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# The electrodynamics of anisotropic media in the special theory of relativity ([1])

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### Notation

**E** ($E_1$, $E_2$, $E_3$) – electric field strength

**H** ($H_1$, $H_2$, $H_3$) – magnetic field strength

**D** ($D_1$, $D_2$, $D_3$) – electric displacement

**B** ($B_1$, $B_2$, $B_3$) – magnetic induction

$F_{ij}$ – covariant electromagnetic tensor:

$$(F_{14}, F_{24}, F_{34}) = \mathbf{E}, (F_{23}, F_{31}, F_{12}) = \mathbf{B}$$

$f^{ij}$ – contravariant electromagnetic tensor:

$$(f^{14}, f^{24}, f^{34}) = -\mathbf{D}, (f^{23}, f^{31}, f^{12}) = \mathbf{H}$$

Although all of the recent progress in physics has come about in the domain of the theory of electrons, nonetheless, the complete, macroscopic description of electromagnetic phenomena in material bodies has not lost its significance, and that description is made in a realm in which the electromagnetic properties of a body are characterized by the values of its dielectric constant $\varepsilon$ and magnetic permeability $\mu$. As far as we know, the macroscopic equations of electrodynamics have not yet been extended to the case of moving, *anisotropic* media. In what follows, we shall attempt to justify the corresponding generalization of the famous Minkowski equation by following a path that was pointed out to the author of this article by prof. L. I. Mandelstam. The author would once again like to express his deep feeling of gratitude to prof. L. I. Mandelstam for both posing the present problem and his constant help and attention while the author was carrying out the work.

In the classical theory of anisotropic media, the values of $\varepsilon_{ij}$ and $\mu_{ij}$ are regarded as the components of two symmetric, three-dimensional tensors:

$$\varepsilon_{ij} = \left\| \begin{matrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{matrix} \right\|,$$

([1]) Zh. R. F. Kh. O., fiz. dep., (1924), 56, vyp. 2-3, 248.

and analogously for $\mu_{ij}$. In the derivation of the equations of moving anisotropic media, it becomes necessary to combine these two three-dimensional tensors of rank two into one four-dimensional tensor of rank four. This agrees completely with the fact that in relativistic electrodynamics, two three-dimensional tensors of rank one (i.e., two vectors: the first of which is the electric field strength **E**, and the second of which is the magnetic field strength **B**) are combined into one four-dimensional electromagnetic tensor of rank two $F_{ij}$. The interpretation of the values of $\varepsilon_{ij}$ and $\mu_{ij}$ as a relationship between $F_{ij}$ (viz., **E** and **B**) and $f^{ij}$ (viz., **D** and **H**) will not involve the velocity tensor $u^i$, because the role of the velocity components will be reduced to a subordinate role as the parameters of a Lorentz transformation. § 1 will be dedicated to the interpretation of those Minkowski equations for which the values of $\varepsilon$ and $\mu$ are regarded as the components of a four-dimensional tensor, not as numbers.

In this article, we shall confine ourselves to the special theory of relativity. However, the tensor form of the equations that we will obtain will allow us to convert them into the general theory. In the near future, we expect to present the results of that generalization of the equations, and in particular, to apply them to the optics of anisotropic media.

**§ 1.** First of all, we shall consider the electromagnetic equations for isotropic media and give them an interpretation that is somewhat different from the usual one, but which will be necessary if we are to address the underlying problem of deriving the equations of anisotropic media.

In this paragraph, for the sake of simplicity, we shall suppose that we are dealing with a dielectric in which there are no free electric charges.

The differential equations of the field – viz., div $f = 0$, rot $F = 0$ – remain valid for anisotropic media, so we shall not need to return to them. The main interest for us in the equations that establish relations between the tensors $f^{ij}$ and $F_{ij}$, namely, the Minkowski equations:

$$\mathbf{D}' + [\mathbf{v}\,\mathbf{H}'] = \varepsilon(\mathbf{E}' + [\mathbf{v}\,\mathbf{B}']), \qquad \mathbf{B}' - [\mathbf{v}\,\mathbf{E}'] = \mu(\mathbf{H}' - [\mathbf{v}\,\mathbf{D}']). \tag{1}$$

Here, as well as in all of what follows, a value that is given a prime will be measured in the moving system (with respect to the body in question).

