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FASCICLE XXIX

THE EUCLIDIAN ACTION OF DEFORMATION AND MOTION

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PREFACE

The aspiration to contribute to the diffusion of the theory of the Euclidian action was what led us to take on the production of this fascicle. This did not come about without two reservations: The reader who judges it from the purely formal viewpoint and who begins the *Théorie des corps déformables* of E. and F. Cosserat will find us to be insufficiently faithful to the way of looking at things of those authors. In order to give our presentation some concrete support, we have made use of the notions of energy and force. Now, one knows about the difficulties that the definition of energy presents; as for force, it can give rise to interminable discussions. Lazare Carnot was sufficiently hostile to that notion being given *a priori* that he deemed any proof that contained the word "force" to be absurd!

We can remark that the method of exposition that is called *formal* is not recognized today without some inconvenience. It is universally accepted when one defines, for example, the length of an arc of a curve by an integral that this is the way to make the common notion of length for a rectilinear segment persist; however, when, by a tyrannical lack of logic, one extends to some other notions, one must meet up with some setbacks. One no longer accepts the definition of velocity for a rectilinear motion that Lagrange gave in his analytical mechanics, which amounts to taking that quantity to be the coefficient of t in the Maclaurin series that represents the space that is traversed as a function of time. Therefore, here are some *natural* notions, as Ch. de Freycinet would say, that, to be suitably precise, necessitate the previous knowledge of the development of the functions into series as if the idea of velocity and that of force are not, in our sense, previous to the study of algebra.

Eminent professors do not hesitate to utilize the notion of force in their teaching. However, there is more: In the work that we shall present, it will be a question of a function W playing the primordial role; it depends upon geometric and kinetic arguments. It is defined by E. and F. Cosserat as a scalar function that must remain invariant when one subjects these arguments to a transformation of the group of Euclidian displacements; i.e., to speak simply, when one gives the body, which is assumed to be instantaneously undeformable, an elementary helicoidal displacement. How does one content oneself with this definition? If W is such a function then the same will be true of $f(W)$, where f is an arbitrary function. Now, in the course of the theory, W will take on a precise significance. In the case of a static deformation, it will coincide with the energy of deformation per unit volume.

One will then understand our infidelity. In order to justify it, we will say that the excess of abstraction has probably retarded progress in the theory. At its debut, it received the warmest of welcomes, and P. Appel's *Cours de Mécanique* (2nd edition) carried a note in its third volume that was a detailed discussion of it that was edited by E. and F. Cosserat ([1]). After the premature death of the latter, an engineer for the eastern railroads, E. Cosserat, who has been the eminent director of the Toulouse Observatory up to these latter years, dedicated himself to other work, and, despite his generosity, it has

([1]) The same discussion is reproduced in the French translation of volume two of Chwolson's *Physique* (Hermann, editor).

not been easy for him to recall the path that he followed before without a certain tragic sense of loss.

Another scruple makes us hesitate before the purely mathematical aspect of a theory that is destined for the *Mémorial des Sciences physiques.* The desire for a concrete presentation that we already expressed made us simplify the algebraic apparatus. We have naturally taken explicit pains to make use of the vectorial calculus and its innumerable pages of equations, which is heresy for the theory of deformable bodies, so one finds it reduced considerably. The unpublished study that is carried out in the first chapter avoids the repetition (up to five times) of an argument that is, moreover, insufficient in form and occasionally in its conclusions ([1]).

One should not be deceived by the spirit of these observations; we cannot admire that *Théorie* enough in all of its various parts as it was described by E. and F. Cosserat. If those authors left anything further to be gleaned from it, it is for them to recognize. Their collaboration was arduous: The Cartesian calculations, which were in favor only in France at the beginning of the century, were tedious whenever they applied them to quadruple integrals. The use of vectorial calculus, which condenses the *mathematical content*, permits us to perceive some imperfections. Nothing will diminish our admiration for the creators of the theory.

The method of Euclidian action was first introduced in dynamics, and in this field of study it gives a generalization of ordinary mechanics; it seems to be a first-order approximation ([2]). Whereas the mass of a material element in the general theory is a function of velocity, it is constant in the first approximation. By this important detail, one sees the possibility of rejoining the modern physical theories, since the Euclidian character can be replaced by another that better conforms to these theories. In the case of the determination of the function W that was mentioned above, this happens by writing that this function remains invariant for any transformation, not just the group of Euclidian displacements, but, in fact, the group that one has, by definition, chosen in order to explain the universe ([3]).

This fascicle is limited to the theory of deformable bodies by the method of Euclidian action, and does not pretend to establish the principles of a new mechanics.

Here, one will find a generalization of the theory of elasticity in the sense that one no longer postulates the reduction of the actions that are exerted upon a material element to a unique force; that reduction likewise involves a couple.

In addition, the theory is not confined to the consideration of infinitely small deformations. It thus presents itself as a double generalization of the classical study of deformations.

Finally, one can see a liberating effort of all metaphysics, as one said at the time of L. Carnot and Lagrange. We must confess that it is not this aspect of the study that can seduce the reader. "The artifices that are put to work by the greatest minds do not suffice to replace the concepts that are suggested by the nature of things itself." It is with that

([1]) For example, in the chapter on action at a distance that we treated completely in a previous paper ["Contribution à l'étude de l'action euclidienne," Annales de la Faculté des Sciences de Toulouse (1926)].

([2]) V. CHWOLSON, *Traité de Physique*, French translation by E. DAVAUX, marine engineer. Tome I, pp. 236, *et seq.* Note of E. and F. Cosserat (librairie Hermann).

([3]) Another comparison can be deduced from this is that Planck's quanta are fragments or "grains of action."

thought by Ch. de Freycinet that we conclude by adding that the procedures that are placed at our disposal by the method are indeed very precious so that one should not seek to augment their value by giving them a purely abstract basis.

PRELIMINARIES.

1. – Notations adopted. – In principle, we adopt the vectorial notations of Burali-Forti and Marcolongo, which have been recommended in France for some years now by Bricard (*Nouvelles Annales de Mathématiques,* 1923).

In general, a vector will be denoted by just one letter (most often, Egyptian). In some exceptions, it can be a set of two letters. The elementary displacement vector of a point M from M to M' will be denoted by ΔM.

The scalar and vector products of two vectors **u** and **v** are written $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \wedge \mathbf{v}$, respectively.

Finally, for a vector **u** that is a function of one parameter t, we utilize the notion of relative derivative of a moving reference system (notation: $d\mathbf{u} / dt$). The absolute derivative, when taken in a fixed reference system (notation: $D\mathbf{U} / Dt$), will be linked with the preceding by the relation:

$$\frac{D\mathbf{U}}{Dt} = \frac{d\mathbf{u}}{dt} + \mathbf{\Omega} \wedge \mathbf{U},$$

$\mathbf{\Omega}$ being the instantaneous rotational velocity vector of the moving reference system, while the parameter t plays the role of time ([1]).

2. The displacements D **and** Δ. – Consider a curve $A_0\, M_0\, B_0$, at each point of which a trirectangular trihedron that has that point as its summit is attached. The orientation of that trihedron varies with the point M_0 in a continuous and known manner.

Imagine that one deforms the curve in such a manner that each point M_0 of the initial state corresponds to one and only one point M of the deformed one. The orientation of the trihedron that is attached to that point is likewise modified in a continuous manner from one extremity of the curve to the other. This comes about through a succession of deformed states.

If **u** is a uniform vector function of the point M_0 of the curve (which is the summit of the trihedron that is attached to that point) then one can establish the equality:

(1) $$D(\Delta\mathbf{u}) = \Delta(D\mathbf{u})$$

by an argument that is utilized in the calculus of variations, where D denotes an absolute variation that corresponds to an arbitrary displacement of the summit at M along one of the deformed curves and Δ denotes an absolute variation that provides the passage from a point of one deformed curve to the corresponding point of another deformed one.

([1]) J. SUDRIA, "Sur la dérivée relative d'un vecteur," Nouvelles Annales de Math. (5), t. 11.

3. Special formulas. – If one envisions two elementary displacements D and Δ and if one lets α and θ denote the elementary rotation vectors of the trihedron in each of these two cases then one can establish some results that are very useful for what follows:

a. Take the vector **u** to be the vector OM that joins a fixed origin to the summit of the trihedron. The preceding relation can be written:

(2) $$d(\Delta M) + \alpha \wedge \Delta M = \delta(DM) + \theta \wedge DM,$$

where d and δ now denote the relative variations when they are evaluated using the trihedron whose summit is at M.

If ds_0 is the arc of the undeformed curve that corresponds to the displacement that is envisioned along that curve, and ω is the instantaneous rotational velocity of the trihedron when one makes s_0 play the role of time then one can again transform the relation (2) thus:

$$\frac{d}{dx_0}(\Delta M) + \omega \wedge \Delta M = \delta\left(\frac{DM}{Ds_0}\right) + \theta \wedge \frac{DM}{Ds_0},$$

or furthermore:

(3) $$(\Delta M) + \mathbf{V}_0 \wedge \theta = \delta \mathbf{V}_0,$$

$\mathbf{v}_0$ being the velocity of displacement of the summit of the trihedron along the undeformed line when s_0 plays the role of time.

b. If one chooses the vector **u** of the fundamental relation to be a well-defined vector that is at rest in the moving trihedron then one has:

$$\Delta \mathbf{u} = \theta \wedge \mathbf{U} \qquad \text{and} \qquad D\mathbf{u} = \alpha \wedge \mathbf{V}.$$

Equality (1) then gives, successively ([1]):

$$\begin{aligned} D\theta \wedge \mathbf{U} + \theta \wedge D\mathbf{U} &= \Delta\alpha \wedge \mathbf{U} + \alpha \wedge \Delta\mathbf{U}, \\ D\theta \wedge \mathbf{u} + \theta \wedge (\alpha \wedge \mathbf{u}) &= \Delta\alpha \wedge \mathbf{u} + \alpha \wedge (\theta \wedge \mathbf{u}), \\ D\theta \wedge \mathbf{u} - \Delta\alpha \wedge \mathbf{u} &= \alpha \wedge (\theta \wedge \mathbf{u}) + \theta \wedge (\mathbf{u} \wedge \alpha) = (\alpha \wedge \theta) \wedge \mathbf{u}. \end{aligned}$$

Since the last relation must be true for any **u**, one then infers that:

$$D\theta = \Delta\alpha + \alpha \wedge \theta,$$

or furthermore:

$$\frac{D\theta}{Ds_0} = \Delta\omega - \theta \wedge \omega = \delta\omega$$

or

([1]) The second equation follows from the identity $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) + \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a}) + \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b}) = 0$.

$$(4) \qquad \delta\omega = \frac{d\theta}{ds_0} + \theta \wedge \omega \qquad (^1).$$

4. Classical formulas. – One thus finds two relations (3) and (4′) proved and attached to a common origin that were obtained by E. and F. Cosserat by means of Cartesian calculations.

We point out, in passing, that when D and Δ denote two displacements of a point on two coordinate lines of a surface the relations that were obtained give, in one case, the Kirchhoff formulas, and in the other, those of Combescure-Darboux (*see* G. Darboux, *Théorie des surfaces*, t. I, pp. 55 and 49).

We finally point out the following relation (5):

Consider a vector **u** that is united with a moving trirectangular trihedron. One has:

$$\mathbf{u} = \mathbf{u} \times \mathbf{I} \cdot \mathbf{I} + \mathbf{u} \times \mathbf{J} \cdot \mathbf{J} + \mathbf{u} \times \mathbf{K} \cdot \mathbf{K}$$

and

$$(5) \qquad \begin{aligned} D\mathbf{u} &= \mathbf{u} \times \mathbf{I} \cdot \alpha \wedge \mathbf{I} + \mathbf{u} \times \mathbf{J} \cdot \alpha \wedge \mathbf{J} + \mathbf{u} \times \mathbf{K} \cdot \alpha \wedge \mathbf{K}, \\ &= \mathbf{u} \times \mathbf{I} \cdot D\mathbf{I} + \mathbf{u} \times \mathbf{J} \cdot D\mathbf{J} + \mathbf{u} \times \mathbf{K} \cdot D\mathbf{K}. \end{aligned}$$

5. Partial gradients. – We extend the notion of gradient, which is defined in the context of a scalar function of a point M, to the case of a scalar function of several vectors. (The ordinary case can be considered to be that of a function of a vector OM.)

If one is given a scalar function f of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ then we refer to the *partial gradients* of f, which we denote by grad $f_{\mathbf{v}_1}$, grad $f_{\mathbf{v}_2}$, …, when we are dealing with the vectors that are defined by the equality:

$$df = \operatorname{grad} f_{\mathbf{v}_1} \times d\mathbf{v}_1 + \operatorname{grad} f_{\mathbf{v}_2} \times d\mathbf{v}_2 + \ldots + \operatorname{grad} f_{\mathbf{v}_p} \times d\mathbf{v}_p ,$$

where the infinitesimal variations $d\mathbf{v}_1, d\mathbf{v}_2, \ldots, d\mathbf{v}_p$ are arbitrary.

For example, if the function f is written:

([1]) In order to facilitate the reading of the book by E. and F. Cosserat, upon starting with the intrinsic formulas above, recall that $\Delta(OM)$ is represented by its projections δ'_x, δ'_y, δ'_z onto the axes of the moving trihedron. $\delta I'$, $\delta J'$, $\delta K'$, are the components of Θ, p, q, r are those of ω, and finally ξ, η, ζ are the components of V. Equation (3) gives:

$$\frac{d}{ds}\delta' x + q\,\delta' y - r\,\delta' z = \delta\xi + z\,\delta J' - \eta\,\delta K'$$

and two other analogous relations, while equation (4) gives:

$$\delta p = \frac{d}{ds_0}\delta I' + q\,\delta K' + r\,\delta J'$$

and two other analogous relations.

These formulas are used continually in *Théorie*.

$$f(X_1, Y_1, Z_1, X_2, Y_2, Z_2, \ldots, X_p, Y_p, Z_p),$$

in which X_i , Y_i, Z_i are the components of the vector $\mathbf{v}_i$, then one will have:

$$\frac{\partial f}{\partial X_i}, \frac{\partial f}{\partial Y_i}, \frac{\partial f}{\partial Z_i}$$

for the components of the partial gradient $f_{\mathbf{v}_i}$.

6. Relativity of partial gradients. – We say that a vector is referred to a trihedron when the variations of that vector are evaluated in that trihedron when it is taken as the reference.

Let there be given a vector that we denote by **U** when it is referred to a fixed trihedron and by **u** when it is observed in a moving trihedron.

