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The crystal-optical theory of relativity, as it relates to the geometry of bi-quadratic forms ([1])

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§ 1. Introduction and brief summary

In collaboration with prof. L. I. Mandelstam, I arrived at the invariant equations of the electrodynamics of anisotropic media ([2]). In that article, they are used to determine the laws of propagation of light in crystals. These formulas are quite analogous to the known equations for determining the propagation of light in gravitational fields *in vacuo*. That analogy permits one to give the law of crystal optics a geometrical interpretation, which amounts to the statement that in a material medium, as well as in a vacuum, light propagates along null lines ($ds = 0$). In that case, one must determine the element of length by means of the following formula:

$$ds^4 = h^{ijhk}\, dx_i\, dx_j\, dx_h\, dx_k = h_{ijhk}\, dx^i\, dx^j\, dx^h\, dx^k, \tag{1}$$

in which h^{ijhk} (h_{ijhk}, resp.) is a tensor *of rank four* whose components are functions of the magnetic permeability $\mu_{\alpha\beta}$ and the dielectric constants $\varepsilon_{\alpha\beta}$. In other words, in an anisotropic medium, one must assume that the element of length is determined by a *bi-quadratic* form in the coordinate differentials, while in a vacuum and in isotropic media, it will be a quadratic form.

Physically, a bi-quadratic element of length corresponds to the presence of double refraction. Thus, in anisotropic bodies, the Riemannian geometry of quadratic forms is no longer valid, but one must deal with the more general geometry of bi-quadratic forms.

([1]) Zh. R. F. Kh. O., v. **3-4** (1925), 1.
([2]) Zh. R. F. Kh. O., **56** (1924), 248. (cf., esp. volume 1, first ed.)

Further development of the aforementioned theory of crystal-optical relativity will allow us to elucidate some of the features of that geometry. For example, the major role that is played by the (metric) tensor g_{ij} in Riemannian geometry gets distributed among the three tensors h_{ijhk}, k_{ij}, and s_{ijhk} in bi-quadratic geometry. The first of them plays the role of the metric tensor, which defines an element of length, while the tensor k_{ij} of "refraction indices" establishes a connection between the covariant and contravariant components of the same tensor. Finally, the tensor s_{ijhk} establishes a connection between the fundamental electromagnetic field tensors F_{ij} and f^{ij} (cf., § 2). In isotropic media, bi-quadratic geometry goes to quadratic, and these three aforementioned tensors will become identical with each other.

In anisotropic media, there is a difference between the propagation of light waves and the propagation of light rays. That fact is in apparent contradiction with the reality that in the "natural" bi-quadratic geometry of anisotropic media both of these phenomena are described by the same equation $ds = 0$. The significance of that distinction lies in the fact that the difference between the speed of the wave and the speed of the ray can be irrelevant for a Cartesian interpretation of arbitrary Riemannian (i.e., non-Cartesian) coordinates, which is analogous to the way that one explains the bending of a light ray in a gravitational field by means of a similar interpretation of Riemannian coordinates, moreover. For a more detailed explanation of that topic, § 8 is devoted to the consideration of the optical anisotropy of gravitational fields *in vacuo*.

The present theory has a macroscopic character. A discussion of its relationship to a deeper physical interpretation in terms of the microscopic (electron) theory of a material medium will be postponed to a later article.

The present article is an attempt to sketch out a path to the construction of a general theory of geometrical optics that would admit one to assume a single viewpoint on the laws of propagation for light in various media (gravitational fields *in vacuo*, material media, both isotropic and anisotropic, homogeneous and inhomogeneous).

§ 2. Basic equations of electrodynamics for anisotropic media

In this article, we shall restrict ourselves to the consideration of dielectric media, and in addition, assume that the medium has three mutually-perpendicular principal axes of anisotropy. (This is true in all crystalline systems, except for monoclinic and triclinic ones.) ([3])

Electromagnetic fields in dielectrics are determined by anti-symmetric tensors F_{ij} and f^{ij}, whose components have the following physical interpretation:

$$\begin{aligned} &(F_{14}, F_{24}, F_{34}) = (E_1, E_2, E_3), && (F_{23}, F_{31}, F_{12}) = (B_1, B_2, B_3), \\ &(f^{14}, f^{24}, f^{34}) = (-D_1, -D_2, -D_3), && (f^{23}, f^{31}, f^{12}) = (H_1, H_2, H_3). \end{aligned} \tag{2}$$

([3]) This assumption is introduced only for the sake of simplifying the calculations. The main results of this article – in particular, all of § 6 – will remain valid in the general case. (For the importance of the components of the tensor s that relates to that case, cf., the article that was cited above.)

Here, E_α, H_α, D_α, and B_α are the components of the usual three-dimensional electromagnetic vectors (electric force and magnetic field, electric displacement and magnetic induction, resp.).

The tensors F_{ij} and f^{ij} must satisfy three fundamental equations. The first of them establishes a connection between F_{ij} and f^{ij}. That equation was derived by me in the aforementioned article and has the following form:

$$f^{ij} = s^{ijpq} F_{pq} , \tag{3}$$

or the corresponding form that relates to F_{ij}:

$$F_{ij} = s_{ijpq} f^{pq}. \tag{3a}$$

Here, s_{ijpq} and s^{ijpq} are two tensors of rank four, or rather, two general forms (covariant and contravariant, resp.) of the same tensor of "dielectricity and magnetic permeability." The following relations exist between them:

$$s^{ijpq} s_{hkpq} = \delta_h^i \delta_k^j , \tag{4}$$

in which $\delta_h^i = 1$ when $i = h$ and $\delta_h^i = 0$ when $i \neq h$.

In the sequel, we shall often have to use Cartesian coordinate systems that are at rest with respect to the region of the dielectric considered and whose axes coincide with its principal axes of anisotropy. We shall refer to these coordinate systems as *A* systems, for the sake of brevity.

One can show (*loc. cit.*) that in *A* systems the only components of the tensor s^{ijpq} (s_{ijpq}, resp.) that are non-zero are the ones for which $i = p$ and $j = q$. When viewed in an *A* system, the values of the components of the tensor s^{ijpq} can be displayed in the form of a square matrix:

$$s^{ijpq} = \begin{bmatrix} \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_3} & \dfrac{1}{\mu_2} & -\varepsilon_1 \\ \dfrac{1}{\mu_3} & \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_1} & -\varepsilon_2 \\ \dfrac{1}{\mu_2} & \dfrac{1}{\mu_1} & \left[\dfrac{1}{\mu}\right] & -\varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & [\varepsilon^2\mu] \end{bmatrix}. \tag{5}$$

The elements of this matrix that are found at the locations (1, 1), (1, 2), (1, 3), … are equal to the components s^{1111}, s^{1212}, s^{1313}, …, respectively. I was not able to determine the meaning of the elements of the main diagonal ([4]), and the respective locations of these elements in the matrix contain only the dimensions of the elements. As was pointed out before, all of the components of s^{ijpq} that are not included in the matrix are zero.

([4]) These elements will drop out of the equations that we shall have to use in what follows.

By means of equation (4), it is easy to see that the following relation is valid for all non-zero components:

$$s_{ijpq} = 1 / s^{ijpq} \tag{6}$$

in an *A* system.

In *isotropic* media, the tensor s^{ijpq} of rank four (s_{ijpq}, resp.) will reduce to the square of the tensor s^{ip} (s_{ip}, resp.) of rank two. In other words:

$$s^{ijpq} = s^{ip}\, s^{jq}, \quad s_{ijpq} = s_{ip}\, s_{jq}, \tag{7}$$

and in the rest system of Cartesian coordinates, the components s^{ip} will have the following interpretations:

$$s^{ip} = \begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mu}} & 0 \\ 0 & 0 & 0 & -\varepsilon\sqrt{\mu} \end{bmatrix}. \tag{5a}$$

Equations (4) and (6) reduce to analogous formulas in isotropic media, namely:

$$s^{ip}\, s_{iq} = \delta_q^p \tag{4a}$$

and (in a stationary Cartesian system):

$$s_{ij} = 1 / s^{ij}. \tag{6a}$$

We now turn to the other two equations of electromagnetic fields. In the special theory of relativity, they have the following form:

$$\text{rot } F_{ij} = \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{jk}}{\partial x^i} = 0, \tag{8}$$

$$\text{div } f^{ij} = \frac{\partial f^{ij}}{\partial x^j} = 0. \tag{9}$$

The first of these equations is invariant with respect to any coordinate transformation. However, in the general theory of relativity, the second one is replaced with the following tensor equation:

$$\frac{\partial \sqrt{-g}\, f^{ij}}{\partial x^j} = 0. \tag{10}$$

However, we have no right to introduce the "microscopic" tensor g_{ij} into the development of our macroscopic theory, so we shall deal with only the average, collective values of the physical magnitudes. We can give equation (9) an invariant form without having to resort to the use of the gravitational potentials g_{ij}. Indeed, $\sqrt{-g}$ transforms by the formula:

$$\sqrt{-g} = J\sqrt{-g'},$$

in which J is the Jacobian of the transformation:

$$J = \frac{\partial(x^{1'}, x^{2'}, x^{3'}, x^{4'})}{\partial(x^1, x^2, x^3, x^4)}. \tag{11}$$

On the other hand, $\sqrt{-g'} = 1$ in a Cartesian coordinate system. Therefore, if we are to understand J to mean *the Jacobian of the transformation that takes the coordinate system* x^1, x^2, x^3, x^4 to the Cartesian (primed) system then we will get $\sqrt{-g} = J$, and the invariant equations (10) will take the form:

$$\partial J f^{ij} / \partial x^j = 0. \tag{10a}$$

We can transfer this equation directly to our macroscopic theory in this form ([5]). Later on, we shall give this equation a slightly more convenient form.

