"Variétés différentiables cobordant," Colloque Int. du C. N. R. S., v. LII, *Géométrie différentielle*, Strasbourg (1953), pp. 143-149.

Cobordant differentiable manifolds

By R. THOM (Strasbourg)

Translated by D. H. Delphenich

All of the manifolds envisioned here are assumed to be compact and differentiable of class C^∞; any submanifold is assumed to be differentiably embedded of class C^∞.

1. Definitions. A space M^{n+1} of dimension $n + 1$ is a *manifold with boundary* V^n if:

1. The complement $M^{n+1} - V^n$ is a (paracompact) open subset of dimension $n+1$.
2. The boundary V^n is a manifold of dimension n.
3. At any point x of V^n, there exists a local chart (that is compatible with the given differential structures on $M^{n+1} - V^n$ and on V^n) in which the image of M^{n+1} is a half space of $\mathbb{R}^{n+1}$ that is bounded by an $\mathbb{R}^n$ that is the image of V^n.

If M^{n+1} is orientable then the boundary V^n of M^{n+1} is likewise orientable, and any orientation of M^{n+1} canonically induces an orientation on V^n. One may define that induced orientation thanks to the boundary operator in homology:

$$\partial: H_{n+1}(M^{n+1}, V^n) \to H_n(V^n).$$

Let V^n be a – not necessarily connected, but orientable and *oriented* – manifold. If there exists a compact, orientable manifold with boundary M^{n+1}, with boundary V^n, and if M^{n+1} may be endowed with an orientation such that $\partial M^{n+1} = V^n$ then one says that V^n is a *bounding manifold*. If one repeats this definition with no condition of orientability for V^n or M^{n+1} then one says that V^n is a *bounding manifold mod 2*.

For a long time now, it has been known that there exist manifolds that do not bound, notably, the ones whose Euler-Poincaré characteristic is odd. Steenrod, in [**2**], posed the question of giving the necessary and sufficient conditions for such a manifold to be a bounding manifold. We begin this problem by generalizing it as follows: Two *orientable* manifolds V^n, V'^n of the same dimension n are called *cobordant* if the manifold $V'^n - V^n$, which is the union of V'^n and V^n, when it is endowed with the opposite orientation to the given one, is a *bounding manifold*. Two manifolds that are cobordant to a third are cobordant to each other. The set of equivalence classes thus defined between oriented manifolds of dimension n will be denoted by Ω_n . The union of two manifolds represents two classes that define a law of addition on the elements of Ω_n that makes it an Abelian group (viz., the *cobordism group of dimension n*). The null class is the class of bounding manifolds. One verifies that $V + (-V) = 0$, because $V \cup (-V)$ is the boundary of the product $V \times I$, where I is the segment [0, 1]. If V^n is cobordant to V'^n, and if W^r is another manifold then it is easy to see that the product manifolds $V^n \times W^r$ and $V'^n \times W^r$ are *cobordant*. The topological product thus defines a multiplication on the direct sum of the Ω_n that is anti-commutative and distributive with respect to addition. One will denote the graded ring thus defined by Ω.

Likewise, with no condition of orientability, one defines two manifolds to be cobordant mod 2, the cobordism group mod 2 $\mathfrak{N}_k$, and the ring $\mathfrak{N}$ that is the direct sum of the $\mathfrak{N}_k$. Any element of $\mathfrak{N}$ is order 2.

Invariants of cobordism classes. – From a theorem of Pontrjagin [**3**], all of the characteristic numbers of a bounding manifold are null. (Recall that a characteristic number of an oriented manifold is the value that is taken by a characteristic class of maximum dimension on the fundamental cycle of the manifold.) As a result, if two manifolds are cobordant then their characteristic numbers are equal. These numbers are as good as the "characters" of the group Ω_n (or $\mathfrak{N}_k$). They amount to the *characteristic Pontrjagin numbers* $<\pi(P^{4r}), V^{4m}>$ that are defined for the oriented manifold of dimension $\equiv 0$ mod 4. In cobordism mod 2, they are the characteristic Stiefel-Whitney numbers $<\pi(W^i), V>$, which are integers mod 2, the fundamental class $<W^n, V^n>$ giving precisely the Euler-Poincaré characteristic reduced mod 2. Finally, we note that for an oriented manifold of dimension $4k$ the excess τ of the number of positive squares over the negative squares of the quadratic form that is defined by the intersection matrix of $2k$-cycles (in real coefficients) *is an invariant of the cobordism class.* This results with no difficulty from duality theorems for manifolds with boundaries, where the duality at issue is Poincaré-Lefschetz.