If F is a scalar function of the vectors **U**, **V**, … then one has, on the one hand:

$$d\mathbf{V} = \text{grad } F_{\mathbf{u}} \times D\mathbf{U} + \text{grad } F_{\mathbf{v}} \times D\mathbf{U} + \ldots$$

If one refers the vectors to a moving trihedron whose pair-wise rectangular axes carry the unit vectors **I**, **J**, and **K** then F(**U**, **V**, …) becomes Φ(**u**, **v**, …, **I**, **J**, **K**), and:

$$\begin{gathered} dF = d\Phi = \text{grad } \Phi_{\mathbf{u}} \times D\mathbf{u} + \text{grad } \Phi_{\mathbf{v}} \times D\mathbf{v} + \ldots \\ + \text{grad } \Phi_{\mathbf{I}} \times D\mathbf{I} + \text{grad } \Phi_{\mathbf{J}} \times D\mathbf{J} + \text{grad } \Phi_{\mathbf{K}} \times D\mathbf{K}. \end{gathered}$$

On the other hand (formula 5):

$$D\mathbf{U} = d\mathbf{u} + a \wedge \mathbf{u} = d\mathbf{u} + \mathbf{U} \times \mathbf{I} \cdot D\mathbf{I} + \mathbf{U} + \mathbf{U} \times \mathbf{J} \cdot D\mathbf{J} + \mathbf{U} + \mathbf{U} \times \mathbf{K} \cdot D\mathbf{K}$$

and

$$dF \text{ or } d\Phi = \text{grad } F_{\mathbf{U}} \times (d\mathbf{u} + \mathbf{U} \times \mathbf{I} \cdot D\mathbf{I} + \mathbf{U} \times \mathbf{J} \cdot D\mathbf{J} + \mathbf{U} \times \mathbf{K} \cdot D\mathbf{K}) + \ldots$$

Finally:

$$\begin{aligned} &\text{grad } \Phi_{\mathbf{u}} = \text{grad } F_{\mathbf{U}}, \\ &\text{grad } \Phi_{\mathbf{I}} = \text{grad } F_{\mathbf{U}} \cdot \mathbf{U} \times \mathbf{I} + \text{grad } F_{\mathbf{V}} \cdot \mathbf{V} \times \mathbf{I} + \ldots \end{aligned}$$

Remark. – If the function F already depends upon the vectors **I**, **J**, **K** then

$$\text{grad } \Phi_{\mathbf{I}} = \text{grad } F_{\mathbf{I}} + \text{grad } F_{\mathbf{U}} \cdot \mathbf{U} \times \mathbf{I} + \ldots$$

7. Functions that are invariant under a Euclidian displacement. – If one is given a scalar function W of several vectors $\mathbf{V}_1$, $\mathbf{V}_2$, …, $\mathbf{V}_p$ then the equation:

$$\text{grad } W_{\mathbf{V}_1} \wedge \mathbf{V}_1 + \text{grad } W_{\mathbf{V}_2} \wedge \mathbf{V}_2 + \ldots + \text{grad } W_{\mathbf{V}_p} \wedge \mathbf{V}_p = 0,$$

or, more simply:

$$(E) \qquad \sum_{i=1}^{p} \operatorname{grad} W_{\mathbf{V}_i} \wedge \mathbf{V}_i = 0,$$

translates into the fact that the function W is invariant under any elementary Euclidian rotation of the system of vectors around one axis and also under any Euclidian displacement of the set of vectors.

Indeed, taking the scalar product of the left-hand side of the preceding equation with $\theta\, dt$ gives:

$$\sum_{i=1}^{p} \operatorname{grad} W_{\mathbf{V}_i} \wedge \mathbf{V}_i \times \theta\, dt = 0,$$

or

$$\sum_{i=1}^{p} \operatorname{grad} W_{\mathbf{V}_i} \wedge \Delta\mathbf{v}_i = 0.$$

Here, $\Delta\mathbf{v}_i$ is the variation of the vector $\mathbf{v}_i$ under a rotation of the set with an angular velocity of θ; the latter result amounts to the equality $dW = 0$.

Remark. – Equation (E) is equivalent to the system of three partial differential equations:

$$(8) \qquad \begin{cases} \sum_i \left(y_i \dfrac{dW}{dz_i} - z_i \dfrac{dW}{dy_i} \right) = 0, \\ \sum_i \left(z_i \dfrac{dW}{dx_i} - x_i \dfrac{dW}{dz_i} \right) = 0, \\ \sum_i \left(x_i \dfrac{dW}{dy_i} - y_i \dfrac{dW}{dx_i} \right) = 0, \end{cases}$$

in which x_i, y_i, z_i are the components of $\mathbf{v}_i$; the number of scalar variables is then $3p$. In order to solve the system of three equations above, it will suffice to find $3(p - 1)$ independent solutions:

$$\varphi_1(x_i, y_i, z_i), \quad \varphi_2(x_i, y_i, z_i), \ \ldots, \ \varphi_{3(p-1)}(x_i, y_i, z_i).$$

The general solution of the system will be:

$$W = F(\varphi_1, \varphi_2, \ldots, \varphi_{3(p-1)}),$$

in which F is an arbitrary function of the arguments φ_1, φ_2, … One can take these arguments to be:

1. $x_i^2 + y_i^2 + z_i^2$ or $\mathbf{V}_i^2$ $(i = 1, 2, \ldots, p)$,
2. $x_1\, x_2 + y_1\, y_2 + z_1\, z_2$ or $\mathbf{V}_1 \cdot \mathbf{V}_2$,
3. The $2(p - 2)$ solutions:

$$x_1\,x_i + y_1\,y_i + z_1\,z_i\,, \qquad x_2\,x_i + y_2\,y_i + z_2\,z_i\,,$$

or

$$\mathbf{V}_1 \times \mathbf{V}_i \qquad\text{and}\qquad \mathbf{V}_2 \times \mathbf{V}_i \qquad (i = 1, 2, \ldots, p),$$

so one has, in all:

$$p + 1 + 2(p - 2) = 3(p - 1)$$

independent functions.

8. Case where certain vectors satisfy some scalar relations. – The function thus obtained is the general integral that satisfies equation (E) identically.

In the sequel, it will be necessary to know how to find the most general function that satisfies the same equation, no longer identically, but now only under the hypothesis that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ satisfy certain scalar relations.

For our ultimate applications, we set aside the case where these relations are not intrinsic – i.e., independent of the coordinate axes.

Let:

$$\begin{gathered} f_1(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) = C_1, \\ f_2(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) = C_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_n(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) = C_n \end{gathered}$$

be these constraint equations.

They are equivalent to n relations between x_i, y_i, z_i, which are the components of $\mathbf{v}_i$, namely:

$$(L) \qquad \begin{cases} \varphi_1(x_i, y_i, z_i) = C_1, \\ \varphi_2(x_i, y_i, z_i) = C_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_n(x_i, y_i, z_i) = C_n. \end{cases}$$

$\varphi_1, \varphi_2, \ldots, \varphi_n$ are solutions of equation (E), since relations (L) are indeed independent of the axes, so this must persist when one gives the reference system an elementary rotation θ – i.e., the functions φ_k must verify the relation:

$$\sum \left(\underset{\mathbf{v}_i}{\operatorname{grad}}\, \varphi_{\mathbf{K}} \times \Delta\mathbf{v}_i \right) = 0$$

for $\Delta\mathbf{v}_i = \theta \wedge \mathbf{V}_i$, which gives, after one has factored out the vector θ:

$$\sum \left(\underset{\mathbf{v}_i}{\operatorname{grad}}\, \varphi_{\mathbf{K}} \times \mathbf{V}_i \right) = 0 \qquad \text{for } \mathbf{K} = 1, 2, \ldots, n; \qquad i = 1, 2, \ldots, p.$$

Having said this, take $2p - 3 - n$ other solutions of (E) that are *independent* of φ_1, φ_2, …, φ_n (with the meaning that is given to that word in the theory of functions of several variables), namely:

$$\varphi_{n+1}, \varphi_{n+1}, \ldots, \varphi_{3p-3} .$$

Finally, if α, α', α'', β, β', β'', and γ, γ', γ'' are the components of three vectors that appear in equation (E) (for example, the last three $\mathbf{V}_{p-2}$, $\mathbf{V}_{p-1}$, $\mathbf{V}_p$) then keep the components α, β', and γ'', in particular. They are independent of each other and of the functions φ_k, since otherwise α, for example, would be a solution of the system (2), and one can see that this is impossible unless $\alpha' = \alpha'' = 0$, which are not intrinsic relations, and are thus rejected ([1]).

Having said this, let $\Phi(x_i, y_i, z_i)$ be a function that satisfies the system (S), thanks to equations (L).

9. The solution in this case, as deduced from the general solution. – We shall show that Φ can be deduced from the general solution F when one takes the conditions (L) into account.

Make a change of variables in Φ by taking the new arguments, which number $3p$ and are independent:

$$\varphi_1, \varphi_2, \ldots, \varphi_{3p-3}, \alpha, \beta', \gamma''.$$

Φ will then be of the form:

$$\Phi(\varphi_1, \varphi_2, \ldots, \varphi_{3p-3}, \alpha, \beta', \gamma''),$$

or, upon taking equations (L) into account:

$$\Phi(\varphi_{n+1}, \varphi_{n+2}, \ldots, \varphi_{3p-3}, \alpha, \beta', \gamma'').$$

We now show that α, β', γ'' do not figure explicitly in Φ.

In order to simplify, let $E(W)$ denote the left-hand side of the vectorial equation E.

We remark that:

$$\alpha = \mathbf{V}_{p-2} \times \mathbf{X},$$
$$\beta = \mathbf{V}_{p-1} \times \mathbf{Y},$$
$$\gamma = \mathbf{V}_p \quad \times \mathbf{Z},$$

in which **X**, **Y**, **Z** are unit vectors that are carried by the fixed axes. Therefore:

([1]) Indeed, one can, as we will do later on, set $\alpha = \mathbf{V}_{p-2} \times \mathbf{X}$, where **X** is a unit vector that is carried by the x-axis, so it results that $\mathrm{grad}(\alpha)\ \mathbf{v}_{p-2} = \mathbf{X}$.

For $W = \alpha$, the vectorial differential equation (E) reduces to $\mathbf{X} \wedge \mathbf{V}_{p-2} = 0$.

If we project this relation onto the y-axis and then onto the z-axis then we get:

$$\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{V}_{p-2} = 0, \qquad \mathbf{X} \wedge \mathbf{Z} \wedge \mathbf{V}_{p-2} = 0,$$

or

$$\mathbf{Z} \times \mathbf{V}_{p-2} = 0, \qquad \mathbf{Y} \times \mathbf{V}_{p-2} = 0;$$

i.e., one finally has:

$$\alpha' = 0 \quad \text{and} \qquad \alpha'' = 0.$$

$$\text{grad } \alpha_{\mathbf{v}_{p-2}} = \mathbf{X}, \qquad \text{grad } \beta'_{\mathbf{v}_{p-2}} = \mathbf{Y}, \qquad \text{grad } \gamma''_{\mathbf{v}_{p-2}} = \mathbf{Z},$$

and

$$E(\alpha) = \mathbf{X} \wedge \mathbf{V}_{p-2}\,, \qquad E(\beta') = \mathbf{Y} \wedge \mathbf{V}_{p-2}\,, \qquad E(\gamma'') = \mathbf{Z} \wedge \mathbf{V}_{p-2}\,.$$

It is easy to see that:

$$E(\varphi) = \frac{\partial \Phi}{\partial \varphi_{n+1}} E(\varphi_{n+1}) + \ldots + \frac{\partial \Phi}{\partial \varphi_{3p-3}} E(\varphi_{3p-3})$$
$$+ \frac{\partial \Phi}{\partial \alpha} E(\alpha) + \frac{\partial \Phi}{\partial \beta'} E(\beta') + \frac{\partial \Phi}{\partial \gamma''} E(\gamma''),$$

in such a way that if one writes that Φ is one solution then what remains is:

$$\frac{\partial \Phi}{\partial \alpha} E(\alpha) + \frac{\partial \Phi}{\partial \beta'} E(\beta') + \frac{\partial \Phi}{\partial \gamma''} E(\gamma'') = 0.$$

Upon taking the vector product with **X**, **Y**, **Z**, successively, one gets:

$$\beta'' \frac{\partial \Phi}{\partial \beta'} - \gamma' \frac{\partial \Phi}{\partial \gamma''} = 0,$$
$$\gamma \frac{\partial \Phi}{\partial \gamma''} - \alpha'' \frac{\partial \Phi}{\partial \alpha} = 0,$$
$$\alpha' \frac{\partial \Phi}{\partial \alpha} - \beta \frac{\partial \Phi}{\partial \beta'} = 0,$$

and these equations in:

$$\frac{\partial \Phi}{\partial \alpha}, \frac{\partial \Phi}{\partial \beta'}, \frac{\partial \Phi}{\partial \gamma''}$$

can have no other solutions than:

$$\frac{\partial \Phi}{\partial \alpha} = \frac{\partial \Phi}{\partial \beta'} = \frac{\partial \Phi}{\partial \gamma''} = 0,$$

unless the determinant:

$$\begin{vmatrix} 0 & \beta'' & -\gamma' \\ -\alpha'' & 0 & \gamma \\ \alpha' & \beta & 0 \end{vmatrix}$$

is zero, i.e., unless $\alpha'\beta''\gamma = \alpha''\beta\gamma'$, a relation that is not intrinsic, because it can be written:

$$\alpha'(\beta''\gamma - \beta\gamma'') = \beta(\alpha''\gamma' - \alpha'\gamma'').$$

If **X**, **Y**, **Z** are, as always, three unit vectors that are carried by the reference axes and **I**, **J**, **K** are the three vectors ([1]) whose components are α, α', α'', β, β', β'', γ, γ', γ'', respectively, then one has:

$$\alpha' = \mathbf{I} \times \mathbf{Y},$$

and $\beta''\gamma - \beta\gamma''$ is the component of $\mathbf{J} \wedge \mathbf{K}$ along OY, so:

$$\beta''\gamma - \beta\gamma'' = \mathbf{JKY}.$$

Upon similarly transforming the right-hand side, one will finally have:

$$\mathbf{I} \times \mathbf{Y} \cdot (\mathbf{JKY}) = \mathbf{J} \times \mathbf{X} \cdot (\mathbf{KIX}).$$

Now, if one changes the *Y*-axis without changing the *X*-axis then the left-hand side varies, but not the right-hand side.

The relation $\alpha'\beta''\gamma = \alpha''\beta\gamma'$ is not intrinsic, and one cannot find solutions of the system (*S*) that satisfy that system identically, in which one must take equations (1) into account, other than the ones that are deduced from the solution *W*.

The size of the introduction above is justified by the following considerations:

The problem of determining a function *W* that is invariant under the group of Euclidian displacements was first examined by E. and F. Cosserat in the note by these authors that appeared in Chwolson's *Traité de Physique* (t. I). [French translation by Devaux, a marine engineer (librairie Hermann)]

The solution to the first question is easily found and leaves no doubt (pp. 246). However, the result is then extended to a different case by analogy (pp. 268). It alludes to a simple calculation that, in fact, was given only in another work, and which was, in fact, *Théorie des corps déformables.*

We have shown that this calculation was based upon an inexact assertion ([2]). Since the argument is repeated several times and it leads to erroneous results in certain cases, it is necessary to give it a firmer foundation.

([1]) They are the vectors that were previously denoted by $\mathbf{V}_{p-2}$, $\mathbf{V}_{p-1}$, $\mathbf{V}_p$.

([2]) That assertion amounts to this: If a vector is framed with respect to two trihedra with the same summit then one knows the position of one of these trihedra with respect to the other one. (See "Contribution à la Théorie de l'action euclidienne," Nouvelles Annales de la Faculté des Sciences de Toulouse, 1926.)

CHAPTER I.

THE EUCLIDIAN ACTION OF DEFORMATION

10. Concrete representation of a deformable line. – One will understand the theory more easily if one represents a deformable body as an agglomeration of undeformable particles.

Without going into the refinements to that notion that are appropriate in order to account for the progress in modern physics, we can take the preceding particles to be atoms and imagine that one has linked each one of them to an undeformable reference system – for example, a tri-rectangular trihedron.

The study of the deformation of a body amounts to the consideration of the modification of the situation that relates to the trihedra. No matter how large, if not inconceivable, the number of elements is, they do not constitute a *continuum* in the mathematical sense of the word; however, it will be entirely advantageous to treat them as a continuous set. This will notably permit the utilization of the procedures of the infinitesimal calculus.