The set of equations (3) [or (3a)], (8), and (10a) represents the total system of equations for electromagnetic fields in dielectrics. It can be simplified by introducing the tensor potentials Φ_i into consideration and setting:

$$F_{ij} = \frac{\partial \Phi_i}{\partial x^j} - \frac{\partial \Phi_j}{\partial x^i}. \tag{12}$$

Equation (8) would then be satisfied identically, and f^{ij} would be defined by equation (3):

$$f^{ij} = s^{ijpq}\left(\frac{\partial \Phi_i}{\partial x^j} - \frac{\partial \Phi_j}{\partial x^i}\right). \tag{13}$$

§ 3. Electromagnetic fields of light waves

The electromagnetic fields of light waves are characterized by the fact that their tensor-potentials are periodic functions of the coordinates (viz., space and time); i.e., that:

([5]) In the terminology of H. Weyl, the quantity $J f^{ij} = \mathfrak{f}^{ij}$ is called a "tensor density." These arguments boil down to the fact that the invariant divergence of the tensor density $\mathfrak{f}^{ij}$ should be equal to zero. We determine the value of $\mathfrak{f}^{ij}$ from the condition that its components in a Cartesian coordinate system must be equal to the components of the tensor f^{ij}.

$$\Phi_i = A_i \; e^{\sqrt{-1}Q}, \tag{14}$$

where Q is a scalar function of the coordinates.

Strictly speaking, the functions Φ_i can be called periodic only under the condition that the components of the tensor A_i must have a constant value (in space and time). However, that condition is not invariant under coordinate transformations and must be replaced with the requirement that the derivatives of A_i must be small in comparison to the derivatives of Q.

In the present article, we shall confine ourselves to geometrical optics. That means that we shall exclude from consideration, first of all, the *dispersion* of light and secondly its *diffraction*. The first exclusion is achieved by assuming that the light ray is monochromatic. (For a given frequency, ε_α and μ_α will then be well-defined functions of the coordinates.) The influence of diffraction can be neglected only if the medium is not rapidly-varying; i.e., only if the properties of the medium do not change appreciably at distances that are comparable to the wavelength (and for a period of time that is comparable to the period of oscillation). In other words, it is necessary that the derivatives of s^{ijpq} must be small in comparison to the derivatives of Q.

Finally, it is necessary to impose the restriction on the derivatives of the Jacobian J that obviously amounts to excluding those coordinate systems in which $\sqrt{-g}$ varies significantly over distances that are comparable to the wavelength of the light. This restriction is rooted in the very essence of all *macroscopic* theories.

As described above in regard to the differentials of the potentials Φ_i, the terms that contain the derivatives of A_i can be neglected in comparison to the terms that contain the derivatives of Q. Therefore, equation (12) will take on the form:

$$F_{ij} = \sqrt{-1}\;(A_i\, Q_j - A_j\, Q_i)\; e^{\sqrt{-1}Q}, \tag{15}$$

in which we have introduced the notation $Q_i = \partial Q / \partial x^i$. Moreover, we will get:

$$f^{ij} = s^{ijpq}\;\sqrt{-1}\;(A_i\, Q_j - A_j\, Q_i)\; e^{\sqrt{-1}Q} \tag{16}$$

from equation (13).

Turning now to equation (10a), and differentiating the expression $J f^{ij}$, we must once more confine ourselves to terms that contain the highest power of Q (in the present case, the second power) ([6]). It is obvious that thanks to the wave factor $e^{\sqrt{-1}Q}$, differentiating with respect to x^j will reduce to multiplication by $\sqrt{-1}\,Q_j$ in this case. Hence:

$$\frac{\partial\, J\, f^{ij}}{\partial x^j} = \sqrt{-1}\; J f^{ij}\, Q_j = 0,$$

or:

([6]) When differentiating, it is assumed that Q is constant to a first approximation; i.e., that Q is a linear function of the coordinates. That assumption can always be satisfied is one restricts oneself to a sufficiently small coordinate change. The laws of geometrical optics are applicable only within that domain; outside of it, the period of oscillation will vary as the wave propagates.

$$f^{ij} Q_j = 0. \tag{17}$$

Equation (17) can be regarded as a set of four homogeneous, linear equations in the four unknowns Q_j . In order for these equations to have non-zero solutions, it is necessary that the determinant of the coefficients must be zero ([7]):

$$| f^{ij} | = 0.$$

It is known that the determinant $| f^{ij} |$, which is composed of the components of the anti-symmetric tensor of rank two, is expressed as follows:

$$| f^{ij} | = | f^{12} f^{34} + f^{13} f^{42} + f^{14} f^{23} |^4.$$

Hence:

$$f^{12} f^{34} + f^{13} f^{42} + f^{14} f^{23} = 0.$$

One can express f^{ij} in terms of the components of the vectors **H** and **D** [cf., equation (2)]:

$$H_1 D_1 + H_2 D_2 + H_3 D_3 = 0. \tag{18}$$

We then have the:

Theorem: *In any coordinate system in which we measure the electromagnetic field of a light wave, we can combine the magnetic field vector* **H** *and the electric induction vector* **D** *in such a way that the relation* (18) *will always be valid, which is equivalent to the condition that those vectors must be perpendicular in a Cartesian coordinate system. It is assumed that the values of the constituent vectors* **H** *and* **D** *are determined from the tensor* f^{ij} *by using formula* (2).

Let us go back to equation (17). It is easy to show that only three of the four equations are independent of each other. We shall show, first of all, that the equations $f^{ij} Q_j = 0$ is a consequence of the equations $f^{1j} Q_j = 0$, $f^{2j} Q_j = 0$, and $f^{3j} Q_j = 0$. For the sake of brevity, in the sequel we shall use Greek letters to denote indices that take on one of the three values 1, 2, 3. By contrast, indices that take on any of the four values 1, 2, 3, 4 will be denoted by Latin letters. In order to prove that the equation $f^{4j} Q_j = 0$ is a consequence of the equations $f^{\alpha j} Q_j = 0$, multiply them by Q_α , respectively, and add them. The result of that will be:

$$f^{\alpha i} Q_i Q_\alpha = f^{\alpha\beta} Q_\beta Q_\alpha + f^{\alpha 4} Q_4 Q_\alpha = 0.$$

The first sum is equal to zero, since $f^{\alpha\beta} = - f^{\alpha\beta}$, while $Q_\alpha Q_\beta = Q_\beta Q_\alpha$. Therefore, $f^{\alpha 4} Q_4 Q_\alpha = 0$. If one cancels Q_4 and permutes the indices of $f^{\alpha 4}$ then that will give $f^{4\alpha} Q_\alpha =$

([7]) This theorem is taken from the theory of linear equations. It will remain valid for the case in which the coefficients f^{ij}, along with the unknowns Q_j , are themselves unknown functions [cf., equation (16)].

0. However, $f^{4i} Q_i = f^{4\alpha} Q_\alpha + f^{44} Q_4 = f^{4i} Q_i$, because $f^{44} = 0$. Therefore, $f^{4i} Q_i = 0$, which was to be proved ([8]).

Thus, all of the restrictions that are imposed upon the equations of electrodynamics regarding the values of the potentials of the fields of light waves will reduce to the three equations ([9]):

$$f^{\alpha i} Q_i = 0 \qquad (\alpha = 1, 2, 3). \tag{19}$$

Inserting the values of $f^{\alpha i}$ into equation (16) and cancelling $e^{\sqrt{-1}Q}$ will give:

$$s^{\alpha ipq} (A_p Q_q - A_q Q_p) Q_i = 0 \quad (\alpha = 1, 2, 3). \tag{20}$$

§ 4. Fresnel equations

In this and subsequent paragraphs, we shall restrict ourselves to the consideration of *A* systems (viz., stationary Cartesian coordinate systems that are parallel to the principal axes of anisotropy). Due to the fact that in *A* systems only those components of the tensor s^{ijpq} for which one simultaneously has $i = p$ and $j = q$ (cf., § 2) will be non-zero, equation (20) will take the following simple form:

$$\sum_i s^{\alpha i\alpha j} (A_\alpha Q_i - A_i Q_\alpha) Q_i = 0. \tag{21}$$

There is obviously no need to sum these equations over the index α.

In what follows, we shall write the summation sign with the index that is being summed over in those and only those cases in which there is some departure from the general rule of summation over the same pairs of indices.

In order to simplify the calculations, we shall replace the amplitudes of the potentials A_i with new unknowns B_i that are defined by the equations:

$$A_i = \frac{A_4}{Q_4} Q_i + B_i . \tag{22}$$

It is clear from this that $B_4 = 0$. On the other hand, all B_i cannot be zero simultaneously, since otherwise A_i would be proportional to Q_i, and according to equation (15), all F_{ij} would then be equal to zero: i.e., there would be no electromagnetic field. Thus, the inequality:

$$B_\alpha \neq 0 \tag{22a}$$

will be true for at least one of the indices α.

From equation (22), we should have that:

([8]) $Q_4 = \partial Q / \partial x^4$ cannot be zero, since we are considering electromagnetic fields of *light waves*, and therefore the factor $e^{\sqrt{-1}Q}$ cannot be independent of time.

