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General-relativistic quantum theory of the electron

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In the following, an extension of the Dirac theory (*) will be proposed that satisfies the requirement of general covariance and likewise admits a direct derivation of the impulse-energy theorem, as was possible following the path of special relativity (**).

§ 1. We replace the Dirac condition equations for the four four-rowed matrices γ_μ :

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_\mu^\nu \qquad (\delta_\mu^\nu = 1 \text{ for } \mu = \nu \text{ and } = 0 \text{ for } \mu \neq \nu) \tag{1}$$

with

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = g_{\mu\nu} , \tag{2}$$

where $g_{\mu\nu}$ are the components of the fundamental metric tensor. As in the classical theory, they shall always be “ordinary” quantities; i.e., they shall differ from the identity matrix by only a factor, and shall therefore commute with all matrices of the group of γ_μ under multiplication. From (2), the γ_μ , just like the $g_{\mu\nu}$, are spacetime functions, in general, while they previously could be regarded as constants (***). Along with the covariant vector components γ_μ , we define the contravariant γ^μ by:

$$\gamma^\mu = \sum_\nu g^{\mu\nu}\gamma_\nu , \tag{3}$$

where the $g^{\mu\nu}$ have their well-known meaning from the general theory of relativity. It follows from (2) and (3) that:

(*) P. A. M. Dirac, Proc. Roy. Soc. (A) **117** (1928), 610 and **118** (1928), 351; Friedrich Möglich, Zeit. f. Phys. **48** (1928), 852; J. v. Neumann, *ibidem*, pp. 868.

(**) H. Tetrode, Zeit. f. Phys. **49** (1928), 858, cited as I.

(***) Dirac occasionally said that the γ_μ were functions of time; however, he meant the volume integral of $\bar{\psi}\gamma_\mu\psi$ by that.

$$\left.\begin{aligned}\gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu &= 2\delta^\mu_\nu , \\ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2g^{\mu\nu} .\end{aligned}\right\} \tag{4}$$

We define the covariant six-vector with the components:

$$\alpha_{\mu\nu} = - \alpha_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \tag{5}$$

whose square is given using (2):

$$\alpha_{\mu\nu}^2 = \tfrac{1}{2}(\gamma_\mu \gamma_\nu - g_{\mu\nu})(g_{\mu\nu} - \gamma_\nu \gamma_\mu) = g_{\mu\nu}^2 - g_{\mu\mu}\, g_{\nu\nu} . \tag{6}$$

These are then ordinary quantities, and are likewise assumed to be the negative second-order sub-determinants of the determinant g of the fundamental tensor.

We further define the scalar density:

$$\gamma = \frac{1}{4!}\sum_{\mu,\nu,\rho,\sigma} \delta_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \tag{7}$$

($\delta_{\mu\nu\rho\sigma} = +1$ or $= -1$, according to whether $\mu\nu\rho\sigma$ is an even or odd permutation of the indices 1, 2, 3, 4; $\delta_{\mu\nu\rho\sigma} = 0$ whenever μ, ν, ρ, σ are not all different.) Its square γ^2 consists of the sum of products of every eight factors γ_μ in which each γ_μ appears twice. By repeated application of the commutation relations (2), we can thus ultimately arrive at an expression in which only equal γ_μ appear next to each other, and then replace the γ_μ^2 with $g_{\mu\nu}$. This makes γ^2 a sum of products of four factors $g_{\mu\nu}$, amongst whose eight indices each of the numbers 1, 2, 3, 4 must appear twice. Since γ^2 has the transformation character of the square of a scalar density, we conclude that it must equal the determinant g of $g_{\mu\nu}$, up to a numerical factor. For the special case (1), the factor is determined to be 1, and one therefore has:

$$\gamma^2 = g. \tag{8}$$

The covariant structure that corresponds to γ is:

$$\beta = \frac{1}{4!}\sum_{\mu,\nu,\rho,\sigma} \delta_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma , \tag{9}$$

whose square is obtained in the same way as above:

$$\beta^2 = \frac{1}{g}. \tag{10}$$

Furthermore, one has:

$$\beta\,\gamma = \frac{1}{(4!)^2}\sum_{\mu,\nu,\rho,\sigma,\alpha,\beta,\kappa,\lambda} \delta_{\mu\nu\rho\sigma}\delta_{\alpha\beta\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\alpha\gamma_\beta\gamma_\kappa\gamma_\lambda$$

$$= \frac{1}{(4!)^2} \sum_{\mu,\nu,\rho,\sigma,\alpha,\beta,\kappa,\lambda,a,b,k,l} g_{\alpha a} g_{\beta b} g_{\gamma\kappa} g_{\lambda l} \delta_{\mu\nu\rho\sigma} \delta_{\alpha\beta\kappa\lambda} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^a \gamma^b \gamma^k \gamma^l \, .$$

Since:

$$\frac{1}{(4!)^2} \sum_{\alpha,\beta,\kappa,\lambda} \delta_{\alpha\beta\kappa\lambda} g_{\alpha a} g_{\beta b} g_{\gamma\kappa} g_{\lambda l} = \delta_{abkl} \, g \, ,$$

this yields:

$$\beta \gamma = \beta^2 \gamma = 1, \tag{11}$$

when one observes (10).

Finally, we define the contravariant vector density with the components:

$$\beta^\mu = \frac{1}{3!} \sum_{\nu,\rho,\sigma} \delta_{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \, , \tag{12}$$

and find, when an argument that is similar to the one above is applied to the derivation of (8):

$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = - 2 \, g \, g_{\mu\nu} \, . \tag{13}$$

§ 2. The general-covariant generalizations of the previous equations I, (8) and (7) are obviously:

$$\sum_\mu \gamma^\mu \left(\frac{h}{2\pi} \frac{\partial \chi}{\partial x^\mu} + \frac{ie}{c} \varphi_\mu \chi \right) - mc \, \chi = 0 \tag{14}$$

and

$$\sum_\mu \left(- \frac{h}{2\pi} \frac{\partial \varpi}{\partial x^\mu} + \frac{ie}{c} \varphi_\mu \varpi \right) \gamma^\mu - mc \, \varpi = 0, \tag{15}$$

where the latter equation was written with the other sequence of factors.

Under a canonical transformation:

$$\gamma_\mu \to S \, \gamma_\mu \, S^{-1}, \qquad \chi \to S \, \chi, \qquad \omega \to \omega S^{-1}, \tag{16}$$

(14) goes to:

$$S \left\{ \sum_\mu \gamma^\mu \left(\frac{h}{2\pi} \frac{\partial \chi}{\partial x^\mu} + \frac{h}{2\pi} S^{-1} \frac{\partial S}{\partial x^\mu} \chi + \frac{ie}{c} \varphi_\mu \chi \right) - mc\chi \right\} = 0,$$

or, after multiplying by S^{-1}:

$$\sum_\mu \gamma^\mu \left(\frac{h}{2\pi} \frac{\partial \chi}{\partial x^\mu} + \frac{h}{2\pi} S^{-1} \frac{\partial S}{\partial x^\mu} \chi + \frac{ie}{c} \varphi_\mu \chi \right) - mc \, \chi = 0, \tag{17}$$

and analogously for (15). Therefore, if an arbitrary canonical transformation is allowed, our system of equations would no longer be invariant, in general. In the previous case of constant $g_{\mu\nu}$, it seemed sensible to restrict to constant transformation matrices S, from which (17) becomes identical with (14). Now, since the γ_μ are spacetime functions, one must also assume the same thing for S. Thus, since the invariance of (14) must be demanded, we can only allow those S for which one has:

$$\sum_\mu \gamma^\mu S^{-1} \frac{\partial S}{\partial x^\mu} = \sum_\mu \gamma^\mu \frac{\partial f}{\partial x^\mu}, \tag{18}$$

with an "ordinary" quantity f. (16) then simply means the replacement of χ with $\chi\, e^f$, which is only a change of notation, and is therefore physically inessential.