The derivation of formula (1) is based upon two assumption or postulates. First of all, one postulates that the usual relations are valid in the stationary system (with respect to the moving body), namely:

$$B_i = \mu\, H_i\,, \qquad E_i = \frac{1}{\varepsilon} D_i\,. \tag{2}$$

These equations can be put into the following form:

$$F_{12} = \mu f^{12}, \qquad F_{14} = -\frac{1}{\varepsilon} f^{14} \tag{3}$$

by using the components of the tensors $F_{ij}$ and $f^{ij}$, along with analogous expressions for $F_{21}$, $F_{13}$, $F_{24}$, etc.

In the second place, it is, of course, necessary to postulate that $F_{ij}$ and $f^{ij}$ are essentially components of tensors, which will determine the character of their transformations. The argument that is based upon these assumptions and will result in formula (1) proceeds as follows:

Assume that some Lorentz transformation $A$ will take the moving system $S'$ to a state of rest (relative to the moving body); the components of the tensor considered in the system $S$ will then be expressed in terms of the components of the tensor in the system $S'$ by way of some functions $\varphi$ and $\psi$:

$$F_{ij} = \varphi_{ij}\,(F'_{pq}, A), \qquad f^{ij} = \psi^{ij}\,(f'^{pq}, A), \tag{4}$$

whose arguments will include parameters of the transformation $A$. However, we get the desired relations between the components of the tensors $F'_{ij}$ and $f'^{ij}$ in the moving system from equations (3) and (4) in the components $F_{ij}$ and $f^{ij}$; these relations will include both the values of $\varepsilon$ and $\mu$ and the parameters of the transformation $A$. However, those parameters are nothing but the components of the velocity of the body relative to the system $S'$. If one takes that into account then it will be easy to get formula (1) by a simple calculation.

It is known that these formulas can be expressed in tensorial form:

$$f'_{ik}u^k = \varepsilon F'_{ik}u^k,$$
$$F'_{ik}u_l + F'_{kl}u_i + F'_{li}u_k = \mu\,(f'_{ik}u_l + f'_{kl}u_i + f'_{li}u_k),$$

in which $u^k$ is the four-dimensional (rank-one) velocity tensor ([2]):

$$u^\alpha = \frac{v^\alpha}{\sqrt{1-v^2}} \quad (\alpha = 1, 2, 3), \quad u^4 = \frac{1}{\sqrt{1-v^2}}.$$

In these formulas, the relationship between the tensors $f^{ij}$ and $F_{ij}$ is established by using a *third* tensor, namely, the velocity tensor. This is understandable: After all, every linear, tensorial dependency between two tensors (except for simple proportionalities in their components) can be established by using only a third tensor; the velocity tensor is usually chosen to be that "auxiliary" tensor. However, that is not mandatory. Indeed, in all of the foregoing, it was tacitly assumed that $\varepsilon$ and $\mu$ were numbers – i.e., *scalars*. Meanwhile in the theory of anisotropic media the corresponding values of $\varepsilon_{ij}$ and $\mu_{ij}$ are regarded as the components of two three-dimensional tensors. Of course, one can take a further step and interpret $\varepsilon$ and $\mu$ – viz., the coefficients in formula (3) – as the components of some *four-dimensional* tensor that will serve to establish a link between $F$ and $f$; the components of the velocity $v^i$ or $u^i$ will then revert to their former roles as the parameters of a Lorentz transformation.

To that end, we write equation (3) in the following form:

([2]) The speed of light *in vacuo* is assumed to be unity.

$$F_{ij} = s_{ij\alpha\beta} f^{\alpha\beta}, \tag{5}$$

where, obviously, one has the following values:

$$\begin{aligned} &s_{1212} = \mu, && \text{and the rest of the} \quad s_{12\alpha\beta} = 0, \\ &s_{1414} = -\frac{1}{\varepsilon}, && \text{and all the remaining} \quad s_{14\alpha\beta} = 0, \text{ etc.} \end{aligned} \tag{6}$$

These equations are valid only in the stationary system, for if, under the transition to another system, we transform the coefficients $s_{ij\alpha\beta}$ according to the rules of covariant tensors of rank four then the equation that will be obtained in that way:

$$F'_{ij} = s'_{ij\alpha\beta} f'^{\alpha\beta} \tag{5a}$$

will express the correct relationship between $F'_{ij}$ and $f'^{ij}$ in any coordinate system. Indeed, in view of the tensoriality of $F$ and $f$:

$$F'_{ij} = \frac{\partial x^h}{\partial x'^i}\frac{\partial x^k}{\partial x'^j} F_{hk}, \qquad f^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^\kappa}\frac{\partial x^\beta}{\partial x'^\mu} f'^{\lambda\mu};$$

substituting this into (5) will give:

$$F'_{ij} = \left( s_{hk\alpha\beta} \frac{\partial x^h}{\partial x'^i}\frac{\partial x^k}{\partial x'^j}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu} \right) f'^{\lambda\mu},$$

which are obviously equivalent to equations (5a).