Therefore, suppose that a tri-rectangular trihedron is given at each point of a space (M_0) that is described by the point M_0, whose edges carry the unit vectors a_0, b_0, c_0, respectively. We suppose that these vectors are continuous functions of the point M_0 by the intermediary of their direction cosines.

If the space (M_0) is the natural state of the body then give each point M_0 a displacement ΔM_0 and imprint an elementary rotation on the trihedron $M_0\, a_0\, b_0\, c_0$; it will become a trihedron $Mabc$. The continuous, three-dimensional set of all such trihedra constitutes the deformed state.

For more simplicity in the presentation, we first consider the case of infinitely slender body that we call a deformable curve. We then show how the calculations and the results that are obtained are reproduced in the study of a surface or a three-dimensional body.

If the natural state of the curve is given then one considers a continuous sequence of trihedra whose summits are the various points of the curve. The positions of the summits, as well as the orientation of the axes, are known as functions of the argument s_0, which is the arc length from one of the extremities (A_0, for example) up to the point M_0 .

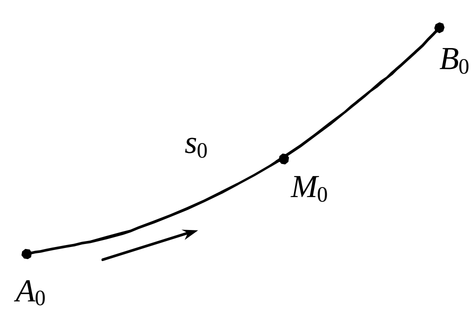

Fig. 1.

One can follow the displacement of a point M and the variations of the orientation of the axes on the deformed state of the curve (*see* above) by varying s_0 .

In order to characterize the deformation of the line at a point M that comes from a point M_0 with argument s_0 on the line before deformation, E. and F. Cosserat take a function W of two infinitely close positions of the trihedron $Mabc$ – i.e., a function of s_0, $x, y, z, \alpha, \alpha', \alpha'', \ldots, \gamma''$ and their first derivative with respect to s_0 . ($x, y, z, \alpha, \alpha', \alpha'', \ldots, \gamma''$ are the components of OM, a, b, c.)

11. The Euclidian action. – These authors consider the integral:

$$\int_{A_0}^{B_0} W\, ds_0 ,$$

which is taken along an arbitrary portion of the line (M_0), and then impose the condition on that integral that it have "a zero variation when one subjects the set of all trihedra of the deformable line, when taken in its deformable state, to an arbitrary infinitesimal transformation of the group of Euclidian displacements." This integral is called the *action of deformation* on the deformed line between the points A and B, which correspond to the points A_0 and B_0 of (M_0).

What must one have in mind with the term *Euclidian action*? The question is paramount. The words employed recall the principle that was obscurely-stated by Maupertuis and then repeated by several mathematicians, to the extent that Jacobi found it so incomprehensible that he completely recast it.

Here, the risk is not the same, since the obscurity does not come from the complexity of the statement, but from an inadequacy that is initially quite shocking. The definition of Euclidian action is not only appropriate to the notion that E. and F. Cosserat had in mind, but to an infinitude of other ones.

It is curious to remark, as we will do later on, that it suffices for the function W to satisfy the condition of invariance that is demanded in order for the curve to receive an arbitrary elementary deformation, so the integral $\int_{A_0}^{B_0} W\, ds_0$ takes on a variation of the form:

$$\begin{aligned} &- \int_{A_0}^{B_0} X\,\delta x + Y\,\delta y + Z\,\delta z + L\,\delta I + M\,\delta J + N\,\delta K \\ &+ [F\,\delta x + G\,\delta y + H\,\delta z + R\,\delta I + S\,\delta J + T\,\delta K]_{A_0}^{B_0} , \end{aligned}$$

in which δx, δy, δz are the components of the displacement of a point whose argument is s_0 , and δI, δJ, δK are the components of the elementary rotation of the corresponding trihedron.

This expression is, up to sign, the one that gives the work done by the forces that are applied to the curve (X, Y, Z being the components of the force per unit length at a point and L, M, N being those of the external moment at this point).

F, G, H, and R, S, T are the components of the forces and moments that act on the extremities. More precisely, if one makes an imaginary cut in the line at M then F, G, H,

R, S, T are the components of the force and the moment that are exerted by the part MB on the part MA.

Among the functions W that answer to the single condition that was imposed above, one must then find the energy of deformation of the curve (with its sign changed), and it is that particular solution that will be truly the Euclidian action per unit length.

This definition clarifies the notion *in the particular case where the forces simply give rise to the static deformation* (i.e., when they take their values by starting with zero and progressing infinitely slowly).

However, in the general case (viz., deformation and motion) it will be simple to extend the notion *without it being confused with the energy of the deformed body* (*see* pp. 30).

12. Expression for the Euclidian action. – We thus seek the condition that must be imposed upon the integral $\int_{A_0}^{B_0} W\,ds_0$ in order for it to be invariant under all transformations of the group of Euclidian displacements.

This demands that $\delta\int_{A_0}^{B_0} W\,ds_0 = 0$ for such a transformation, or, in a more concrete manner, that the value of W does not change when one displaces the line (M) in the manner of the undeformable lines of rational mechanics. In order to find the expression for W, we shall utilize the study that we carried out in paragraph 7, in place of the argument that was employed by the authors of *Théorie* and insufficiently explained.

Let:

$$W\left(s_0, x, y, z, \alpha, \alpha', \ldots, \gamma'', \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \frac{d\alpha}{ds_0}, \frac{d\alpha'}{ds_0}, \ldots, \frac{d\gamma''}{ds_0}\right),$$

which we write with vector notation, letting **I**, **J**, **K** be three unit vectors that are carried by the axes Ma, Mb, Mc:

$$W\left(s_0, \overrightarrow{OM}, \mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{V}_0, \frac{d\mathbf{I}}{ds_0}, \frac{d\mathbf{J}}{ds_0}, \frac{d\mathbf{K}}{ds_0}\right).$$

The vectors $d\mathbf{I} / ds_0$, $d\mathbf{J} / ds_0$, $d\mathbf{K} / ds_0$ can be replaced with:

$$\Omega_0 \wedge \mathbf{I}, \qquad \Omega_0 \wedge \mathbf{J}, \qquad \Omega_0 \wedge \mathbf{K},$$

where $\mathbf{V}_0$ and Ω_0 are the velocity of the point M and the instantaneous rotation of the trihedron when one makes s_0 play the role of time. The function W can then be written:

$$W(s_0, OM, \mathbf{V}_0, \Omega_0, \mathbf{I}, \mathbf{J}, \mathbf{K}).$$

Imagine that one gives the curve a translation, so the variation of W reduces to:

$$\text{grad } W_{OM} \times \Delta M = 0,$$

which demands that grad W_{OM} must be identically zero, or furthermore that the function W does not depend upon OM explicitly. (In the form that was proposed by E. and F. Cosserat, it did not contain x, y, z.)

On the other hand, we can suppose that one has introduced $\mathbf{V}_0$ and Ω_0 into W, where these vectors are referred to the trihedron $Mabc$ and are denoted by $\mathbf{v}_0$ and ω_0, which amounts to modifying the influence of the vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, as we explained before (§ 6).

Once the function has been put into the form:

$$W(s_0, \mathbf{v}_0, \omega_0, \mathbf{I}, \mathbf{J}, \mathbf{K}),$$

we write down that it does not vary when one gives the set of trihedra $Mabc$ a rotation θ, which gives, since $\delta\mathbf{v}_0 = \delta\omega_0 = 0$:

$$\text{grad } W_{\mathbf{I}} \times \Delta\mathbf{I} + \text{grad } W_{\mathbf{J}} \times \Delta\mathbf{J} + \text{grad } W_{\mathbf{K}} \times \Delta\mathbf{K} = 0,$$

or

$$\text{grad } W_{\mathbf{I}} \wedge \Delta\mathbf{I} + \text{grad } W_{\mathbf{J}} \wedge \Delta\mathbf{J} + \text{grad } W_{\mathbf{K}} \wedge \Delta\mathbf{K} = 0,$$

a vector differential equation for which we must find $3 \cdot (3 - 1) = 6$ solutions. They are obviously:

$$\mathbf{I}^2, \quad \mathbf{J}^2, \quad \mathbf{K}^2, \quad \mathbf{J} \times \mathbf{K}, \quad \mathbf{K} \times \mathbf{I}, \quad \mathbf{I} \times \mathbf{J},$$

and the general solution is:

$$W(s_0, \mathbf{w}_0, \omega_0, \mathbf{I}^2, \quad \mathbf{J}^2, \quad \mathbf{K}^2, \quad \mathbf{J} \times \mathbf{K}, \quad \mathbf{K} \times \mathbf{I}, \quad \mathbf{I} \times \mathbf{J}).$$

Upon taking into account the relations between the cosines of the edges of a tri-rectangular trihedron, this relation reduces to:

$$W(s_0, \mathbf{v}_0, \omega_0).$$

Recall that $\mathbf{v}_0$ and ω_0 must be framed relative to the moving trihedron $M(abc)$. In other words, W contains only the components ξ, η, ζ, p, q, r of these vectors along the moving axes.

Remark. – The function W is called the *density of action per unit length along the line before deformation.* One deduces the density per unit length of the deformed line from it, namely, $W\, ds_0 / ds$.

13. The external force and moment. The external effort and moment of deformation. The effort and moment of deformation at a point of the deformed line. – We write down the Euclidian action of deformation between two points A and B of the line (M) for an arbitrary variation. Upon suppressing the index 0 in the vectors $\mathbf{v}$ and ω, one has:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \int_{A_0}^{B_0} (\text{grad}\, W_{\mathbf{v}} \times \delta\mathbf{v} + \text{grad}\, W_{\omega} \times \delta\omega)\, ds_0 .$$

Recall that:

$$\delta\mathbf{v} = \frac{D}{Ds_0}\Delta M + \mathbf{v} \wedge \theta,$$

which permits one to write:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \int_{A_0}^{B_0}\left[\operatorname{grad} W_{\mathbf{v}} \times \left(\frac{D}{Ds_0}\Delta M + \mathbf{v}\wedge\theta\right) + \operatorname{grad} W_{\omega} \times \frac{D}{Ds_0}\theta\right] ds_0 ,$$

and upon integrating by parts:

$$= [\operatorname{grad} W_{\mathbf{V}} \times \Delta M + \operatorname{grad} W_{\omega} \times \theta\,]_{A_0}^{B_0} - \int\left(\frac{D}{Ds_0}\operatorname{grad} W_{\mathbf{v}} \times \Delta M\right) ds_0$$
$$- \int\left(\frac{D}{Ds_0}\operatorname{grad} W_{\omega} + \mathbf{V} \wedge \operatorname{grad} W_{\mathbf{V}}\right) \times \theta \cdot ds_0 .$$

If we set:

$$\mathcal{E} = \operatorname{grad} W_{\mathbf{V}} , \qquad \mathcal{M} = \operatorname{grad} W_{\omega} ,$$

$$\frac{D\mathcal{E}}{Ds_0} = \varphi, \qquad \frac{D\mathcal{M}}{Ds_0} + \mathbf{v} \wedge \mathcal{E} = \mu$$

then we get:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \mathcal{E}_{\mathbf{B}} \times \Delta\mathbf{B} - \mathcal{E}_{\mathbf{A}} \times \Delta\mathbf{A} + \mathcal{M}_{\mathbf{B}} \times \theta_{\mathbf{B}} - \mathcal{M}_{\mathbf{A}} \times \theta_{\mathbf{A}}$$
$$- \int \varphi\; ds_0 \times \Delta M + \mu\, ds_0 \times \theta.$$

At a point whose argument is s_0, $\mathcal{E}$ is called the *external effort of deformation*, $\mathcal{M}$, the *external moment of deformation*, φ, the vector of the *external force per unit length* of the undeformed line, and μ is the vector of the *external moment or couple per unit length* of the undeformed line.

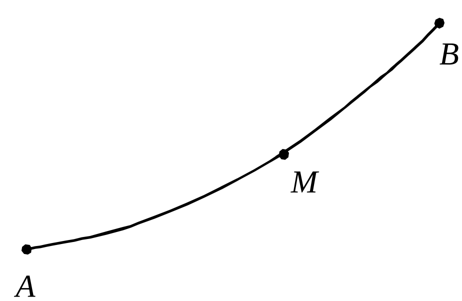

Fig. 2.

In a more precise manner, consider an arbitrary point M between A and B. Suppose that one has separated the curve into two portions on one side of M and the other. If one imagines that the portion MB has been isolated then $\mathcal{E}$ and $\mathcal{M}$ are the force and moment that are exercised by the part AM on the portion MB.

(These quantities constitute the generalizations of the sectional effort and bending moment of the *strength of materials*.)

The variation of the action with the sign changed is written:

$$- \delta\int_{A_0}^{B_0} W\, ds_0 = \mathcal{E}_{\mathbf{A}} \times \Delta\mathbf{A} - \mathcal{E}_{\mathbf{B}} \times \Delta\mathbf{B} + \mathcal{M}_{\mathbf{A}} \times \theta_{\mathbf{A}} - \mathcal{M}_{\mathbf{B}} \times \theta_{\mathbf{B}}$$

$$+ \int_0^t \varphi\; ds_0 \times \Delta\mathbf{M} + \mu\, ds_0 \times \theta.$$

On the right-hand side, one finds the elementary work done by the forces and couples that are distributed along the length of the curve and the forces and moments that act on the extremities.

14. Rigorous definition of the notion of Euclidian action. – *This permits us to make the thoughts of the authors of Théorie more precise. W must not be merely a function that satisfies the formal conditions that were studied above. It is necessary that – W ds_0 is, up to an additive constant, the energy of deformation on the element ds of the deformed curve that is due to an element ds_0 of the curve before deformation.*

Indeed, imagine the curve in two states, one of which is the state before deformation and the other of which is the an arbitrary current state, and suppose that the passage from the first one to the second one happens progressively and infinitely slowly by means of a continuous succession of equilibrium states. One can suppose that the quantities define a state that depends upon one parameter h, which will give the curve before deformation for the value zero.

The work that is done by external forces for a variation h is:

$$\delta\mathcal{T}_e = - \int_{A_0}^{B_0} \delta W\, ds_0 \;= - \int_{A_0}^{B_0} \frac{\partial W}{\partial h} dh\, ds_0 \,.$$

The total work will be:

$$\mathcal{T}_e = - \int_{A_0}^{B_0} (W - W_0)\, ds_0 \,.$$

Without changing anything in the preceding, one can always suppose that the function $W_0 = W(v_0, \omega_0, s_0)$ is identically zero for the natural state. The quantity – $W(v_0, \omega_0, s_0)\, ds_0$ will then be the energy stored in the portion of the deformable curve that is due to an element ds_0 whose argument is s_0 in the natural state.

One further says that – W is the density of energy of deformation per unit length of the undeformed curve at the point whose argument is s_0 .

However, that definition is valid only for deformed curves that are in equilibrium. We will give a more general definition later that applies to the case of a body in motion (*see* § 18).

Nevertheless, we point out that the following relations were given by Thomson and Tait only for the infinitely small deformations and by means of *a posteriori* hypotheses.

The condensed form that we gave to them makes these relations obvious in the equations of definition:

At a point of the deformed line, one has:

$$\varphi = \frac{D\mathcal{E}}{Ds_0} \qquad \text{and} \qquad \mu = \frac{D\mathcal{M}}{Ds_0}.$$

CHAPTER II

THE EUCLIDIAN ACTION OF DEFORMATION AND MOTION ([1]).