We will now show that this condition is fulfilled when the rotation and divergence of γ_μ remain unchanged under (16); in formulas:

$$\frac{\partial \gamma_\mu}{\partial x^\nu} - \frac{\partial \gamma_\nu}{\partial x^\mu} = \frac{\partial (S\gamma_\mu S^{-1)}}{\partial x^\nu} - \frac{\partial (S\gamma_\nu S^{-1})}{\partial x^\mu}$$

and

$$\sum_\mu \frac{\partial (\gamma^\mu \sqrt{-g})}{\partial x^\mu} = \sum_\mu \frac{\partial (S\gamma^\mu S^{-1}\sqrt{-g})}{\partial x^\mu},$$

or (*):

$$\left.\begin{aligned} S^{-1}\frac{\partial S}{\partial x^\nu}\gamma_\mu - \gamma_\mu S^{-1}\frac{\partial S}{\partial x^\nu} &= S^{-1}\frac{\partial S}{\partial x^\mu}\gamma_\nu - \gamma_\nu S^{-1}\frac{\partial S}{\partial x^\mu}, \\ \sum_\mu S^{-1}\frac{\partial S}{\partial x^\mu}\gamma^\mu - \sum_\mu \gamma^\mu S^{-1}\frac{\partial S}{\partial x^\mu} &= 0. \end{aligned}\right\} \tag{19}$$

In order to prove this, it obviously suffices to restrict oneself to infinitesimal canonical transformations. For them, one has $S = 1 + \zeta$, with an infinitely small matrix ζ, and therefore one has, in the first approximation:

$$S^{-1} \frac{\partial S}{\partial x^\mu} = (1 - \zeta)\frac{\partial \zeta}{\partial x^\mu} = \frac{\partial \zeta}{\partial x^\mu}. \tag{20}$$

We further choose the coordinate system for the point in question in such a way that $g_{\mu\nu} = \delta_\mu^\nu$, such that one does not need to distinguish between covariance and contravariance, and equations (1) are true. An arbitrary four-rowed matrix ζ can be represented as a linear form in the 16 linearly-independent matrices 1, $\overset{o}{\gamma}_\mu$, $\overset{o}{\gamma}_\mu \overset{o}{\gamma}_\nu$, $\overset{o}{\gamma}_\mu \overset{o}{\gamma}_\nu \overset{o}{\gamma}_\rho$, $\overset{o}{\gamma}_1 \overset{o}{\gamma}_2 \overset{o}{\gamma}_3 \overset{o}{\gamma}_4$ ($\mu \neq \nu \neq \rho \neq \mu$), where $\overset{o}{\gamma}_\mu$ is the value of γ_μ at the point in question, a value that is then to be considered as constant and must not be differentiated. On the contrary, the coefficients take the form of arbitrary functions of the coordinates. One then has:

(*) Use is made of the identity: $\dfrac{\partial S^{-1}}{\partial x^\mu} = -S^{-1}\dfrac{\partial S}{\partial x^\mu}S^{-1}$.

$$\zeta = a + a_1 \overset{\circ}{\gamma}_1 + \ldots + a_{12} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 + \ldots + a_{123} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 \overset{\circ}{\gamma}_3 + \ldots + a_{1234} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 \overset{\circ}{\gamma}_3 \overset{\circ}{\gamma}_4 ,$$

and

$$\frac{\partial \zeta}{\partial x^\mu} = \frac{\partial a}{\partial x^\mu} + \frac{\partial a_1}{\partial x^\mu} \overset{\circ}{\gamma}_1 + \ldots + \frac{\partial a_{12}}{\partial x^\mu} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 + \ldots + \frac{\partial a_{123}}{\partial x^\mu} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 \overset{\circ}{\gamma}_3 + \ldots + \frac{\partial a_{1234}}{\partial x^\mu} \overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 \overset{\circ}{\gamma}_3 \overset{\circ}{\gamma}_4 , \qquad (21)$$

where the $a_{\mu\nu}$, $a_{\mu\nu\rho}$, $a_{\mu\nu\rho\sigma}$ can be conveniently regarded as anti-symmetric in the indices, since, from (10), the same thing is true for the products $\overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2$, $\overset{\circ}{\gamma}_1 \overset{\circ}{\gamma}_2 \overset{\circ}{\gamma}_3$, etc., and therefore do not depend upon the sequence of factors as long as the indices of the corresponding a are associated. Since they are not differentiated further, we can now omit the index "o" over the γ_μ in (21), and get, from the first equation of (19) for $\mu = 1$, $\nu = 2$, with the use of (20) and (21), and when we express all products of the γ_μ in terms of the 16 linearly-independent 1, γ_1, …, $\gamma_1 \gamma_2$, etc., with the help of (1):