Thus, the relationship between the $F_{ij}$ and $f'^{ij}$ can be established by way of the tensor $s_{ij\alpha\beta}$, whose components in the stationary system satisfy the conditions (6); the velocity tensor does not appear explicitly in (5). The assumptions that underlie our conclusions are entirely consistent with the generally-accepted assumptions that were listed at the beginning of this paragraph.

I must point out that equation (3) is not the only possible way of writing equation (2) by means of the components of the tensors $F$ and $f$. For example, we could choose to write it in the following form: $F_{12} = \frac{\mu}{2}(f^{12} - f^{21})$, $F_{14} = \frac{1}{2\varepsilon}(f^{14} - f^{41})$, etc., which would change the form of the tensor $s_{ij\alpha\beta}$. However, we shall focus on equation (3), because they have the following advantages: In this, and only this, form will the fourth-rank tensor $s_{ij\alpha\beta}$ reduce to the square of a second-rank tensor; in other words, there will exist a tensor $s_{i\alpha}$, such that:

$$s_{ij\alpha\beta} = s_{i\alpha}\, s_{j\beta}. \tag{7}$$

The proof of the uniqueness of the tensor $s_{i\alpha}$ is given in the Appendix; as for the form of its components, they are easily determined from (6) and (7) in the stationary system, namely:

$$s_{ij} = \begin{Vmatrix} \sqrt{\mu} & 0 & 0 & 0 \\ 0 & \sqrt{\mu} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\varepsilon\sqrt{\mu}} \end{Vmatrix}. \tag{8}$$

Due to that fact, the tensor equation (5) is properly written as follows:

$$F_{ij} = s_{i\alpha}\, s_{j\beta}\, f^{\alpha\beta}. \tag{9}$$

Solving this equation for $f^{ij}$ will give:

$$f^{ij} = s^{i\alpha}\, s^{j\beta}\, F_{\alpha\beta}\,, \tag{10}$$

in which the coefficients $s^{ij}$ will have the following values:

$$s^{ij} = \begin{Vmatrix} \dfrac{1}{\sqrt{\mu}} & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{\mu}} & 0 \\ 0 & 0 & 0 & -\varepsilon\sqrt{\mu} \end{Vmatrix} \tag{11}$$

in the rest system.

Under the transition to another coordinate system, the values $s^{ij}$ will transform by the rule for the transformation of the components of a contravariant tensor of rank two. This follows from the fact that the $s^{ij}$ are the minor determinants that are formed from the given $s_{ij}$. In other words, the relationship between $s^{ij}$ and $s_{ij}$ is completely analogous to the one between $g^{ij}$ and $g_{ij}$ in the general theory of relativity:

$$s^{ij}\, s_{ij} = \delta^i_k \qquad (\delta^i_k = 0 \text{ when } j \neq k, \quad \text{and} \quad \delta^i_k = 1 \text{ when } j = k).$$

As an example, we use the foregoing to determine the relationship between the vectors **B**, **E**, and **H**, **D** in a uniformly-moving system. Let the speed of the coordinate system with respect to a material medium equal $q$ and let it be directed along the $x$-axis. Applying the Lorentz transformation:

$$a^i_j = \left\| \begin{matrix} \beta & 0 & 0 & -q\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q\beta & 0 & 0 & -\beta \end{matrix} \right\|, \qquad \left( \beta = \frac{1}{\sqrt{1-q^2}} \right) \tag{12}$$

to the tensor $s^{ij}$ will give $s'^{ij} = a^i_\alpha a^j_\beta s^{\alpha\beta}$. If we perform the calculations then that will give:

$$s'^{ij} = \left\| \begin{matrix} \frac{+\beta^2}{\sqrt{\mu}}(1-q^2\varepsilon\mu) & 0 & 0 & \frac{q\beta^2}{\sqrt{\mu}}(1-\varepsilon\mu) \\ 0 & \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ \frac{q\beta^2}{\sqrt{\mu}}(1-\varepsilon\mu) & 0 & 0 & \beta^2\varepsilon\sqrt{\mu}\left(1-\frac{q^2}{\varepsilon\mu}\right) \end{matrix} \right\| .$$

Substituting these values of $s'^{ij}$ into, e.g., the expression for $D'_2$:

$$D'_2 = f'^{42} = s'^{4i} s'^{2j} F'_{ij} = s'^{22}(s'^{41}F'_{12} + s'^{44}F'_{42})$$

will give:

$$\frac{1}{\beta^2} D'_2 = \varepsilon \left(1 - \frac{q^2}{\varepsilon\mu}\right) E'_2 - \frac{q}{\mu}(1-\varepsilon\mu) B'_3 ,$$

which is one of Minkowski's equations, when it is solved for $D'_2$.