([9]) Of course, one could choose any other three of the four equations (17).

$$A_\alpha Q_i - A_i\, Q_\alpha = B_\alpha Q_i - B_i\, Q_\alpha\,.$$

Substituting this expression into equation (21) will give:

$$\sum_i s^{\alpha i \alpha i}(B_\alpha Q_i - B_i Q_\alpha)\, Q_i = 0,$$

or, since $B_4 = 0$:

$$\sum_\beta s^{\alpha\beta\alpha\beta}(B_\alpha Q_\beta - B_\beta Q_\alpha)\, Q_\beta + s^{\alpha 4 \alpha 4} B_\alpha Q_4^2 = 0. \tag{23}$$

It is easy to ascertain the physical interpretation of the quantities B_α. From (22), one will have that $Q_4\, B_\alpha = A_\alpha\, Q_4 - A_4\, Q_\alpha$. If we compare these expressions with equation (15) then, on the basis of equation (2), that will give:

$$B_1 : B_2 : B_3 = F_{14} : F_{24} : F_{34} = E_1 : E_2 : E_3\,. \tag{24}$$

Thus, B_α are proportional to the direction cosines of the electric force vector **E**.

It is just as easy to ascertain the physical interpretation of the quantity Q. In a Cartesian coordinate system of the kind that we are considering in this paragraph, the factor $e^{\sqrt{-1}Q}$ will take the form:

$$\exp\left[\sqrt{-1}\,\frac{2\pi}{\lambda}(wt - m_\alpha x^\alpha + \varphi)\right],$$

in which w is the speed of light, λ is the wave length, and m_α are the direction cosines of the wave normal, which are related by the condition that $\sum_\alpha m_\alpha^2 = 1$.

Therefore:

$$Q_1 : Q_2 : Q_3 : Q_4 = m_1 : m_2 : m_3 : -\, w. \tag{25}$$

Here, as in all of what follows, the unit of speed that will be adopted is the speed of light *in vacuo*, such that:

$$Q_4 = \frac{\partial Q}{\partial x^4} = \frac{\partial Q}{\partial t} \qquad \left(\text{and not } \frac{1}{c}\frac{\partial Q}{\partial t}\right).$$

Before we go onto the solution of a more general problem, we show that the well-known Fresnel equation, which determines the speed of propagation of light waves in crystals, represents the special case of equations (23) for which $\mu = 1$. In fact, in that case, all $s^{\alpha\beta\alpha\beta} = 1$ and $s^{\alpha 4 \alpha 4} = -\,\varepsilon_\alpha$ [cf., equation (5)]. If one substitute this expression into equation (23), and in addition replaces Q_i with m_α and w, then from (25) that will give:

$$B_\alpha \sum_\beta m_\beta^2 - m_\alpha \sum_\beta B_\beta m_\beta - \varepsilon_\alpha B_\alpha w^2 = 0. \tag{26}$$

If one multiplies these equations by m_α, respectively, and sums over α then one will verify that the first two sums cancel each other, and therefore:

$$\sum_\alpha \varepsilon_\alpha B_\alpha m_\alpha = 0. \tag{27}$$

By dint of equation (24), it is easy to see that this formula expresses the idea that the displacement vector **D** is perpendicular to the direction of the wave normal. On the other hand, from (26) and the fact that $\sum_\beta m_\beta^2 = 1$, one will have:

$$B_\alpha = \frac{m_\alpha \sum_\beta B_\beta m_\beta}{1 - \varepsilon_\alpha w^2} .$$

If one multiplies this by $m_\alpha\, \varepsilon_\alpha$, sums over α, and takes equation (27) into account, moreover, then that will give:

$$\sum_\alpha m_\alpha \varepsilon_\alpha B_\alpha = \sum_\alpha \frac{m_\alpha^2 \varepsilon_\alpha \sum_\beta B_\beta m_\beta}{1 - \varepsilon_\alpha w^2} = 0,$$

or, when one cancels $\sum_\beta m_\beta B_\beta$:

$$\sum_\alpha \frac{m_\alpha^2}{\dfrac{1}{\varepsilon_\alpha} - w^2} = 0. \tag{28}$$

This expression is nothing but the one that is known as *Fresnel's equation*.

§ 5. General equations of light wave propagation

Fresnel's equation is valid only in *A* coordinate systems, and then only in the case of non-magnetic crystals ($\mu = 1$). In order to find the general equation of propagation for light waves, one must analyze the case of $\mu \neq 1$ using the former *A* system and then show that the equation that is obtained will be invariant with respect to any coordinate transformation.

Let us return to equation (23). That equation can be put into the following form:

$$B_\alpha \left(\sum_\beta s^{\alpha\beta\alpha\beta} Q_\beta^2 + s^{\alpha 4 \alpha 4} Q_4^2 \right) - \sum_\beta B_\beta s^{\alpha\beta\alpha\beta} Q_\alpha Q_\beta = 0,$$

or

$$\sum_\beta B_\beta p^{\alpha\beta} = 0, \tag{29}$$

in which it easy to see that the coefficients $p^{\alpha\beta}$ are the following expressions: $p^{\alpha\beta} = -s^{\alpha\beta\alpha\beta} Q_\alpha Q_\beta$, in which $\alpha \neq \beta$ (there is no need to sum over α and β) and:

$$p^{\alpha\alpha} = \sum_\beta s^{\alpha\beta\alpha\beta} Q_\beta^2 + s^{\alpha 4\alpha 4} Q_4^2 - s^{\alpha\alpha\alpha\alpha} Q_\alpha^2 .$$

Equations (29) represent three linear, homogeneous equations in the three unknowns B_β. Since all three of the quantities B_β cannot be simultaneously zero [cf., equation (22a)], the determinant that is composed of the coefficients $p^{\alpha\beta}$ must be equal to zero: $| p^{\alpha\beta} | = 0$.

In order to simplify the calculations, we represent the elements of the determinant of $p^{\alpha\beta}$ as sums of elements of two other determinants $u^{\alpha\beta}$ and $v^{\alpha\beta}$, namely:

$$u^{\alpha\beta} = - s^{\alpha\beta\alpha\beta} Q_\alpha Q_\beta + \delta_\alpha^\beta \sum_\gamma s^{\alpha\gamma\alpha\gamma} Q_\gamma^2 , \qquad v^{\alpha\beta} = \delta_\alpha^\beta s^{\alpha 4\alpha 4} Q_4^2 .$$

Hence: $| u^{\alpha\beta} + v^{\alpha\beta} | = | p^{\alpha\beta} | = 0$.

Calculating the determinant $| u^{\alpha\beta} + v^{\alpha\beta} |$ will present no difficulties if we take into account the facts that the in the determinant $| v^{\alpha\beta} |$, only elements of the main diagonal are non-zero and, on the other hand, that the determinant $| u^{\alpha\beta} |$ is zero. It now becomes clear from the fact that if the columns of the determinant $| u^{\alpha\beta} |$ are multiplied by Q_β, respectively, and then added then the result will be that all of the rows will become zero. In fact:

$$\sum_\beta u^{\alpha\beta} Q_\beta = - \sum_\beta s^{\alpha\beta\alpha\beta} Q_\alpha Q_\beta^2 + \sum_\beta \delta_\alpha^\beta Q_\beta \sum_\gamma s^{\alpha\gamma\alpha\gamma} Q_\gamma^2 ,$$

or

$$\sum_\beta u^{\alpha\beta} Q_\beta = - \sum_\beta s^{\alpha\gamma\alpha\gamma} Q_\alpha Q_\gamma^2 + Q_\alpha \sum_\gamma s^{\alpha\gamma\alpha\gamma} Q_\gamma^2 = 0.$$

If one takes advantage of these properties of the determinants $| u^{\alpha\beta} |$ and $| v^{\alpha\beta} |$ and expands the determinant $| u^{\alpha\beta} + v^{\alpha\beta} |$ according to certain rules into sums of eight determinants that are composed of elements of $u^{\alpha\beta}$ and $v^{\alpha\beta}$ then that will give:

$$\begin{aligned} | p^{\alpha\beta} | &= | u^{\alpha\beta} + v^{\alpha\beta} | \\ &= v^{11} (u^{22} u^{33} - u^{23} u^{32}) + v^{22} (u^{33} u^{11} - u^{31} u^{13}) + v^{33} (u^{11} u^{22} - u^{12} u^{21}) \\ &\quad + v^{11} v^{22} u^{33} + v^{11} u^{22} v^{33} + u^{11} v^{22} v^{33} = 0. \end{aligned}$$

We insert the values of $u^{\alpha\beta}$ and $v^{\alpha\beta}$ into this and express the quantities $s^{\alpha i \alpha i}$ in terms of μ_α and ε_α using equation (5), and finally divide the equations by the common non-zero factor Q_4^2. The result of that will give the following equation:

$$\mathrm{S}\,Q_1^4\frac{\varepsilon_1}{\mu_2\mu_3}+\mathrm{S}\,Q_1^2Q_3^2\left(\frac{\varepsilon_1}{\mu_1}+\frac{\varepsilon_2}{\mu_2}\right)-Q_2^4\,\mathrm{S}\,Q_1^2\varepsilon_1\left(\frac{\varepsilon_2}{\mu_2}+\frac{\varepsilon_3}{\mu_3}\right)+Q_4^4\varepsilon_1\varepsilon_2\varepsilon_3=0, \tag{30}$$

in which the symbol S means a sum of elements that have been subjected to a cyclic permutation of the indices 1, 2, 3. This equation represents a generalization of Fresnel's formula (28) to the case of magnetic crystals ($\mu \neq 1$). Indeed, if one expresses Q_i in terms of m_α and w using equation (25) then one will get an equation that determines the speed w of light waves in terms of the direction of (m_α). That equation is equal to the corresponding formula that was derived Heaviside by an argument of an entirely different character ([10]). Equation (30) will reduce to Fresnel's equation (28) when $\mu = 1$.