2. Classification of submanifolds. Let W_0^k, W_1^k be two oriented submanifolds of an oriented manifold V^n. Form the product $V^n \times I$, where I is the segment [0, –1]. If there exists a submanifold with boundary X^{k+1} that is embedded in $V^n \times I$, and whose boundary, which is entirely contained within boundary $(V^n, 0) \cup (V^n, 1)$ of $V^n \times I$, is composed of W_0^k, which is embedded in $(V^n, 0)$ and W_1^k, which is embedded in $(V^n, 1)$, then one says that W_0^k and W_1^k are *L-equivalent.* If W_0 and W_1 are L-equivalent to the same submanifold Y then they are L-equivalent to each other. This results from the fact that one may assume, with no restriction on generality, that the submanifold with boundary X^{k+1} meets the boundary $(V^n, 0) \cup (V^n, 1)$ of $V^n \times I$ orthogonally (for a Riemannian metric that is given in advance). One will denote the set of L-equivalence classes for oriented submanifolds of dimension k by $L_k(V)$ and the set of L-equivalence classes mod 2 for oriented submanifolds of dimension k, with no orientability conditions, by $L_k(V^n; \mathbb{Z}_2)$. If $k < n/2$ then the representatives of two classes may be assumed to be disjoint, and their union defines a law of addition on $L_k(V^n)$ that makes it an Abelian group. (Indeed, here again, $W^k + (-W^k)$ is the boundary of $W^k \times I$, which is embedded as a neighborhood of W^k.) Two L-equivalent submanifolds are both *cobordant* and *homologous.* If two submanifolds W, W' form the boundary in V^n of a submanifold with boundary X that is embedded in V^n; they are obviously L-equivalent.

It is easy to verify that the characteristic numbers of submanifolds that are defined by either starting with characteristic classes of the fiber bundle of normal vectors (*normal* characteristic numbers) or starting with classes of the tangent bundle (*tangent* characteristic numbers) give essentially numerical invariants of the L-equivalence classes.

Map associated to a submanifold. One denotes the orthogonal group in k variables by $O(k)$ and the subgroup of $O(k)$ that is formed from transformations that preserve the orientation (the rotation group) by $SO(k)$. G_k will denote the Grassmannian of unoriented k-planes, and $\hat{G}_k$ will denote the Grassmanian of oriented k-planes, which is a covering with two sheets. $A_{SO(k)}$ will denote the universal bundle of k-balls with base $\hat{G}_k$ that is obtained by associating any k-plane with the unit ball that is contained in it. $A_{SO(k)}$ is a manifold with boundary whose boundary $E_{SO(k)}$ is the universal fiber bundle that is fibered into (k–1)-spheres. Let Φ be the map that is defined by identifying the boundary $E_{SO(k)}$ of $A_{SO(k)}$ to a point. The image space $\Phi(A_{SO(k)})$ will be denoted $M(SO(k))$. One has analogous definitions for $E_{SO(k)}$, $A_{SO(k)}$, and $M(O(k))$.

Let W^{n-k} be a submanifold of the manifold V^n, and endow V^n with a Riemannian metric. The set of points that are situated at a geodesic distance from W^{n-k} that is less than $\mathcal{E}$ is, for a sufficiently small $\mathcal{E} > 0$, a fiber bundle on W^{n-k} that is fibered into geodesic normal k-balls. This set N – which is a normal tubular neighborhood of W^{n-k} in V^n – is a manifold with boundary whose boundary T is fibered over W^{n-k} into spheres S^{k-1}. Suppose that the manifold V^n is embedded in Euclidian space $\mathbb{R}^{n+m}$. At any point x of W^{n-k}, let H_x be the k-plane that is tangent to V^n and normal to W^{n-k}, and endowed with an orientation that is compatible with the given orientations of V^n and W^{n-k}. Choose a k-plane that is parallel to H_x at the origin O of $\mathbb{R}^{n+m}$. This defines a map:

$$g: W^{n-k} \to \hat{G}_k .$$

Upon associating any normal geodesic at x with its tangent vector at x and the unit vector that issues from O and is parallel to it, one defines a map:

$$G: N \to A_{SO(k)} ,$$

where g is the projection of the fibration into k-balls of N and $A_{SO(k)}$.

