"Théorie pentadimensionnelle de la gravitation et de l'électromagnétisme," Sém. Janet. Méc. anal. et méc. céleste 2 (1958-1959), 1-15.

# Penta-dimensional theory of gravitation and electromagnetism 

By Yves THIRY<br>Translated by D. H. Delphenich

1. Axiomatic presentation of the theory. - The basic element of the unitary theory is comprised of a five-dimensional differentiable manifold $V_{5}$ of class $C^{2}$, such that the second derivatives of the coordinate changes on the intersection of two admissible coordinate domains are piece-wise $C^{2}$ (so $V_{5}$ has class [ $C^{2}$, piece-wise $C^{4}$ ]).

A Riemannian metric $d \sigma^{2}$ of normal hyperbolic type everywhere is defined on that manifold, and its local expression in an admissible coordinate system is:

$$
d \sigma^{2}=\gamma_{\alpha \beta} d x^{\alpha} d x^{\beta} \quad(\alpha, \beta, \text { any Greek index }=0,1,2,3,4)
$$

The fundamental tensor, whose components $\gamma_{\alpha \beta}$ are called the potentials for the coordinate system in question, and which have class $C^{1}$ with first derivatives that are piece-wise $C^{2}$, is said to determine the elementary unitary phenomenon; i.e., the motion of a charged material particle.

The following set of hypotheses constitutes the cylindricity postulate for $V_{5}$ :
The Riemannian manifold $V$ admits a connected one-parameter group of global isometries of $V_{5}$ with trajectories $z$ that are oriented to have $d \sigma^{2}<0$ and leave no point of $V_{5}$ invariant. The family of those trajectories satisfies the following hypotheses:

- They are homeomorphic to a circle $T^{1}$.
- One can find a differentiable manifold $V_{4}$ that has class [ $C^{2}$, piece-wise $C^{4}$ ], like $V_{5}$, such that there exists a differentiable homeomorphism of class [ $C^{2}$, piece-wise $C^{4}$ ] from $V_{5}$ onto the topological product $V_{4} \times T^{1}$, under which the trajectories $z$ map to the circle factors.
$V_{4}$, which can be identified with the space whose elements are the trajectories $z$, is the quotient manifold of $V_{5}$ by the equivalence relation that is defined by the isometry group.

There will then exist local coordinates $x^{\alpha}$ in $V_{5}$ that are said to be adapted to the group, which will always be used in what follows and are such that:

1. The $x^{i}$ are an arbitrary local coordinate system on $V_{4}$. The manifolds $x^{0}=$ const. are manifolds that are defined globally on $V_{5}$ and homeomorphic to $V_{4}$. The
homeomorphism of $V_{5}$ onto $V_{4} \times T^{1}$ can be assumed to map each $x^{0}=$ const. onto the manifolds that are homeomorphic to $V_{4}$ in $V_{4} \times T^{1}$.
2. The $\gamma_{\alpha \beta}$ that relate to adapted coordinates are independent of $x^{0}$. The vector $\xi$ that is the infinitesimal generator of the isometry group has contravariant components:

$$
\left\{\begin{array}{l}
\xi^{i}=0 \quad(i, \text { any Latin index }=1,2,3,4) \\
\xi^{0}=1
\end{array}\right.
$$

and its square is $\gamma_{00}$, which is negative. One sets:

$$
\xi=\sqrt{-\gamma_{00}} .
$$

3. Those coordinates are defined up to a change of coordinates:

$$
\left\{\begin{aligned}
x^{i^{\prime}} & =\psi^{i^{\prime}}\left(x^{j}\right), \\
x^{0^{\prime}} & =x^{0}+\psi\left(x^{j}\right),
\end{aligned}\right.
$$

in which $\psi$ represents the restriction of an arbitrary function $\psi(z)$ that is defined on $V_{4}$ to a local chart in $V_{4}$, and one calls this transformation a change of gauge.

