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The electrodynamics of anisotropic media in the special theory of relativity ([1])

By

L. Mandelstam and I. Tamm in Leningrad.

Translated by D. H. Delphenich

In what follows, we shall report on the generalization of Minkowski’s equations to the case of an anisotropic medium.

In § 1, we present a new notation for Minkowski’s equations. Ordinarily, the tensor connection between the field tensor F_{ik} and f^{ik} is:

$$(F_{14}, F_{24}, F_{34}) = (E_1, E_2, E_3); \qquad (F_{23}, F_{31}, F_{12}) = (\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3);$$
$$(f^{41}, f^{42}, f^{43}) = (D_1, D_2, D_3); \qquad (f^{23}, f^{31}, f^{12}) = (H_1, H_2, H_3);$$

It will be presented [eq. (1) and (2)] with the help of the velocity tensor u^i. The dielectric constant ε and the permeability μ will thus be regarded as scalar quantities. However, one can also interpret ε and μ as the determining data for a four-dimensional dielectricity-magnetization tensor s^{ij}, eq. (3) [s_{ij}, eq. (6) resp.] (which will be called briefly the D-M tensor in the sequel). The Minkowski equations [(1) and (2)] can then be brought into a very simple form (4) [(5), resp.]. The velocity tensor u^i will no longer enter into these equations explicitly.

In § 2, the equations will then be given for anisotropic bodies. The D-M tensor will no longer be of rank two there, as it is in the case of isotropic bodies, but of rank *four*. We shall denote it by $s_{ij\alpha\beta}$ ($s^{ij\alpha\beta}$, resp.) [(11) and (13)].

The tensorial form of the equations that we arrive at will allow us to carry over these equations into the domain of general relativity. We hope to report on the application of the results obtained to some questions of optics and the metrics of ponderable media in a further publication.

§ 1.

If we first restrict our considerations to non-conducting media then we will need to deal with only the equations:

([1]) One of us reported more thoroughly on the question to be addressed here in *loc. cit.* (Journ. Russ. Phys.-Chem. Society).

After this treatise was already completed, we learned that Prof. Fredriks (Leningrad) had employed analogous considerations to ours.

(1) $$f_{ij}\, u^j = \varepsilon\, F_{ik}\, u^k,$$

(2) $$F_{kl}\, u_i + F_{li}\, u_k + F_{ik}\, u_l = \mu\, (f_{kl}\, u_i + f_{li}\, u_k + f_{ik}\, u_l),$$

and the remaining two basic equations:

$$\mathrm{Div}\, f^{\,ij} = 0, \qquad \mathrm{Rot}\, F_{ij} = 0$$

will take on their usual form that is also true for the vacuum. In addition to the field tensors F_{ik} and $f^{\,ik}$ and the material constants ε and μ, the velocity tensor u^i also appears in equations (1) and (2). If one regards the quantities ε and μ as scalars then the addition of such an auxiliary tensor will be essential. A linear, tensorial coupling of two tensors (here, F_{ik} and $f^{\,ik}$) can be exhibited only by means of a third tensor (here, u^i, and if one excludes the trivial case of a pure proportionality). However, if one regards the quantities ε and μ, not as scalars, but as components of a *four-dimensional* tensor, then the inclusion of the velocity tensor will be superfluous. Such a viewpoint will seem completely reasonable when one recalls the usual *three-dimensional* tensors ε_{ij} and μ_{ij} of crystal physics.

We will define a "dielectricity-magnetization tensor" s_{ij} when we ascribe the following values to its components in the rest system:

(3) $$(s_{ij}) = \begin{bmatrix} -\sqrt{\mu} & 0 & 0 & 0 \\ 0 & -\sqrt{\mu} & 0 & 0 \\ 0 & 0 & -\sqrt{\mu} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\varepsilon\sqrt{\mu}} \end{bmatrix}.$$

With the help of this tensor, one can combine the two equations (1) and (2) into one equation:

(4) $$F_{ij} = s_{i\alpha}\, s_{j\beta}\, f^{\,\alpha\beta}.$$

Equation (4) then gives the correct connection between F_{ij} and $f^{\,ij}$ in the rest system, as one easily confirms by means of a simple substitution. Due to its tensor character, the equation will preserve its validity in *any* arbitrary reference system ([2]).

The velocity tensor components u^i no longer enter into equation (4) explicitly, but only implicitly, since the coefficients of the Lorentz transformation depend upon the velocity of the medium.

If one solves equation (4) for $f^{\,ij}$ then one will arrive at an equation of the following kind:

(5) $$f^{\,ij} = s^{\,i\alpha}\, s^{\,j\beta}\, F_{\alpha\beta}\,,$$

([2]) If necessary, one can confirm the validity of the last statement by actually performing the coordinate transformation.

in which the $s_{i\alpha}$ are covariant tensor components that possess the following values in the rest system:

(6) $$(s^{ij}) = \begin{bmatrix} -\frac{1}{\sqrt{\mu}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{\mu}} & 0 \\ 0 & 0 & 0 & \varepsilon\sqrt{\mu} \end{bmatrix}.$$

The tensors s_{ij} and s^{ij} have the same tensorial relationship to each other that the gravitational potentials g_{ij} and g^{ij} have in general relativity theory; namely, one has:

(7) $$s^{i\alpha} s_{k\alpha} = \delta_k^i \qquad \begin{cases} \delta_k^i = 0, & \text{when } i \neq k, \\ \delta_k^i = 1, & \text{when } i = k. \end{cases}$$

In other words: The s^{ij} are equal to the corresponding sub-determinants that are constructed from s_{ij}, divided by the determinant $| s_{ij} |$.