We form the composed map:

$$N \xrightarrow{G} A_{SO(k)} \xrightarrow{\Phi} M(SO(k)).$$

Its restriction to the boundary T of N maps T onto $\Phi(E_{SO(k)}) = a$, a singular point of $M(SO(k))$. As a result, there exists an obvious prolongation of $\Phi \circ G$ to any V^n. It suffices to map any point of the complement $V^n - N$ onto the point *a*. The map thus obtained:

$$f : V^n \to M(SO(k))$$

is, by definition, the map associated with the submanifold W^{n-k}. One remarks that if one considers $\hat{G}_k$ to be embedded in $M(SO(k))$ (as the image by Φ of the central section of $A_{SO(k)}$) then the reciprocal image *f* of $\hat{G}_k$ is nothing but the submanifold W^{n-k}, and the map *f*, when prolonged to tangent vectors, induces an isomorphism of the fiber bundle of vectors transverse to W^{n-k} with the bundle of vectors transverse to $\hat{G}_k$ in $M(SO(k))$. One may easily show that the *homotopy class* of the map *f* depends upon neither the choice of Riemannian metric on V^n nor the immersion of V^n in $\mathbb{R}^{n+m}$. Conversely, being given a map f: $V^n \to M(SO(k))$, there exists an approximation f' to f such that $f'^{-1}(\hat{G}_k)$ is a submanifold W^{n-k} of V^n, the prolonged map f' inducing an isomorphism of the spaces of transverse vectors. Moreover, one may show that if *f* and *g* verify these conditions and they are two *homotopic* maps of V^n into $M(SO(k))$ then the submanifolds $W^{n-k} = f^{-1}(\hat{G}_k)$, $W'^{n-k} = g^{-1}(\hat{G}_k)$ *are L-equivalent*. (It suffices to conveniently regularize the map of $V^n \times I$ that defines the homotopy.) Finally, in any class of maps $f : V^n \to M(SO(k))$ there exists an *f* that may be obtained by the construction that was described above. Let $C_k(V^n)$ the set of maps of V^n into $M(SO(k))$. One then proceeds to show that there is a bijective correspondence between elements of $L_{n-k}(V^n)$ and elements of $C_k(V^n)$; the class of submanifolds that are *L*-equivalent to *O* corresponds to the class of inessential maps. On the other hand, if $k > n/2$ then $C_k(V^n)$ may be endowed with an Abelian group structure as the *cohomotopy group*. Indeed, one easily shows that $M(SO(k))$ is aspherical for dimensions $< k$, in such a way that the classes of maps of a space of dimension $< 2k - 1$ into $M(SO(k))$ may be endowed with an Abelian group structure.

One finally obtains:

Theorem 1. *The set $L_{n-k}(V^n)$ of L-classes of dimension $n-k$ may be identified with the set $C_k(V^n)$ of classes of maps f: $V^n \to M(SO(k))$. For $k > n/2$, this identification is an isomorphism of Abelian groups $L_{n-k}(V^n)$ and $C_k(V^n)$. Likewise, $L_{n-k}(V^n; \mathbb{Z}_2)$ is identified with the set of classes of maps f: $V^n \to M(SO(k))$.*

3. Maps. There exists a canonical map *J* of the set $L_k(V^n)$ into the homology group $H_k(V^n)$; for any $k > n/2$, it is an isomorphism. The image of *J* in $H_k(V^n)$ is comprised of only those homology classes that are realizable by a submanifold; Theorem 1 allows us to resolve that question to a certain degree. One recovers the essence of these results in (1). Here, we shall occupy ourselves with only the kernel of the map *J*; this kernel is non-zero, in general. We meanwhile point out the following special case: The kernel of *J* is zero on $L_{n-1}(V^n)$, $L_{n-2}(V^n)$, and $L_i(V^n)$, $i \leq 3$, and similarly on $L_{n-1}(V^n; \mathbb{Z}_2)$. One deduces, for example:

Any oriented submanifold of dimension n–2 that is homologous to 0 in V^n is L-equivalent to 0. In particular, it is a manifold with boundary. We now place ourselves in the case where V^n is the sphere S^n. One obtains:

Lemma. *If* $n > 2k + 2$ *then the groups* $L_k(S^n)$ *and* $L_k(S^n; \mathbb{Z}_2)$ *are identified with the cobordism groups* Ω_k *and* $\mathfrak{N}_k$, *respectively*.