**§ 2.** If we turn to the more general case of an arbitrary material medium with conductivity $\sigma$, and in which there are free charges $\rho$, then our problem will be to express the relationship between the current tensor $I^i$, electric field strength $E$, conductivity $\sigma$, and the charge density $\rho$ in tensorial form. Considerations that are quite similar to the ones in the preceding paragraph will lead to the following form for that relationship:

$$I^i = \sigma^{i\alpha\beta} F_{\alpha\beta} + \rho^i , \tag{13}$$

in which the speed clearly does not appear. In this, $\rho^i$ denote the components of the tensor that has the following values in the stationary system:

$$(\rho^i) = (0, 0, 0, \rho); \tag{14}$$

In the same system, the components of the tensor rank three $\sigma^{i\alpha\beta}$ can be defined by one of the following expressions: Either:

$$\sigma^{114} = \sigma^{224} = \sigma^{334} = \sigma, \qquad \text{and the remaining} \qquad s^{i\alpha\beta} = 0, \tag{15}$$

or

$$\sigma^{114} = \sigma^{224} = \sigma^{234} = -\sigma^{141} = -\sigma^{242} = -\sigma^{344} = \tfrac{1}{2}\sigma, \quad \text{and the remaining } s^{i\alpha\beta} = 0. \tag{15a}$$

It is easy to show that such a tensorial interpretation of the values of $\sigma$ and $\rho$ will lead to Minkowski's equations. Indeed, in the moving coordinate system, equations (13) will be replaced with the following ones:

$$I'^i = \sigma'^{i\alpha\beta} F'_{\alpha\beta} + \rho'^i. \tag{13a}$$

Express $\sigma'^{i\alpha\beta}$ and $\rho'^i$ in terms of $\sigma$ and $\rho$, make use of the relations (14) and (15) and the transformation formula:

$$\sigma'^{i\alpha\beta} = a^i_j a^\alpha_\lambda a^\beta_\mu \sigma^{j\lambda\mu} \quad \text{and} \quad \rho'^i = a^i_j \rho^j$$

[in which $a^i_j$ are the coefficients of the transformation (12)], and insert the values thus found into (13a); the result will be:

$$I'^1 = \beta\sigma E'_1 - q\beta\rho, \quad I'^2 = \beta\sigma(E'_1 + qB'_3),$$
$$I'^4 = -\beta\sigma E'_1 + \beta\rho, \quad I'^3 = \beta\sigma(E'_3 - qB'_2).$$

Here, $I'^4$ is nothing but the charge density, as measured in the moving system, i.e., with the usual notation, $I'^4 = \rho'$. Eliminating $\rho$ from the two remaining equations and replacing $I'^4$ with $\rho'$, will give:

$$\sigma E'_1 = \beta(I'^1 + q\,\rho'),$$

and from the equation for $I'^4$:

$$\rho' = \beta(\rho - q\sigma E'_1).$$

It is easy to see that these equations, instead of the two relations on the right above, will coincide with existence of the Minkowski system of equations.

**§ 3.** The interpretation of Minkowski's equations above greatly facilitates the task of generalizing to the case of an *anisotropic* medium. Strictly speaking, we can only repeat the arguments in a slightly-modified form.

For simplicity, we first assume that the axes of the three-dimensional tensors $\varepsilon$ and $\mu$ coincide with each other (i.e., that the corresponding axes of the ellipses agree). Choose a stationary coordinate system whose spatial axes coincide with those of that tensor. The relationship between the components of the tensors $F_{ij}$ and $f^{ij}$ can obviously be written in the following way:

$$F_{12} = \mu_3 f^{12}, \qquad F_{14} = -\frac{1}{\varepsilon_1} f^{14}, \tag{16}$$

etc. ([3]). We continue to write this system of equations in tensorial form:

$$F_{ij} = s_{ij\alpha\beta} f^{\alpha\beta}. \tag{17}$$

The components of the tensor $s_{ij\alpha\beta}$ obviously have the following values in the coordinate system that we chose:

$$\begin{aligned} &s_{1212} = s_{2121} = \mu_3, \quad && s_{2323} = s_{3232} = \mu_1, \quad && s_{1313} = s_{3131} = \mu_2, \\ &s_{1414} = s_{4141} = \frac{-1}{\varepsilon_1}, \quad && s_{2424} = s_{4242} = \frac{-1}{\varepsilon_2}, \quad && s_{3434} = s_{4343} = \frac{-1}{\varepsilon_3}. \end{aligned} \tag{18}$$