One remarks that the electric and magnetic quantities ε and μ appear in (3) and (6) as the basic data of a single tensor, which corresponds completely to the well-known blending of the electric and magnetic field vectors E and $\mathfrak{B}$ (D and H, resp.) into *one* four-dimensional tensor F_{ij} (f^{ij}, resp.).

One can further remark that the considerations that were discussed above can also be extended to the case of conducting media. The dependency of the four-current J^i on the charge density and the field strengths can be expressed in the following form:

(8) $$J^i = \lambda_{i\alpha\beta} F^{\alpha\beta} + \rho^i.$$

By definition, the tensor components $\lambda^{i\alpha\beta}$ and ρ^i possess the following values in the rest system:

$$\rho^1 = \rho^2 = \rho^3 = 0, \qquad \rho^4 = \rho$$

and $\lambda^{i\alpha\beta} = 0$, with the exception of $\lambda^{114} = \lambda^{224} = \lambda^{334} = \lambda$ ([3]). Here, the charge density (conductance, resp.), as measured in the rest system, is denoted by ρ (λ, resp.).

It can be easily shown that eqs. (4), (5), and (8) agree with the corresponding Minkowski equations.

([3]) We shall leave it undecided whether one might not have, say, $\lambda^{i\alpha\beta} = 0$, with the exception of $\lambda^{114} = \lambda^{224} = \lambda^{334} = -\lambda^{141} = -\lambda^{242} = -\lambda^{343} = \frac{1}{2}\lambda$.

§ 2.

It is now easy to generalize the foregoing arguments to the case of anisotropy. We again seek to bring the coupling equations between F_{ij} and f^{ij} into a form that is analogous to equation (4). However, one will not arrive at a D-M tensor of rank *two*, but one must appeal to a tensor of rank *four* $\delta_{ij\alpha\beta}$, moreover ([4]). We set:

$$F_{ij} = \delta_{ij\alpha\beta}\, f^{\alpha\beta}. \tag{9}$$

The values of the tensor components $\delta_{ij\alpha\beta}$ in the rest system are determined by the demand that equation (9) must be equivalent to the usual equations:

$$D_i = \varepsilon_{ij}\, E_j\,, \qquad \mathfrak{B}_i = \mu_{ij}\, H_j$$

in that rest system. Indeed, this requirement does not suffice to determine the necessary values uniquely, but it is entirely reasonable to demand the following: Equation (9) shall go to the corresponding equation (4) under a continuous transition to isotropy, and therefore, $\delta_{ij\alpha\beta}$ shall go to $s_{ia} \cdot s_{j\beta}$. The calculations lead to the following values of the components [in the rest system whose coordinate axes coincide with the principal directions of anisotropy, and such a coordinate system will be referred to briefly in what follows as an "*A*-system"]:

$$\begin{aligned} &s_{1212} = \delta_{2121} = \mu_3\,, \qquad s_{1313} = \delta_{3131} = \mu_2\,, \qquad s_{2323} = \delta_{3232} = \mu_1\,, \\ &s_{1414} = \delta_{4141} = -\frac{1}{\varepsilon_1}\,, \quad s_{2424} = \delta_{4242} = -\frac{1}{\varepsilon_2}\,, \quad s_{3434} = \delta_{4343} = -\frac{1}{\varepsilon_3}\,; \end{aligned} \tag{10}$$

all of the remaining $s_{ij\alpha\beta}$ vanish in the *A*-system with the exception of the terms in the principal diagonal. However, as far as the terms in the principal diagonal are concerned, our requirements only succeed in determining their dimensions, but not their values. Those terms are then of no interest to us, since they will drop out of equations (9) in any arbitrary system of reference ([5]).

([4]) If one establishes some symmetry properties of the D-M tensor then it will immediately come to light that the ten basic data of a symmetric tensor of rank two are not sufficient to express the twelve quantities ε_{ij} and μ_{ij} uniquely. We hope to show in a later publication that this fact (viz., that the D-M tensor is a tensor of rank four in an anisotropic medium) can be closely connected with the capacity of the medium to exhibit double refraction.