15. Concrete representation of a deformable curve in motion. – Consider a curve (M_0) that is described by a point M_0 and attach a tri-rectangular trihedron $M_0\ a_0\ b_0\ c_0$ to each point M_0 of the curve whose axes are defined in direction and sense by means of the unit vectors $\mathbf{I}_0$, $\mathbf{J}_0$, $\mathbf{K}_0$, which are functions of the point M_0. The continuous set of these trihedra can be considered to be the position at the epoch t_0 of a *deformable curve* that is defined in the following manner:

Give a displacement M_0M to the point M_0 that is a function of time t and the point M_0 and is, in addition, annulled for $t = t_0$. On the other hand, imprint a rotation on the trihedron $M_0\ a_0\ b_0\ c_0$ that finally brings the axes into coincidence with those of the trihedron $Mabc$ that we attach to the point M. We define that rotation by saying that the unit vectors $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ that are carried by the trihedron $Mabc$, respectively, are functions of M_0 and t.

The continuous set of trihedra $Mabc$ for a given value of t will be called the *deformed state* of the curve at the time t. The doubly-infinite continuous set that is composed of the sets thus defined for all values of t will be the trajectory of the deformed state of the deformable curve.

We continue to let $\mathbf{v}_0$ and ω_0 denote the velocity of the point M_0 and the instantaneous rotation of the trihedron $M_0\ a_0\ b_0\ c_0$ ([2]) when only s_0 varies and plays the role of time. We let $\mathbf{v}$ and ω denote the analogous vectors that relate to the point M and the trihedron $M_0\ a_0\ b_0\ c_0$, where s_0 is always the derivation parameter.

On the other hand, we let $\mathbf{v}_t$ and ω_t denote the velocity of the point M and the instantaneous rotation, properly speaking, of the trihedron $Mabc$ – i.e., while preserving the character of a derivation parameter for t.

If one imprints an infinitely small displacement upon each of the trihedra of the *trajectory* of the deformed state that varies in a continuous manner with these trihedra then we have, with the notations that are already employed, as we have said:

$$\delta\mathbf{v} = \frac{D}{Ds_0}\Delta M + \mathbf{v} \wedge \theta,$$

$$\delta\omega = \frac{D}{Ds_0}\theta,$$

$$\delta\mathbf{v}_t = \frac{D}{Dt}\Delta M + \mathbf{v}_t \wedge \theta,$$

$$\delta\omega_t = \frac{D}{Dt}\theta.$$

([1]) This chapter is not explicitly a part of the *Théorie* of E. and F. Cosserat. It will prepare the reader for the study of a medium in motion, which appears to be more complicated.

([2]) I.e., in the undeformed state.

16. Euclidian action of deformation and motion for a deformable curve in motion. – Consider a function W of two infinitely close positions of the trihedron *Mabc* – i.e., a function of M_0 and t, and of **I**, **J**, **K** and their first derivatives with respect to s_0 and t.

We propose to determine what the form of W must be in order for the double integral:

$$\iint W \, ds_0 \, dt \,,$$

which is extended over an arbitrary portion of the curve (M_0) and an interval of time that is comprised of the instants between t_1 and t_2, to have a zero variation when one subjects the set of all the trihedra that we have called the trajectory of the deformed medium *to the same arbitrary infinitesimal transformation of the group of Euclidian displacements.*

The argument that we have already employed and the study that we made on pp. 14 permit us to assert that W will have the following form:

$$W(s_0, t, \mathbf{v}, \omega, \mathbf{v}_t, \omega_t).$$

The integral:

$$\int_{t_1}^{t_2}\int_{A_0}^{B_0} W \, ds_0 \, dt$$

is the action of deformation and motion on the curve (or on a portion of it, if A_0 and B_0 are not the extremities of the given curve). W is the density of action at a given point and instant when it is referred to the unit of length of the undeformed curve and a unit of time. The density when referred to the unit of length of the deformed curve and the unit of time is:

$$W \frac{ds_0}{ds} .$$

17. The external force and moment. The external effort and moment of deformation. The external effort and moment of deformation, quantity of motion, and kinetic moment of the deformed medium in motion at a given point and instant. – Consider an arbitrary variation of the action on the line and in the time interval t_1, t_2, namely:

$$\delta\int_{t_1}^{t_2}\int_{A_0}^{B_0} W \, ds_0 \, dt = \int_{t_1}^{t_2}\int_{A_0}^{B_0} (\operatorname{grad} W_{\mathbf{v}} \times \delta\mathbf{v} + \operatorname{grad} W_{\omega} \times \delta\omega$$
$$+ \operatorname{grad} W_{\mathbf{v}_t} \times \delta\mathbf{v}_t + \operatorname{grad} W_{\omega_t} \times \delta\omega_t) \, ds_0 \, dt \, .$$

By replacing $\delta\mathbf{v}$, $\delta\omega$, $\delta\mathbf{v}_t$, $\delta\omega_t$ with their expressions above, and then integrating the terms that contain the derivatives with respect to s_0 by parts over s_0, and then the ones that contain the derivatives with respect to time over t, one obtains:

$$\int_{t_1}^{t_2} [\operatorname{grad} W_{\mathbf{v}} \times \Delta M + \operatorname{grad} W_{\omega} \times \Delta\theta]_{A_0}^{B_0} dt$$
$$+ \left[\int_{A_0}^{B_0} (\operatorname{grad} W_{\mathbf{v}_t} \times \Delta M + \operatorname{grad} W_{\omega_t} \times \Delta\theta)\, ds_0 \right]_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \int_{A_0}^{B_0} \left[\left(\frac{D}{Ds_0} \operatorname{grad} W_{\mathbf{v}} + \frac{D}{Dt} \operatorname{grad} W_{\mathbf{v}_t} \right) \times \Delta M \right.$$
$$+ \left(\frac{D}{Ds_0} \operatorname{grad} W_{\omega} + \mathbf{v} \wedge \operatorname{grad} W_{\mathbf{v}} \right.$$
$$\left. \left. + \frac{D}{Dt} \operatorname{grad} W_{\omega_t} + \mathbf{v}_t \wedge \operatorname{grad} W_{\mathbf{v}_t} \right) \times \Delta\theta \right] ds_0\, dt\,.$$

We call the vectors:

$$\varphi = \frac{D}{Ds_0} \operatorname{grad} W_{\mathbf{v}} + \frac{D}{Dt} \operatorname{grad} W_{\mathbf{v}_t}\,,$$

$$\mu = \frac{D}{Ds_0} \operatorname{grad} W_{\omega} + \mathbf{v} \wedge \operatorname{grad} W_{\mathbf{v}} + \frac{D}{Dt} \operatorname{grad} W_{\omega_t} + \mathbf{v}_t \wedge \operatorname{grad} W_{\mathbf{v}_t}$$

the *external force* and *external moment* at the point M and instant t, when referred to the unit of length of the curve before deformation. Upon considering the integral taken over time, we call the vectors grad $W_{\mathbf{v}}$ and grad W_{ω} the *external effort* and *moment of deformation* at the point M, respectively, when referred to the unit of length of the curve before deformation; more precisely, they are the effort and moment that the part AM exerts one the part MB. We denote these vectors by $\mathcal{E}$ and $\mathcal{M}$, in such a way that:

$$\mathcal{E} = \operatorname{grad} W_{\mathbf{v}},$$
$$\mathcal{M} = \operatorname{grad} W_{\omega}$$

at the points A and B. These efforts and moments are called *external*.

Finally, upon considering the integral taken over the length A_0B_0 , and taking the difference between the values at the epochs t_1 and t_2, we call the vectors grad $W_{\mathbf{v}_t}$ and grad W_{ω_t} the *quantity of motion* and the *kinetic moment* at the point M and the epoch t; one denotes them by $\mathbf{Q}$ and $\mathcal{H}$.

The relations below results from the preceding definitions ([1]):

([1]) With the notation of E. and F. Cosserat (notations that one must not lose sight of if one desires to go into the presentation of those authors in detail, which is strongly suggested), one will have, for the first formula:

$$\mathbf{X}_0' = \frac{d}{ds_0} \frac{\partial W}{\partial \xi} + q \frac{\partial W}{\partial \zeta} - r \frac{\partial W}{\partial \eta} + \frac{d}{dt} \operatorname{grad} \frac{\partial W}{\partial \xi} + q_t \frac{\partial W}{\partial \zeta_t} - r_t \frac{\partial W}{\partial \eta_t}\,,$$

$$\varphi = \frac{D\mathcal{E}}{Ds_0} + \frac{D}{Dt}\mathbf{Q},$$

$$\mu = \frac{D\mathcal{M}}{Ds_0} + \frac{D}{Dt}\mathcal{H} + \mathbf{v} \wedge \mathcal{E} + \mathbf{v}_t \wedge \mathbf{Q}.$$

The presence of the arguments $\mathbf{v}_t$ and ω_t in W implies some expressions for φ and μ that, in fact, reduce to the ones that we already found in the study that we made (§ 13) in the case of equilibrium. In the general case, these expressions bring us into the presence of "*the notion of kinetic anisotropy that was already imagined by Rankine, and which was then introduced into several theories of physics in the theory of double refraction and in that of rotatory polarization, for example*" (E. and F. Cosserat).

By making an idea of the authors of *Théorie* more precise, we have shown that the arguments $\mathbf{V}^2$, $\mathbf{V}_t^2$, and $\mathbf{V} \times \mathbf{V}_t$. ("Contribution à la Théorie")

"When the mixed argument $\mathbf{V} \times \mathbf{V}_t$ does not occur in W, one must, in general, consider the state of deformation and motion that is infinitely close to the natural state in order to find where the action of deformation is completely separate from the kinetic action in the case of classical mechanics."

18. Notion of energy of deformation and motion. – We now propose to determine the work that is done by the external forces and moments and external efforts and moments of deformation during an arbitrary time interval for a real deformation.

The elementary work done during a time dt is:

$$\int_{A_0}^{B_0} [\varphi \times \mathbf{v}_t \, dt + \mu \times \omega_t \, dt$$

$\mathbf{Y}'_0$ = ……………………………………………………………,

$\mathbf{Z}'_0$ = ……………………………………………………………,

and for the second one:

$$\mathbf{L}'_0 = \frac{d}{ds_0}\frac{\partial W}{\partial p} + \frac{d}{dt}\frac{\partial W}{\partial p_t} + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + q_t\frac{\partial W}{\partial r_t} - r_t\frac{\partial W}{\partial q_t}$$
$$+ \eta\frac{\partial W}{\partial \zeta} - \zeta\frac{\partial W}{\partial \eta} + \eta_t\frac{\partial W}{\partial \zeta_t} - \zeta_t\frac{\partial W}{\partial \eta_t},$$

$\mathbf{M}'_0$ = ……………………………………………………,

$\mathbf{N}'_0$ = …………………………………………………….

The advantage of the vector notations is even greater when one considers a deformable surface or three-dimensional medium.

$$+ \mathcal{E}_{\mathbf{A}} \times \mathbf{v}_t^{\mathbf{A}}\, dt + \mathcal{M}_{\mathbf{A}} \times \omega_t^{\mathbf{A}}\, dt - \mathcal{E}_{\mathbf{B}} \times \mathbf{v}_t^{\mathbf{B}}\, dt + \mathcal{M}_{\mathbf{B}} \times \omega_t^{\mathbf{B}}\, dt]\, ds_0 \,.$$

By replacing φ, μ with their expressions that were given above as functions of W and by a calculation that we have detailed (§ 13), and which one can characterize by saying that it is the inverse of the one that led to the definition of φ and μ, one gets:

$$dt \int_{A_0}^{B_0} \left(\frac{dE}{dt} + \frac{\partial W}{\partial t} \right) ds_0 \,,$$

in which E is, by definition, the scalar function:

$$\text{grad } W_{\mathbf{v}_t} \times \mathbf{V}_t + \text{grad } W_{\omega_t} \times \omega_t - W.$$

In the particular case where $\partial W / \partial t \equiv 0$ – i.e., where W does not contain t explicitly – the expression found is the differential with respect to time of the quantity:

$$U = \int_{A_0}^{B_0} E \, ds_0 \,.$$

E is called the *energy of deformation and motion per unit length* of the original line; the definition of U is deduced from it. U remains constant when the external work done is zero; this leads to the notion of the conservation of energy under the hypothesis that the deformable curve is isolated from the external world.

In the preface, we said how the definition of W was hardly acceptable to a physicist. Since that definition involves only the invariance of W under the group of Euclidian substitutions, it cannot be made precise, since an arbitrary function of W will also be invariant. The significance becomes definitive when one sees that, as the authors of *Théorie* reasoned, W coincides with the energy of deformation per unit length of the original line, with the sign changed, in the case of static deformations. In the case where the deformable line is in motion, the function W, when imagined in reality, is, among all of the functions that satisfy the conditions of invariance that were proposed, the one that is, in addition, a solution of the partial differential equation:

$$\text{grad } W_{\mathbf{v}_t} \times \mathbf{V}_t + \text{grad } W_{\omega_t} \times \omega_t - W = E,$$

or furthermore:

$$\frac{\partial W}{\partial \xi}\xi + \frac{\partial W}{\partial \eta}\eta + \frac{\partial W}{\partial \zeta}\zeta + \frac{\partial W}{\partial p}p + \frac{\partial W}{\partial q}q + \frac{\partial W}{\partial r}r - W = E,$$

in which E is the energy of deformation and motion per unit length of the original line; this will be made more precise later on.

We compare the method that was just presented with the one that was recently employed by R. Ferrier ([1]) in order to construct a theory of the ether. After having shown that there was an original sin being committed in the conceptions of the old theories, Ferrier was led to consider a function:

$$\Omega(r_{12}, \ldots, r_{ij}, \ldots, r'_{12}, \ldots, r'_{ij}, \ldots),$$

where r_{ij} represents the distance between two arbitrary points that belong to the frames of a medium; r'_{ij} is the derivative of r_{ij} with respect to time. The law of motion of the frames is given by the condition that the integral:

$$\int_{t_2}^{t_1} \Omega(r_{ij}, r'_{ij})\, dt$$

be an extremum, which produces the same equations that are obtained by the Lagrange method and a remarkably simple interpretation when one takes into account the interdependence of the r_{ij}. *Ferrier was then led to consider a function W that is defined by the equation:*

$$W = \Omega - \sum r'_{ij} \frac{\partial \Omega}{\partial r'_{ij}},$$

and to call it the energy of the system.

From what was said, it is painfully necessary to remark that if a function Ω_1 obeys equation (1) then the same thing will be true for $\Omega_1 + u$, where u satisfies the equation:

$$u - \sum r'_{ij} \frac{\partial u}{\partial r'_{ij}} = 0.$$

This equation is a particular case of the one that is known in analysis under the name of the "equation of homogeneous functions." More precisely, it is the equation that is obeyed by homogeneous functions of first order relative to the arguments r'_{mn}.

19. Comparison with other formulations. – One can make the same remark for the function W of E. and F. Cosserat ([2]), which, from the standpoint of mathematical analysis, plays the same role as Ω in R. Ferrier's theory of the ether.

We return to the equations that we are occupied with, and which we write:

([1]) *Quelques idées sur l'Électrodynamique* (librairie A. Blanchard). Here, it amounts to a formal comparison of the two theories, which do not seem to have any point in common; we simply point out the similar use of the generalized Legendre transformation.

([2]) In several places in the *Théorie des corps déformables*, E. and F. Cosserat likewise made use of the calculus of variations.

$$\frac{\partial W}{\partial \xi}\xi + \frac{\partial W}{\partial \eta}\eta + \frac{\partial W}{\partial \zeta}\zeta + \frac{\partial W}{\partial p}p + \frac{\partial W}{\partial q}q + \frac{\partial W}{\partial r}r - W = E.$$

If W_1 is a solution to that equation in which E is the unitary energy then the general solution for W will be:

$$W = W_1 + u,$$

where u is an arbitrary homogeneous function of first order in ξ, η, ζ, p, q, r that can contain the arguments ξ_0, η_0, ζ_0, p_0, q_0, r_0, in addition. One thus sees that there is a precision that one can bring to this that is, moreover, very possible, and in several ways.