$$\frac{\partial a_1}{\partial x^1} = -\frac{\partial a_2}{\partial x^2}, \qquad \frac{\partial a_3}{\partial x^1} = \frac{\partial a_4}{\partial x^2} = 0, \qquad \frac{\partial a_{12}}{\partial x^1} = \frac{\partial a_{12}}{\partial x^2} = 0,$$

$$\frac{\partial a_{32}}{\partial x^1} = \frac{\partial a_{31}}{\partial x^2}, \qquad \frac{\partial a_{42}}{\partial x^2} = \frac{\partial a_{41}}{\partial x^1}, \qquad \frac{\partial a_{341}}{\partial x^1} = -\frac{\partial a_{342}}{\partial x^2}, \qquad \frac{\partial a_{1234}}{\partial x^1} = 0.$$

Since the corresponding expressions for the other five index combinations of μ, ν must be true, it follows that:

$$\frac{\partial a_1}{\partial x^1} = -\frac{\partial a_2}{\partial x^2} = \frac{\partial a_3}{\partial x^3} = -\frac{\partial a_1}{\partial x^1},$$

and therefore:

$$\frac{\partial a_1}{\partial x^1} = 0;$$

furthermore:

$$\frac{\partial a_{32}}{\partial x^1} = \frac{\partial a_{31}}{\partial x^2} = -\frac{\partial a_{13}}{\partial x^2} = -\frac{\partial a_{12}}{\partial x^3} = \frac{\partial a_{21}}{\partial x^3} = \frac{\partial a_{23}}{\partial x^1} = -\frac{\partial a_{32}}{\partial x^1},$$

so

$$\frac{\partial a_{32}}{\partial x^1} = 0.$$

Therefore, in general, one has:

$$\frac{\partial a_\mu}{\partial x^\rho} = \frac{\partial a_{\mu\nu}}{\partial x^\rho} = \frac{\partial a_{1234}}{\partial x^\rho} = 0 \qquad (\rho \text{ arbitrary}), \qquad (22)$$

$$\frac{\partial a_{\mu\nu\rho}}{\partial x^\rho} = -\frac{\partial a_{\mu\nu\sigma}}{\partial x^\sigma} \qquad (\mu, \nu, \rho, \sigma \text{ arbitrary}). \qquad (23)$$

From (21) and (22), one must then set:

$$\frac{\partial \zeta}{\partial x^\mu} = \frac{\partial a}{\partial x^\mu} + \frac{\partial a_{123}}{\partial x^\mu}\gamma_1\gamma_2\gamma_3 + \frac{\partial a_{234}}{\partial x^\mu}\gamma_2\gamma_3\gamma_4 + \frac{\partial a_{341}}{\partial x^\mu}\gamma_3\gamma_4\gamma_1 + \frac{\partial a_{412}}{\partial x^\mu}\gamma_4\gamma_1\gamma_2 . \tag{24}$$

From (20), when one substitutes this in the second of equations (19), one finds, while observing (1), that this equation fulfills (19) when:

$$\frac{\partial a_{123}}{\partial x_4} - \frac{\partial a_{234}}{\partial x_1} + \frac{\partial a_{314}}{\partial x_2} - \frac{\partial a_{412}}{\partial x_3} = 0. \tag{25}$$

From (23), (24), and (25), one finally gets:

$$\sum_\mu \gamma_\mu \frac{\partial \zeta}{\partial x^\mu} = \sum_\mu \gamma_\mu \frac{\partial a}{\partial x^\mu} . \tag{26}$$

Because the two sides of this equation are scalars after replacing γ_μ with γ^μ, it is true for any coordinate system. Therefore, (18) is fulfilled.