In an adapted frame [which is a frame that has a point $x$ of $V_{5}$ for its origin, is orthonormal, and its first vector $\mathbf{e}_{0}$ is a unit vector that is tangent at $x$ to the trajectory $z(x)$ that passes through $x$ ], the metric on $V_{5}$ is expressed with the Pfaff forms $\omega^{\alpha}$ :

$$
d \sigma^{2}=-\left(\omega^{0}\right)^{2}+\left(\omega^{4}\right)^{2}-\left(\omega^{1}\right)^{2}-\left(\omega^{2}\right)^{2}-\left(\omega^{3}\right)^{2}
$$

in which:

$$
\omega^{0}=\xi\left(d x^{0}+\frac{\gamma_{0 i}}{\gamma_{00}} d x^{i}\right)
$$

We set:

$$
\begin{aligned}
d s^{2} & =\xi\left[\left(\omega^{4}\right)^{2}-\left(\omega^{1}\right)^{2}-\left(\omega^{2}\right)^{2}-\left(\omega^{3}\right)^{2}\right] \\
& =g_{i j} d x^{i} d x^{j},
\end{aligned}
$$

with

$$
g_{i j}=\xi\left(\gamma_{i j}-\frac{\gamma_{0 i} \gamma_{0 j}}{\gamma_{00}}\right)
$$

If one considers the vector $\varphi_{\lambda}$ on $V_{5}$ that is defined by:

$$
\beta \varphi_{\lambda}=\frac{\gamma_{0 \lambda}}{\gamma_{00}}, \quad \text { in which } \beta \text { is a constant }
$$

and its rotation $F_{\lambda \mu}$ then one can intrinsically define the following objects on the manifold $V_{4}$ :

- A scalar $\xi$,
- A metric of normal hyperbolic type:

$$
d s^{2}=g_{i j} d x^{i} d x^{j},
$$

which is conformal to the quotient metric $\left(\omega^{4}\right)^{2}-\left(\omega^{1}\right)^{2}-\left(\omega^{2}\right)^{2}-\left(\omega^{3}\right)^{2}$ that is induced by $d \sigma^{2}$ on $V_{4}$, and which we simply call the conformal metric.

- An antisymmetric tensor $F_{i j}$ and on each section $W_{4}$ that is canonically homeomorphic to $V_{4}$, one has a vector field $\varphi_{i}$ such that:

$$
\begin{aligned}
& g_{i j}=\xi \gamma_{i j}+\beta^{2} \xi^{3} \varphi_{i} \varphi_{j}, \\
& F_{i j}=\partial_{i} \varphi_{j}-\partial_{j} \varphi_{i} .
\end{aligned}
$$

That being the case, the manifold $V_{4}$, when endowed with the conformal metric $d s^{2}$ can be interpreted as the space-time of general relativity, while the $g_{i j}$ are the gravitational potentials.

The non-canonical reciprocal image on $V_{4}$ of a vector $\varphi_{i}$ that is associated with a section $W_{4}$ will be interpreted as the electromagnetic potential vector. More simply, one says that $\varphi_{i}$ is the electromagnetic potential vector, so the tensor $F_{i j}$ can be interpreted as the electromagnetic field.

The hypotheses that were made on $V_{5}$ will induce the metric $d s^{2}$ on $V_{4}$, as well as inducing the classical hypotheses of the axiomatics of the provisional theory of electromagnetism on the electromagnetic field tensor.
2. The provisional theory of electromagnetism. - Before commencing with our elaboration of the equations of the theory, we shall present a rapid critique of the provisional theory of electromagnetism, since that study and the ambition to unify the gravitational field and the electromagnetic field (at least in the broad sense) is the basis for the genesis of the penta-dimensional theory that is presented here.

In the provisional theory of electromagnetism, the gravitational field that is defined by the geometric structure that is adopted for the universe will take on a satisfactory explanation. By contrast, the electromagnetic field is simply superimposed over that geometric structure by its introduction into the right-hand side of the Einstein equation with the aid of an energy-impulse tensor that is defined by starting with some concepts of special relativity.

In order to write down the equations of the provisional theory, let us now introduce some inductions that are distinct from the field, and let $F_{i j}$ represent the tensor of magnetic induction (B) and electric field (E), while $H_{i j}$ represents the tensor of the magnetic field $(\mathbf{H})$ and the electric induction $(\mathbf{D})$. Upon assuming the following relations between the fields and the inductions:

$$
\begin{aligned}
& \mathbf{H} \tau=\mathbf{B} . \\
& \mathbf{D}=\varepsilon \mathbf{E},
\end{aligned}
$$

in which $\tau$ is the magnetic permeability and $\varepsilon$ is the dielectric strength, with $\varepsilon \tau=1$, then one will have the relation:

$$
H_{i j}=\varepsilon F_{i j}
$$

between the tensors $H_{i j}$ and $F_{i j}$.
Under those conditions, we write out the energy-impulse tensor of the pure electromagnetic field schema in the form:

$$
\tau_{i j}=\frac{1}{4} g_{i j} H_{k l} F^{k l}-\frac{1}{2}\left(H_{i}{ }^{k} F_{j k}+H_{j k} F_{i}^{k}\right) .
$$