([5]) Example:

$$\frac{\partial F'_{ij}}{\partial \mathfrak{B}'_3} = s'_{ij12} - s'_{ij21}\,, \qquad \mathfrak{B}'_3 = f'^{12} = -f'^{21},$$

$$\frac{\partial (s'_{ij12} - s'_{ij21})}{\partial S_{pppp}} = \frac{\partial x^p}{\partial x'^i}\frac{\partial x^p}{\partial x'^j}\left(\frac{\partial x^p}{\partial x'^1}\frac{\partial x^p}{\partial x'^2} - \frac{\partial x^p}{\partial x'^2}\frac{\partial x^p}{\partial x'^1}\right) = 0.$$

As is easy to see in (10), the tensor $s_{ij\alpha\beta}$ is symmetric with respect to the index permutations (i, α), (j, β), and ($i\alpha$, $j\beta$). In the A-system, its *non-zero* components can be arranged into a quadratic matrix:

$$(11) \qquad (s_{ij\alpha\beta}) = \begin{bmatrix} [\mu] & \mu_3 & \mu_2 & -\dfrac{1}{\varepsilon_1} \\ \mu_3 & [\mu] & \mu_1 & -\dfrac{1}{\varepsilon_2} \\ \mu_2 & \mu_1 & [\mu] & -\dfrac{1}{\varepsilon_3} \\ -\dfrac{1}{\varepsilon_1} & -\dfrac{1}{\varepsilon_2} & -\dfrac{1}{\varepsilon_3} & \left[\dfrac{1}{\varepsilon^2\mu}\right] \end{bmatrix}.$$

The components (11), (12), (13), etc., of this matrix correspond to the components s_{1111}, s_{1212}, s_{1313}, etc.; only the dimensions are given for the terms in the main diagonal.

The corresponding contravariant tensor $s^{ij\alpha\beta}$, which is defined by the equation:

$$(12) \qquad (s^{ij\alpha\beta}) = \begin{bmatrix} \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_3} & \dfrac{1}{\mu_2} & -\varepsilon_1 \\ \dfrac{1}{\mu_3} & \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_1} & -\varepsilon_2 \\ \dfrac{1}{\mu_2} & \dfrac{1}{\mu_1} & \left[\dfrac{1}{\mu}\right] & -\varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \left[\varepsilon^2\mu\right] \end{bmatrix},$$

can be found in an entirely similar manner.

One has the relation:

$$s^{ij\alpha\beta}\, s^{\alpha\beta hk} = \delta_h^i \delta_k^j ,$$

which is analogous to (7).

Equations (9) and (11) [(12) and (13), resp.] resolve the problem that was posed and allow one to ascertain the connection between F_{ij} and f^{ij} in an arbitrary reference system.

In a reference system that moves, e.g., with a velocity of – $\mathfrak{v}$ relative to the medium considered, and indeed in such a way that *the velocity – $\mathfrak{v}$ is directed parallel to a principal axis of anisotropy*, the basic equations (9) [(12), resp.] can be brought into the following form ([6]):

$$(14) \qquad \left\{ \begin{aligned} D + [\mathfrak{v}\, H] &= \varepsilon(E + [\mathfrak{v}\,\mathfrak{B}]), \\ \mathfrak{B} - [\mathfrak{v}\, E] &= \mu(H - [\mathfrak{v}\, D]). \end{aligned} \right.$$

([6]) We shall pass over the intermediate calculations here.

These equations are then entirely similar to the usual Minkowski equations for isotropic media. The single difference consists in the fact that what one denotes by the symbols ε and μ are not scalars here, but tensors:

$$(\varepsilon) = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \qquad (\mu) = \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}.$$

For that reason, e.g., the product εE will have the following meaning:

$$\varepsilon E = i\,\varepsilon_{1\alpha} E_\alpha + j\,\varepsilon_{2\alpha} E_\alpha + k\,\varepsilon_{3\alpha} E_\alpha\,,$$

in which i, j, k are the basic unit vectors.

In conclusion, let it be remarked that we tacitly made the assumption above that the principal axes of the two spatial tensors ε_{ij} and μ_{ij} coincided with each other, which does not necessarily need to be the case in crystals of the lowest symmetry classes. One can remove that restriction if one imagines that any four-dimensional tensor can be decomposed into a series of spatial tensors when one restricts oneself to purely spatial coordinate transformations ([7]). Generally speaking, a tensor of rank four will decompose into 16 spatial tensors. Only four of those spatial tensors will be non-zero for the special tensor $s^{ij\alpha\beta}$ ($s_{ij\alpha\beta}$, resp.), namely, an "internal" tensor of rank four, two identical "outer" tensors of rank two, and finally a tensor s_{4444} of rank zero (i.e., a scalar). In that way, one shows that the components of the "internal" tensor depend exclusively upon the μ_{ij}, while the "external" ones depend upon only the ε_{ij} .

Should the principal axes of the ellipsoids ε_{ij} and μ_{ij} not coincide, one would then form the numerical matrix (11) and then subject its internal (external, resp.) terms to a purely-spatial coordinate transformation, while leaving its external (internal, resp.) terms untouched. If one chooses that transformation in such a way that it brings the axes of the ellipsoids ε_{ij} and μ_{ij} [which were assumed to coincide in the definition of the numerical matrix (11)] into the oppositely-directed attitude then one would arrive at the correct values of the tensor components $s_{ij\alpha\beta}$.

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([7]) Cf., e.g., Weyl, *Raum-Zeit-Materie*, 5th ed., pp. 183, *et seq.*