§ 3. We would now like to derive the differential equations (14) and (15) from a single variational principle:

$$\delta \iiint\int \mathfrak{H}\, dx^1 dx^2 dx^3 dx^4 = 0, \tag{27}$$

so we then set:

$$\mathfrak{H} = \omega \sqrt{-g} \left\{ \sum_\mu \gamma^\mu \left(\frac{h}{2\pi}\frac{\partial}{\partial x^\mu} + \frac{ie}{c}\varphi_\mu \right) - mc \right\} \chi , \tag{28}$$

which is obviously a scalar density, as required. The variation of ω then immediately yields equation (14), while the variation of χ, by contrast, yields, by partial integration:

$$\sum_\mu \left(-\frac{h}{2\pi}\frac{\partial}{\partial x^\mu} + \frac{ie}{c}\varphi_\mu \right)(\varpi\, \gamma^\mu \sqrt{-g}) - mc\,\varpi\sqrt{-g} = 0, \tag{29}$$

which, after dividing by $\sqrt{-g}$, differs from (15) by the term:

$$-\frac{h}{2\pi}\frac{\varpi}{\sqrt{-g}} \sum_\mu \frac{\partial(\gamma^\mu \sqrt{-g})}{\partial x^\mu} .$$

Nothing stands in the way now of deriving the conservation law for electricity, as well as the energy-impulse law, from equations (14) and (29) [from the variational principle (27), (28), resp.], which is precisely what we will do later on. However, because (29) is not the "transpose" of equation (14), as (15) is, we will not, as we did in I, be able to prove the correct reality character of our physical quantities (*), and this would then also go away, in general. For that reason, we subject the γ_μ to not only the algebraic conditions (2), but also:

$$\frac{1}{\sqrt{-g}}\sum_\mu \frac{\partial(\gamma^\mu\sqrt{-g})}{\partial x^\mu} = 0, \tag{30}$$

from which (29) becomes equivalent to (15).

In § 2, we demanded that the allowed canonical transformations should leave the divergence and the rotation of γ_μ unchanged. Starting with that, we now demand the vanishing of the divergence, a condition that naturally cannot be violated by any allowed canonical transformation. One can then also imagine arriving at the vanishing of the rotation of γ_μ . However, although we will not go into this further here, this condition means a restriction of the required freedom to choose the $g_{\mu\nu}$-field, since algebraic relations between the components of the Riemann curvature tensor would result from it. We thus merely prescribe the condition (30), which is indeed also satisfied, in order to be able to derive both equations (14) and (15) from the variational principle (27), (28).

§ 4. Upon multiplying (14) by $\omega\sqrt{-g}$, (15) by $\chi\sqrt{-g}$, and subtracting, what follows, while observing (30), is the conservation law for electricity:

$$\sum_\mu \frac{\partial \mathfrak{f}^\mu}{\partial x^\mu} = 0, \tag{31}$$

with

$$\mathfrak{f}^\mu = -\,\varepsilon\,\omega\,\gamma^\mu\,\sqrt{-g}\,\chi, \tag{32}$$

which would, moreover, also follow from (14) and (29), without (30).

We shall now derive the impulse-energy law by means of an infinitesimal deformation of the coordinate system (**). If we let δa denote variation of a quantity a for a fixed point, and let $\delta^* a$ denote the variation for a point that is taken along with a coordinate system then one has for the scalar density $\mathfrak{H}$:

$$\delta\int \mathfrak{H}\,dx = \int \delta^*\mathfrak{H}\,dx \equiv 0 \tag{33}$$

for arbitrary variations δx^μ that should only vanish at the boundary of the world-domain in question $\int dx = \iiiint dx^1\,dx^2\,dx^3\,dx^4$. With:

$$\delta^*\gamma^\mu = \sum_\nu \frac{\partial\,\delta x^\mu}{\partial x^\nu}\gamma^\nu - \sum_\nu \delta x^\nu\,\frac{\partial\gamma^\mu}{\partial x^\nu},$$

(*) See the addendum at the conclusion.