We put ourselves in the case of classical mechanics.

If one evaluates the kinetic energy of an element ds whose argument is s_0 then one finds precisely:

$$\tfrac{1}{2}\, ds_0\, [\mathbf{Q} \times \mathbf{v}_t + \mathbf{K} \times \mathbf{v}_t],$$

so it suffices to apply Koenig's theorem to that element: It is the *vis viva* that the element ds_0 will have if the matter that it is composed of is concentrated and its center of inertia is $ds_0\, \mathbf{Q} \times \mathbf{v}_t$. In addition, in the motion around the center of inertia, the relative *via viva* is the scalar product of the kinetic moment $\mathbf{K}$ with the vector ω_t, a scalar product that is, moreover, independent of the mode of framing of these vectors.

One thus has:

$$\tfrac{1}{2}\, ds_0\, [\text{grad } W_{\mathbf{v}_t} \times \mathbf{v}_t + \text{grad } W_{\omega_t} \times \omega_t]$$

for the kinetic energy of the element and:

$$C = \tfrac{1}{2}\int_{A_0}^{B_0} [\text{grad}\, W_{\mathbf{v}_t} \times \mathbf{v}_t + \text{grad}\, W_{\omega_t} \times \omega_t]\, ds_0$$

for the total kinetic energy. Having said this, from the relation between W and E, one deduces that:

$$2C = \int_{A_0}^{B_0} E\, ds_0 + \int_{A_0}^{B_0} W\, ds_0\,.$$

The first integral is the total energy U of the curve, and since $U = V + C$, V being the potential energy (here, it is the work done by molecular forces, with the sign changed), one finally has:

$$-\int_{A_0}^{B_0} W\, ds_0 = V - C = H.$$

Finally, the action of deformation and motion will be:

$$\int_{t_1}^{t_2}\int_{A_0}^{B_0} W\, ds_0\, dt = -\int_{t_1}^{t_2} H\, dt\,,$$

H being the kinetic potential that was considered by Helmholtz ([1]).

One confirms that in the case of static deformations the action $\int W\, ds_0$ reduces to $-V$, which is the potential energy that results from the deformation, with the sign changed.

In summation ([2]): *In the general case (deformation and motion), the action is homogeneous as the product of an energy with time, so it is exactly the integral* $\int_{t_1}^{t_2} -H\, dt$*, H being Helmholtz's kinetic potential.*

20. Use of the fixed trihedron. – It results from what we said on pp. 15 that if one expresses W as a function of $\mathbf{V}$ and W, when referred to a fixed axis, then W will contain scalar functions of unit vectors that are carried by the axes of the moving trihedron, or furthermore, functions of the cosines of the angles that the moving axes make with the fixed axes. These cosines can be expressed by means of three parameters λ_1, λ_2, λ_3; for example, the Euler angles.

On the other hand, p, q, and r can be expressed by means of these same parameters and their derivatives:

$$p = \varpi_1' \frac{d\lambda_1}{ds_0} + \varpi_2' \frac{d\lambda_2}{ds_0} + \varpi_3' \frac{d\lambda_3}{ds_0},$$
$$q = \chi_1' \frac{d\lambda_1}{ds_0} + \chi_2' \frac{d\lambda_2}{ds_0} + \chi_3' \frac{d\lambda_3}{ds_0},$$
$$r = \sigma_1' \frac{d\lambda_1}{ds_0} + \sigma_2' \frac{d\lambda_2}{ds_0} + \sigma_3' \frac{d\lambda_3}{ds_0},$$

or, in summation:

$$\dot{\Omega} = \dot{\Omega}_1 \frac{d\lambda_1}{ds_0} + \dot{\Omega}_2 \frac{d\lambda_2}{ds_0} + \dot{\Omega}_3 \frac{d\lambda_3}{ds_0},$$

in which $\dot{\Omega}_h$ is the instantaneous rotation of the moving trihedron when one makes only λ_i vary, and ϖ_i', χ_i', σ_i' is the projections onto the moving trihedron.

If $\mathbf{I}'$, $\mathbf{J}'$, $\mathbf{K}'$ are the projections of the external moment of deformation $\mathcal{M}$ onto the same axes then we set:

$$\mathcal{I} = \varpi_1' \mathbf{I}' + \chi_1' \mathbf{J}' + \sigma_1' \mathbf{K}' = \mathcal{M} \times \Omega_1,$$
$$\mathcal{J} = \varpi_2' \mathbf{I}' + \chi_2' \mathbf{J}' + \sigma_2' \mathbf{K}' = \mathcal{M} \times \Omega_2,$$
$$\mathcal{K} = \varpi_3' \mathbf{I}' + \chi_3' \mathbf{J}' + \sigma_3' \mathbf{K}' = \mathcal{M} \times \Omega_3 .$$

If one considers the fact that, with the new variables:

([1]) More precisely, in a definition of action that would characterize the evolution of an isolated system. In order to justify this analysis, we cite Léon Brillouin: "Among the physicists, who dares to boast that they have a clear idea of action?" (*R. G. E.*, 17 November 1934).

([2]) One will find the result above exactly by studying the deformable surface and medium in motion.

$$\frac{dx}{dx_0}, \ldots, \lambda_1, \ldots, \frac{d\lambda_1}{ds_0}, \ldots$$

then the external work that is done by deformation can be written:

$$\begin{aligned}\delta\mathcal{T}_e &= -\int_{A_0}^{B_0} \delta W \, ds_0 \\ &= -\int_{A_0}^{B_0} \left(\frac{\partial W}{\partial \frac{dx}{ds_0}} \delta \frac{dx}{ds_0} + \cdots + \frac{\partial W}{\partial \lambda_1} \delta\lambda_1 + \cdots + \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} \delta \frac{d\lambda_1}{ds_0} + \cdots \right) ds_0 ,\end{aligned}$$

so the integration by parts of the terms that contain the derivatives dx / ds_0, $d\lambda_1 / ds_0$ permits us to transform $\delta\mathcal{T}_e$ into:

$$\delta\mathcal{T}_e = -\left[\frac{\partial W}{\partial\left(\frac{dx}{ds_0}\right)} \delta x + \frac{\partial W}{\partial\left(\frac{dy}{ds_0}\right)} \delta y + \frac{\partial W}{\partial\left(\frac{dx}{ds_0}\right)} \delta x + \frac{\partial W}{\partial\left(\frac{d\lambda_1}{ds_0}\right)} \delta\lambda_1 + \cdots \right]_{A_0}^{B_0}$$

$$+ \int_{A_0}^{B_0} \left[\frac{d}{ds_0} \frac{\partial W}{\partial\left(\frac{dx}{ds_0}\right)} \delta x + \cdots + \left(\frac{d}{ds_0} \frac{\partial W}{\partial\left(\frac{d\lambda_1}{ds_0}\right)} - \frac{\partial W}{\partial \lambda_1} \right) \delta\lambda_1 + \cdots \right] ds_0 ,$$

which is a form that we compare to:

$$\begin{aligned}\delta\mathcal{T}_e = &- [F\, \delta x + G\, \delta y + H\, \delta z + I\, \delta I + J\, \delta J + K\, \delta K]_{A_0}^{B_0} \\ &+ \int_{A_0}^{B_0} (X\, \delta x + Y\, \delta y + \cdots)\, ds_0 ,\end{aligned}$$

δI, δJ, δK being the components of the elementary rotation of the moving trihedron that is attached to the infinitely small displacement considered, namely:

$$\begin{aligned}\delta I &= \varpi_1\, \delta\lambda_1 + \varpi_2\, \delta\lambda_2 + \varpi_3\, \delta\lambda_3 , \\ \delta J &= \chi_1\, \delta\lambda_1 + \chi_2\, \delta\lambda_2 + \chi_3\, \delta\lambda_3 , \\ \delta K &= \sigma_1\, \delta\lambda_1 + \sigma_2\, \delta\lambda_2 + \sigma_3\, \delta\lambda_3 ,\end{aligned}$$

in such a way that:

$$\mathbf{I}\, \delta I + \mathbf{J}\, \delta J + \mathbf{K}\, \delta K = (\varpi_1\, \mathbf{I} + \chi_1\, \mathbf{J} + \sigma_1\, \mathbf{K})\, \delta\lambda_1 + \ldots = \mathcal{I}\, \delta\lambda_1 + \mathcal{J}\, \delta\lambda_2 + \mathcal{K}\, \delta\lambda_3 .$$

The complete integral term is written:

$$[F\,\delta x + G\,\delta y + H\,\delta z + \mathcal{I}\,\delta\lambda_1 + \mathcal{J}\,\delta\lambda_2 + \mathcal{K}\,\delta\lambda_3]_{A_0}^{B_0},$$

and the comparison that we mentioned gives:

$$F = \frac{\partial W}{\partial\left(\dfrac{dx}{ds_0}\right)}, \qquad G = \frac{\partial W}{\partial\left(\dfrac{dy}{ds_0}\right)}, \qquad H = \frac{\partial W}{\partial\left(\dfrac{dz}{ds_0}\right)},$$

$$\mathcal{I} = \frac{\partial W}{\partial\left(\dfrac{d\lambda_1}{ds_0}\right)}, \qquad \mathcal{J} = \frac{\partial W}{\partial\left(\dfrac{d\lambda_2}{ds_0}\right)}, \qquad \mathcal{K} = \frac{\partial W}{\partial\left(\dfrac{d\lambda_3}{ds_0}\right)}.$$

In other words, whereas the external effort of deformation is found directly by the first three formulas, which can be summarized into the vector equality:

$$\mathcal{E} = \text{grad } W_{\mathbf{v}},$$

the external moment of deformation is given by the linear combinations:

$$\mathcal{I} = \xi_1\,\mathbf{I} + \chi_1\,\mathbf{J} + \sigma_1\,\mathbf{K}, \ldots,$$

or, if one prefers, by the scalar products:

$$\mathcal{I} = \mathcal{M} \times \mathbf{\Omega}_1, \quad \mathcal{J} = \mathcal{M} \times \mathbf{\Omega}_2, \quad \mathcal{K} = \mathcal{M} \times \mathbf{\Omega}_3.$$

One will likewise have the external effort and moment at the point M by considering the integral that figures in the expressions for $\delta\mathcal{T}_e$ and comparing it with the one that we have already given, namely:

$$X_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \dfrac{dx}{ds_0}}\right), \qquad Y_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \dfrac{dy}{ds_0}}\right), \qquad Z_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \dfrac{dz}{ds_0}}\right),$$

and

$$\mathcal{L}_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \dfrac{d\lambda_1}{ds_0}}\right) - \frac{\partial W}{\partial \lambda_1}, \qquad \mathcal{M}_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \dfrac{d\lambda_2}{ds_0}}\right) - \frac{\partial W}{\partial \lambda_2},$$

$$\mathcal{N}_0 = \frac{d}{ds_0}\left(\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}\right) - \frac{\partial W}{\partial \lambda_3},$$

upon setting:

$$\mathcal{L}_0 = \mu \times \Omega_1, \quad \mathcal{M}_0 = \mu \times \Omega_2, \quad \mathcal{N}_0 = \mu \times \Omega_3 .$$

CHAPTER III.

VARIOUS APPLICATIONS – DISPLACEMENTS IN BODIES WITH A MEAN FIBER.

21. The flexible and inextensible curve. – The preceding results permit us to recover all of the properties of funicular curves, which amount to flexible curves that are or are not inextensible, or even the deformable line that was studied by Lord Kelvin and Tait, in particular. The consideration of the latter is of paramount importance in the *theory of the strength of materials.*

The advantage of the method consists in precisely that generality; in addition, it goes further than the predecessors had gone, since they had considered the infinitely small deformation exclusively, in most cases.

The introduction that we gave permits us to expand upon the details of all the examples that were contained in the work of E. and F. Cosserat. Always with the goal of giving us guidance, we shall recall an important case, for which we appeal to the notations of vector calculus; it will be extremely simple for the reader to transpose them to the other examples.

Consider the case where W depends upon only ξ, η, ζ, which will correspond to the particular nature of the curve. (This would entail that grad $W_\omega = 0$.)

Suppose that $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero for all deformations.

Finally, assume that X_0, Y_0, Z_0 are given functions of s_0, x, y, z, dy / ds_0, dx / ds_0, dz / ds_0, λ_1, λ_2, λ_3 .

The fact that $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero implies the same condition for L_0, M_0, N_0 , and one has, consequently, by virtue of the vector equation:

$$\mathcal{M}_0 = \text{grad } W_\omega + \text{grad } W_{\mathbf{v}} \wedge \mathbf{V},$$
$$\text{grad } W_{\mathbf{v}} \wedge \mathbf{V} = 0,$$

so

$$\text{grad } W_{\mathbf{v}} = \lambda \, \mathbf{V},$$

in which λ is a scalar.

Give W a virtual variation without modifying s_0; one gets:

$$\delta W = \text{grad } W_{\mathbf{v}} \times \delta\mathbf{v} = \lambda \, \mathbf{V} \times \delta\mathbf{v} = \frac{\lambda}{2}\, \delta(\mathbf{V}^2),$$

which shows that W depends upon just the argument mod $\mathbf{V}$, and perhaps, on s_0 .

Set:

$$\mu = \frac{ds}{ds_0} - 1$$

and

$$W = W(\mu, s_0).$$

The external effort is written ([1]):

$$\mathcal{E} = \frac{\partial W}{\partial \mu} \text{grad } \mu_{\mathrm{v}} ,$$

and the gradient of μ is nothing but ds / ds_0 , when considered as a function of the vector **V**; it is a unit vector **I** that is carried by the positive semi-tangent to the deformed curve.

Finally:

$$\mathcal{E} = - T \cdot \mathbf{I},$$

in which T is the scalar $- \partial W / \partial \mu$. Hence, one has equations that are analogous to the ones that classical mechanics gives for filaments:

$$\frac{d\mathcal{E}}{ds_0} + \varphi_0 = 0,$$

or

$$\frac{d}{ds_0}\left(T \frac{dx}{ds}\right) + X_0 = 0,$$
$$\frac{d}{ds_0}\left(T \frac{dy}{ds}\right) + Y_0 = 0,$$
$$\frac{d}{ds_0}\left(T \frac{dz}{ds}\right) + Z_0 = 0,$$

equations in which one can suppress the index zero, from the definitions of X_0, Y_0, Z_0 .

In the *Théorie des corps déformables*, one will find the study of several other interesting cases (flexible and inextensible filament; deformable line, when one supposes that one edge of the trihedron that is attached to each point of the curve *remains* tangent to the deformed one ([2]); deformable line when a plane of the trihedron remains osculating to the deformed one ([3]); deformable line subject to constraints, etc.).

22. Calculation of the displacements. – The theory of the Euclidian action permits us to evaluate the elements of the deformation by starting with the expression for W. One will find a general method in the book of E. and F. Cosserat. We have presented it in

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([1]) These equations were given by Lagrange in a particular case. In our presentation, they correspond to the hypothesis that W does not contain s_0 explicitly, which can be interpreted by saying that the material curve is homogeneous.

The general procedure above avoids an obscurity in Lagrange's argument (*Mécanique analytique*, Part I, Section V, § 11) that was pointed out by E. and F. Cosserat.

([2]) This is the case that was studied by Lord Kelvin and Tait, in particular, for an infinitely small deformation.

