have two kinds of pencils $\lambda'\mathfrak{u}' + \lambda''\mathfrak{u}''$, and three kinds of contact to distinguish:

(*a*) The plane of intersection $\mathfrak{U}$ of $\mathfrak{u}'$ and $\mathfrak{u}''$ can meet M_3^2 in a pair of different lines. *The pencil $\lambda'\mathfrak{u}' + \lambda''\mathfrak{u}''$ will then contain two coincident null spheres.* The regular spheres of the pencil can be oriented in two ways, and they will then yield two analytically non-coincident overlapping pencils of oriented spheres. *Both pencils each contain the same null sphere once.* One can then have that:

(a_1) The given spheres are both regular, or
(a_2) One of them is the null sphere of the pencil.

([1]) But not exactly with the definition that LIE gave.

In the second case, one is dealing with *improper contact*, which consists of saying that the vertex of the null sphere is contained on the other sphere.

(*b*) The lines that were mentioned in (*a*) coincide. *The entire pencil then consists of null spheres, and any two of them will contact each other infinitely many times. By contrast, the associated point-spheres contact each other only simply.* Now, as a union of leaves, a point-sphere is the same thing as a point:

We see that two distinct points in sphere space can contact each other. Namely, they will always do that when their connecting line is a null line (which lies on M_3^2 *).*

That then shows once more the very essential difference between the concepts that are linked with the word "contact": In projective geometry, two distinct points can never contact each other.

We also already know the condition for two unoriented spheres to contact each other. It consists of the demand that their plane of intersection should contact M_3^2. We then come to the theorem:

The algebraic expression whose vanishing indicates to the contact of two spheres will be decomposed into two factors by the orientation process.

In fact, on the basis of what we have established:

$$\begin{aligned}
&\mathfrak{U}_{24}^2+\mathfrak{U}_{35}^2+\mathfrak{U}_{51}^2+\mathfrak{U}_{13}^2-\mathfrak{U}_{21}^2-\mathfrak{U}_{23}^2-\mathfrak{U}_{25}^2-\mathfrak{U}_{41}^2-\mathfrak{U}_{43}^2-\mathfrak{U}_{45}^2\\
&= \mathfrak{Z}_{24}^2+\mathfrak{Z}_{35}^2+\mathfrak{Z}_{51}^2+\mathfrak{Z}_{13}^2-\mathfrak{Z}_{21}^2-\mathfrak{Z}_{23}^2-\mathfrak{Z}_{25}^2-\mathfrak{Z}_{41}^2-\mathfrak{Z}_{43}^2-\mathfrak{Z}_{45}^2\\
&=\{\mathfrak{z}_0'\,\mathfrak{z}_0''-\mathfrak{z}_1'\,\mathfrak{z}_1''+\mathfrak{z}_2'\,\mathfrak{z}_2''-\mathfrak{z}_3'\,\mathfrak{z}_3''+\mathfrak{z}_4'\,\mathfrak{z}_4''-\mathfrak{z}_5'\,\mathfrak{z}_5''\}\\
&\quad\cdot\{\mathfrak{z}_0'\,\mathfrak{z}_0''+\mathfrak{z}_1'\,\mathfrak{z}_1''+\mathfrak{z}_2'\,\mathfrak{z}_2''+\mathfrak{z}_3'\,\mathfrak{z}_3''-\mathfrak{z}_4'\,\mathfrak{z}_4''+\mathfrak{z}_5'\,\mathfrak{z}_5''\}\\
&=2\,\{Z_{01}'\,Z_{23}''+Z_{02}'\,Z_{31}''+Z_{03}'\,Z_{12}''+Z_{23}'\,Z_{01}''+Z_{31}'\,Z_{02}''+Z_{12}'\,Z_{03}''\}\\
&\quad\cdot 2\,\{Z_{01}'\,Z_{23}''-Z_{02}'\,Z_{31}''-Z_{03}'\,Z_{12}''+Z_{23}'\,Z_{01}''-Z_{31}'\,Z_{02}''-Z_{12}'\,Z_{03}''\}
\end{aligned}\tag{33}$$

(cf., no. 13b). The meaning of the vanishing of the second of those factors is immediate.

Finally, we might draw attention to the following situation:

Whereas, up to five unoriented spheres can alternately contact each other in such a way that no two of the ten contact points lie on a null line ([1]), for more than two oriented spheres that contact each other pair-wise, either the contact points must always be contained along the same $\mathfrak{X}$-line or the same $\mathfrak{Y}$-line, or both of them must occur at the same time.

We already have that triples of mutually-contacting sphere might not always be oriented in such a way that contact still takes place after their orientation.

([1]) The corresponding 2 · 5 lines in R_3 define a "double-five" with a collineation group of icosahedral type.

The ideas that were just presented are also so simple that other authors, beginning with S. LIE, have also created some confusion that cannot persist without consequences. One correctly recognizes that oriented or unoriented regular spheres must correspond to one or two lines in image space, respectively. However, those spheres would not be regarded as unions of leaves (which is a concept that is lacking), but as unions of oriented or unoriented "elements." Now, as such, the null spheres are *reducible figures* in both cases; the lines in image space that allegedly correspond to them must then likewise decompose into two pieces! Instead of inferring the conclusion from this that something was amiss, one will find that the cone that the null-sphere consists of has been almost completely thrown out, and merely its vertex seems to remain on the image surface, which actually seems very much like sleight-of-hand. The fact that this mistake has not come to light so far is probably due, for the most part, to the incompleteness in the algebraic apparatus that excludes any effective controls. (See, e.g., KLEIN-BLASCHKE, §§ 70, 71.) In addition, one would still have to consider that from the elementary definition of the concept of a sphere, two arbitrary spheres will already contact each other, at least doubly, and even infinitely many times in some situations, and that this type of contact cannot also be simply spirited away.

Things are no better with a different process by which one comes to grips with these things. (See, e.g., KLEIN-BLASCHKE, §§ 65, 66.) Even in that case, one has bridled the horse from behind. Meanwhile, the seed of a useful thought is also present this time. It can perhaps be phrased as:

Instead of first intersecting the manifold M_4^2 with a linear R_4 and then linking sphere geometry with Euclidian or non-Euclidian geometry by projecting the M_3^2 that is obtained, one can invert the sequence of operations of intersecting and projecting without altering the results.

In that way, one will arrive at a MÖBIUS point-continuum $\mathfrak{M}_4^2$, with a group G'_{15}, H'_{15} of conformal transformations ([1]), and then at a sphere space that can likewise be regarded as a MÖBIUS continuum $\mathfrak{M}_3^2$. For example, Euclidian geometry can just as well find its place there as in projective geometry. However, one finds distinctions there that arise from just the various extensions of the "proper point" to a closed continuum. For example, a plane in $\mathfrak{M}_3^2$ is something completely different from a plane in the projective continuum R_3, although both concepts completely overlap each other in the domain of proper points. *Those figures are already topologically different:* As point-loci, non-isotropic planes have the same connection to sphere geometry as regular spheres, and isotropic planes have the same connection as null-spheres. (However, the associated union of $\infty^{2\cdot 2}$ leaves has the same connection as a regular sphere.) The CHASLES mapping process (viz., "isotropic projection"), which derives a point (x, y, z, ir) in space R_4 from the usual representation of a sphere (x, y, z, r) in "space" (R_3), is achieved only for fragments of the continua $\mathfrak{M}_4^2$ and $\mathfrak{M}_3^2$, which is as it should be. For example, planes cannot be oriented with the help of their "radii."

([1]) Math. Zeit. **18**, **21** (1923-24). Journal für Mathematik **157** (1926), 35-39.