Equation (30) can be put into the following form, which will be more convenient for our later discussions:

$$\sum_{i,j,h,k} h^{ijhk} Q_i\, Q_j\, Q_h\, Q_k = 0. \tag{30a}$$

If one compares equation (30) with equation (30a) then one will see that of the 256 coefficients h^{ijhk}, the only non-zero ones are the ones for which one has both $i = h$ and $j = k$. Their values are written most conveniently in the form of the symbolic quadratic matrix (5):

$$h^{ijhk}=\begin{bmatrix}
\dfrac{\varepsilon_1}{\mu_2\mu_3} & \dfrac{1}{2\mu_3}\left(\dfrac{\varepsilon_1}{\mu_1}+\dfrac{\varepsilon_2}{\mu_2}\right) & \dfrac{1}{2\mu_2}\left(\dfrac{\varepsilon_3}{\mu_3}+\dfrac{\varepsilon_1}{\mu_1}\right) & -\dfrac{\varepsilon_1}{2}\left(\dfrac{\varepsilon_2}{\mu_2}+\dfrac{\varepsilon_3}{\mu_3}\right)\\
\dfrac{1}{2\mu_3}\left(\dfrac{\varepsilon_1}{\mu_1}+\dfrac{\varepsilon_2}{\mu_2}\right) & \dfrac{\varepsilon_2}{\mu_3\mu_1} & \dfrac{1}{2\mu_1}\left(\dfrac{\varepsilon_2}{\mu_2}+\dfrac{\varepsilon_3}{\mu_3}\right) & -\dfrac{\varepsilon_2}{2}\left(\dfrac{\varepsilon_3}{\mu_3}+\dfrac{\varepsilon_1}{\mu_1}\right)\\
\dfrac{1}{2\mu_2}\left(\dfrac{\varepsilon_3}{\mu_3}+\dfrac{\varepsilon_1}{\mu_1}\right) & \dfrac{1}{2\mu_1}\left(\dfrac{\varepsilon_2}{\mu_2}+\dfrac{\varepsilon_3}{\mu_3}\right) & \dfrac{\varepsilon_3}{\mu_1\mu_2} & -\dfrac{\varepsilon_3}{2}\left(\dfrac{\varepsilon_1}{\mu_1}+\dfrac{\varepsilon_2}{\mu_2}\right)\\
-\dfrac{\varepsilon_1}{2}\left(\dfrac{\varepsilon_2}{\mu_2}+\dfrac{\varepsilon_3}{\mu_3}\right) & -\dfrac{\varepsilon_2}{2}\left(\dfrac{\varepsilon_3}{\mu_3}+\dfrac{\varepsilon_1}{\mu_1}\right) & -\dfrac{\varepsilon_3}{2}\left(\dfrac{\varepsilon_1}{\mu_1}+\dfrac{\varepsilon_2}{\mu_2}\right) & \varepsilon_1\varepsilon_2\varepsilon_3
\end{bmatrix}. \tag{31}$$

The elements of this matrix, when referred to as (1, 1), (1, 2), (1, 3), … are equal to the coefficients h^{1111}, h^{1212}, h^{1313}, …, respectively. All of the elements of the matrix of coefficients h^{ijhk} that were not included are equal to zero.

We used an A coordinate system for the derivation of equation (30a). However, it is obvious that the equation will preserve its form under the transition to any other coordinate system as along as we also transform the coefficients h^{ijhk} according to the rules of transformation for the components of a contravariant tensor of rank four ([11]) at

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([10]) O. Heaviside, *Electromagnetic Theory*, v. II, London, Benn, 1922, pp. 522, equation (6).

([11]) If we denote the values of quantities that are measured in the transformed coordinate system by primes (x'^i) then we will have $Q_i = \dfrac{\partial x'^p}{\partial x^i} Q'_p$.

Substituting this into equation (30a) will give:

the same time that we transform the coordinates. (Needless to say, the components of the covariant tensor $Q_i = \partial Q / \partial x^i$ must also transform according to the rules of tensor calculus under it.) In other words, equations (30a) will lead to the correct dependency between the components of the tensor Q_i in any coordinate system, as long as we understand that h^{ijhk} are the components of a *fourth-rank tensor* whose values are determined by equation (31) in *an A system*. Thus, in order to define the tensor h^{ijhk}, we shall use an *A* system as a "natural" coordinate system.

However, it is possible to determine the values of the tensor h^{ijhk} in terms of the fundamental tensor s^{ijpq} without referring to the original *A* system. To that end, we introduce the unit anti-symmetric tensor e_{ijhk} of rank four. The components of that tensor are non-zero only if all four indices are different. Its components are equal to + 1 or – 1 depending upon whether the given sequence of indices *i*, *j*, *h*, *k* is obtained from the normal sequence 1, 2, 3, 4 by an even or odd number of transpositions, respectively.

The relationship between the tensor h^{ijhk} and s^{ijpq} can be expressed in the following form by means of the tensor e_{ijhk} :

$$h^{qrst} = -\frac{1}{6}(s^{qisj}\, s^{rhkl} - s^{qsij}\, s^{rhkl} + s^{qrik}\, s^{shjl})\, s^{imnp}\, e_{ihkm}\, e_{jlnp}\,. \tag{32}$$

The validity of this formula in an *A* system can be verified by directly substituting the values of the components h^{ijhk} and s^{ijpq} into equations (5) and (31) ([12]). Since both sides of equation (32) are tensors, formula (32) will remain valid under any coordinate transformations. Thus, the values of the tensor h^{ijhk} are determined from the values of the tensor s^{ijpq} completely in any coordinate system.

Strictly speaking, the last statement will be true only if we modify equation (32) slightly. The fact is that the values of the components of the tensor e_{ijhk} are not invariant: Under a coordinate transformation, their numerical values will be divided by *J*, where *J* is the Jacobian of the transformation (11):

$$\varepsilon'_{ijhk} = \frac{1}{J}\, e_{ijhk}\,. \tag{33}$$

$$h^{ijhk}\,\frac{\partial x'^p}{\partial x^i}\frac{\partial x'^q}{\partial x^j}\frac{\partial x'^r}{\partial x^h}\frac{\partial x'^s}{\partial x^k}\,Q'_pQ'_qQ'_rQ'_s = 0$$

or

$$h'^{ijhk}\,Q'_pQ'_qQ'_rQ'_s = 0, \tag{30b}$$

in which:

$$h'^{ijhk} = h^{ijhk}\,\frac{\partial x'^p}{\partial x^i}\frac{\partial x'^q}{\partial x^j}\frac{\partial x'^r}{\partial x^h}\frac{\partial x'^s}{\partial x^k}$$

are the expressions for the *h* coefficients in the primed coordinate system.

([12]) In equation (5), the elements s_{iiii} of the main diagonal were not defined, but in equation (32), all elements of that kind cancelled each other out.

Therefore, formula (32) implicitly refers to the original A system: It is assumed that the non-zero components of e_{ijhk} are equal to ± 1 in this system. In order to avoid such a reference to the original system, in the general theory of relativity, the components e_{ijhk} are usually defined by saying that the non-zero components of the fundamental tensor e_{ijhk} are equal to $\pm\sqrt{-g}$, where g is the determinant that is formed from the covariant components g_{ij} of the metric tensor.

The validity of such a definition for the components e_{ijhk} is derived from the fact that the value of $\sqrt{-g}$ transforms according to the same rules as the numerical values of the components e_{ijhk}, namely:

$$\sqrt{-g'} = \frac{1}{J}\sqrt{-g}\ .$$

As we already pointed out, we have no right to introduce the "microscopic" tensor g_{ij} into the development of our macroscopic theory. However, we can give an invariant definition to the tensor e_{ijhk} if we consider the *four-dimensional* determinant $\underline{s}$ that is defined by the components of the macroscopic covariant tensor s_{ijhk} *of rank four*. In the theory of multi-dimensional determinants, it is proved that the values of four-dimensional determinants transform according to the following rule ([13]):

$$\underline{s}' = \frac{1}{J}\underline{s}\ . \tag{33a}$$

Hence:

$$\sqrt[4]{\underline{s}'} = \frac{1}{J}\sqrt[4]{\underline{s}'}\ . \tag{33b}$$

If one compares formulas (33) and (33b) then one will verify that the non-zero components of e_{ijhk} can be defined invariantly if one equates them to $\pm\sqrt[4]{\underline{s}}$. However, we shall not equate e_{ijhk} to $\pm\sqrt[4]{\underline{s}}$, but to $\pm\sqrt[4]{\underline{s}/24}$. The reason for this will become clear later on ([14]).

([13]) Cf., above all, L. Gegenbauer, Wien. Akad. Denkschr. **43** (part 2) (1882), 17; **46** (part 2) (1883), 291. Since the dimension of the determinant in question is even, he gets only *one* value for the determinant, as opposed to the odd (e.g., cubic) dimensions.

Note that if one uses equation (33) then the third of the fundamental equations of electrodynamics (10a) can be put into the following form:

$$\frac{\partial}{\partial x^j}\left(\sqrt[4]{\underline{s}/\underline{s}'}\,f^{ij}\right) = 0, \tag{10b}$$

in which $\underline{s}'$ denotes the value of the determinant $\underline{s}$ in the A coordinate system.

([14]) We present a purely formal argument in support of that choice. In isotropic media, the tensor s_{ijhk} is the square of a second-rank tensor $s_{ijhk} = s_{ip}\,s_{jq}$ [cf., equation (7)]. In that case, one must naturally equate e_{ijhk} to the square root of the two-dimensional determinant $\underline{\underline{s}} = |\,s_{ip}\,|$ (in analogy with $\sqrt{-g}$). In isotropic bodies $\underline{\underline{s}} = 24\,(-\underline{s}\,)^2 = 4!\,(-\underline{s}\,)^2$; hence, $\sqrt{-\underline{s}} = \sqrt[4]{\underline{\underline{s}}/24}$.

