

## GENERALIZATION OF MORSE THEORY TO FOLIATED MANIFOLDS

By **René THOM, Strasbourg.**

Translated by D.H. DELPHENICH

### **Introduction.**

One knows that Morse theory gives a powerful means of classifying differentiable structures, as was generalized by authors such as A. Wallace, S. Smale, etc. It is natural to think that this method will likewise reveal its efficacy in the study of finer structures, such as foliated manifolds. In this article I make an attempt in that direction, which must hardly be considered conclusive, and, in any case, is certainly incomplete. Indeed, I shall stop short of the essential problem of the theory, which is that of "recurrence." An essential difference will separate out theory from the classical one: whereas, in the latter theory, one must consider only a finite number of critical points (which are classified by an integer, the index) and the corresponding critical values may be considered to be isolated, the same is no longer true in the present case. On the one hand, it will be difficult to classify the singularities that are presented "generically; on the other hand, the set of critical values is generally infinite – not discrete – and this entails that in a neighborhood of a critical value,  $c$ , the topological type of the foliated manifold  $f \leq c + \varepsilon$  must present an infinitude of variations (a non-denumerable infinitude, in general). At the end of the article, I will make several remarks on this question that are obviously related to the problem of the structural stability of dynamical systems. I nevertheless hope that the method described will provide a means of attacking this difficult question.

### **1. Relation of compact equivalence.**

Let  $M^n$  be a compact  $n$ -dimensional differentiable manifold, possibly with boundary. A foliation  $(X)$  of codimension  $k$  in  $M$  is defined by the given of an atlas  $(U_j)$  on  $M$ , and for each  $U_j$ , the given of a map of rank  $k$ :  $q_j: U_j \rightarrow \mathbb{R}^k$ , such that on an intersection  $U_i \cap U_j$  the  $q_j$  must satisfy the obvious commutation conditions with the diffeomorphisms of the change of chart from the source to the target. At a point  $x$  of the boundary  $V$  of  $M^n$ , if it exists, one may assume that the manifold  $M$  is embedded in a regular open neighborhood. In the product  $M^n \times M^n$  of the manifold with itself, one forms the graph  $(G)$  that is associated with the foliation  $(X)$ : a pair  $(x, y)$  belongs to  $(G)$  if  $x$  and  $y$  are two points of  $M^n$  from the same leaf. The graph  $(G)$  is a submanifold of codimension  $k$  that contains the diagonal of  $M \times M$  and is invariant under the symmetry  $(x, y) \rightarrow (y, x)$ . Nevertheless,  $(G)$  is not generally a proper submanifold, i.e., i.e., the topology that is induced by the embedding of  $M \times M$  does not generally coincide with the topology of the manifold. This is true only if  $(G)$  is a closed submanifold of  $M \times M$ , hence, compact. In this case, the

adherence of a leaf agrees with the leaf itself. In this case, in order to simplify the discussion, one says that the equivalence relation, or furthermore, that the foliation  $(X)$  is *compact* ( $e$ ).

The conditions that are imposed on the graph of a “compact” foliation are very restrictive: notably, from the homological viewpoint, one may deduce some conditions for a section of the bundle of  $(n - k)$ -planes on  $M$  to be the set of tangent planes to a foliation. Suppose, for the moment, that  $M$  is compact without boundary. Let  $T(\Delta)$  be a tubular neighborhood of the diagonal  $\Delta$  in  $M \times M$ . The quotient of the complement  $M \times M - T(\Delta)$  under the symmetry  $(x, y) \rightarrow (y, x)$  is the reduced symmetric product  $M_2$  of the manifold  $M$ .  $M_2$  is a manifold with boundary whose boundary  $Q$  is a fiber space with base  $D$  and fiber  $PR(n - 1)$  the real projective space of dimension  $n - 1$ . The given of a section of the bundle of  $(n - k)$ -planes of  $M$  is equivalent to the given in  $Q$  of a submanifold  $W$  of codimension  $k$  that is fibered into  $(n - k + 1)$ -planes that project on the diagonal  $(\Delta)$ : Each fiber of  $Q$  contains the fiber of  $W$  as a projective subspace of codimension  $k$ . By considering a regular neighborhood of  $W$  in  $Q$ , one may associate  $W$  with a map  $g: Q \rightarrow MO(Q)$ , where  $MO(k)$  is the Alexandroff compactification of the universal bundle into  $k$ -balls. Therefore, if there exists a “compact” foliation such that  $W$  is the set of tangent  $(n - k)$ -planes then the quotient under symmetry of the graph  $(G)$ , namely,  $(g)$ , is a manifold with boundary in  $M_2$  with boundary  $W$ . Therefore, a necessary condition for there to exist such a compact foliation is that the map  $g: Q \rightarrow MO(k)$  admit an extension  $g_1 = M_2 \rightarrow MO(k)$ . One may further refine this condition by the following remark: Let  $A$  be the subspace of the reduced symmetric product of  $M_2$  with itself that is composed of pairs  $(x, y), (y, x)$  such that the two components have one and only one common coordinate. There exists a canonical map of  $A$  into  $M_2$  that is defined by the formula:  $\psi: (x, y), (x, z) \rightarrow (y, z)$ . The graph  $(g)$  then has the property that its intersection with  $A$  has  $\psi(g)$  itself for its image, a property that may be expressed in terms of homology classes. If  $z$  is the intersection class of  $(g \times g)$  with  $A$  in  $(M_2)_2$ , then  $\psi(z) = g$ .