The equations of the provisional theory are composed of:

- The Einstein equations:

$$
S_{i j}=\lambda T_{i j},
$$

- The first group of Maxwell equations:

$$
\nabla_{j} H^{j i}=J^{i},
$$

- The second group of Maxwell equations:

$$
\frac{1}{2} \eta^{j k l i} \nabla_{j} F_{k l}=0,
$$

which is equivalent to saying that there locally exists a potential vector $\varphi_{i}$ (whose global existence we shall assume) such that:

$$
F_{i j}=\partial_{i} \varphi_{j}-\partial_{j} \varphi_{i} .
$$

In the pure electromagnetic field schema, $T_{i j}$ reduces to $\tau_{i j}$, and $J^{i}=0$.
In the electromagnetic field-matter schema, one takes:

$$
T_{i j}=\rho u_{i} u_{j}+\tau_{i j}
$$

and one assumes the Lorentz transport equation:

$$
J^{i}=\mu u^{i},
$$

in which $\mu$ is the pure electric charge density.
One can then show that $k=\mu / \rho$ remains constant along a streamline, as we consider only the homogeneous schema, for which $k$ is an absolute constant.

For a given field that corresponds to a pure electromagnetic schema, if one considers a small charged test mass with charge / mass that is equal to $k$ and one represents its interior field by a pure matter schema then it will result from a study of the matching
conditions on its world-tube, which is generated by streamlines of its interior field, and passing to the limit that the trajectory of that mass will satisfy the differential system:

$$
u^{i} \nabla_{i} u_{j}=k F_{j i} u^{i} .
$$

We shall now state a result that is more general then the one that we shall appeal to, but which can play a role in the interpretation of the penta-dimensional theory:

For any motion of a homogeneous-charged perfect fluid, the streamlines will be timelike lines that realize an extremum for the integral:

$$
\int_{x_{0}}^{x_{1}} F d s+k \varphi
$$

for variations with fixed extremities, in which $\varphi$ is the vector potential form, and $F$ is the index of the fluid; i.e.:

$$
\exp \int_{p_{0}}^{p} \frac{d p}{\rho+p}
$$

when one assumes an equation of state $\rho=\varphi(p)$, with $k=\mu / \rho^{*}, \rho^{*}=(\rho+p) / F$.
Upon using that result for the charged pure matter schema, which one can pass to by setting:

$$
p=0, \quad F=1, \rho^{*}=\rho,
$$

and upon returning to the test mass, one will see that the trajectories of that charged particle are the time-like lines that realize the extremum of the integral:

$$
\int_{x_{0}}^{x_{1}}\left[\left(g_{i j} \dot{x}^{i} \dot{x}^{j}\right)^{1 / 2}+k \varphi_{i} \dot{x}^{i}\right] d u, \quad \dot{x}^{i}=\frac{d x^{i}}{d u} .
$$

Those trajectories are then geodesics of the Finsler manifold that admits the metric:

$$
L\left(x^{i}, \dot{x}^{i}\right)=\left(g_{i j} \dot{x}^{i} \dot{x}^{j}\right)^{1 / 2}+k \varphi_{i} \dot{x}^{i} .
$$

3. The geodesics of $V_{5}$ and their projections onto $V_{4}$. - Now return to the Riemannian manifold $V_{5}$ that was envisioned in paragraph 1. Set:

$$
\mathcal{L}^{2}=\gamma_{\alpha \beta} \dot{x}^{\alpha} \dot{x}^{\beta}
$$

and suppose that $\gamma_{00} \neq 0$.
One can then establish the following result:

The extremals on $V_{5}$ of the integral $\int_{x_{0}}^{x_{1}} \mathcal{L} d u$ that correspond to the value $h$ of the first integral $\partial_{0} \mathcal{L}=h$ project onto $V_{4}$ along extremals of the integral:

$$
\int_{x_{0}}^{x_{1}} L\left(x^{i}, \dot{x}^{j}, h\right) d u,
$$

in which $h$ has the same value and $L$ is given by:

$$
L=\sqrt{\frac{1+h^{2} / \xi^{2}}{\xi} g_{i j} \dot{x}^{i} \dot{x}^{j}}+h \beta \varphi_{i} \dot{x}^{i},
$$

in which $g_{i j}$ and $\varphi_{i}$ have the meanings that they were given in paragraph 1.
That expression for $L$ is valid for the geodesics of $V_{5}$ that make $\mathcal{L}^{2}$ positive. In order to make $\mathcal{L}^{2}$ negative, which must likewise play a role in the interpretation of the theory, one sets $i h=h_{1}$ and considers the function:

$$
L_{1}=i\left[\sqrt{\frac{h_{1}^{2} / \xi^{2}-1}{\xi} g_{i j} \dot{x}^{i} \dot{x}^{j}}+h_{1} \beta \varphi_{i} \dot{x}^{i}\right]
$$

instead of $L$. The functions $L$ have the type of the square root of a quadratic form that is augmented by a linear form of the kind that defined the trajectories of the charged particle of the provisional theory in the preceding paragraph.

The penta-dimensional unitary theory was presented axiomatically, and its properties were presented as things that are implicit from those axioms. However, the genesis of the theory must naturally follow along a different path: A problem in the calculus of variations lead to a "descent" process from a function $\mathcal{L}$ to a function $L$ and an "ascent" process from $L$ to $\mathcal{L}$. In the case where $\gamma_{00} \neq 0$, when the descent process is applied to a Riemannian metric $\mathcal{L}$, it will lead to a function $L$ that has the type of a square root of a quadratic form plus a linear form. Historically, one developed a cylindrical manifold $V_{5}$ by the inverse ascent process, which starts with the function that defines the trajectories of a charged particle in the provisional theory.

Return to the function $L$ and try to relate it to the provisional theory. One will then be led to set:

$$
\begin{equation*}
k=\beta h \sqrt{\frac{\xi}{1+h^{2} / \xi^{2}}}, \quad \text { or for } L_{1}, \quad k=\beta h_{1} \sqrt{\frac{\xi}{h_{1}^{2} / \xi^{2}-1}} \tag{3.1}
\end{equation*}
$$

and to study the variations of $x^{0}$ by specifying $\partial_{0} \mathcal{L}=h$, which gives:

$$
\begin{equation*}
d x^{0}=\frac{k}{\beta \xi^{3}} d s-\beta \varphi \tag{3.2}
\end{equation*}
$$

One asserts that if one supposes that $\xi$ is constant then one will recover the trajectories of the charged particle in the space-time $V_{4}$ of general relativity precisely as projections of the geodesics of $V_{5}$ along which $x^{0}$ vary according to formula (3.2). One will be in the presence of the Kaluza-Klein penta-dimensional theory, which adds nothing new to the provisional theory of electromagnetism, but simply represents it is a different formalism, which is nonetheless useful and interesting, and especially for the treatment of the global theorems in the provisional theory.

When one rejects the hypothesis that $\xi=$ const., that will lead to the pentadimensional theory whose axioms I have presented.

Consider the geodesics in $V_{5}$ along which $x^{0}$ varies according to formula (3.2), in which $h$ is a well-defined constant, and consider the projection of such a geodesic. It will result from the field equations and the matching conditions that will be adopted that the projection defines the spatio-temporal trajectory of a charged particle. That projection is extremal in $V_{4}$ of the integral that is associated with the function:

$$
\frac{1}{k} d s+\varphi,
$$

in which $k$ has the value that was given by (3.1).
If $h$ is constant, but $\xi$ varies, then that function will differ from the function $d s+k \varphi$ of the provisional theory by more than just a multiplicative constant. The variations of $\xi$ will then be regarded as weak, and $\xi$ will be regarded as close to 1 , which permits one to introduce the constant $\beta$.
4. The equations of the theory. - We take the field equations of the pentadimensional theory to be the formal generalizations of the field equations of general relativity.

In the "external unitary case," which corresponds to the pure electromagnetic field schema of general relativity, the equations are written:

$$
\mathcal{S}_{\alpha \beta}=0,
$$

in which $\mathcal{S}_{\alpha \beta}=\mathcal{R}_{\alpha \beta}-\frac{1}{2} g_{\alpha \beta} \mathcal{R}$ is the Einstein tensor on $V_{5}$. They will be characterized by the following variational principle:

They define the extremum of the integral:

$$
\int_{C} \mathcal{R}_{\alpha \beta} \gamma^{\alpha \beta} \sqrt{|\gamma|} d x^{0} \wedge d x^{1} \wedge d x^{2} \wedge d x^{3} \wedge d x^{4}
$$

which is taken over a five-dimensional differentiable C for variations of the 15 potentials and their first derivatives that are zero on the boundary of $C$.