As is obvious from equation (32), the replacement of the former values $e_{ijhk} = \pm 1$ with the values $e_{ijhk} = \pm\sqrt[4]{\underline{s}/24}$ is equivalent to the coupling of the components of the tensor h^{ijhk} with $\sqrt[2]{\underline{s}/24}$. Hence, the values of h^{ijhk} in an A system will no longer be determined by the matrix (31); every element of that matrix must be multiplied by $\sqrt{\underline{s}/24}$.

The new definition of the tensor h^{ijhk} does not change our previous argument, since our basic equation is homogeneous with respect to the components h^{ijhk}, and all of those components are multiplied by the same factor $\sqrt{\underline{s}/24}$, which will not violate its validity. In other words, it can be shown that it is only with the new definition of the tensor h^{ijhk} that it will take on the following remarkable property: *In isotropic media, the tensor* h^{ijhk} *will become equal to the tensor* s^{ijhk} *identically* ([15]).

Let us summarize the results of this paragraph: The propagation of light in anisotropic media is determined by the invariant equation (30a), in which h^{ijhk} are the components of a tensor of rank four. That tensor is a function of the tensor s^{ijhk} and is determined by equation (32), in which e_{ijhk} should be understood to mean the completely anti-symmetric tensor whose non-zero components are equal to $\pm\sqrt[4]{\underline{s}/24}$. In the A systems, the components of the tensor h^{ijhk} are equal to the elements of the matrix (31), multiplied by $\sqrt{\underline{s}/24}$.

§ 6. Double refraction and bi-quadratic geometry

The equation Q = const. determines an arbitrary surface (viz., the surface of equal phase). The propagation of light waves will occur along world lines that are perpendicular to its surfaces; i.e., along lines that satisfy the equations:

$$dx_i = \lambda \operatorname{grad}_i Q = \lambda \frac{\partial Q}{\partial x^i}, \tag{34}$$

in which λ is a proportionality factor. Inserting this into equation (30a) will give the equation of light rays:

$$h^{ijhk}\, dx_i\, dx_j\, dx_p\, dx_q = 0. \tag{35}$$

It is known that light propagates in a vacuum along null lines ([16]), whose equation takes the form:

$$ds^2 = g^{ij}\, dx_i\, dx_j = 0, \tag{35a}$$

in which g^{ij} are the components of the metric tensor.

The complete analogy of the last two equations allows us to give the following interpretation to equation (35):

([15]) Cf., the Appendix.
([16]) We shall pass over the question of the geometry of light rays.

Light propagates along null lines in material media, just as it does in vacuo. Equation (35) is equivalent to the equation $ds = 0$. Thus, *the metric tensor of a material medium is a tensor h^{ijhk} of rank four.* In other words, in material media, the line element is determined by the following expression:

$$ds^4 = h^{ijhk}\,dx_i\,dx_j\,dx_h\,dx_k = h_{ijhk}\,dx^i\,dx^j\,dx^h\,dx^k\,. \tag{36}$$

Of course, these suggestions are of a formal character and are essentially definitions. However, in order to maintain them, one must deal with a number of criticisms.

The values of the components of the metric tensor g^{ij} in vacuum can be found from three kinds of measurements: Optical [based upon equation (35a)], mechanical (the free motion of material bodies), and direct metrical ones (the behavior of rods and clocks); the last two methods are inapplicable in material media. The "geometrization" of the macroscopic theory of material media is possible only with the "optical" method. In other words, the only way to determine the metric tensor in a material medium is from the conditions for light rays to have zero length. Meanwhile, the application of well-developed geometrical methods to crystal optics, as well as to general electrodynamics, has greatly facilitated the further development of those theories.

Finally, as we shall see later on, our assumption that equation (35) is equivalent to the equation $ds = 0$ will lead to a number of physically-interesting consequences whose accuracy can be confirmed independently of our recently-proposed hypothesis.

Therefore, we shall assume that the line element is not determined by a quadratic form in the coordinate differentials, but a *bi-quadratic* one. That situation could have been predicted *a priori*, because it corresponds to the presence of *double refraction* in crystals; every spatial direction (dx^1, dx^2, dx^3) corresponds to not two ($\pm w$), but four ($\pm w_1$, $\pm w_2$), possible values of the speed of light that satisfy the fourth-degree equation $ds = 0$ ([17]).

It is obvious that in the case of the absence of double refraction – i.e., the case of isotropy – the line element must still be determined by a quadratic form in the coordinate differentials, not a bi-quadratic one. Indeed, it was already mentioned in the previous paragraph that $h^{ijhk} \equiv s^{ijhk}$ in isotropic media; in other words, in that case, the fourth-rank tensor s^{ijpq} will reduce to square of the second-rank tensor s^{ip} [cf., equation (7)]. Thus, equation (35) will take the following forms:

$$ds^4 = s^{ih}\,s^{jk}\,dx_i\,dx_h\,dx_j\,dx_k = (s^{ih}\,dx_i\,dx_k)^2,$$

or

$$ds^2 = s^{ih}\,dx_i\,dx_k, \tag{36b}$$

which was to be proved.

Thus, the "natural" geometry of anisotropic media is the geometry of bi-quadratic forms, which is a direct generalization of the Riemannian geometry of quadratic forms. It is easy to understand that the need for such a generalization might be superfluous in anisotropic media. Indeed, Helmholtz has shown that quadratic geometry is the only geometry in which a solid body preserves three rotational degrees of freedom around a

([17]) In the general case, the values of the speed of light differ, not only in their signs, but also in their absolute values.

fixed point. Now, the general theory of relativity is restricted essentially to the consideration of physical processes *in vacuo*, to the extent that it is based upon the concept of the rotation of rigid-bodies (of infinitesimal size). However, it is obvious that this concept is based upon a general assumption of the isotropy of space. There is no reason to believe that the freedom of rigid-body rotation would be preserved in anisotropic space; rather, we must reject such a possibility. The existence of double refraction resolves the issue in favor of bi-quadratic geometry.

§ 7. Waves and rays

The equation of the null lines $ds = 0$ can be written down in either contravariant or covariant coordinates. Previously, we saw that in the former case it determines the line of propagation of a light wave. What is the physical meaning of the equation:

$$ds^4 = h_{ijhk}\, dx^i\, dx^j\, dx^h\, dx^k = 0, \tag{37}$$

when it is written in contravariant coordinates dx^i ?

If the main provisions of the preceding paragraph are correct then that equation must also determine the line of propagation of light. In order to look into the matter more deeply, we express the differentials of the contravariant coordinates in terms of the speed of light rays and their direction cosines l^α:

$$v = \sqrt{\left(\frac{dx^1}{dx^4}\right)^2 + \left(\frac{dx^2}{dx^4}\right)^2 + \left(\frac{dx^3}{dx^4}\right)^2},$$

or

$$\frac{dx^\alpha}{dx^4} = l^\alpha v, \qquad \sum_\alpha (l^\alpha)^2 = 1, \quad \alpha = 1, 2, 3. \tag{38}$$

The admissibility and validity of this interpretation of the coordinate differentials will be justified in the following paragraph; in the meantime, I will confine myself to referring to Einstein, who used equation (38) in his article on "The fundamentals of the general theory of relativity." ([18])

In order to commence with the physical interpretation of equation (37), we still need to determine the values of the *covariant* components of the tensor h_{ijhk} . The contravariant components of that tensor are determined in terms of the tensors s^{ijhk} and e_{ijhk} by using equation (32). In order to get the covariant components of h_{ijhk} , we obviously need to replace the superscripts on both sides of that equation with subscripts. If we take advantage of an *A* coordinate system then, in view of equations (6) and (5), the replacement of s^{ijhk} with s_{ijhk} will come down to the replacement of μ_α and ε_α with their reciprocal values $1 / \mu_\alpha$ and $1 / \varepsilon_\alpha$ in the final result.

If one makes that substitution into the matrix (31) then that will give:

([18]) Ann. Phys. **49** (1916), 821 (cf., *Albert Einstein.* Collected Scientific Papers. v. 1, M. "Science," 1965, pp. 452, 1st ed.)

$$h_{ijhk} =$$

$$\begin{bmatrix} \dfrac{\mu_2\mu_3}{\varepsilon_1} & \dfrac{\mu_3}{2}\left(\dfrac{\mu_1}{\varepsilon_1}+\dfrac{\mu_2}{\varepsilon_2}\right) & \dfrac{\mu_2}{2}\left(\dfrac{\mu_3}{\varepsilon_3}+\dfrac{\mu_1}{\varepsilon_1}\right) & -\dfrac{1}{2\varepsilon_1}\left(\dfrac{\mu_2}{\varepsilon_2}+\dfrac{\mu_3}{\varepsilon_3}\right) \\ \dfrac{\mu_3}{2}\left(\dfrac{\mu_1}{\varepsilon_1}+\dfrac{\mu_2}{\varepsilon_2}\right) & \dfrac{\mu_3\mu_1}{\varepsilon_2} & \dfrac{\mu_1}{2}\left(\dfrac{\mu_2}{\varepsilon_2}+\dfrac{\mu_3}{\varepsilon_3}\right) & -\dfrac{1}{2\varepsilon_2}\left(\dfrac{\mu_3}{\varepsilon_3}+\dfrac{\mu_1}{\varepsilon_1}\right) \\ \dfrac{\mu_2}{2}\left(\dfrac{\mu_3}{\varepsilon_3}+\dfrac{\mu_1}{\varepsilon_1}\right) & \dfrac{\mu_1}{2}\left(\dfrac{\mu_2}{\varepsilon_2}+\dfrac{\mu_3}{\varepsilon_3}\right) & \dfrac{\mu_1\mu_2}{\varepsilon_3} & -\dfrac{1}{2\varepsilon_3}\left(\dfrac{\mu_1}{\varepsilon_1}+\dfrac{\mu_2}{\varepsilon_2}\right) \\ -\dfrac{1}{2\varepsilon_1}\left(\dfrac{\mu_2}{\varepsilon_2}+\dfrac{\mu_3}{\varepsilon_3}\right) & -\dfrac{1}{2\varepsilon_2}\left(\dfrac{\mu_3}{\varepsilon_3}+\dfrac{\mu_1}{\varepsilon_1}\right) & -\dfrac{1}{2\varepsilon_3}\left(\dfrac{\mu_1}{\varepsilon_1}+\dfrac{\mu_2}{\varepsilon_2}\right) & \dfrac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \end{bmatrix}. \qquad (39)$$

This matrix, like the matrix (31), gives the values of the components h_{ijhk} in an A coordinate system, up to a common factor that equals $\sqrt{\underline{s}/24}$ for h^{ijhk} and $\sqrt{\bar{s}/24}$ for h_{ijhk} . Here, $\bar{s}$ denotes the four-dimensional determinant that is defined by the *contravariant* components s^{ijhk}.