These homological conditions on the section  $(W)$  are actually of little interest; indeed, their use is difficult, since the calculation of the homology of the reduced symmetric product  $M_2$  is no easy task. Furthermore, it gives only necessary conditions for the existence of a compact foliation, which is an excessive restriction one is interested only in the existence of a foliation that is associated with the class of the section of the bundle of  $(n - k)$ -planes.

## 2. Morse theory on a foliated manifold.

Let  $M^n$  be a differentiable manifold that is endowed with a foliation  $(X)$  of codimension  $k$ . First of all, one chooses a standard function on  $M^n$  (a “nice function” in the sense of Smale). The critical points of  $f$  are non-degenerate and the values of  $f$  at these points are ordered by increasing index (by assuming that they are all distinct, here). Let  $(G)$  be the graph of the foliation  $(X)$  in the product  $M$ . A first approach to the problem of how to generalize Morse theory consists of studying the function  $g$  that is defined by  $g(x, y) = \text{Sup}(f(x), f(y))$ , on the manifold  $(G)$ .

This function, (which is, moreover, only piece-wise differentiable) must present critical points on  $(G)$  in the following cases:

1.  $f(x) = f(y)$ , and  $df(x) = 0$ ,
2.  $f(x) = f(y)$ , and  $f$  is stationary at  $x$  on the leaf of  $x$  and at  $y$  on the leaf of  $y$ .

When one displaces  $(x)$  transversally to the foliation in the sense of decreasing  $f$ , it might happen that  $y$  is displaced in the sense of increasing  $f$ , in such a way that  $g$  admits  $(x, y)$  as a relative minimum. One thus sees that other than the critical points of  $g$  that are naturally associated with those of  $f$  there exist critical points of  $g$  – in the topological sense – which correspond to *certain* leaves that are bitangent to a level manifold  $f = c$ . Some simple examples (differential systems on the torus) show that if  $O$  is the minimum value of  $f$  then it is possible that  $g$  has an infinitude of critical points for  $g \leq \varepsilon$ . In order to avoid the appearance of an accumulation of an infinitude of critical values, at least to begin with, it is preferable to adopt the following convention: in the manifold with boundary  $f \leq c$  one considers only the equivalence relation  $(X_c)$  that is defined by the restriction of the foliation  $(X)$  to the manifold  $f \leq c$ . One earns the right to adopt this manner of operating since the relation  $(X_c)$  is *compact* for sufficiently small values of  $c$ . Indeed, if  $c$  is sufficiently close to zero the inequality  $f \leq c$  defines a ball that is completely situated in a chart  $U_i$  of the foliation and is convex in this chart. Furthermore, the projection  $\pi$  is a compact equivalence relation on this ball.