(**) See, e.g., W. Pauli, *Relativitätstheorie*, no. 23.

$$\delta^* \varphi_\mu = -\sum_\nu \frac{\partial \delta x^\mu}{\partial x^\nu} \varphi_\nu - \sum_\nu \delta x^\nu \frac{\partial \varphi_\mu}{\partial x^\nu},$$

one finds for the variation of the action quantity (27), from (28), after one has removed the derivatives of δx^μ by partial integration:

$$\begin{aligned}\int \delta^* \mathfrak{H}\, dx = \sum_{\lambda,\mu} \Bigg[& -\frac{\partial}{\partial x^\mu} \left\{ \varpi \sqrt{-g}\, \gamma^\mu \left(\frac{h}{2\pi} \frac{\partial}{\partial x^\lambda} + \frac{ie}{c} \varphi_\lambda \right) \chi \right\} \\ & - \varpi \sqrt{-g}\, \frac{\partial \gamma^\mu}{\partial x^\lambda} \left(\frac{h}{2\pi} \frac{\partial}{\partial x^\mu} + \frac{ie}{c} \varphi_\mu \right) \chi + \frac{\partial}{\partial x^\mu} \left(\varpi \sqrt{-g}\, \gamma^\mu \frac{ie}{c} \varphi_\mu \chi \right) \\ & - \varpi \sqrt{-g}\, \gamma^\mu \frac{ie}{c} \frac{\partial \varphi_\mu}{\partial x^\lambda} \chi \Bigg] \delta x^\lambda . \end{aligned} \tag{34}$$

In this, use has been made of the fact that the coefficient of $\delta^*(\varpi \sqrt{-g})$ vanishes as a result of (14), just as that of $\delta^* \chi$ vanishes as a result of (15) by partial integration. The factor of δx^λ in (34) must then be zero, and upon consideration of (31) and (32), one finds:

$$\begin{aligned} & \sum_\mu \frac{\partial}{\partial x^\mu} \left\{ \frac{c}{i} \varpi \sqrt{-g}\, \gamma^\mu \left(\frac{h}{2\pi} \frac{\partial \chi}{\partial x^\lambda} + \frac{ie}{c} \varphi_\lambda \chi \right) \right\} \\ + & \sum_\mu \frac{c}{i} \varpi \sqrt{-g}\, \frac{\partial \gamma^\mu}{\partial x^\lambda} \left(\frac{h}{2\pi} \frac{\partial \chi}{\partial x^\mu} + \frac{ie}{c} \varphi_\mu \chi \right) \\ + & \sum_\mu \mathfrak{f}^\mu \left(\frac{\partial \varphi_\lambda}{\partial x^\mu} - \frac{\partial \varphi_\mu}{\partial x^\lambda} \right) = 0. \end{aligned} \tag{35}$$

The last term represents the electromagnetic four-force on the unit volume, the curly bracket in the first term will thus be regarded as the impulse-energy tensor of the electron, while the second term will be the four-force density that is exerted by the gravitational field, except that the last two quantities mentioned still do not have the correct reality character. In order to arrive at that, we replace $\mathfrak{H}$ with the quantity $\mathfrak{H}'$ that arises by partial integration and which, by similar computations, can be considered to be the density of the action function:

$$\mathfrak{H}' = \left\{ \sum_\mu \left(-\frac{h}{2\pi} \frac{\partial \omega}{\partial x^\nu} + \frac{ie}{c} \varphi_\mu \varpi \right) \gamma^\mu - mc\varpi \right\} \sqrt{-g}\, \chi, \tag{36}$$

and obtain:

$$\begin{aligned} & \sum_\mu \frac{\partial}{\partial x^\mu} \left\{ \frac{c}{i} \left(-\frac{h}{2\pi} \frac{\partial \varpi}{\partial x^\lambda} + \frac{ie}{c} \varphi_\lambda \varpi \right) \sqrt{-g}\, \gamma^\mu \chi \right\} \\ + & \sum_\mu \frac{c}{i} \left(-\frac{h}{2\pi} \frac{\partial \varpi}{\partial x^\mu} + \frac{ie}{c} \varphi_\mu \varpi \right) \sqrt{-g}\, \frac{\partial \gamma^\mu}{\partial x^\lambda} \chi \end{aligned}$$

$$+\sum_{\mu}\mathfrak{f}^{\mu}\left(\frac{\partial\varphi_{\lambda}}{\partial x^{\mu}}-\frac{\partial\varphi_{\mu}}{\partial x^{\lambda}}\right)=0. \tag{37}$$