That comes down to specifying the equations $\mathcal{S}_{\alpha \beta}=0$ solely in terms of $V_{4}$ when it is endowed with the conformal metric. The calculation of $S_{i 0}$ involves the covariant derivative, not of the electromagnetic field tensor $F_{i j}$ that was defined in paragraph 1, but of $\xi^{3} F_{i j}$. One will then be led to introduce some inductions that are distinct from the fields by setting:

$$
H_{i j}=\xi^{3} F_{i j}
$$

and to interpret $\xi^{3}$ as the dielectric constant.
In $\mathcal{S}_{i j}$, the tensor $\tau_{i j}$ is constructed in the form that was indicated in the provisional theory and only the first derivatives of $\xi$ intervene.

The equations of the external unitary case in local adapted coordinates on $V_{4}$, when it is endowed with the conformal metric:

$$
\begin{align*}
& S_{i j}+\frac{3}{2} K_{i j}-\chi \tau_{i j}=0, \\
& \nabla_{j} H_{i}^{j}=0,  \tag{4.1}\\
& \Delta_{2} \sigma+2 \Delta_{1} \sigma+\chi H_{i j} F^{i j}=0,
\end{align*}
$$

with:

$$
\begin{aligned}
K_{i j} & =\frac{1}{2} g_{i j} \Delta_{1} \sigma-\partial_{i} \sigma \partial_{j} \sigma, \\
\sigma & =\log \xi
\end{aligned}
$$

$\chi=\beta^{2} / 2$ is interpreted as the gravitational constant.
If one gives the value of 1 to $\xi$ and suppresses the fifth equation then one will recover the equations of the provisional theory of electromagnetism when they are adapted to the scope of Klein's theory.

If $\xi$ is variable then they will generalize the equations of the pure electromagnetic field schema of general relativity, and the variations of the fifteenth field variable (viz., dielectric strength) will be governed by a fifteenth equation.

In the "internal unitary case," the equations of the penta-dimensional theory are written:

$$
\mathcal{S}_{\alpha \beta}=\Theta_{\alpha \beta} .
$$

At best, $\Theta_{\alpha \beta}$ is chosen to represent a charged matter distribution.
If one uses the formal generalization of a perfect fluid schema $\Theta_{\alpha \beta}=r v_{\alpha} v_{\beta}-\Phi \gamma_{\alpha \beta}$ then the introduction of the index of the fluid (which will undoubtedly be interesting for the physical interpretation) will possibly permit one to narrow down the choice of trajectories of the provisional theory that are extremals of the integral:

$$
\int F d s+k \varphi
$$

We shall use only the formal generalization of the pure matter schema $\Theta_{\alpha \beta}=r v_{\alpha} v_{\beta}$, which is a particular case of the preceding one. $r$ is a positive scalar, in which $v_{\alpha}$ are the covariant components of a vector whose square is +1 or -1 and whose trajectories are call penta-dimensional streamlines. It results from the conservation equations that in $V_{5}$ those streamlines will be geodesics of that Riemannian manifold, and one can just as well use geodesics that are oriented to have $d \sigma^{2}<0$ as ones that are oriented to have $d \sigma^{2}>0$.

The penta-dimensional streamlines (oriented to have $d \sigma^{2}>0$ ) that satisfy the integral $\partial_{0} \mathcal{L}=0$, which translates into $v_{0}=h$, project onto $V_{4}$ along time-like streamlines such that along those lines:

$$
\frac{1+h^{2} / \xi^{2}}{\xi} d \sigma^{2}=d s^{2}
$$

and they are extremals of:

$$
\int_{z_{0}}^{z_{1}} \sqrt{\frac{1+h^{2} / \xi^{2}}{\xi}} d s+\beta h \varphi .
$$