If we let  $(M_c)$  designate the set  $f \leq c$  then one must study the variation of the topological type  $(M(c), X_c)$  when  $c$  varies. With this objective, it is first of all necessary to specify the set  $(C)$  of points where the level hypersurface  $f = c$  is tangent to the foliation  $(X)$ . I say that this set  $(C)$  is “generically” a submanifold without singularities of dimension  $k - 1$ . Indeed, in a local chart  $U$  one must consider the restriction of the projection  $\mathbb{R}^n \rightarrow \mathbb{R}^k$  to a portion of the level hypersurface  $f = c$  at  $x$ . One must therefore consider maps of  $\mathbb{R}^{n-1}$  into  $\mathbb{R}^k$  that admit a factorization of the form:  $\mathbb{R}^{n-1} \xrightarrow{i} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^k$ , in which  $i$  is an injection and  $\pi$  is a linear projection. Since the corank at the source of the map  $\pi \cdot i$  may not exceed  $n - k$ , the dimension of the kernel of  $\pi$  and the corank at the target may not exceed one. Conversely, any map of into has a corank that is smaller than one (at the target) then it locally admits a factorization of the form  $\pi \cdot i$ . It results from the transversality lemma that the critical locus of such a map is generically without singularities; indeed, the singularities of the critical locus possess only points of corank 2 (an  $S_2$  singularity, in the terminology of [1]). By contrast, there are good reasons to distinguish subspaces of the “contact” manifold  $C$  in which the contact is not ordinary. These subspaces correspond to the symbols  $S_j, \dots, S_1$  of [1], and their classification is not known in full generality. Nevertheless, in low dimensions ( $n < 6$ , for example), this classification may be made precise: there then exists only singularities of the form  $(S_1)^j$ , which correspond to submanifolds without singularities of codimension  $j$  in  $(C)$ . The first case of more complicated singularities presents itself with the singularity  $S_2 S_1$  of  $\mathbb{R}^5$  into  $\mathbb{R}^4$ , hence, for a foliation of  $\mathbb{R}^6$  with codimension 4.

Of course, when the value ( $c$ ) varies, the contact manifold ( $C$ ) must not remain isotopic to itself. It must be subjected to modifications, which we specify when we cross a critical point of the function  $f$ . However, it may likewise be transformed when it crosses certain regular values ( $a$ ) of  $f$ . This corresponds to a generic transition between two generic forms of ( $C$ ). These generic transitions happen at isolated points, which correspond to generic singularities of maps  $\mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}$  that admit a factorization of the form:

$$\mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^k \times \mathbb{R},$$

with conservation of the last coordinate  $t$ . The description of these transition points may be made explicit only for low dimensions ( $n < 6$ ); later on, we shall do this briefly for the case of  $n \leq 3$ .

### 1. Critical values of the first species.

These are critical values of  $f$  in the usual sense. One supposes that each critical value,  $c$ , corresponds to just one critical point  $O$ , which is non-degenerate quadratic. One calls the number,  $p$ , of negative squares in the second differential  $d^2f$  at  $O$  the *type* of the critical point, and the difference  $|-p + (n - p)|$  between the number of positive squares and the number of negative ones, the *index*. One assumes, moreover, that the leaf of  $X$  that passes through  $O$  cuts the quadratic tangent cone to level hypersurface  $f = f(O)$  transversally. This condition, which is realized trivially when  $p = 0$  or  $n$ , may be satisfied, moreover, for an arbitrarily small deformation of the function  $f$ . One may then find a local coordinate chart about  $O$ ,  $x_1, \dots, x_r, u_1, \dots, u_q$ , in which the equations of the foliation ( $X$ ) are:  $u_j = u_{j_0}$ , and for which  $f - f(O)$  takes the form:

$$f - f(O) = -x_i^2 - \dots - x_s^2 - u_1 \dots - u_t^2 + x_{s+1}^2 + \dots + x_r^2 + \dots + u_{i+2}^2 \dots + u_q^2.$$

The integer  $s$ , which is the smaller of  $r$  and  $p$ , will be called the *X-type* of  $O$ . The “attaching sphere” that is associated with  $O$ , with the equation:

$$x_1 + \dots + x_s^2 + u_1 + \dots + u_t^2 = \varepsilon, \quad x_{s+j} = u_{t+i} = 0,$$

takes the form of the join of two spheres: an  $(s - 1)$ -sphere that is contained in the leaf of  $O$ , and a  $(t - 1)$ -sphere that is situated in a plane transverse to the foliation. In the chart, the contact locus ( $\Gamma$ ) is the  $q$ -plane  $x_1 = x_2 = \dots = x_q = 0$ . The restriction of  $f$  to the manifold ( $\Gamma$ ) possesses a critical point at  $O$  of the type  $t = p - s$ . As a result, this describes the variation of the contact manifold ( $C$ ) when one crosses the critical point  $O$ .