([3]) A case that was studied by Lagrange and generalized by Binet.

vector notation while rectifying a conclusion that was insufficiently exact that related to the case where W is quadratic in F, G, H, I, J, K ([1]).

We shall examine this useful case directly:

Since the function W is defined up to a constant, one can always suppose that its value is zero for the natural state of the body considered; in other words, one can suppose that $W(\xi_0, \eta_0, \ldots, r_0) = 0$.

On the other hand, $\partial W / \partial \xi_0$, $\partial W / \partial \eta_0$, $\partial W / \partial \zeta_0$ are the components of $\mathcal{E}$ in the undeformed state. These partial derivatives are therefore zero, as well as:

$$\frac{\partial W}{\partial p_0}, \quad \frac{\partial W}{\partial q_0}, \quad \frac{\partial W}{\partial r_0}.$$

Therefore, if one develops $W(\xi, \eta, \ldots)$ in powers of $\xi - \xi_0$, $\eta - \eta_0$, $\zeta - \zeta_0$, …, $r - r_0$ then the terms that are independent of $\xi, \eta, \zeta, p, q, r$ and the terms of first degree are separately zero. For these sufficiently small deformations, one will have, upon limiting oneself to terms of second degree:

$$W = \text{quadratic function of } (\xi - \xi_0, \eta - \eta_0, \zeta - \zeta_0, \ldots, r - r_0).$$

Now, the relations:

$$\mathcal{E} = \text{grad } W_{\mathbf{v}}, \qquad \mathcal{M} = \text{grad } W_\omega$$

can be just as well written:

$$\mathcal{E} = \text{grad } W_{\mathbf{v}-\mathbf{v}_0}, \qquad \mathcal{M} = \text{grad } W_{\omega-\omega_0},$$

in which the notations $\mathbf{v}_0$ and ω_0 refer to the undeformed state.

These equations amount to the following ones, in which F', G', H' are the components of $\mathcal{E}$, and I', J', K', those of L along the axes of the moving trihedron whose origin is at the point considered:

$$F' = \frac{\partial W}{\partial(\xi - \xi_0)}, \qquad G' = \frac{\partial W}{\partial(\eta - \eta_0)}, \qquad H' = \frac{\partial W}{\partial(\zeta - \zeta_0)},$$

$$I' = \frac{\partial W}{\partial(p - p_0)}, \qquad J' = \frac{\partial W}{\partial(q - q_0)}, \qquad K' = \frac{\partial W}{\partial(r - r_0)},$$

equations that are linear in the differences $\xi - \xi_0$, $\eta - \eta_0$, …, $r - r_0$, and which permit one to express them as linear functions $F', G', \ldots, K'$.

([1]) "Contribution à l'étude des corps déformables," Annales de la Faculté de Toulouse (1926), §§ 44 and 45.

Upon substituting the values thus found in W, that quantity is expressed as a quadratic function of F', G', …, K'.

It is easy to see that when this substitution has been carried out, one has:

$$\xi - \xi_0 = \frac{\partial W}{\partial F'}, \quad \ldots, \qquad r - r_0 = \frac{\partial W}{\partial K'}.$$

This results from a property of quadratic functions. Indeed, let $\mathcal{F}(\lambda, \mu, \nu)$ be such a function of the variables λ, μ, ν.

From a theorem of Euler, it results that:

$$2\mathcal{F} = \lambda\frac{\partial \mathcal{F}}{\partial \lambda} + \mu\frac{\partial \mathcal{F}}{\partial \mu} + \nu\frac{\partial \mathcal{F}}{\partial \nu} = \lambda\, u + \mu\, v + \nu\, w,$$

if one sets:

$$u = \frac{\partial \mathcal{F}}{\partial \lambda}, \qquad v = \frac{\partial \mathcal{F}}{\partial \mu}, \qquad w = \frac{\partial \mathcal{F}}{\partial \nu};$$

hence:

$$2(u\,d\lambda + v\,d\mu + w\,d\nu) = u\,d\lambda + v\,d\mu + w\,d\nu + \lambda\,du + \mu\,dv + \nu\,dw,$$

and upon simplifying this, one gets:

$$d\mathcal{F} = \lambda\,du + \mu\,dv + \nu\,dw.$$

Therefore, if one expresses $\mathcal{F}$ as a function of u, v, w then one has, in fact ([1]):

$$\lambda = \frac{\partial \mathcal{F}}{\partial u}, \qquad \ldots$$

Applications to the displacements in bodies with a mean fiber ([2]). – Equation (1) amounts to two vector relations:

([1]) In the general case – i.e., when W is not quadratic – one obtains some general formulas for the displacements by considering the function:

$$E = \xi\frac{\partial W}{\partial \xi} + \eta\frac{\partial W}{\partial \eta} + \cdots + r\frac{\partial W}{\partial r} - W;$$

Upon making an inversion that is analogous to the Hamilton-Poisson transformation, which is a generalization of the Legendre transformation, one finds that:

$$\frac{dx}{ds_0} = \frac{\partial E}{\partial F}, \qquad \frac{dy}{ds_0} = \frac{\partial E}{\partial G}.$$

See J. S., *loc. cit.*, pp. 78.

([2]) This topic is not treated in the *Théorie* of E. and F. Cosserat.

$$\delta\mathbf{V} = \text{grad } W_{\mathcal{E}} ,$$
$$\delta\boldsymbol{\Omega} = \text{grad } W_{\mathcal{M}} ,$$

in which $\delta\mathbf{V}$ and $\delta\boldsymbol{\Omega}$ are the variations of **V** and Ω that were defined at the beginning of our study – i.e., with respect to the moving trihedron. It then results that the absolute variation of **V** is:

$$\Delta\mathbf{V} = \text{grad } W_{\mathcal{E}} + \theta \wedge \mathbf{V}.$$

The left-hand side is nothing but $\Delta(DM / Ds_0)$ or $D(\Delta M) / Ds_0$.

Finally, upon taking the geometric integral from a point A to the point M:

$$\Delta M = \Delta A + \int_A^M \text{grad}\, W_{\mathcal{E}}\, ds_0 + \int_A^M \theta \wedge \frac{DP}{Ds_0} ds_0 .$$

The P inside the integral sign denotes the point that describes the curve from A to M.

On the other hand, one knows that:

$$\delta\boldsymbol{\Omega} = \frac{D\theta}{Ds_0} \qquad (\textit{see}, \S\ 3),$$

so:

$$(S_1) \qquad \theta_M = \theta_A + \int_A^M \text{grad}\, W_{\mathcal{M}}\, ds_0 .$$

Integrating by parts gives:

$$(S_2) \qquad \Delta M = \Delta A + \int_A^M \text{grad}\, W_{\mathcal{E}}\, ds_0 + \left[\dot{\theta} \wedge OP\right]_A^M - \int_A^M (\text{grad}\, W_{\mathcal{M}} \wedge OP)\, ds_0 .$$

This is the fundamental vector equality from which we shall infer a general method for the calculation of deformations.

In the case where the curve remains planar during the deformation, if the applied forces are in the plane then the equality (S_2) gives the classical formulas of Bresse for the deformation of the mean fiber of the bodies that are considered in the strength of materials.

It suffices to project the vector equality onto two coordinate axes whose origin will be at M. If u_M and u_A are the components along one of the axes of the displacement of the points M and A then one gets:

$$u_M = u_A + \int_A^M \text{grad}\, W_{\mathcal{E}} \times i\, ds_0 + \theta_0 (z_M - z_A) - z_0 \int_A^M \frac{M}{EI} ds_0 + \int_A^M \frac{M_{\mathcal{E}}}{EI} ds_0 ,$$

in which i is a unit vector that is carried by the projection axis.

The part of W that depends upon the external effort is written:

$$-\left[\frac{N^2}{E\Omega}+\frac{T^2}{G\Omega}+\cdots\right],$$

in which Ω is the normal section to the mean fiber at the point considered. The gradient of W with respect to $\mathcal{E}$ thus has the components $N / GE\Omega$ along the tangent and $T / G\Omega$ along the normal; the projection onto the x-axis, or $i \times \text{grad } W_{\mathcal{E}}$, is obtained immediately.

23. Body with a mean fiber. – It is intuitive that the study of small displacements could be utilized in the strength of materials.

Indeed, that discipline studies the deformations of bodies with mean fibers that one can assume to be generated in the natural state by a planar section that is deformed and which displaces while remaining normal to the mean line (the center of the section describes the latter line).

To each point of the mean line, one can therefore attach a tri-rectangular trihedron such that one of its edges Gx is tangent to the line, while the other two are in the plane of the generating section and can coincide with the principal axes of inertia of the section.

One can abstract from the body and only take into consideration the mean line and the set of trihedra, and then repeat the argument that was already made on the deformable line.

It is very simple to give a generalized Bresse equation that applies to the most general skew system and the most general elementary deformation by translating the general equality (S_2) by means of projections onto the three axes. In order to underscore the possible interest in that extension, recall the following lines of Mesnager (Bull. de la Société d'encouragement pour l'industrie nationale, t. CXXXIII, no. 4, April 1921):

"As far as the three-dimensional systems are concerned, the use of geometry and kinematics gives rise to some difficulties and complications such that the authors have, in general, eluded these difficulties and complications by the introduction of unjustified hypotheses into the calculations that eliminate any value in the results thus obtained."

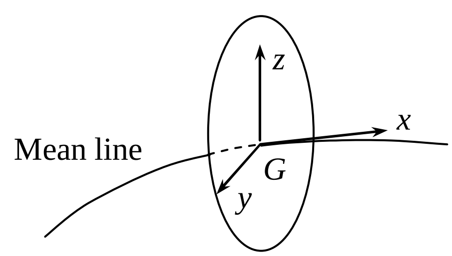

Fig. 3.

24. New formulas. – There have been other criticisms of the Bresse formulas, notably, *the inevitable introduction of auxiliary unknowns, in the form of linear and angular displacements* ([1]).

We shall deduce some formulas from the general equation (S_2) that are, once and for all, devoid of any auxiliary unknowns. The formulas, which apply to a deformed skew system, involve only two coefficients (just one for planar systems) that can be calculated in advance, independently of the form of the system and the applied forces.

This method also seems to be as easy to apply as the method that Bertrand de Fontviolant has made known. We will show this by means of some examples. In addition, it does not involve fictitious forces.

We shall content ourselves with the following obvious remark: The body with a skew mean fiber has at least three fixed points, while the body with a planar mean fiber has two fixed points ([2]).

We treat the first case, which is the most general:

Call the fixed points A_1, A_2, A_3 . We wish to find the component Δx of the displacement of a point M along a direction that is defined by a vector $\mathbf{I}$ that will be taken to have unit modulus in what follows.

When one takes the point O to be the fixed point A_i then equation (S) is written:

$$\Delta M = \int_{A_i}^{M} \operatorname{grad} W_{\mathcal{E}}\, ds_0 + \theta_M \wedge A_i M - \int_{A_i}^{M} \operatorname{grad} W_{\mathcal{M}} \wedge A_i P\, ds_0 .$$

Multiply both sides of the equation by λ_i and add the corresponding sides of the three analogous equations, upon choosing the coefficients in such a manner that:

$$\sum_i (\lambda_i A_i M) = \mathbf{I}.$$

One gets:

$$\Delta x \sum \lambda_i = \sum \left(\lambda_i \int_{A_i}^{M} [\operatorname{grad} W_{\mathcal{E}} + A_i P \wedge \operatorname{grad} W_{\mathcal{M}}] \right) \times \mathbf{I}\, ds_0 .$$

This is the formula that we have in mind. It can also be written:

$$\Delta x = \sum \mu_i \int_{A_i}^{M} [\operatorname{grad} W_{\mathcal{E}} + A_i P \wedge \operatorname{grad} W_{\mathcal{M}}] \times \mathbf{I}\, ds_0 ,$$

with

$$\mu_i = \frac{\lambda_i}{\sum \lambda_i} .$$

In the most general case, there are, in reality, only two independent coefficients because $\sum \mu_i = 1$.

([1]) MESNAGER, *loc. cit.*

([2]) Upon giving this manner of speaking a general sense, this case includes both recessed (encastrés) systems and ones with rolling supports.

25. Calculation of the coefficients. – I. It is easy to geometrically obtain the coefficients, or rather, some proportional quantities.

Let S be the oblique projection (made parallel to **I**) from the point M onto the plane A_1, A_2, A_3.

The relation:

$$\lambda_1 A_1M + \lambda_2 A_2M + \lambda_3 A_3M = \mathbf{I}$$

implies that:

$$\lambda_1 A_1S + \lambda_2 A_2S + \lambda_3 A_3S = 0.$$

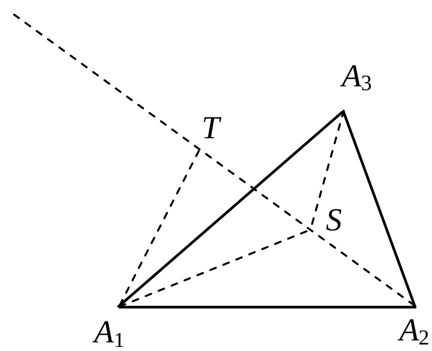

Fig. 4.

Draw a parallel A_1T to SA_3 through the point A_1 until it meets SA_2.

If S is interior to the triangle then one can take:

$$\lambda_1 = 1,$$

$$\lambda_2 = \frac{\text{mod } ST}{\text{mod } A_2S},$$

$$\lambda_3 = \frac{\text{mod } TA_1}{\text{mod } A_3S},$$

and indeed:

$$A_1S + ST + TA_1 = 0.$$

II. If the point S is outside of the triangle A_1, A_2, A_3 then one of the λ coefficients will be negative.

The figure shows this immediately.

III. If the direction of **I** is in the plane MA_1, A_2, for example then the formula simplifies: Indeed, one has seen that $\lambda_1, \lambda_2, \lambda_3$ are chosen in such a manner as to annul the trivector or mixed product:

$$(\lambda_1 A_1 P + \lambda_2 A_2 P + \lambda_3 A_3 P)\ \theta \mathbf{I},$$

but here $\lambda_3 = 0$, and it suffices to take λ_1 and λ_2 in such a manner that:

$$\lambda_1 A_1 P + \lambda_2 A_2 P = \mathbf{I},$$

or

$$\lambda_1 A_1 S + \lambda_2 A_2 S = 0,$$

so if the point S is interior to the segment $A_1 A_2$ then:

$$\frac{\lambda_1}{\lambda_2} = \frac{\text{mod } A_2 S}{\text{mod } A_1 S},$$

$$\mu_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\text{mod } A_2 S}{\text{mod } A_1 A_2},$$

and

$$\mu_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\text{mod } A_1 S}{\text{mod } A_1 A_2}.$$

26. Algebraic translation of some vector equalities. –

1. In order to simply the presentation, take the case of a planar system that is subject to forces that are situated in the plane. One then has:

$$W = -\frac{1}{2}\left[\frac{N^2}{E\Omega} + \frac{T^2}{G\Omega} + \frac{M^2}{EI}\right].$$

The gradient of W with respect to the effort has the components $N / E\Omega$, $T / G\Omega$, so it is easy to project them onto an arbitrary axis, a projection that we have denoted by grad $W_{\mathcal{E}} \times \mathbf{I}$ ($\mathbf{I}$ is a unit vector that is carried by the axis).

2. As far as grad $W_{\mathcal{M}}$ is concerned, it reduces to its component M / EI along the normal to the plane of the system. It is measured positively according to the convention that links the positive sense of a semi-normal to the plane to the positive sense of rotation in the plane.