The rest of the components are equal to zero, except for the elements of the main diagonal. As for those elements of the main diagonal, their values cannot be determined from the relations (16) and (17), since when one considers the anti-symmetric tensors $F_{ij}$ and $f^{ij}$, those elements will drop out of the dependency (17) in any coordinate system ([4]).

A significant different between the equations that we have considered for anisotropic media and the equations of isotropic media consists of the fact that the main electromagnetic constitutive tensor $s$ is a tensor of rank four in that case that, as is easy to see, *cannot be reduced to the square of a tensor of rank two.* In particular, it is obvious that the tensor cannot be reduced to the square of a symmetric tensor of rank two, as it can in the isotropic case. Indeed, the 12 independent values $\varepsilon_{ij}$ and $\mu_{ij}$ cannot be brought into agreement with the 10 components of a symmetric tensor of rank two. Generally speaking, a tensor of rank four should play a significant role in the relativistic theory of anisotropic media. The following article will show that the phenomenon of double refraction in such a medium leads to the conclusion that there is some other tensor that characterizes the properties of anisotropic media and is also a tensor rank four.

We return to the question of the components of the tensor $s_{ij\alpha\beta}$ in the stationary coordinate system (whose axes coincide with the principal axes of anisotropy). As is obvious in (18), the tensor $s_{ij\alpha\beta}$ is symmetric with respect to permutations of the indices in the groups $(i\alpha)$ and $(j\beta)$ and permutations of those groups with each other. Due to the fact that the only non-zero components $s_{ij\alpha\beta}$ are the ones for which one simultaneously

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([3]) There are other possible ways of writing this; e.g., $F_{12} = (\mu_3 / 2)(f^{12} - f^{21})$, etc. However, we have stopped with equation (16), because that is the system of equations that reduces to the system of equations (3) under the transition to the isotropic case.

([4]) One shows, for example, that the elements of the form $s_{pppp}$ drop out of the relationship between $F'_{ij}$ and $H'_3$ in any coordinate system:

$$\frac{\partial F'_{ij}}{\partial H'_3} = s'_{ij12} - s'_{ij21}$$

(because $H'_3 = -f'^{12} = -f'^{21}$):

$$\frac{\partial (s'_{ij12} - s'_{ij21})}{\partial s_{pppp}} = a_i^p a_j^p a_1^p a_2^p - a_i^p a_j^p a_2^p a_1^p = 0.$$

has $i = \alpha$ and $j = \beta$, we can symbolically denote the tensor $s_{ij\alpha\beta}$ in the form of a square matrix:

$$s_{ij\alpha\beta} = \left\| \begin{matrix} [\mu] & \mu_3 & \mu_2 & -\dfrac{1}{\varepsilon_1} \\ \mu_3 & [\mu] & \mu_1 & -\dfrac{1}{\varepsilon_2} \\ \mu_2 & \mu_1 & [\mu] & -\dfrac{1}{\varepsilon_3} \\ -\dfrac{1}{\varepsilon_1} & -\dfrac{1}{\varepsilon_2} & -\dfrac{1}{\varepsilon_3} & \left[\dfrac{1}{\varepsilon^2\mu}\right] \end{matrix} \right\|, \tag{19}$$

in which the elements in the positions (1, 1), (1, 2), (1, 3), … correspond to the components $s_{1111}$, $s_{1212}$, $s_{1313}$, …, resp. The form of the elements in the main diagonal, as we have already said, remains undetermined, and the corresponding locations of those elements contain only their dimensions, and are defined by the following conditions: Under the transition from an anisotropic medium to an isotropic one, the tensor $s_{ij\alpha\beta}$ [formula (18)] will go to the square of the tensor $s_{i\alpha}$ [formula (8)]. Analogously, it is possible to find the corresponding contravariant tensor, which satisfies the relations:

$$f^{ij} = s^{ij\alpha\beta} F_{\alpha\beta}, \tag{20}$$

namely:

$$s^{ij\alpha\beta} = \left\| \begin{matrix} \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_3} & \dfrac{1}{\mu_2} & -\varepsilon_1 \\ \dfrac{1}{\mu_3} & \left[\dfrac{1}{\mu}\right] & \dfrac{1}{\mu_1} & -\varepsilon_2 \\ \dfrac{1}{\mu_2} & \dfrac{1}{\mu_1} & \left[\dfrac{1}{\mu}\right] & -\varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & [\varepsilon^2\mu] \end{matrix} \right\|. \tag{21}$$