### 2. Critical values of the second species.

We shall always let  $(x_1, \dots, x_r, u_1, u_{2i}, \dots, u_q)$  represent a local chart in which  $u_i = c_i$  represents the leaves of ( $X$ ). If  $O$  is a regular point of the contact set ( $\Gamma$ ) then one may assume that  $f$  is taken in the form:

$$f = u_1 + g_2(x_j) + \sum c_j^r u_r x_j \dots$$

The tangent plane to  $(\Gamma)$ , which is defined by the equations:

$$\frac{\partial g_2}{\partial x_i} + \sum c_i^s u_s + \dots = 0,$$

contains the leaf  $u_j = 0$  only if the quadratic form  $g_2$  is degenerate. In the generic situation  $g_2$  has rank  $(r - 1)$  and may be written:

$$g_2 = -\sum (x_i)^2 + \sum (x_j)^2 + (x_{r+1})^3 + \dots$$

and

$$f - f(O) = u_1 + g_2 + u_1 x_{r+1} + \sum c_j^r u_r x_j .$$

One may then locally parameterize  $(\Gamma)$  by the  $q$  variables,  $u_2, u_3, \dots, u_q$ , and  $x_{r+1}$ . To second order, the restriction of  $f$  to  $(\Gamma)$  begins with the expression:

$$f = -2x_r^2 - \sum_i \left( \sum_s c_i^s u_s \right)^2 + \sum_j \left( \sum_t c_j^t u_t \right)^2 + \dots$$

As a result,  $f(\Gamma)$  admits a critical point at  $O$  (non-degenerate, in general), whose type may not exceed that of the form  $g_2$  (and similarly for the index). Moreover,  $O$  is a point of the transition manifold  $s_1(\Gamma)$  that separates the strata of type  $r$  from the strata of type  $(r+1)$  in  $(\Gamma)$ . Also, the restriction of  $f$  to this manifold  $S^1(\Gamma)$  likewise admits 0 as a non-degenerate critical point.

This gives a description of the simplest type (generically 1) for a critical point of the second species. There may be others, which are likewise “generic” and situated on “exceptional” manifolds  $(S_1)^k(G)$ ; this case occurs notably for vector fields.

### 3. Critical values of the third species.

In the manifold with boundary  $M(c)$  ( $f \leq c$ ), let  $K(C)$  be the subset of the contact manifold  $C$  that is saturated by  $(X_c)$ . The set  $K(C)$  thus obtained is a “stratified set” (See [2] for the definition). In what follows, we always assume that  $K(C)$  is compact, and form the intersection of  $K(C)$  with  $C$  in the boundary  $f^1(c)$ , of  $M(c)$ . In general, it is a stratified set that one combines with the stratification of  $K(C)$  by subdivision. One iterates this construction, and, by reason of dimension, the process terminates. Again, one lets  $K(C)$  designate the saturated set thus stratified. One calls any value  $(c)$  for which the stratification of  $K(C)$  changes its topological type a *critical value of the third species*. More precisely, if one forms the set  $K(\Gamma) = \cup_c K(C)$  for all values  $c$  of an interval  $]a, b[$ , in which there exists no critical value, and if the map (which is induced by  $f$ )  $K(\Gamma) \rightarrow ]a,$

$b[$  is stratified (in the sense [2]), then the stratification of the target-interval is trivial (has no summits between  $a$  and  $b$ ).

When the set  $K(C)$  is no longer compact, the notion of critical value itself ceases to be defined. Nevertheless, in certain cases, in which the set  $K(C)$  admits an infinitely structurally stable generalization, a generalization does not seem possible.

*Examples. – Differential systems in dimensions two and three.*

More generally, one lets  $n$  denote the dimension of the space and  $k$ , the codimension of the foliation considered ( $n > k$ ).