If x and y are the coordinates of the point P with respect to the rectangular axes that issue from A_1 and one of which – viz., x – is directed along A_1A_2 then the external products such as $A_1P \wedge$ grad $W_{\mathcal{M}}$ have either $(M / EI)\, y$ for their component along ox or $-(M / EI)\, x$ for their component along oy.

The integration by parts, by means of which we have obtained the formula (S) *is not valid if the deformable line is composed of articulated segments.* The rotation Θ varies discontinuously by a finite quantity $\Theta_2 - \Theta_1$ when one passes from the extremity of one segment to the continuous extremity of the following segment.

In order to simplify the presentation, take a planar system with intermediate joints. Upon operating with each segment as we said, one gets:

$$\Delta P = \mathcal{I}_{A_1}^P + \sum_C (\theta_2 - \theta_1) \wedge A_1C + \theta_P \wedge A_1P$$

for an arbitrary point P, upon representing the integral of grad $W_{\mathcal{E}}$ + A_1P ^ grad $W_{\mathcal{M}}$ by $\mathcal{I}_{A_1}^P$, while the sum is taken over all segments that connect A_1 to P. Similarly:

$$\Delta P = \mathcal{I}_{A_2}^P + \sum_C (\theta_1 - \theta_2) \wedge A_2C + \theta_P \wedge A_2P ,$$

where the sum $\sum$ is extended over all of the segments that connect P to A_2 . If one projects onto the line of support A_1A_2 then this becomes the algebraic translation of the vector equality that we obtained by subtracting two sides of the two formulas ([1]):

$$2\sum_i f_i(\theta_2 - \theta_1) = \int_{A_1}^{A_2} \frac{M}{EI}\, y\, ds_0 ,$$

in which f_i denotes the height of the segment C_i above the line A_1A_2 .

Recall, the example that Mesnager ([2]) treated by the Bresse formulas and then those of Bertrand de Fontviolant, successively.

Calculate the vertical displacement dh of an arbitrary point P of an arc of three segments that is subject arbitrary loads that are situated in the plane of the mean line of that arc. Although it is very easy to take into account the external effort, we shall neglect it, to simplify:

$$\delta h = \mu_1 \mathbf{Y} \times \mathcal{I}_{A_0}^P + (\theta_2 - \theta_1) \wedge A_1C + \mu_2 \mathcal{I}_{A_0}^P \times \mathbf{Y},$$

in which $\mathbf{Y}$ is an ascending vertical unit vector:

$$\mu_1 = \frac{l - x_P}{l}, \qquad \mu_2 = \frac{x_P}{l}.$$

However, from the preceding remark:

$$\theta_2 - \theta_1 = \frac{1}{f}\int_{A_1}^{A_2} \frac{M}{EI}\, y\, ds_0 ,$$

so

$$\delta h = \frac{l - x_P}{l}\int_{A_1}^{P} \frac{M}{EI}\, x\, ds_0 + \frac{x_0}{l}\int_{P}^{A_2} \frac{M(l-x)}{EI}\, ds_0 - \frac{x_P}{2f}\int_{A_1}^{A_2} \frac{M}{EI}\, y\, ds_0 .$$

([1]) To simplify, we neglect the effort that is involved with cutting; it is obviously very easy to take into account.

([2]) Bulletin de la Société d'encouragement pour l'industrie nationale, t. CXXXIII, no. 4, April 1931.

In our previous paper, one finds the solution, based upon this same method, to an example that was mentioned (but not treated) in the same article of Mesnager.

CHAPTER IV.

THE DEFORMABLE SURFACE

27. Extension of the preceding notions to the case of surfaces. – In order to complete this presentation, which must be, above all, a guide, we now pass on to the case of deformable surfaces, and then to deformable bodies.

We suppose that the position of a point M_0 on the undeformed surface (M_0) is defined by means of two curvilinear coordinates ρ_1 and ρ_2 .

Any point M_0 is associated with a tri-rectangular trihedron whose summit is M_0 and whose edges have directions that are continuous functions of M_0 – i.e., the two coordinates ρ_1 and ρ_2 .

The deformed state of the surface can be imagined by supposing that each point M_0 has received a displacement M_0M and that the directions of the trihedron of the modifications are continuous functions of ρ_1 and ρ_2 .

The deformed state and the natural state are thus defined by two sets of trihedra, sets that are continuous in two parameters.

The undeformed surface element can be represented by the expression $\Delta_0\,d\rho_1\,d\rho_2$, in which $\Delta_0 = \sqrt{eg - f^2}$ if one takes the linear element to have the form:

$$[e\,d\rho_1^2 + 2\,f\,d\rho_1\,d\rho_2 + g\,d\rho_2^2]^{1/2}\,.$$

We shall define the action of deformation W per unit of undeformed surface. On a portion Σ of the surface in question, the total action will be:

$$\iint_\Sigma W\,\Delta_0\,d\rho_1\,d\rho_2\,.$$

W is a function of two neighboring positions of the trihedron $Mabc$; i.e., of ρ_1, ρ_2, x, y, z (the coordinates of the points M), of α, β, γ, α', …, γ'', and finally of the derivatives of these twelve quantities with respect to ρ_1 and ρ_2 . In a more condensed manner, one has:

$$W(\rho_i,\ OM,\ \mathbf{V}_i,\ \mathbf{I},\ \mathbf{J},\ \mathbf{K},\ \mathbf{I}'_i,\ \mathbf{J}'_i,\ \mathbf{K}'_i),$$

in which $\mathbf{V}_i$, $\mathbf{I}'_i$, $\mathbf{J}'_i$, $\mathbf{K}'_i$ are the derivatives of OM, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ with respect to the parameter ρ_i.

However, equalities such as:

$$\mathbf{I}'_i = \Omega_i \wedge \mathbf{I}_i\,,$$

where Ω_i is the instantaneous rotation of the trihedron when only the coordinate ρ_i varies and plays the role of time, permit us to put W into the form:

$$W(\rho_i,\ OM,\ \mathbf{V}_i,\ \Omega_i,\ \mathbf{I},\ \mathbf{J},\ \mathbf{K}).$$

As for the deformable curve, we impose the constraint on W that it remain invariant under a transformation of the group of Euclidian displacements. It will suffice to have $\delta W = 0$ for an infinitesimal transformation of the group.

In order to pursue a unique method, suppose that instead of $\mathbf{V}_i$, $\boldsymbol{\Omega}_i$, which are vectors that are framed in a fixed reference system, one makes $\mathbf{v}_i$ and ω_i intervene – i.e., the preceding vectors when framed with respect to the trihedron M, which amounts to modifying the manner in which W depends upon $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ ([1]).

The infinitesimal transformation can be performed in two steps:

1. A translation that alters only OM, since the point O is fixed. One then has:

$$\delta W = \text{grad } W_{OM} \times \Delta M = 0,$$

which demands that grad $W_{OM} = 0$; i.e., that W does not depend upon the point M explicitly.

2. A rotation Θ.

Since the frames $\mathbf{v}_i$ and ω_i remain unchanged, one then gets:

$$\delta W = \text{grad } W_{\mathbf{I}} \times \delta\mathbf{I} + \text{grad } W_{\mathbf{J}} \times \delta\mathbf{J} + \text{grad } W_{\mathbf{K}} \times \delta\mathbf{K} = 0,$$

with

$$\delta\mathbf{I} = \theta \wedge \mathbf{I}, \qquad \ldots,$$

and finally:

$$\text{grad } W_{\mathbf{I}} \times \mathbf{I} + \text{grad } W_{\mathbf{J}} \times \mathbf{J} + \text{grad } W_{\mathbf{K}} \times \mathbf{K} = 0.$$

This differential equation, which is equivalent to three Cartesian equations between the nine components of $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, is solved by first considering the six obvious solutions:

$$\mathbf{I}^2, \quad \mathbf{J}^2, \quad \mathbf{K}^2, \quad \mathbf{J} \times \mathbf{K}, \quad \mathbf{K} \times \mathbf{I}, \quad \mathbf{I} \times \mathbf{J},$$

which are all functions that remain constant. Therefore, W, which is a function of these six solutions, does not contain $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ explicitly.

28. Transformation of the elementary variation of the action. – By analogy with the case of a deformable curve, take the elementary variation of the action on a portion S of the surface (M) that is bounded by a curve C. If S_0 and C_0 are the initial states of that portion and its contour then one will have:

([1]) Therefore:

$$\mathbf{V}_1 = \mathbf{V}_1 \times \mathbf{I} \cdot \mathbf{I} + \mathbf{V}_1 \times \mathbf{J} \cdot \mathbf{J} + \mathbf{V}_1 \times \mathbf{K} \cdot \mathbf{K} = \xi_1\, \mathbf{I} + \eta_1\, \mathbf{J} + \zeta_1\, \mathbf{K},$$

ξ, η, ζ being the frames or projections of $\mathbf{V}_1$ onto the axes of the moving trihedron; on the other hand, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ already figure in W.

$$\delta\iint_{S_0} W\,\Delta_0\,d\rho_1\,d\rho_2 = \iint\sum_i (\text{grad}\,W_{\mathbf{v}_i}\times\delta\mathbf{v}_i + \text{grad}\,W_{\omega_i}\times\delta\omega_i)\,\Delta_0\,d\rho_1\,d\rho_2\,,$$

which, from the formulas of the previous page, can be written:

$$\sum_i\iint\Delta_0\left\{\text{grad}\,W_{\mathbf{v}_i}\times\left[\frac{D(\Delta M)}{D\rho_i}+\mathbf{V}_i\wedge\theta\right]+\text{grad}\,W_{\omega_i}\times\frac{D\theta}{D\rho_i}\right\}d\rho_1\,d\rho_2\ .$$

If one integrates the terms that contain a derivative by parts one time and then applies Green's formula to these terms then one gets:

$$\begin{aligned}\delta\iint_{S_0} W\,\Delta_0\,d\rho_1\,d\rho_2 = &\int_{C_0}(\text{grad}\,W_{\mathbf{v}_1}\times\Delta M + \text{grad}\,W_{\omega_1}\times\theta)\Delta_0 d\rho_2\\ &-\int_{C_0}(\text{grad}\,W_{\mathbf{v}_2}\times\Delta M + \text{grad}\,W_{\omega_2}\times\theta)\Delta_0 d\rho_1\\ &-\sum_i\iint\frac{D}{D\rho_i}(\Delta_0\,\text{grad}\,W_{\mathbf{v}_i})\times\Delta M\\ &+\left(\frac{D}{D\rho_i}\Delta_0\,\text{grad}\,W_{\omega_i}+\mathbf{v}_i\wedge\Delta_0\,\text{grad}\,W_{\mathbf{v}_i}\right)\times\theta\,d\rho_1\,d\rho_2\ .\end{aligned}$$

The curvilinear integral must be taken in the usual direct sense.

Set ([1]):

([1]) One can transform these results by means of Beltrami formulas. We have also given (*loc. cit.*, pp. 71) some other formulas that are based upon the equalities:

$$\frac{d\rho_1}{ds} = -\,v_1\frac{\sqrt{e}}{\Delta_0}, \qquad \frac{d\rho_2}{ds} = +\,v_2\frac{\sqrt{g}}{\Delta_0}\ .$$

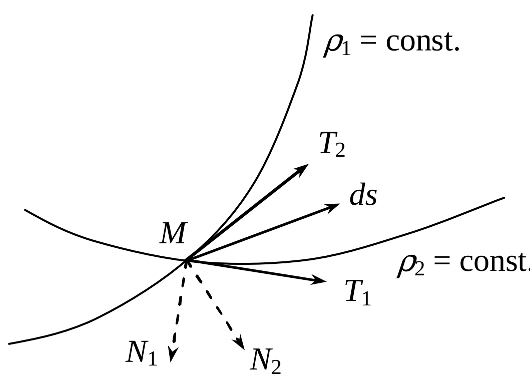

Fig. 5.

v_1 and v_2 are the cosines of ds with respect to the semi-normals N_1 and N_2, respectively, which point along the coordinate curves at the point M. The senses of the semi-normals are obtained by moving (décalant) the positive semi-tangents T_1 and T_2 by $-\pi/2$. This convention seems preferable to that of Beltrami to us, which involves, not the element, but the curve that passes through the element.

$$\Delta_0 \left(\operatorname{grad} W_{\mathbf{v}_1} \frac{d\rho_2}{ds_0} - \operatorname{grad} W_{\mathbf{v}_2} \frac{d\rho_1}{ds_0} \right) = -\,\mathcal{E},$$

$$\Delta_0 \left(\operatorname{grad} W_{\omega_1} \frac{d\rho_2}{ds_0} - \operatorname{grad} W_{\omega_2} \frac{d\rho_1}{ds_0} \right) = -\,\mathcal{M},$$

so

$$\sum_i \left[\frac{D}{D\rho_i} (\Delta_0 \operatorname{grad} W_{\mathbf{v}_i}) \right] = \varphi\, \Delta_0$$

and

$$\sum_i \left[\frac{D}{D\rho_i} (\Delta_0 \operatorname{grad} W_{\omega_i}) + \Delta_0 \operatorname{grad} W_{\mathbf{v}_i} \wedge \mathbf{v}_i \right] = \mu\, \Delta_0\,,$$

and one finally gets:

$$\delta \iint_{S_0} W\, dS_0 = -\int_{C_0} \mathcal{E}\, ds_0 \times \Delta M + \mathcal{M}\, ds_0 \times \theta \;-\; \iint_{S_0} \varphi\, ds_0 \times \Delta M + \mu\, ds_0 \times \theta\,.$$

29. Examples. – As for the deformable curve, one deduces that $\mathcal{E}$ and $\mathcal{M}$ are the external effort and external moment of deformation at a point M of the contour C of the deformed surface, when referred to the unit of length of the contour C_0 .

The results above comprise, in particular, the equations that relate to the infinitely small deformation of a planar surface that were utilized by Lord Kelvin and Tait. One can, moreover, point out several aspects of these results (see *Théorie*, pp. 77, *et seq.*). Notably, one will find equations that are obtained by introducing, as in the example of Poisson, the coordinates x and y as independent variables in place of ρ_1 and ρ_2 . One then introduces new auxiliary variables that are provided by considering the non-tri-rectangular trihedra that are defined by the normal to the deformed surface and the tangents to the conjugate curves.

It is painfully necessary to confirm the analogy between the given formula $\delta \iint_{S_0} W\, dS_0$ and the analogous formula that relates to deformable curves. Also, the integral:

$$\iint_{S_0} -W\, dS_0$$

will represent the energy of deformation. Indeed, by changing the signs in the expression for $\delta \iint_{S_0} W\, dS_0$, one finds δT_e (elementary work done by external forces). We say that – W represents the density of energy of deformation with respect to the unit of undeformed area.

We again point out that if one supposes that W does not depend on ω_1 and ω_2 then in the case of infinitely small deformation the surface behaves like the membrane that was studied by Poisson and Lagrange.

The theory also studies the flexible and inextensible surface of the geometers and shows how one must take W in order to be dealing with reinforced (armée) surfaces, such as those of aerostats; i.e., elastic surfaces in which a fabric of inextensible filaments is embedded.

CHAPTER V

THE DEFORMABLE MEDIUM.
OUTLINE OF THE PROBLEMS THAT ARE POSED IN THE SUBJECT OF DEFORMABLE MEDIA.

30. The preceding notions can be extended to the case of three-dimensional media. – The extension of the theory of action to the case of a deformable medium is of paramount importance for mathematical physics. That extension leads one to envision a more general medium that the one that is usually considered in the theory of elasticity. We have already spoken of the two differences between the theories in the preface.

Let a three-dimensional space (M_0) be described by the point M_0 that is framed with respect to a fixed reference system (by means of arbitrary curvilinear coordinates or simply Cartesian coordinates).