If we insert the values of h_{ijhk} and dx^α / dx^4 from (38) and (39) into the equation of the null lines (37) then we will get the following equation:

$$v^4\,\mathrm{S}\,(l^1)^4\,\frac{\mu_2\mu_3}{\varepsilon_1}+v^4\,\mathrm{S}\,(l^1l^2)^2\,\mu_3\left(\frac{\mu_1}{\varepsilon_1}+\frac{\mu_2'}{\varepsilon_2}\right)-v^2\,\mathrm{S}\,(l^1)^2\,\frac{1}{\varepsilon_1}\left(\frac{\mu_2}{\varepsilon_2}+\frac{\mu_3'}{\varepsilon_3}\right)+\frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}=0, \qquad (40)$$

in which the S denotes a sum over cyclic permutations of the indices 1, 2, 3.

Formula (40) coincides completely with the formula that determines the speed of propagation of *light rays* in magnetic crystals ([19]) and is a generalization of the known formula of elementary crystal optics to the case of $\mu \neq 1$.

In this situation, we find our first confirmation of the basic assumptions of the preceding paragraph. Except for the tensor *h*, which is the metric tensor in material media, there is no reason to assume that there exists any strong physical connection between (35) and (37), especially if we take into account the way by which we have determined the values of the covariant components of the tensor *h*.

Here, the null line equation (37) *in contravariant* coordinates determines the propagation of light rays, while the null line equation (35) in *covariant* coordinates determines the propagation of *light waves.* In this situation, it is essential that the covariant and contravariant coordinate differentials must be interpreted in entirely different ways: For the covariant coordinates, we use equations (25) and (34), which can be put into the following form:

([19]) Cf., E. Cohn, *Das Elektromagnetische Feld*, Leipzig, S. Hirzel, 1900, pp. 570 [equation (37b)]. Cohn’s term “Wellenfläche” corresponds to our term “ray surface.” (cf., *infra*).

$$\frac{dx_\alpha}{dx_4} = \frac{Q_\alpha}{Q_4} = -\frac{m_\alpha}{w}, \qquad \sum_\alpha (m^\alpha)^2 = 1, \tag{41}$$

in which w is the speed of waves, and m_α are its direction cosines. In other words, equation (38) will take the form:

$$\frac{dx_\alpha}{dx_4} = l^\alpha v, \quad \sum_\alpha (l^\alpha)^2 = 1, \tag{41a}$$

in which v is the ray speed and l^α are its direction cosines.

The question naturally arises: How does one find a consistent and reasonable interpretation of such different values dx^i and dx_i? Now, equation (41) was only derived in an A coordinate system, and even then, only under the assumption that it was a Cartesian system. However, now that we have identified the element of length by means of formula (36), we no longer have no right to assume that the A system is Cartesian. Furthermore, equation (38) is certainly true for a Cartesian system. But is it applicable in any coordinate system? Finally, equations (35) and (37) represent, in essence, two different ways of writing down *the same* equation; namely, $ds = 0$. How can *the same* equation represent such diverse things as the propagation of waves and the propagation of rays?

In order to unravel these issues, one must recall the cases in which there is a difference between the speed of rays v and the speed of waves w.

The values of v and w can be determined in two different ways. With the first one, one can start with the wave equation [viz., equation (14)]. Assume that the wave surfaces Q = const. at two consecutive moments in time t_1 and t_2 define the surfaces P_1 and P_2. If the time interval $t_2 - t_1$ is equal to 1 sec then the speed of the wave would be numerically equal to the length of the segment that is normal to the surfaces P_1 and P_2 and included between them. In other words, the speed of the ray will be numerically equal to the length of the line segment that is between the surfaces, whose direction will coincide with the direction of the flow of energy. The difference between the ray speed and the wave speed arises only in the case in which the direction of the flow of energy is not perpendicular to the wave surface.

The desired speeds v and w can also be determined by examining the so-called *ray* surface – i.e., the geometry of the locus of points that are illuminated at the time t by a light pulse that has the initial coordinates at the moment t_0. Let the ray surface coincide with the surfaces P_1 and P_2 at two consecutive moments t_1 and t_2. If the time interval $t_2 - t_1$ equals 1 sec then the ray speed will be numerically different from the length of the segment of the radius vector that is included between the surfaces P_1 and P_2. Furthermore, the wave speed will be equal to the length of the normal segment that is include between the tangent planes to the surfaces P_1 and P_2. The wave speed is different from the ray speed if and only if the normal direction to the ray surface does not coincide with the direction of the radius vector to that surface.

Thus, the problem of determining the speeds v and w reduces to the problem of constructing a normal to a given surface P. The concept of a normal is meaningful only if one has defined a metric, which will define a geometry.

It is possible to given an answer to the question above on the basis of the following conditions:

1. In a natural geometry that enjoys an invariant definition of line elements, the normal to the ray surface will always coincide with the direction of the radius vector to that surface, and the normal to the wave surface will coincide with the direction of the flow of energy. Therefore, $v = w$, so there will be no difference between the laws of propagation for light waves and those of light rays, and both phenomena will be described by the same equation $ds = 0$.

2. The difference between waves and rays will arise only in the case in which a non-Cartesian coordinate system (or any other non-invariant system) uses a Cartesian metric.

Naturally, these contravariant Riemannian coordinates can be directly identified with Cartesian coordinates [according to equation (41a)]. As for the covariant coordinates, they have no direct geometric interpretation in Cartesian geometry. However, it is not difficult to insure that when one is considering the laws of the propagation of light, the covariant coordinate differentials should be interpreted according to our equation (41). Indeed, within an infinitesimal region of the wave, the factor $e^{\sqrt{-1}Q}$ can always be represented in the following form:

$$e^{\sqrt{-1}Q} = e^{\sqrt{-1}(Q)_0} e^{\sqrt{-1}Q_i\,dx^i},$$

in which $(Q)_0$ = const.

If one identifies the factor $e^{\sqrt{-1}Q_i\,dx^i}$ with the wave factor $e^{\sqrt{-1}\frac{2\pi}{\lambda}(wt - m_\alpha x^\alpha)}$ of a plane wave in Cartesian space then we can repeat all of the arguments that led up to equations (25) and (41) in § 4 and 5, respectively.

3. We claim that if the invariant equation $ds = 0$ is written in contravariant (covariant, resp.) coordinates then if we eliminate those coordinates by using the relations (41a) [(41), resp.], the equation that will be obtained will be tantamount to the Cartesian equation of a light ray, in which case, one would have a *wave* surface ([20]). In order to prove the correctness of this assertion, we still need to show that there is a proper geometric relationship between certain surfaces, which amounts to the fact that the wave surface is the locus of the feet of the perpendicular that is dropped from the center of the ray surface to a tangent plane to it.

In order to prove the validity of the statement above, I shall turn to the optics of the gravitational field *in vacuo*, which is also a Cartesian viewpoint on anisotropic space, and therefore there is a difference between a light ray and a light wave. All of the results that we arrive at in this simple case can be transferred to the optics of material media directly.

([20]) By definition, the length of the radius vector of the wave surface in a given direction is the speed of a plane wave that propagates in that direction.

§ 8. The optics of gravitational fields

Although the equations of light rays in vacuum are generally known, I shall still give a brief derivation of those equations here, while adhering to the method that was outlined in § 3.

In a vacuum, the relationship between the tensors F_{ij} and f^{ij} is established by using the metric tensor ([21]):

$$f^{ij} = g^{ip}\, g^{jq}\, F_{pq}\,.$$

Equations (14) and (15) still remain in force, while equation (16) will take the form:

$$f^{ij} = \sqrt{-1}\; g^{ip}\, g^{jq}\, (A_p\, Q_q - A_q\, Q_p)\; e^{\sqrt{-1}Q}\,, \tag{16a}$$

and equation (17) can be written in the stated form as follows (after removing $\sqrt{-1}\; e^{\sqrt{-1}Q}$):

$$g^{ip}\, g^{jq}\, (A_p\, Q_q - A_q\, Q_p)\, Q_j = 0. \tag{17a}$$

Multiplying this equation by g_{ik} and summing over i will give (since $g^{ip}\, g_{ik} = \delta_k^p$):

$$A_k\, (g^{jq}\, Q_q\, Q_j) - Q_k\, (g^{jq}\, A_q\, Q_j\,) = 0.$$

These four equations can be satisfied in only the following two cases: Either A_k is proportional to Q_k, or the following equalities are satisfied simultaneously:

$$g^{jq}\, Q_q\, Q_j = 0, \quad g^{jq}\, A_q\, Q_j = 0. \tag{42}$$

The first situation is impossible [cf., (23*a*)]; therefore, one must have equations (42).