The set  $(C)$  consists of isolated points on the manifold  $f = c$ . The critical manifolds of the second species correspond to the creation (or annihilation) of a pair of points of  $(C)$ . They are associated with points where the leaves have an inflectional contact with the level lines of  $f$ .

$$(1) \quad n = 2, \quad k = 1.$$

The intersection  $K(C) \cap f^{-1}(c)$  generically consists of isolated points. There is a *critical value of the third species* when two of these points coincide.

$$(2) \quad n = 3, \quad k = 2.$$

The critical set  $(C)$  is a regular curve that contains isolated points where the field is tangent to  $(C)$ . These points correspond to the singularity  $(S_1)^2$  of a fold. The critical values of the second species correspond to the appearance (or the disappearance) of a simple curve of  $(C)$  that possess two fold points, or furthermore, to the join of two such curves by the intermediary of point of collar type.

The set  $K(C) \cap f^{-1}(c)$  is a set of curves that possess ordinary regression points on the trajectories of the fold. They will have a *critical value of the third species* when that set of curves cuts the curve  $(C)$  non-transversally.

$$(3) \quad n = 3, \quad k = 1.$$

For  $k = 1$  (foliations of dimension two in  $\mathbb{R}^3$ ), the set  $(C)$  decomposes into isolated points. There is a critical value of the second species when two of these points coincide (when they are destroyed or created). The set  $K(C) \cap f^{-1}(c)$  is a set of simple curves (which possibly admit quadratic double points at the points of  $(C)$ ). There is a *critical value of the third species* when these points of  $(C)$  are triple points of  $K(C) \cap f^{-1}(c)$ .

It is important to observe that the set  $K(C) \cap f^{-1}(c)$  must be assumed to be saturated with respect to the relation  $(X_c)$ . For example, in the case of  $n = 3, k = 2$ , the locus  $(C)$  is assumed to be a simple curve. In general,  $K(C) \cap f^{-1}(c)$  is then another curve  $C'$  that cuts  $(C)$  transversally. Let  $x$  be the point of intersection that is so defined. There is good reason to add the trajectory of  $x$  to the stratification of  $K(C)$ , hence, to also consider the point  $x' \in (C')$  that is the other extremity of this trajectory. If the point  $x'$  coincides with  $x$  for a certain value of  $c$  then one will have a critical value of the third species of a

particularly important type; indeed, this must say that the trajectory that issues from  $x$  is closed. Such critical values, which are characterized by the coincidence of two points  $x, x'$  that are situated on the same leaf of  $(X_c)$ , will be called *coincidence values*.

### Morse Theory on a foliated manifold $(M, X)$ .

Let  $f: M \rightarrow \mathbb{R}$  be a differentiable function that is “generic” on the foliated manifold  $(M, X)$ . The contact locus  $(\Gamma)$  of the leaves with the level manifolds of  $f$  is a manifold without singularities (except for the critical points of  $f$ ) of dimension  $q$ , which is equal to the codimension of  $X$ . There exists a submanifold  $S_1(\Gamma)$  such that the type of the restriction (stratum) of  $f$  to the leaves of  $X$  remains constant on the complement  $\Gamma - S_1(\Gamma)$ . As a result, one may associate an integer  $k$  to each component of  $\Gamma - S_1(\Gamma)$ , which falls between 0 and  $n - q$  and is called the  $X$ -type of this stratum.

One then endows  $M$  with a Riemannian metric, which one normalizes in a neighborhood of  $(\Gamma)$  in such a manner that, in each leaf of  $(X)$ , the metric becomes the Euclidean metric of the associated chart. With these conditions, to any point  $m$  of  $M$ , one attaches the endpoint of the trajectory of the gradient of  $f$ , when restricted to the leaf of  $m$ . Such a point is necessarily a point of  $(\Gamma)$ . One thus defines a stratification of the manifold  $M$  whose strata are of three species:

- 1) The endpoint of the trajectory  $g(m)$  is a critical point of  $f$  of  $X$ -type  $k$ . The corresponding stratum is an open  $k$ -ball.
- 2) The extremity point  $g(m)$  is a regular point of  $\Gamma$ , which is situated in an open strata  $U$  ( $c$  is the connected component of  $\Gamma - S_1(\Gamma)$ ). Therefore, the corresponding stratum, which is formed of all of the points whose trajectories end in  $U$ , is isomorphic to the product  $U \times D_s$ , where  $D$  is an open  $s$ -ball, if the type of the stratum  $U$  is equal to  $(s)$ .
- 3) The extremity of  $g(m)$  is a point of  $S_1(\Gamma)$ ; one is dealing with an exceptional stratum. At an ordinary point of  $S_1(\Gamma)$ , the set of trajectories that end at this point form a semi-ball whose cohomology with compact supports is null.