By taking the arithmetic mean of (35) and (37), one ultimately finds that:

$$\sum_{\mu}\frac{\partial\mathfrak{T}_{\lambda}^{\mu}}{\partial x^{\mu}}+\sum_{\mu}\frac{e}{2}\left\{\varpi\frac{\sqrt{-g}}{i}\frac{\partial\gamma^{\mu}}{\partial x^{\mu}}\left(\frac{h}{2\pi}\frac{\partial\chi}{\partial x^{\mu}}+\frac{ie}{c}\varphi_{\mu}\chi\right)+\left(-\frac{h}{2\pi}\frac{\partial\varpi}{\partial x^{\mu}}+\frac{ie}{c}\varphi_{\mu}\varpi\right)\frac{\sqrt{-g}}{i}\frac{\partial\gamma^{\mu}}{\partial x^{\lambda}}\chi\right\}$$
$$+\sum_{\mu}\mathfrak{f}^{\mu}\left(\frac{\partial\varphi_{\lambda}}{\partial x^{\mu}}-\frac{\partial\varphi_{\mu}}{\partial x^{\lambda}}\right)=0, \tag{38}$$

with

$$\mathfrak{T}_{\lambda}^{\mu}=\frac{c}{2}\varpi\frac{\sqrt{-g}}{i}\gamma^{\mu}\left(\frac{h}{2\pi}\frac{\partial\chi}{\partial x^{\lambda}}+\frac{ie}{c}\varphi_{\lambda}\chi\right)+\frac{c}{2}\left(-\frac{h}{2\pi}\frac{\partial\varpi}{\partial x^{\lambda}}+\frac{ie}{c}\varphi_{\lambda}\varpi\right)\frac{\sqrt{-g}}{i}\gamma^{\mu}\chi, \tag{39}$$

an expression that agrees with I, (14) for $g_{\mu\nu}=\delta_{\mu}^{\nu}$.

In the classical theory, one has a term:

$$-\frac{1}{2}\sum_{\mu,\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\mathfrak{T}^{\mu\nu}=\frac{1}{2}\sum_{\mu,\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}}\mathfrak{T}_{\mu\nu}, \tag{40}$$

in place of the second term in (38). One can also attempt to bring the corresponding term here into this form by varying the tensor $\mathfrak{T}$ in (39), when one confers the general-covariant generalization of the surface tensor $T_{\lambda\mu}-T_{\mu\lambda}$ in I, (16). However, this attempt does not succeed, as we would now like to show.

We choose a coordinate system for which $g_{\mu\nu}=\delta_{\mu}^{\nu}$ at the point in question, and that is, moreover, geodetic there. Equations (1) are true here for two infinitely neighboring points, such that we can also apply them when the γ_{μ} are differentiated once. By precisely the same reasoning that led from (14) to (16) in I, we now get:

$$\frac{2}{c}(T_{\lambda\mu}-T_{\mu\lambda})_{\lambda\neq\mu}=-\frac{h}{2\pi}\sum_{\substack{\nu\\ \lambda\neq\nu\neq\mu}}\left(\frac{\partial\varpi}{\partial x^{\nu}}\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu}\chi+\varpi\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu}\frac{\partial\chi}{\partial x^{\nu}}\right), \tag{41}$$

and for the divergence of this:

$$\frac{2}{c}\sum_{\mu}\frac{\partial}{\partial x^{\mu}}(T_{\lambda\mu}-T_{\mu\lambda})=-\frac{h}{2\pi}\sum_{\substack{\mu,\nu\\ \lambda\neq\nu\neq\mu\neq\lambda}}\left(\frac{\partial\varpi}{\partial x^{\nu}}\frac{\partial(\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu})}{\partial x^{\mu}}\chi+\varpi\frac{\partial(\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu})}{\partial x^{\mu}}\frac{\partial\chi}{\partial x^{\nu}}\right). \tag{42}$$

From the fact that here the φ_{μ} that appear in (39), and which indeed cannot be transformed away by a special choice of the coordinate system, drop out, one concludes that it is not possible to bring the four-force density of the gravitational field into the form (40) by the addition of an anti-symmetric term:

$$\text{const.}\ (\mathfrak{T}^{\lambda\mu} - \mathfrak{T}^{\mu\lambda})$$

to our original tensor $\mathfrak{T}^{\lambda\mu}$ in (39).