The derivation of equations (42) that is contained here differs from the results that are known to me (e.g., Laue, Eddington, etc.) by the fact that those authors did not prove the validity of the second of equations (42) *completely*, or rather, the equivalence of those equations with div $\Phi_i = 0$, where Φ_i is the tensor potential.

If one replaces Q_j in equations (42) with the coordinate differentials of a light ray [cf., equations (34)] then one will finally get:

$$g^{jk}\, dx_j\, dx_k = 0. \tag{43}$$

The same equations in contravariant coordinates are written as follows:

$$g_{jk}\, dx^j\, dx^k = 0. \tag{43a}$$

In terms of Riemannian geometry, equations (43) and (43*a*) are identical. The anisotropy of space is out of the question, if only because the expression for the line element can

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([21]) Thus, F_{ij} and f^{ij} are components of *the same* tensor in a vacuum (as opposed to a material medium).

always be reduced to a sum of quadratic differentials by a suitable coordinate transformation ([22]).

By contrast, one always enjoys Cartesian geometry in experimental physics, and any deviation from Cartesian relationships is explained by the influence of external (viz., gravitational) forces. Hence, for example, the speed of light will depend upon the direction of propagation in gravitational field *coordinates*. Experimental physics seeks to explain that fact by way of an anisotropy of space that is created by gravitation. The presence of anisotropy, in turn, will cause differences between the wave speed and the ray speed.

We now turn to the proof of the provisions that were outlined at the end of the preceding paragraph. For simplicity, assume that we have chosen a coordinate system in which time is perpendicular to space; i.e., $g_{14} = g_{24} = g_{34} = 0$. As is known, that condition will imply the validity of the following equality:

$$g^{44} = 1 / g_{44} . \tag{44}$$

In addition, to simplify, we drop the signs in front of the differentials of the coordinate x^i and x_i in equations (43) and (43a), which is, however, limited to a coordinate change over an infinitesimal region; i.e., a region in which the values of the components g_{ij} can be regarded as constant.

The equation of the ray surface $G(x^\alpha)$ will be obtained equation (43a) by setting $t = x^4$ = const.:

$$G(x^\alpha) = g_{\alpha\beta}\, x^\alpha x^\beta + g_{44}\, t^2 = 0, \qquad t = \text{const.}, \qquad \alpha = 1, 2, 3. \tag{45}$$

Draw a spatial segment whose components will be denoted by ξ^α at an arbitrary point (x^α) of the surface $G(x^\alpha)$. That segment can lie in the tangent plane to the surface considered as long as $(\partial G / \partial x^\alpha)\, \xi^\alpha = 0$; i.e., as long as $g_{\alpha\beta}\, x^\beta\, \xi^\alpha = 0$. From the standpoint of Riemannian geometry, the equality the must be shown is that the radius vector (x^α) to the ray surface G will always be perpendicular to that surface ([23]), which agrees with the first part of condition 1 that we stated in the previous paragraph.

Moreover, the direction of energy flow is determined by the components T_α^4 of the energy-impulse tensor T_i^k. For electromagnetic fields, the tensor T_i^k is determined from the equality:

$$T_i^k = F_{ir}\, f^{kr} - \frac{1}{4}\delta_i^k\, F_{rs}\, f^{rs}.$$

When F_{rs} and f^{rs} are expressed in terms of A_i and Q_i by using equations (15) and (16a) and taking equation (42) into account, we confirm that the second term and three terms in the first term on the right hand side of this equation will be equal to zero for the

([22]) In contrast, if that element is determined by a bi-quadratic form in the coordinate differentials then the anisotropy of space will have an essential character, and cannot be eliminated by a suitable choice of coordinates.

([23]) Because it is perpendicular to any of the tangents to that surface.

field of the light wave. Omitting the wave factor $e^{\sqrt{-1}Q}$, which is unimportant for us, will give $T_i^k = g^{kp} Q_p g^{rs} A_r A_s Q_i$. Thus, $T_\alpha^4 = \lambda Q_\alpha$, in which λ is a proportionality factor.

In other words, the direction of the normal to the wave surface will be determined by the values of $dx_\alpha = \lambda \operatorname{grad}_\alpha Q = \lambda Q_\alpha$. Therefore, the direction of energy flow in the field of a light wave will coincide with the direction of the normal to the wave surface. This illustrates the second part of our condition 1.

We now proceed to the proof of condition 3. To that end, we take the viewpoint of Cartesian geometry and identify the contravariant Riemannian coordinates x^i with the Cartesian ones and find the geometric locus of the bases of the perpendiculars that are dropped from the center of ray surface (45) to the tangent plane to that surface.

The tangent plane to the ray surface $G(x^\alpha) = 0$ is determined from the equation:

$$\frac{\partial G}{\partial x^\alpha}(X^\alpha - x^\alpha) = 0,$$

in which X^α are the current coordinates of the tangent plane, and x^α are the coordinates of the point of tangency.

From (45), we will have:

$$\frac{\partial G}{\partial x^\alpha} = 2\, g_{\alpha\beta}\, x^\beta.$$

If we substitute this into the previous equation then we will find that:

$$g_{\alpha\beta} X^\alpha x^\beta - g_{\alpha\beta} x^\alpha x^\beta = 0,$$

or, on the basis of equation (45):

$$g_{\alpha\beta} X^\alpha x^\beta + g_{44}\, t^2 = 0.$$

To simplify, we introduce the notation ([24]):

$$x_\alpha = g_{\alpha\beta}\, x^\beta, \qquad a = g_{44}\, t^2, \tag{46}$$

and with their help, the equations of the tangent planes will be given in the form $X^\alpha x_\alpha + a = 0$.

As is known, the coordinates ξ^α of the base of the perpendicular that is dropped from the coordinate origin to the plane are determined from the equation:

$$\xi^\alpha = -\frac{a\, x_\alpha}{\sum_\beta x_\beta^2}.$$

In order to find the equation that relates the values of ξ^α to t, form the expression:

([24]) Of course, since we are using Cartesian geometry, the quantities x_α cannot be considered to be covariant coordinates. We have introduced the notation in a purely formal way.

$$\sum_{\alpha\beta} g^{\alpha\beta}\xi^\alpha\xi^\beta = \frac{a^2 g^{\alpha\beta}x_\alpha x_\beta}{\left(\sum_\beta x_\beta^2\right)^2}.$$

On the basis of equations (45) and (46), that will easily give:

$$g^{\alpha\beta}\,x_\alpha\,x_\beta = g_{\alpha\beta}\,x^\alpha\,x^\beta = -\,a.$$

Therefore:

$$\sum_{\alpha\beta} g^{\alpha\beta}\xi^\alpha\xi^\beta = \frac{-a^3}{\left(\sum_\beta x_\beta^2\right)^2}.$$

In other words, the length of the desired perpendicular will be given by the formula:

$$r^2 = \sum_\alpha (\xi^\alpha)^2 = \frac{a^2\sum_\alpha x_\alpha^2}{\left(\sum_\beta x_\beta^2\right)^2} = \frac{a^2}{\sum_\beta x_\beta^2}.$$

Hence:

$$\sum_{\alpha\beta} g^{\alpha\beta}\xi^\alpha\xi^\beta = -\,r^4 / a,$$

or

$$\sum_{\alpha\beta} g^{\alpha\beta}\xi^\alpha\xi^\beta + \frac{r^4}{g_{44}t^2} = 0.$$

If we divide both sides of this equation by r^2, introduce the notation $w = r / t$ for the wave speed and $m_\alpha = \xi^\alpha / r$ for the direction cosines, and finally, express g_{44} in terms of g^{44} using equation (44) then we will get the equation of the desired (wave) surface in the following form:

$$g^{\alpha\beta}\,m_\alpha\,m_\beta + g^{44}\,w^2 = 0. \tag{47}$$

It is easy to see that this will lead us to the same equation, and equation (43) of null lines, if we eliminate the differentials of covariant coordinates from it by using the relation (41).

Thus, the equations of the ray and wave surfaces, which are derived from the equation $ds = 0$ by the method that was discussed in the preceding paragraph, are connected by the proper geometric relationship ([25]). Thus, we have proved all of the statements that were

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([25]) Riemannian coordinates will reduce to oblique-angled Cartesian coordinates (with a different length scale along each coordinate axis) within a region that is sufficiently small that the values of the components g_{ij} can be regarded as constants. The contravariant coordinates will equal the projections of the radius vector onto the coordinate axes In an oblique-angled coordinate system. In other words, the wave speed made at the end of the last paragraph in the case of gravitational fields *in vacuo*. In this article, we shall not dwell upon the proof of those statements for more complex cases of the propagation of light in anisotropic material media and shall confine ourselves to the assumption that the aforementioned statements that are valid in quadratic geometry will also remain valid in bi-quadratic geometry.

§ 9. The three fundamental tensors of bi-quadratic geometry

In quadratic geometry, by definition, there are well-known relations between the covariant and contravariant components of the same tensor that are established by the metric tensor.

In particular, they are known to be $dx_i = g_{ij}\, dx^j$ and $dx^i = g^{ij}\, dx_j$. A similar kind of relationship should exist in bi-quadratic geometry. However, these relationships cannot be established using any metric tensor h or s as the tensor that plays the role of "index changer," unless it satisfies the following two conditions:

1) It should be a *second-rank* tensor, since otherwise the transition from the contravariant components of an arbitrary tensor T to the covariant components (and conversely) would change the *rank* of the tensor T. (In particular, that would apply to the coordinate differentials dx^i and dx_i .)

2) The components of the "index-changing" tensor (which will be denoted by k_{ij} and k^{ij}) must satisfy the relations:

$$k_{jh}\, k^{ij} = \delta^i_h . \tag{48}$$

which follow from the fact that the equalities $dx^i = k^{ij}\, dx_j = k^{ij}\, k_{jh}\, dx^h$ should be satisfied identically for any differentials dx^i .