This results from the facts that any manifold V^k may be embedded in $\mathbb{R}^n$ and that two cobordant manifolds in it are L-equivalent.

Moreover, as is known, the cohomotopy groups $C_k(S^n)$ are identified with the homotopy groups $\pi_n(M(SO(k))$. Theorem 1 thus gives:

Theorem 2. – *The cobordism groups* Ω_k *and* $\mathfrak{N}_k$ *are isomorphic to the homotopy groups* $\pi_{n+k}(M(SO(n))$ *and* $\pi_{n+k}(M(O(n))$.

It then results from this that the homotopy groups $\pi_{n+k}(M(SO(r))$ are independent of r for $k < 2r - 2$. One may, moreover, show directly that these complexes $M(SO(k))$ and $M(O(k))$, like the sphere and the Eilenberg-MacLane complexes $\pi_{n+k}(M(SO(n))$, verify a "suspension" theorem.

Theorem 2 thus reduces the calculation of the groups Ω_k *and* $\mathfrak{N}_k$ to that of homotopy groups of a space. This latter problem may be approached by a method that was pointed out by H. Cartan and J. P. Serre: Construct a complex that is homotopically equivalent to the space that is given by successive fibrations of Eilenberg-MacLane complexes. The method arrives at the complexes $M(O(r))$; it then collides with some algebraic difficulties that I cannot surmount, for the moment, at least, not in the case of complexes $M(SO(r))$. Here are the results:

4. The ring $\mathfrak{N}$. Up to a dimension $2r$, the complex that is homotopically equivalent to $M(O(r))$ is a product Y of Eilenberg-MacLane complexes $K(\mathbb{Z}_2, i)$ of the form:

$$Y = K(\mathbb{Z}_2, i) \times (K(\mathbb{Z}_2, r+2))^2 \times \ldots \times (K(\mathbb{Z}_2, r+h))_x^{d(h)}, \qquad h \leq r,$$

where $d(h)$ denotes the number of partitions of the integer h into integers that do not have the form $2^m - 1$.

One may show that the generators of the Eilenberg-MacLane space $K(\mathbb{Z}_2, r+h)$ that are factors of Y correspond to certain characteristic classes of the universal fibration $A_{O(k)} \to G_{k'}$ that is defined as follows: Let the "Stiefel-Whitney polynomial" be defined:

$$1 + W_1\, t + W_2\, t^2 + \ldots + W_r\, t^r,$$

in which $t_1, t_2, \ldots, t_i, \ldots$ denote the symbolic roots of that polynomial. The $d(h)$ generators in dimension $r + h$ correspond to $d(h)$ characteristic classes that are defined as symmetric functions of t_i :

$$X_\omega = \sum (t_1)^{a_1} (t_2)^{a_2} \cdots (t_m)^{a_m} ,$$

where the integers $a_1, a_2, \ldots, a_m$, none of which have the form $2\lambda - 1$, define the $d(h)$ possible partitions ω of the integer h.

This permits us to show that if f is a map of S^{n+h} into $M(O(n))$, when it is regularized on G_k in such a fashion that the reciprocal image $W^h = f^{-1}(G_k)$ is a subspace, and that if f is not inessential then at least one of the normal characteristic numbers X_ω of the manifold W^h is non-zero. This gives:

Theorem 3. – *If the $d(h)$ normal characteristic Stiefel-Whitney numbers that are associated with the classes X_ω of a manifold W^h are zero then all of the characteristic Stiefel-Whitney numbers (both normal and tangent) of W^h are zero, and W^h is a bounding manifold mod 2.*

This contains the converse of the theorem of Pontrjagin that was cited above. From that, one may deduce the structure of the ring $\mathfrak{N}$.

Theorem 4. – *The ring $\mathfrak{N}$ of cobordism classes mod 2 is isomorphic to an algebra of polynomials over the field $\mathbb{Z}_2$ that admits a generator U_k for any dimension k that is not of the form* $2^k - 1$.