Added in proof: In order to be able to establish the correct reality properties of the $\mathfrak{f}^{\mu}$ and $\mathfrak{T}_{\lambda}^{\mu}$, we choose the coordinate system (x^1, x^2, x^3 real, x^4 pure imaginary) such that:

$$g\, g^{44} = 1, \qquad g^{4k} = 0 \qquad (k = 1, 2, 3). \tag{43}$$

In general, this must be possible, since we can choose four spacetime functions arbitrarily (e.g., the new coordinates as functions of the old ones), and define the left-hand sides in (43) as not, perhaps, all components of a tensor, or indeed several tensors. Thus, we tentatively set:

$$\gamma^4 = \frac{1}{\sqrt{g}}\gamma_4^0, \qquad \gamma^k = \sum_{m=1}^{3} a_m^k\, \gamma_m^0, \tag{44}$$

where the γ_μ^0 (μ = 1, 2, 3, 4) mean constant Hermitian matrices that satisfy equations (1), while the a_m^k (k, m = 1, 2, 3) shall be "ordinary" spacetime functions. From (43), (44), the second equation in (4) is then fulfilled identically for μ = 4, ν = 1, 2, 3, 4. For μ, ν = 1, 2, 3, we get:

$$g^{kl} = \sum_{m=1}^{3} a_m^k\, a_m^l \qquad (k, l = 1, 2, 3). \tag{45}$$

Since the quadratic differential form (i.e., the spatial line element) that corresponds to the g^{kl} (k, l = 1, 2, 3) is positive-definite, (45) can be fulfilled by real a_m^k, as one immediately recognizes when one makes $g^{kl} = \delta_k^l$ for a certain point by a coordinate transformation. Therefore, all of the γ^μ are Hermitian, just like the γ_μ^0. The three remaining ones suffice to satisfy the condition (30), which, from (44), is expressed in our case by the three equations:

$$\sum_{k=1}^{3} \frac{\partial a_m^k \sqrt{g}}{\partial x^k} = 0 \qquad (m = 1, 2, 3). \tag{46}$$

From (15), it now follows upon multiplication by γ_4^0 that:

$$\left(-\frac{h}{2\pi}\frac{\partial w}{\partial x^4} + \frac{ie}{c}\varphi_4 w\right)\frac{1}{\sqrt{g}} + \sum_{k=1}^{3}\left(-\frac{h}{2\pi}\frac{\partial w}{\partial x^k} + \frac{ie}{c}\varphi_k w\right)\gamma^k\gamma_4^0 - mc\, w\, \gamma_4^0 = 0, \tag{47}$$

and from (14), when one sets $\chi = \gamma_4^0\,\psi$:

$$\left(-\frac{h}{2\pi}\frac{\partial\psi}{\partial x^4}+\frac{ie}{c}\varphi_4\psi\right)\frac{1}{\sqrt{g}}+\sum_{k=1}^{3}\gamma^k\gamma_4^0\left(-\frac{h}{2\pi}\frac{\partial\psi}{\partial x^k}+\frac{ie}{c}\varphi_k\psi\right)-mc\gamma_4^0\psi = 0. \qquad (48)$$

Since, from (44), the $i\gamma^k\gamma_4^0$ (k = 1, 2, 3) are also Hermitian, equations (47) and (48) are complex-conjugate to each other, and w can then be chosen to be complex-conjugate to $\psi = \gamma_4^0\chi$. If we then make $g^{\mu\nu} = \delta_\mu^\nu$ for a certain point by a coordinate transformation then one can establish the correct reality behavior, just as in the special-relativistic case, since the derivatives of the γ^μ do not, in fact, enter into $\mathfrak{f}^\mu$ and $\mathfrak{T}_\lambda^\mu$. If we now carry out an arbitrary coordinate transformation then the reality behavior naturally remains correct.

However, from § 2, the canonical transformations are restricted to a group. We therefore cannot assume an arbitrary system of γ_μ that corresponds to a given $g_{\mu\nu}$ by way of (2), but one that must be determined, as we have done just now, and we then can only choose between the ones that arise from it by coordinate transformations, as well as the allowable canonical transformations of § 2. We will not go into certain formal questions that are connected with this notion here.

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