If $u^{i}$ is the unit velocity vector with the conformal metric $d s^{2}$ on $V_{4}$ then one will have:

$$
v^{i}=\sqrt{\frac{1+h^{2} / \xi^{2}}{\xi}} u^{i}
$$

along a streamline.
The first fourteen equations of the internal unitary case are then written:

$$
\begin{gathered}
S_{i j}+\frac{3}{2} K_{i j}-\chi \tau_{i j}=r \frac{1+h^{2} / \xi^{2}}{\xi} u_{i} u_{j}, \\
\nabla_{j} H^{j}{ }_{i}=r \frac{2 h}{\beta} \sqrt{\frac{1+h^{2} / \xi^{2}}{\xi}} u_{i},
\end{gathered}
$$

and one will then be led to set:

$$
\begin{aligned}
& \chi \rho=r \frac{1+h^{2} / \xi^{2}}{\xi}, \\
& \mu=r \frac{2 h}{\beta} \sqrt{\frac{1+h^{2} / \xi^{2}}{\xi}},
\end{aligned}
$$

which will indeed give back:

$$
\frac{\mu}{\rho}=k=\beta h \sqrt{\frac{\xi}{1+h^{2} / \xi^{2}}} .
$$

The fifteenth equation can be put into the form:

$$
\Delta_{2} \sigma+2 \Delta_{1} \sigma+\chi H_{i j} F^{i j}=\frac{r}{\xi}\left(\frac{1}{3}+\frac{h^{2}}{\xi^{2}}\right) .
$$

By definition, we can write the equations of internal unitary case in the following form, which generalizes the case of the electromagnetic field-matter schema of general relativity:

$$
\begin{aligned}
& S_{i j}+\frac{3}{2} K_{i j}-\chi \tau_{i j}=\chi \rho u_{i} u_{j}, \\
& \nabla_{j} H_{i}^{j}=\mu u_{i}, \\
& \Delta_{2} \sigma+2 \Delta_{1} \sigma+\chi H_{i j} F^{i j}=\chi \rho \frac{\frac{1}{3}+\frac{h^{2}}{\xi^{2}}}{1+\frac{h^{2}}{\xi^{2}}} .
\end{aligned}
$$

When one considers the equations that correspond to the geodesics of $V_{5}$ that are oriented to have $d \sigma^{2}<0$, it will suffice to set $h_{1}=i h$ in the preceding formulas.

Remarks. - The geodesics of $V_{5}$ that are oriented to have $d \sigma^{2}>0$ can account for the trajectories in $V_{4}$ only for particles that have a very small absolute value of $\mu / \rho$. For the other ones (and that will be case of the electron and photon), one will then have to appeal to the geodesics that are oriented to have $d \sigma^{2}<0$.

One can give various other forms to the fifteenth equation that are divergence formulas and are useful for the treatment of global problems.

## 5. Study of the penta-dimensional theory. -

1. The Cauchy problem and the geodesic principle. - In the exterior unitary case, consider the following problem statement, which relates to $V_{5}$ :

- One is given the potentials $\gamma_{\alpha \beta}$ and their first partial derivatives on a hypersurface $\Sigma$ in $V_{5}$ that is generated by the trajectories of the isometry group. Determine the values of those potentials outside of $\Sigma$ when one supposes that they satisfy the equations $\mathcal{S}_{\alpha \beta}=0$.

This problem is the translation to $V_{5}$ of the following one:

- One is given the gravitational potentials $g_{i j}$, a potential vector $\varphi_{i}$, and a scalar $\xi$ on a hypersurface $S$ in $V_{4}$, along with their first derivatives. Determine the values of those quantities outside of $S$ when one supposes that they satisfy equations (4.1) and $\Sigma$ is the hypersurface in $V_{5}=V_{4} \times T^{1}$ that projects onto the points of $S$. The problem will then admit one and only solution that is cylindrical under some hypotheses that must be specified (e.g., analyticity or simple differentiability of the givens). It will then result that the characteristic manifolds $S_{c}$ of the equations of the theory in $V_{4}$ are always manifolds that are tangent to the elementary cones in $V_{4}$. The characteristic manifolds $\Sigma_{c}$ in $V_{5}$ are
generated by the trajectories that project onto an $S_{c}$ and are, in turn, tangent to the elementary cones of $V_{5}$. Those manifolds are the wave surfaces of a unitary field such that crossing them can produce discontinuities in the second derivatives of the potentials.

The study of the Cauchy problem in the interior unitary case shows that the circumstances that generalize the ones that are encountered in $V_{4}$ in general relativity for the pure matter case to $V_{5}$ correspond to the ones that are encountered in $V_{4}$ for the case of electromagnetic field-pure matter.