For example, the first generators are:

U_2 : the class of the real projective plane $\mathbb{R}P^2$,

U_4 : the class of the real projective space $\mathbb{R}P^4$,

U_5 : the class of the manifold of Wu Wen-Tsün, which is a fiber bundle over S^1 whose fiber is the complex projective plane $\mathbb{C}P^2$ (Cf. [**4**]).

U_6 : the class of $\mathbb{R}P^6$.

The groups $\mathfrak{N}_i$ are:

$\mathfrak{N}_2 = \mathbb{Z}_2$, which is generated by U_2 ,

$\mathfrak{N}_2 = 0$,

$\mathfrak{N}_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators U_4 and $(U_2)^2$,

$\mathfrak{N}_5 = \mathbb{Z}_2$, which is generated by U_5,

$\mathfrak{N}_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators U_6, U_4, U_2, and $(U_2)^3$.

One may take the generator U_2 of even dimension to be the class of the real projective space U_2. By contrast, I do not know of the general construction of U_i for odd *i*. (The first unknown one is U_{11}.)

5. The ring Ω. One may determine the complex that is equivalent to $M(SO(r))$ for dimensions $r + k$, $k \leq 7$. One thus obtains:

Theorem 5. – *The cobordism groups* Ω_k *are, for* $k \leq 7$:

$$\Omega_1 = \Omega_2 = \Omega_3 = 0, \qquad \Omega_4 = \mathbb{Z}, \qquad \Omega_5 = \mathbb{Z}_2, \qquad \Omega_6 = \Omega_7 = 0.$$

Any class of Ω_4 is characterized, on the one hand, by the value of the characteristic Pontrjagin number $P^4(V)$, and on the other hand, by the index τ that was defined in paragraph 1. For the complex projective plane $\mathbb{C}\mathrm{P}^2$, one has $P^4(V) = 3$ and $\tau = 1$. Therefore, the generator Ω_4 is the class of $\mathbb{C}\mathrm{P}^2$ and:

Theorem 6. – *The characteristic Pontrjagin number* $P^4(V)$ *of an oriented manifold of dimension* 4 *is equal to* 3τ, *where* τ *is the excess of the number of positive squares over that of the negative squares of the quadratic form that is defined by the cup product on* $H^2(V^4; \mathbb{R})$ (Cf., [**5**]).

It is therefore a topological invariant, just like the class of Ω^4, if the same topological manifold V^4 can be endowed with two non-isomorphic differential structures, while that manifold remains cobordant to itself.

The algebra Ω in rational coefficients. – Let $\mathbb{Q}$ be the field of rational numbers. Upon applying the *C*-theory of J. P. Serre [**6**] to the complex $M(SO(r))$ (*C* being a family of finite groups) one obtains:

Theorem 7. – *Any of the groups* Ω_i *are finite for* $i \not\equiv 0 \bmod 4$. *The algebra* $\Omega \oplus \mathbb{Q}$ *is an algebra of polynomials that admits a generator* Y_{4m} *for any dimension that is divisible by* 4.

One may take Y_{4m} to be the class of complex projective space $\mathbb{C}\mathrm{P}^{2m}$. One then obtains:

Corollary 8. – *For any oriented manifold V^n there exists a non-zero integer N such that the multiple manifold is cobordant to a linear combination with integer coefficients m_i of products of complex projective spaces of even complex dimension. The integers m_i are homogeneous linear functions of the characteristic Pontrjagin numbers of the manifold $N \cdot V^n$.*

In particular, if all of these numbers are zero then there exists an $N \neq 0$ such that $N \cdot V$ is a bounding manifold.

Bibliography

[1] "Sous-variétés et classes d'homologie des variétés différentielles," C. R. Paris **236** (1953) pp. 453 and 573.
[2] S. EILENBERG, "Problems in Topology," Ann. of Math. **50** (1945), pp. 247-60.
[3] L. PONTRJAGIN, Math. Sbornik, **21**, 63a, pp. 233-84 (1947), th. 3.
[4] W. WU, C. R. Paris **230**, pp. 508-511.
[5] S. A. ROKHLIN, Doklady **81** (1951), pp. 355, and **84**, no. 2 (1952), pp. 221.
[6] J. P. SERRE, "Groupes d'homotopie et classes de groupes abéliens," to appear in Ann. of Math.