*Morse inequalities.* – Let  $P(U, t)$  be the Poincaré polynomial of the cohomology with compact supports of the stratum  $U$ . The classical Morse inequalities are expressed as follows: let  $P(U, t)$  be the Poincaré polynomial of the manifold  $M$ . The difference  $\sum P(U, t) - P(M, t)$  is divisible by  $(1 + t)$ , and the quotient is a polynomial with positive integer coefficients.

Suppose that, in our case, there only exist ordinary points of the second species on  $S_1(\Gamma)$ . The associated strata then have one null cohomology (because the trajectories that end at each point of  $S_1(\Gamma)$  forms a semi-ball). If one lets  $c_k$  designate the number of critical points (first species) of  $X$ -type  $i$  then the polynomial  $Q(t) = \sum (c_k + d_k)t^k$  is such that  $Q(t) - P(t)$  is divisible by  $(1 + t)$  with a positive integer quotient.

### Simply stratified foliations.

For the manifold  $(M, X)$ , one assumes that the set  $K(C)$ , which saturates the contact set  $(C)$  on the boundary, is stratified. In this manner, the set  $\Gamma$  is likewise found to be stratified. One supposes that the function  $f$  possesses only a finite number of critical points on each leaf. One then constructs a graph  $(G)$  as follows: to each stratum of  $X$ -type zero in  $G$ , one associates a vertex. To each stratum of  $X$ -type one, one associates an edge whose extremities are vertices that were associated with strata of  $X$ -type zero that contain the origins of two gradient trajectories that end at a point of this stratum. One says that the foliation  $(M, X)$  is *simple* if the graph thus obtained is a tree (or a finite union of trees). One may (non-canonically) associate a stratum of  $X$ -type zero  $G$  or one of the boundary  $\partial M$  to each tree of the graph, and there will be incidence relations between these strata whose union defines a stratified set. The equivalence relation defined by  $(X)$  in  $M$  is then defined by a stratified map  $q: M \rightarrow S$  such that any leaf of  $X$  is the inverse image  $q^{-1}(s)$  of a point  $s$  of  $S$ . A leaf  $(F)$  of  $(X)$  obviously corresponds to a tree of the graph  $(G)$ . By construction, this tree is nothing but the 1-skeleton of the cellular subdivision that is defined by the gradient lines of the restriction  $f|_F$ . As a result, any leaf  $F$  is compact and *simply connected*.

Suppose further that any stratum of the stratified set  $K(c)$  is defined transversally, i.e., as a regular intersection or by a transversal auxiliary map into the space of jets. This will also be true if one perturbs the foliation  $(X)$  very slightly, or if one varies the value  $c$  of  $f$  very slightly. The graph  $(G)$  stays isomorphic to itself. One then remarks that it is possible to apply theorem 3 of [2], which relates to the invariance of the topological type for stratified maps that do not blow up. Indeed, the stratified map  $q: M \rightarrow S$  does not blow up, because the corank of  $q$  on any stratum is equal to the dimension  $(n - q)$  of the leaves of  $X$ , except possibly on certain strata of the boundary  $\partial M$  that have corank zero. However, these “extremal” strata form a closed stratified subset that is sent isomorphically by  $q$  into the boundary of  $S$ . This permits us to conclude:

**THEOREM 1.** – *If the foliated manifold  $M(c)/X(c)$  is simply stratified then the manifold  $M(c')/X(c')$ , where  $c'$  is sufficiently close to  $c$ , is likewise simply stratified and has the same topological type as  $M(c)/X(c)$ .*

**THEOREM 2.** – *If  $M/X$  is simply stratified then for any foliation  $X'$  that is sufficiently close to  $X$  (in the  $C^q$  topology)  $M/X'$  is simply stratified and has the same topological type as  $M/X$  ( $q \geq 1$ ).*

These theorems show that when one crosses any critical value after starting with a value  $c$  for which the foliation is simply stratified, the foliated manifold remains simply stratified and has the same topological type. One remarks that in the case of a proper function  $f$  that takes the value 0 at its minimum, we have, for a sufficiently small  $c$ , that the manifold  $(M(c), X(c))$  is simply stratified, because it has the topological type of a convex ball that is linearly projected onto a coordinate plane. Theorem 2 shows that the simply stratified foliations have the property of structural stability.