It is obviously that equations (17) and (20) are equivalent to each other, and that the tensor $s^{ij\alpha\beta}$ has the same symmetry properties as the tensor $s_{ij\alpha\beta}$. It is also easy to verify the following tensorial relation:

$$s^{ij\alpha\beta} s_{hk\alpha\beta} = \delta_h^i \delta_k^j, \qquad \text{where} \quad \begin{cases} \delta_n^i = 0 \ \text{ when } i \neq n, \\ \delta_n^i = 1 \ \text{ when } i = n. \end{cases}$$

We shall use a concrete example in order to explain how to use the formula that we have derived, and to that end, to take advantage of or to determine the relationships between the three-dimensional vectors **B′**, **E′**, and **H′**, **D′** in the moving coordinate system whose velocity $q$ is parallel to one of the principal axes of anisotropy (for example, the $x$-axis).

In order to simply the calculations, we introduce the notation:

$$s'^{\,ij\alpha\beta} - s'^{\,ij\beta\alpha} = s''^{\,ij\alpha\beta}.$$

The dependency of the components $f'^{\,14}$ $(= -D'_1)$ on the components of $B'$ and $E'$ can obviously be expressed in the following formula:

$$f'^{\,14} = \sum_{\beta}\sum_{\alpha}^{\beta>\alpha} s''^{14\alpha\beta} F'_{\alpha\beta}.$$

In other words, if one takes into account the fact that $s^{pqrs}$ is non-zero only when one has $p = r$ and $q = s$ simultaneously then that will give:

$$s''^{14\alpha\beta} = \sum_p\sum_q a_p^1 a_q^4 (a_p^\alpha a_q^\beta - a_p^\beta a_q^\alpha)\, s^{pqrs},$$

in which $a_j^i$ are the coefficients of the Lorentz transformation (12). Considering the values of these coefficients, we see that the only non-zero component of $s''^{14\alpha\beta}$ with $\beta \neq \alpha$ will be the component $s''^{1414}$:

$$s''^{1414} = a_1^1 a_4^4 (a_1^1 a_4^4 - a_4^1 a_1^4)\, s^{1414} + a_4^1 a_1^4 (a_4^1 a_1^4 - a_1^1 a_4^4)\, s^{4141}.$$

Since $s^{1414} = s^{4141} = -\,\varepsilon_1$, and $a_1^1 a_4^4 - a_4^1 a_1^4$ is equal to unity (as the determinant of a Lorentz transformation), one will have:

$$s''^{1414} = -\,\varepsilon_1,$$

and therefore:

$$f'^{\,14} = \sum_{\beta}\sum_{\alpha}^{\beta>\alpha} s''^{14\alpha\beta} F'_{\alpha\beta} = -\,\varepsilon_1 F'_{14},$$

or

$$D'_1 = \varepsilon_1 E'_{14}. \qquad (22)$$

Analogously, one gets:

$$H'_1 = \frac{1}{\mu_1} B'_1, \qquad (23)$$

and a somewhat more lengthy calculation will lead us to the formulas:

$$D'_2 = \beta^2\left(\varepsilon_2 - \frac{q^2}{\mu_3}\right)E'_2 - q\beta^2\left(\frac{1}{\mu_2} - \varepsilon_2\right)B'_3, \qquad (24)$$

$$H_3' = \beta^2\left(\frac{1}{\mu_2} - q^2\varepsilon_2\right)B_3' + q\beta^2\left(\frac{1}{\mu_3} - \varepsilon_2\right)E_2'. \tag{25}$$

Eliminating $E_2'$ from this gives:

$$B_3' = \beta^2\left(\mu_2 - \frac{q^2}{\varepsilon_3}\right)H_3' + q\beta^2\left(\mu_2 - \frac{1}{\varepsilon_2}\right)D_2'. \tag{26}$$

The last equation can be obtained directly using the relations (17), instead of equation (20). Indeed, (26) is obtained from (25) by replacing $F_{ij}$ with $f^{ij}$, and conversely, replacing $s^{ij\alpha\beta}$ with $s_{ij\alpha\beta}$ and changing the sign of $q$ (that sign change corresponds to the transition from the direct Lorentz transformation to the inverse one, which should be applied when transforming *covariant* tensors).