Attach a tri-rectangular trihedron $M_0\, a_0\, b_0\, c_0$ to each point M_0 whose axes have the direction cosines α_0, α_0', α_0'', …, γ_0'' with respect to the fixed axes, the latter being functions of the independent variables. The continuous, three-dimensional set of of these trihedra constitutes the deformable medium.

If one gives the point M_0 a displacement M_0M and one imprints a rotation on the trihedron $M_0\, a_0\, b_0\, c_0$ that takes the trihedron into a position *Mabc* then the continuous, three-dimensional set of trihedra *Mabc* constitutes the deformed state of the medium.

Consider a function W of two infinitely close positions of the trihedron *Mabc*; i.e., of M_0, OM, **I**, **J**, **K**, and the derivatives:

$$\frac{\partial M}{\partial \rho_i},\ \frac{\partial \mathbf{I}}{\partial \rho_i},\ \frac{\partial \mathbf{J}}{\partial \rho_i},\ \frac{\partial \mathbf{K}}{\partial \rho_i}$$

(in which ρ_i denotes one of the coordinates of the point M), so we can represent it as:

$$W(M_0,\ OM,\ \mathbf{I},\ \mathbf{J},\ \mathbf{K},\ \mathbf{I}_i',\ \mathbf{J}_i',\ \mathbf{K}_i').$$

We propose to determine W in such a fashion that the elementary variation δW is zero when one subjects the set of trihedra of the deformed state to the same arbitrary infinitesimal transformation of the group of Euclidian displacements. Here, we again replace $\mathbf{I}_i'$, … with $\Omega_i \wedge \mathbf{I}$, …, so W takes on the form:

$$W(M_0,\ OM,\ \mathbf{I},\ \mathbf{J},\ \mathbf{K},\ \mathbf{V}_i,\ \Omega_i),$$

in which we suppose that $\mathbf{V}_i$, Ω_i figure by their frames of the moving trihedron *Mabc* in such a way that W will finally take on the form:

$$W(M_0,\ OM,\ \mathbf{I},\ \mathbf{J},\ \mathbf{K},\ \mathbf{v}_i,\ \omega_i),$$

in which $\mathbf{v}_i$ and ω_i have the significance that was already given in the preceding case.

It is pointless to repeat the argument that was already made (§ 12) in order to establish that in this case the frames of **I**, **J**, **K** do not figure in W, nor does the vector OM.

W thus has the form that was found before:

$$W(M_0, \mathbf{v}_i, \omega_i),$$

where i can take the values 1, 2, 3.

We shall again exhibit the effort and moment of deformation at a point of the deformed medium.

Take a portion τ of the medium in question that is bounded by a surface S, and let τ_0 and S_0 be the initial states of τ and S. Consider the integral of the volume:

$$\iiint_{\tau_0} W \, d\tau_0 \,,$$

which is extended over the portion in question, while W is calculated for the deformed state of the initial element $d\tau_0$.

An arbitrary elementary variation of that integral will be:

$$\delta \iiint_{\tau_0} W \, d\tau_0 \;=\; \iiint_{\tau_0} \sum_i (\operatorname{grad} W_{\mathbf{v}_i} \times \delta \mathbf{v}_i + \operatorname{grad} W_{\omega_i} \times \delta \omega_i) \, d\tau_0 \,,$$

in which, moreover:

$$\delta \mathbf{v}_i = \frac{D(\Delta M)}{D\rho_i} + \mathbf{v}_i \wedge \theta_i \qquad \text{and} \qquad \delta\omega_i = \frac{D\theta}{D\rho_i},$$

so one integrates the terms that contain a derivative once by parts and then applies Green's formula to these terms. One again finds that:

$$\delta \iiint_{\tau_0} W \, d\tau_0 = - \iint_{S_0} [\mathcal{E} \, d\sigma \times \Delta M + \mathcal{M} \, d\sigma \times \theta] - \iiint_{\tau_0} \varphi \, dr \, d\tau \times \Delta M + \mu \, dr \, d\tau \times \theta \,.$$

$\mathcal{E}$ is the unitary effort at a point of the surface that bounds the portion τ (per unit undeformed surface area). $\mathcal{M}$ is the moment of deformation that is exerted on that surface (always per unit undeformed surface area).

In a more precise manner, if the surface S separates the region τ from an external region τ' then $\mathcal{E}$ and $\mathcal{M}$ are the effort and moment that are exerted at a point of S of the region τ on the region τ'.

φ and μ are the force and moment vectors at a point of τ (per unit of undeformed volume).

As was the result of all of the preceding, the original integral, with its sign changed, will be the energy of deformation of the portion.

31. Interpretation of the elementary variation of the action. – This interpretation is deduced from the fact that if one seeks the elementary work done by the forces and moments that we just enumerated then one finds that:

$$\iint_{S_0} (\mathcal{E}\times\Delta M + \mathcal{M}\times\theta)\, d\sigma + \int_{\tau_0} (\varphi\times\Delta M + \mu\times\theta)\, d\tau_0 ,$$

and since the elementary work done is equal to the increase in energy (or the variation of the action, with the opposite sign), the preceding is justified.

If we recall the first expression for $\delta \iiint_{\tau_0} W\, d\tau_0$ and use identities such as:

$$\frac{D(\Delta M)}{D\rho_i} = \frac{D(\Delta M)}{Dx}\frac{\partial x}{\partial \rho_i} + \frac{D(\Delta M)}{Dy}\frac{\partial y}{\partial \rho_i} + \frac{D(\Delta M)}{Dz}\frac{\partial z}{\partial \rho_i}$$

then we will have:

$$\begin{aligned}\delta \iiint_{\tau_0} W\, d\tau_0 = & \iiint_{\tau} \sum_i \operatorname{grad} W_{\mathbf{v}_i} \\ & \times \left(\frac{D(\Delta M)}{Dx}\frac{\partial x}{\partial \rho_i} + \frac{D(\Delta M)}{Dy}\frac{\partial y}{\partial \rho_i} + \frac{D(\Delta M)}{Dz}\frac{\partial z}{\partial \rho_i} + \mathbf{v}_i \wedge \theta \right) \frac{d\tau_0}{H} \\ & + \iiint_{\tau} \sum_i \operatorname{grad} W_{\omega_i} \\ & \times \left(\frac{D\theta}{Dx}\frac{\partial x}{\partial \rho_i} + \frac{D\theta}{Dy}\frac{\partial y}{\partial \rho_i} + \frac{D\theta}{Dz}\frac{\partial z}{\partial \rho_i} \right) \frac{d\tau}{H} ,\end{aligned}$$

in which H is the functional determinant:

$$\begin{vmatrix} \dfrac{\partial x_0}{\partial x} & \dfrac{\partial y_0}{\partial x} & \dfrac{\partial z_0}{\partial x} \\ \dfrac{\partial x_0}{\partial y} & \dfrac{\partial y_0}{\partial y} & \dfrac{\partial z_0}{\partial y} \\ \dfrac{\partial x_0}{\partial z} & \dfrac{\partial y_0}{\partial z} & \dfrac{\partial z_0}{\partial z} \end{vmatrix} .$$

Set:

$$p_x = \frac{1}{H}\sum \frac{\partial x}{\partial \rho_i} \operatorname{grad} W_{\mathbf{v}_i} ,$$

$$q_x = \frac{1}{H}\sum \frac{\partial x}{\partial \rho_i} \operatorname{grad} W_{\omega_i} ,$$

in which p_y, p_z, q_y, q_z are defined analogously. One then gets:

$$\delta \iiint_{\tau_0} W\, d\tau_0 = \iiint_\tau \left(p_x \times \frac{D(\Delta M)}{Dx} + p_y \times \frac{D(\Delta M)}{Dy} + p_z \times \frac{D(\Delta M)}{Dz} \right) d\tau$$

$$+ \iiint_\tau \left(q_x \times \frac{D\theta}{Dx} + q_y \times \frac{D\theta}{Dy} + q_z \times \frac{D\theta}{Dz} \right) d\tau$$

$$+ \iiint_\tau \left(\sum_i \operatorname{grad} W_{\mathbf{v}_i} \wedge \mathbf{v}_i \right) \times \theta \frac{d\tau}{H},$$

and upon integrating by parts:

$$= \iint_S \; [(c_1\, p_x + c_2\, p_y + c_3\, p_z) \times \Delta M + (c_1\, q_x + c_2\, q_y + c_3\, q_z) \times \theta]$$

$$- \iiint_\tau \left(\frac{Dp_x}{Dx} + \frac{Dp_y}{Dy} + \frac{Dp_z}{Dz} \right) \times \Delta M\, d\tau$$

$$- \iiint_\tau \left(\frac{Dq_x}{Dx} + \frac{Dq_y}{Dy} + \frac{Dq_z}{Dz} + \sum_i \frac{\mathbf{v}_i \wedge \operatorname{grad} W_{\omega_i}}{H} \times \theta \right) d\tau\,.$$

Upon comparing the one result obtained above with the other, one gets:

$$\begin{aligned}
\mathcal{E} &= -\,(c_1\, p_x + c_2\, p_y + c_3\, p_z), \\
\mathcal{M} &= -\,(c_1\, q_x + c_2\, q_y + c_3\, q_z), \\
\varphi &= \frac{D}{Dx} p_x + \frac{D}{Dy} p_y + \frac{D}{Dz} p_z\,, \\
\mu &= \frac{D}{Dx} q_x + \frac{D}{Dy} q_y + \frac{D}{Dz} q_z + \frac{1}{H} \sum_i (\mathbf{v}_i \wedge \operatorname{grad} W_{\mathbf{v}_i})\,.
\end{aligned}$$

In order to interpret the vector $\frac{1}{H} \sum_i (\mathbf{v}_i \wedge \operatorname{grad} W_{\mathbf{v}_i})$, we look at its projections onto the axes.

The projection onto Ox is:

$$\frac{1}{H} \mathbf{I} \times \sum_i (\mathbf{v}_i \wedge \operatorname{grad} W_{\mathbf{v}_i}) = \frac{1}{H} (\mathbf{J} \wedge \mathbf{K}) \times \sum_i (\mathbf{v}_i \wedge \operatorname{grad} W_{\mathbf{v}_i})\,,$$

or furthermore, by virtue of a well-known identity ([1]):

([1]) **I**, **J**, **K** are now three unit vectors that are carried by the projection axes. In addition, one knows that:

$$(a \wedge b) \times (u \wedge v) = a \times u \cdot b \times v - a \times v \cdot b \times u.$$

$$\frac{1}{H}\left[\sum_i (\mathbf{v}_i \times \mathbf{K} \cdot \mathrm{grad}\, W_{\mathbf{v}_i} \times \mathbf{J}) - \sum_i (\mathbf{v}_i \times \mathbf{J} \cdot \mathrm{grad}\, W_{\mathbf{v}_i} \times \mathbf{K})\right].$$

However, $\mathbf{v}_i \times \mathbf{K} = \partial z / \partial \rho_i$, so one sees that the first term of the sum is the projection onto Oy (or the scalar product with $\mathbf{J}$) of $\sum_i \frac{\partial z}{\partial f_i} \mathrm{grad}\, W_{\mathbf{v}_i}$, or furthermore, the projection of p_z onto Cy. From a notation that is frequently employed in the theory of elasticity, one can write:

$$p_{yz} - p_{zy}\,, \qquad p_{zx} - p_{xz}\,, \qquad p_{xy} - p_{yx}$$

for the projections of the vector $\frac{1}{H}\sum_i (\mathbf{v}_i \wedge \mathrm{grad}\, W_{\mathbf{v}_i})$.

32. Formulas of E. and F. Cosserat that are deduced from the preceding vector relations. Problems that are posed by the theory of the Euclidian action. – In the formulas that give $\mathcal{E}$ and $\mathcal{M}$, if one sets $c_1 = 1$, $c_2 = c_3 = 0$, for example, then one sees that $\mathcal{E}$ reduces to p_x, so the latter vector is the effort per unit area on an element whose exterior semi-normal is directed along Ox. Likewise, q_x is the moment of deformation per unit area on the same element; p_y, p_z, and q_y, q_z are similarly interpreted.

In order to recover the Cosserat equations from the preceding ones, it suffices to project the equations of the system onto the three coordinate axes.

One thus has:

$$\begin{aligned}
\mathcal{E}_x &= c_1\, p_{xx} + c_2\, p_{yx} + c_3\, p_{zx}\,, \\
\mathcal{E}_y &= c_1\, p_{xy} + c_2\, p_{yy} + c_3\, p_{zy}\,, \\
\mathcal{E}_z &= c_1\, p_{xz} + c_2\, p_{yz} + c_3\, p_{zz}\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_x &= c_1\, q_{xx} + c_2\, q_{yx} + c_3\, q_{zx}\,, \\
\mathcal{M}_y &= c_1\, q_{xy} + c_2\, q_{yy} + c_3\, q_{zz}\,, \\
\mathcal{M}_z &= c_1\, q_{xz} + c_2\, q_{yz} + c_3\, z_{zz}\,,
\end{aligned}$$

$$\begin{aligned}
\varphi_x &= \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\,, \\
\varphi_y &= \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\,, \\
\varphi_z &= \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\,,
\end{aligned}$$

$$\mu_x = \frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z},$$
$$\mu_y = \frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z},$$
$$\mu_z = \frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z}.$$

These formulas were given for the first time, and in this latter form, by the Cosserats in their *Théorie des corps déformables*. Formerly, Voigt made known some formulas that corresponded to a particular case, namely, the one in which the three vectors q_x, q_y, q_z are zero at all points ([1]).

We repeat some of the inumerable remarks of the creators of the *Théorie* that show how one deduces from the preceding results the equations that relate to the deformable body of the strength of material, the ones concerning the perfect fluid medium, and the theory of ethereal media that one imagines for the study of luminous waves, from McCullagh to Lord Kelvin. These same results lead naturally to the consideration of the vector $\mathcal{B}$ (viz., induction) in a magnetized medium, etc.

We think that, when one is endowed with the precision that we have tried to invest the principles with, the reader will easily approach the original work of E. and F. Cosserat, notably the chapters that we were unable to develop for lack of space ([2]), namely:

1. *The study of the deformable body in motion.* One will ultimately follow the path that was presented in the context of the deformable line ([3]) in motion. One finds, as a particular case, the gyrostatic medium that L. Kelvin imagined in order to account for the elastic properties of the luminiferous ether. In addition, the notion of kinetic anisotropy that we spoke of for a deformable curve served as the basis for the theory of double refraction of light, such as Lord Kelvin and Glazebrook have discussed.

2. *Study of the Euclidian action at a distance, the action of constraint, and the dissipative action.*

3. Finally, the last chapter is very useful for researchers: *the Euclidian action from the Eulerian viewpoint.*

In the preceding presentation, one took the variables to be the initial coordinates of a point and time (i.e., Lagrange variables), but one can also take the current coordinates of the point and time (i.e., Euler variables). One is then led to the Eulerian action that was

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([1]) "Theoretischen Studien über die Elasticitätsverhältnisse der Krystalle."
One will find a very complete bibliography, notably for the case of the deformable surface and deformable media, in the *Théorie* of E. and F. Cosserat.
([2]) The original presentation is already extremely condensed.
([3]) P. LANGEVIN, "Aspect général de la relativité," Bull. scient. des Étudiants de Paris.

imagined by H. Poincaré ([1]), and one arrives at conclusions that are analogous to the ones that Lorentz stated in the context of the theory of electromagnetism in a moving body. It is pointless to emphasize the importance of that assertion.

([1]) "Sur la Dynamique de l'électron," Circolo Matematico di Palermo, t. XXI, pp. 129.