The study of the prolongation from the interior to the exterior and the matching conditions then lead to the geodesic principle, and as was said, that will permit one to interpret the geodesics in $V_{5}$ that are oriented to have $d \sigma^{2}>0$ or $d \sigma^{2}<0$ as the pentadimensional trajectories of charges particles in the unitary field.
2. Global study of unitary fields. - That study is based upon the search for the hypotheses under which the following propositions will be valid:

- The introduction of the matter distribution (charged or not) into a given exterior unitary field can be achieved only in domains where that field is not regular.
- An everywhere-regular exterior unitary field is trivial. The circumstances that one finds will induce circumstances in $V_{4}$ that determine whether the similar propositions in $V_{4}$ are true for the various schemas.

The global study in Klein's theory - and thus, in the provisional theory - leads to less satisfactory results.
3. Equations of motion. - The five conservation identities of the tensor $\Theta_{\alpha \beta}$, namely:

$$
D_{\alpha}\left(\Theta^{\alpha}{ }_{\beta}\right)=0 \quad\left(D_{\alpha}:\right. \text { covariant derivative operator for the Riemannian }
$$ connection on $V_{5}$ )

are equivalent to:

- The integral of the geodesics, which one can write $v_{0}=h$.
- A continuity equation that one can translate into the statement that:

$$
\int n d x^{1} \wedge d x^{2} \wedge d x^{3}, \quad n=r v^{4} \sqrt{\gamma}
$$

remains constant along a streamline.

- Three equations:

$$
v^{\alpha} D_{\alpha} v_{A}=0 \quad(A=1,2,3)
$$

which give the equations of motion in the form:

$$
\frac{d}{d t} \int_{C} \Theta_{A}^{4} \sqrt{\gamma} d V=\frac{1}{2} \int_{C} \Theta^{\alpha \beta} \sqrt{\gamma} \partial_{\alpha} \gamma_{\alpha \beta} d V
$$

(where the integrals are taken over some body $C$ ) that is used in general relativity.
The conservation identities can then be interpreted as being the penta-dimensional equations of motion of a test body.

Those equations are intimately linked with the isothermal conditions.
One can define isothermal coordinates in $V_{5}$ by a formal generalization of the isothermal coordinates that are used in general relativity. The isothermal conditions in $V_{5}$ :

$$
\Phi^{\rho} \equiv \frac{1}{\sqrt{\gamma}} \partial_{\alpha}\left(\gamma^{\alpha \beta} \sqrt{\gamma}\right)=0
$$

translate in a $V_{4}$ that is endowed with the conformal metric into the isothermal conditions:

$$
\partial_{j}\left(g^{i j} \sqrt{-g}\right)=0,
$$

and by fixing the gauge transformation upon which the electromagnetic potential depends, the penta-dimensional isothermal coordinates will be found to be compatible with the cylindrical character of $V_{5}$.

The search for an approximate unitary solution leads one to consider some equations of motion that are approximate of order $p$, and one has the following result:

The equations of motion that are approximate of order $p$ imply some isothermal conditions that are approximate of order p that must be verified.

The calculation in the first approximation shows that it is only when one interprets things in a $V_{4}$ that is endowed with the conformal metric that the potentials that relate to an uncharged schema will be identified with the ones in the pure matter case of general relativity.

The issues that were just presented indeed seem to impose the interpretation that was given here upon the theory.

## Bibliography

1. F. HENNEQUIN, "Étude mathématique des approximations en relativité générale et en théorie unitaire de Jordan-Thiry," Bull. scient. Comm. Trav. hist. et scient. 1 (1956), Part 2: Mathématiques, Gauthier-Villars, Paris, 1957; pp. 73-154 (Thesis Sc. math., Paris, 1956).
2. P. JORDAN, Schwerkraft und Weltall, $2^{\text {nd }}$ ed., Friedr. Wieweg and Son, Braunschweig, 1955. (Die Wissenschaft, 107).
3. O. KLEIN, "Quantentheorie und fünfdimensionale Relativitätstheorie," Zeit. Phys. 37 (1926), 895-906.
4. A. LICHNEROWICZ, Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955.
5. Y. THIRY, "Étude mathématique des équations d'une théorie unitaire à quinze variables du champ," J. Math. pures et appl. 30 (1951), 275-396. (Thesis, Sc. math. Paris, 1950).