It might seem that the role of "index-changer" is purely abstract and can be performed by any second-rank tensor that satisfies the condition (48). However, that is not true. The replacement of the contravariant differentials dx^i with covariant ones (and vice versa) must convert the expression for the line element $ds^4 = h_{ijhk}\, dx^i\, dx^j\, dx^h\, dx^k = h^{ijhk}\, dx_i\, dx_j\, dx_h\, dx_k$ from the first form into the second one (and conversely). Therefore, it is necessary that ([26]):

$$h_{ijhk} = k_{ip}\, k_{jq}\, k_{hr}\, k_{ks}\, h^{pqrs}. \tag{49}$$

Due to the fact that the values of the covariant and contravariant components of the metric tensor h are known from the above, we can use equation (49) to determine the

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will equal the projection of the ray speed onto the normal to the wave plane. The analogy is clear and can serve as a confirmation of our equation (41).

([26]) Unlike the tensor h, the components of the tensor s are coupled by the relation (4), which is analogous to equation (48). The components s^{ijhk} and s_{ijhk} do not satisfy any relationship of the type (49).

values of the components of the tensor *k*. A few simple calculations will lead to the values of these components in an *A* coordinate system ([27]):

$$k_{ij}\sqrt[8]{\frac{\underline{\underline{s}}}{\overline{\overline{s}}}} = \begin{bmatrix} \sqrt{\dfrac{\mu_2\mu_1}{\varepsilon_1}} & 0 & 0 & 0 \\ 0 & \sqrt{\dfrac{\mu_3\mu_1}{\varepsilon_2}} & 0 & 0 \\ 0 & 0 & \sqrt{\dfrac{\mu_1\mu_2}{\varepsilon_3}} & 0 \\ 0 & 0 & 0 & \dfrac{-1}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}} \end{bmatrix}. \tag{50}$$

By means of analogous considerations, we can easily see that the values of the contravariant components of the tensor k^{ij} will be obtained from the matrix (50) by replacing the values ε_α and μ_α with their reciprocal values $1 / \varepsilon_\alpha$ and $1 / \mu_\alpha$ and replacing $\overline{\overline{s}}$ with $\underline{\underline{s}}$, and conversely. It is also easy to see that the tensor *k* satisfies the condition (48).

Thus, one needs to distinguish between the three fundamental tensors *s*, *h*, and *k* in bi-quadratic geometry. The fourth-rank tensor *s* establishes a relationship between the electromagnetic tensors F_{ij} and f^{ij}. The fourth-rank tensor *h* determines the line element. The second-rank tensor *k* plays the role of “index-changer” with respect to any tensor, and in particular, the tensor *s*. All of these functions are performed by the same tensor *g* in quadratic geometry.

It is obvious that these three tensors *s*, *h*, and *k* will reduce to one second-rank tensor that is analogous to the tensor *g* in isotropic media, where quadratic geometry dominates. In fact, it was already observed in § 2 that the fourth-rank tensor s_{ijhk} (s^{ijhk}, resp.) will reduce to the square of the second-rank tensor s_{ij} (s^{ij}, resp.) [cf., equation (7)] in an isotropic medium.

In other words, it was pointed out in § 6 that $h_{ijhk} = s^{ijhk} = s_{ih}\, s_{jk}$ in an isotropic medium, and therefore the line element will be determined by the equation (36b): $ds^2 = s_{ij}\, dx^i\, dx^j$ in that case.

Finally, as will be proved in the Appendix, one has the following relations:

$$\overline{\overline{s}} = 24\,\overline{s}^2 = 24\frac{\varepsilon^2}{\mu^2} \qquad \text{and} \qquad \underline{\underline{s}} = 24\,\underline{s}^2 = 24\frac{\mu^2}{\varepsilon^2}$$

in isotropic media, and as a consequence, as is easily seen, the components of the tensor k_{ij} will be equal to the components of the tensor s_{ij} in an isotropic medium. Thus, in an isotropic medium, the tensor s_{ij} will perform all three of the functions of the metric tensor

([27]) Equation (49) is insufficient for the determination of the signs of the components k_{ij}. We selected the signs of k_{ij} by analogy with the signs of the components g_{ij} and s_{ij}.

g_{ij} that were listed above. Of course, as the medium becomes gradually more rarefied by the transition to a vacuum, the tensor s_{ij} will become identical to the tensor g_{ij} .

We also mention that the job of the tensor s is determined uniquely by the values of the other fundamental tensors of bi-quadratic geometry. Thus, for example, if one is given the values of the components s_{ijhk} then by the use of equation (4), one can determine the components of s^{ijhk}, and by using equation (32), one can determine the components h_{ijhk} and h^{ijhk}. Finally, by using equations (48) and (49), one can calculate the values of the components of the tensor k. Unfortunately, I could not find any relations that would establish a direct link between the tensors s and k.

The further development of this theory would have to be associated with the resolution of a number of physically and mathematically interesting problems. Thus, for example, I suppose that the eventual development of the theory will permit one to determine the form of the electromagnetic tensor of energy and impulse in material media and to thus eliminate the existing questions and inconsistencies in regard to it ([28]). The extension of the theory to *inhomogeneous*, anisotropic media will involve finding the equations of geodesic lines in bi-quadratic geometry. Perhaps, the above physical considerations can also facilitate the solution of this purely mathematical problem of finding the laws of bi-quadratic geometry. Thus, for example, the considerations that were described at the end of § 7 suggest that the expression that defines the cosine of the angle between two directions must be such that the surface that is defined by the equations $ds = 0$ or x^i = const. is always perpendicular to its radius vector.

It is indeed a pleasant task for me to express my sincere appreciation to prof. L. I. Mandelstam for his valuable advice and suggestions that helped me in the preparation of this work.

Appendix

We prove that the metric tensor h_{ijhk} is identically equal to the square of the second-rank tensor s_{ij} in isotropic media.

It is known that the four-dimensional determinant $\underline{\underline{s}}$ can be put into the following form:

$$\underline{\underline{s}} = (s_{ijhk}) = \varepsilon^{i_1i_2i_3i_4}\,\varepsilon^{j_1j_2j_3j_4}\,\varepsilon^{h_1h_2h_3h_4}\,s_{1i_1j_1h_1}\,s_{2i_2j_2h_2}\,s_{3i_3j_3h_3}\,s_{4i_4j_4h_4}\,,$$

in which we let e^{ijhk} denote the antisymmetric tensor of rank four whose non-zero components are equal to ± 1. In isotropic media, due to the equality $s_{ijhk} = s_{ih}\,s_{jk}$, we will get:

$$\underline{\underline{s}} = (\varepsilon^{j_1j_2j_3j_4}s_{1j_1}s_{2j_2}s_{3j_3}s_{4j_4})(\varepsilon^{i_1i_2i_3i_4}\,\varepsilon^{h_1h_2h_3h_4}s_{i_1h_1}s_{i_2h_2}s_{i_3h_3}s_{i_4h_4})\,,$$

in which $\underline{\underline{s}} = (\underline{s})(4!\underline{s}) = 24\underline{s}^2$, where $\underline{s} = |\, s_{ij}\,|$.

According to § 5, when the components of the tensor e_{ijhk} are introduced into formula (32), that will give the following values:

([28]) Cf., W. Pauli, *Relativitätstheorie*, Leipzig, 1921, § 35 (translated into: V. Pauli, *Teoriya otnositelnosti*, M. –L., GITTL, 1947. – 1st ed.)

$$e_{ijhk} = \sqrt[4]{\frac{s}{24}}\ \varepsilon_{ijhk}\ .$$

In isotropic media, that equality will take the following form:

$$e_{ijhk} = \sqrt[4]{\underline{s}^2}\ \varepsilon_{ijhk} = \sqrt{(-\underline{s})}\ \varepsilon_{ijhk}\,.$$

(We include a "minus" sign under the square root since $\underline{s} < 0$.)

Thus, equation (32) will take on the following form:

$$h^{qrst} = -\frac{1}{6}\left(\sqrt{-\underline{s}}\right)^2 \{s^{qs}\, s^{ij}\, s^{rk}\, s^{hl} - s^{qi}\, s^{sj}\, s^{rk}\, s^{hl} + s^{qi}\, s^{rk}\, s^{sj}\, s^{hl}\}\, s^{tn}\, s^{mp}\, \varepsilon_{ihkm}\, \varepsilon_{jlnp}$$

in an isotropic medium.

The last two terms in the parentheses cancel each other out, and as a consequence, the equality can be written in the following form:

$$h^{qrst} = \frac{1}{6}\,\underline{s}\, s^{qs}\, s^{rk}\, s^{tn}\{\varepsilon_{ihmk}\, \varepsilon_{jlnp}\, s^{ij}\, s^{hl}\, s^{mp}\}.$$

It is easy to see that the expression in parentheses is the adjunct determinant $\overline{s} = |\, s^{ij}\,|$, multiplied by 3! (= 6), that corresponds to the term s^{kn} ([29]). As is known, that adjunct is equal to the covariant components of the tensor s_{kn}, multiplied by the determinant $\overline{s}$. Thus:

$$h^{qrst} = \frac{1}{6}\,\underline{s}\, s^{qs}\, s^{rk}\, s^{tn}\ 6\ s_{kn}\ \overline{s}\ .$$

Since $\overline{s}\ \underline{s}$ and $s^{rk}\, s^{tn}\, s^{kn} = s_{rt}$ [due to equation (4a)], we finally get:

$$h^{qrst} = s^{qs}\, s^{rt},$$

which was to be proved.

([29]) The appearance of the factor 3! is due to the fact that the summation inside of the brackets is performed over the indices *i*, *h*, *m*, along with the indices *j*, *l*, *p*.