If one adds another equation in $E_2'$ to the previous three equations:

$$E_2' = \beta^2\left(\frac{1}{\varepsilon_3} - q^2\mu_3\right)D_2' - q\beta^2\left(\mu_3 - \frac{1}{\varepsilon_3}\right)H_3', \tag{27}$$

then that will easily put them into the following form:

$$B_3' + qE_2' = \mu_3(H_3' + qD_2'), \qquad D_2' + qH_3' = \varepsilon_2(E_2' + qB_3').$$

Similarly, we obtain two analogous equations for $B_2'$, $H_2'$, $D_3'$, and $E_3'$. It is easy to see that the formulas that we found can be reduced to the following two vectorial equations, which are entirely analogous to Minkowski's formulas in their outward appearance:

$$\mathbf{D}' + [\mathbf{v}\,\mathbf{H}'] = \varepsilon(\mathbf{E}' + [\mathbf{v}\,\mathbf{B}']), \qquad \mathbf{B}' - [\mathbf{v}\,\mathbf{E}'] = \mu(\mathbf{H}' - [\mathbf{v}\,\mathbf{D}']). \tag{28}$$

Here $v$ denotes the speed of the material medium with respect to the coordinate system considered $v = -q$, while $\varepsilon$ and $\mu$ denote the *three-dimensional tensors:*

$$\begin{Vmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{Vmatrix}, \quad \begin{Vmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{Vmatrix},$$

in which, as usual:

$$(\varepsilon E) = i\sum_{\alpha}^{3}\varepsilon_{1\alpha}E_\alpha + j\sum_{\alpha}^{3}\varepsilon_{2\alpha}E_\alpha + k\sum_{\alpha}^{3}\varepsilon_{3\alpha}E_\alpha, \quad \text{and etc.}$$

It is obvious that formula (28) will remain valid for any rotation of the coordinate axes, provided that the motion takes place parallel to the principal axes of anisotropy.

When considering the general case of a velocity with an arbitrary direction, it will be necessary to return to the fundamental equations (17) and (20).

**§ 4.** In the previous paragraph, we assumed that the three-dimensional axes of the ellipsoids $\varepsilon$ and $\mu$ coincided with each other. (This is true for all systems in crystals, except for monoclinic and triclinic ones.) In order to eliminate that assumption, let us consider the case of a purely spatial rotation of our initial axes (i.e., axes that are at rest relative to the crystal and parallel to its principal axes). It is known that under that constraint, the character of the transformation of every four-dimensional tensor will split into a number of independent three-dimensional spatial tensors ([5]). In particular, the tensor $s_{ij\alpha\beta}$ splits into:

1) A spatial tensor of rank two that is composed of the "internal" components of the tensor $s_{ij\alpha\beta}$ – i.e., none of the indices of the component is equal to four.

2) Two spatial tensors of rank two:

$$\left.\begin{array}{l} s_{i4\alpha 4} \\ s_{4j4\beta} \end{array}\right\} \qquad i, \alpha, j, \beta = 1, 2, 3,$$

whose corresponding indices have identical values, and finally:

3) A spatial tensor of rank zero (i.e., a scalar) that corresponds to the term $s_{4444}$ ([6]).

One should pay attention to the fact that the components of the first tensor depend upon only the $\mu_1$, $\mu_2$, and $\mu_3$, and the components of rank two depend upon only the numbers $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ .

In other words, we can say that in the stationary (with respect to the body in question) coordinate system, the electromagnetic constitutive tensor $s_{ij\alpha\beta}$ will split into one magnetic and two identical electric spatial tensors *that transform independently of each other* (if we ignore the invariant element $s_{4444}$). Whereas the magnetic constitutive tensor has rank four, the electric one has rank two, which is in complete agreement with the well-known distinction between the magnetic vectors **H** and **B** and the electric ones **E** and **D** (since the former are, in essence, tensors of rank two). By simple calculations that we shall not give here, we can show that the character of the transformation of the magnetic components of the *fourth-rank* constitutive tensor is consistent with that of the transformation of the usual *second-rank* tensor:

([5]) Cf., e.g., H. Weyl, *Raum-Zeit-Materie*, 5th ed., Berlin, J. Springer, 1923, pp. 183, or M. Laue, *Relativitätstheorie*, 1st ed., Bd. II, Braunschweig, F. Vieweg and Sohn, 1921, pp. 63.

([6]) Generally speaking, a four-dimensional tensor of rank four will split into 16 spatial tensors, but in our particular case of the tensor $s_{ij\alpha\beta}$ , all of these tensors will be zero identically, except for the ones that were listed in the text.

$$\left\| \begin{matrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{matrix} \right\|.$$

[The proof of that is based upon the well-known theorem that states that every minor determinant is orthogonal to its corresponding adjunct (i.e., its algebraic complement).]

All of the above gives us the right to make the following assumption: In the case for which the ellipsoidal axes of $\varepsilon$ and $\mu$ coincide with each other, the values of the tensor components $s_{ij\alpha\beta}$ (or $s^{ij\alpha\beta}$) can be found in the following way: Let the spatial transformation $A$ correspond to the rotation of the coordinates that makes the axes of the ellipsoid of $\mu$ coincide with the axes of the ellipsoid of $\varepsilon$. The components of the fourth-rank matrix transform by the same rules as the tensor (19), and are then transformed by $A$ *into one with only internal* elements (i.e., ones that depend upon $\mu$), while leaving the remaining terms unchanged. The elements of the matrix thus-obtained will correspond to the components of the tensor $s_{ij\alpha\beta}$ in the coordinate system whose axes coincide with the ellipsoidal axes of $\varepsilon$.

## Appendix (to pp. 5)

Proof of the uniqueness of the tensor $s_{ij}$ : In other words, the proof that the values of the components of that tensor (in the stationary system) are determined uniquely by equation (9): $F_{ij} = s_{i\alpha}\, s_{j\beta}\, f^{\alpha\beta}$.

From a comparison of these equations with the relations $B_1 = \mu\, H_1$, $B_2 = \mu\, H_2$, $B_3 = \mu\, H_3$, one will get the following equalities:

$$s_{11}\, s_{22} - s_{12}\, s_{21} = \mu, \tag{a}$$
$$s_{11}\, s_{33} - s_{13}\, s_{31} = \mu, \tag{b}$$
$$s_{22}\, s_{33} - s_{23}\, s_{32} = \mu. \tag{c}$$

Moreover, since $F_{12}$ does not depend upon $H_2$ – i. e., $\partial F_{12} / \partial H_2 = 0$ – and furthermore, since $\partial F_{12} / \partial H_1 = 0$, $\partial F_{13} / \partial H_1 = 0$, $\partial F_{13} / \partial H_3 = 0$, $\partial F_{23} / \partial H_3 = 0$, and $\partial F_{23} / \partial H_3 = 0$, one will have:

$$s_{11}\, s_{23} - s_{13}\, s_{21} = 0, \quad (d) \qquad s_{11}\, s_{32} - s_{12}\, s_{31} = 0, \quad (g)$$
$$s_{12}\, s_{23} - s_{13}\, s_{22} = 0, \quad (e) \qquad s_{21}\, s_{32} - s_{22}\, s_{31} = 0, \quad (h)$$
$$s_{12}\, s_{33} - s_{13}\, s_{32} = 0, \quad (f) \qquad s_{21}\, s_{33} - s_{23}\, s_{31} = 0. \quad (i)$$

If none of the terms in equations (*d*) and (*e*) are non-zero then those equations will give $s_{11}\, s_{22} = s_{12}\, s_{21}$ , which contradicts equation (*a*). Therefore, at least one of the components of $s_{ij}$ must equal zero. Assume that one of the elements of the main diagonal – e.g., $s_{11}$ – is zero. From (*a*) and (*b*), one will then have $s_{12}\, s_{21} = s_{13}\, s_{31} = -\mu$, and from equation (*d*), one will get $s_{12}\, s_{21} = 0$, which are mutually contradictory. Therefore, none of the components of $s_{ij}$ that are equal to zero can belong to the main diagonal. A consideration of equations (*d*)-(*i*) will show that if the members of the main diagonal are non-zero then if any off-diagonal element equals zero then *all* of the off-diagonal elements must equal zero.

The remaining elements $s_{11}$, $s_{22}$, and $s_{33}$ are determined uniquely by the equalities (*a*), (*b*), and (*c*). (Only the signs of these elements are arbitrary.) An entirely similar argument applies to the elements that have only one or two fours in their indices.

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