One is then forced to specify the effect of crossing critical points of various species on a simply stratified foliated structure.

### 1. Critical point of the first species.

If the foliation  $(M, X)$  is simply stratified for if  $f \leq c - \varepsilon$  then there exists a stratification for the boundary  $V_{c-\varepsilon}$  of  $M(c)$ , and a map  $q: V \rightarrow S$  that induces the equivalence relation that is defined by  $X(c - \varepsilon)$ . Consider the manifold with boundary  $c - \varepsilon < f < c + \varepsilon$  ( $c$  is a critical value  $c = f(0)$ ). In this manifold, the contact set  $(\Gamma)$  is stratified, as well as the saturation  $K(\Gamma)$  in  $c - \varepsilon < f < c + \varepsilon$ . In the generic case, the restriction of  $K(\Gamma)$  to  $V_{c-\varepsilon}$  is a stratified set of  $V_{c-\varepsilon}$  that is transversal to the stratification on  $V$  that is already given by  $X(c - \varepsilon)$ . One forms the intersection stratification on  $V_{c-\varepsilon}$ , which one saturates on  $M(c - \varepsilon)$  for  $X(c - \varepsilon)$ , on the one hand, and on  $\{c - \varepsilon \leq f \leq c + \varepsilon\}$ , on the other. I say that, having done this, one further obtains a finite stratification, except in the case where the point 0 is of  $X$ -type equal to one. Indeed, if the  $X$ -type of 0 is different from one then for  $x$  in  $c - \varepsilon \leq f \leq c + 2$  there exists no stratum of  $G$  of type one, except, perhaps, the extensions of strata of the same type that transversally cut  $V_{c-\varepsilon}$  and that do not contain 0. Now, it is clear that it is only important to consider a neighborhood of the critical point 0. Let  $k$  be the type of 0, and let  $r$  be the  $X$ -type of 0. Crossing 0 is topologically equivalent to adding a product of disks  $D^k \times D^{n-k}$  to  $M(c - \varepsilon)$ , and the portion of the boundary that takes the form  $D^k \times D^{n-k}$  is found to be identified with a normal tubular neighborhood of the “attaching sphere”  $S^{k-1}$  that is defined by the nappe of the gradient that ends at 0. However, this identification must be compatible with the foliation  $(X)$ . The examination of a local chart shows that the quotient by the relation  $(X)$  in the *chart* are:

For the neighborhood  $S^{k-1} \times D^{n-k}$ ; for the product  $D^k \times D^{n-k}$ :

1. $r = 0$	$S^{k-1} \times D^{q-k}$	$D^k \times D^{q-k}$
2. $r = 1$	$S^{k-1} \times D^{q-k}$	$D^{k-1} \times D^{q-k+1}$
3. $r = 2$	$D^{k-r} \times D^{q-k+r}$	$D^{k-r} \times D^{q-k+r}$

One confirms that there is just one case in this table, namely  $r = 1$ , for which the crossing of a critical point induces an identification in the local quotient (here, this is the identification  $S^{k-1} \rightarrow D^{k-1}$  by projection onto the equatorial plane). In case 3, the quotient is not affected by crossing 0. In case 1, one adds another disk  $D^k$  of new leaves to the quotient, with the boundary  $\partial D^k$  being given. As a result, in case 1 and 3, the stratified map  $q: M(c - \varepsilon) \rightarrow S$  extends to a stratified map  $q': M(c - \varepsilon) \rightarrow s'$ , where  $S' = S$  in case 3 and  $S' = S \cup D^q$  in case 1. In case 2, one may say nothing “a priori.” Indeed, it is possible that the global saturation  $K(\Gamma)$  is not a finite stratified set; in that case, there exists an infinitude of critical values of the third species in the interval  $]c - \varepsilon, c[$  that accumulate at  $c$ .

**THEOREM 3.** – *Let  $f$  be a function on the foliated manifold  $(M, X)$ . If  $c$  is a critical value of the first species that is associated with a critical point  $O$  of  $X$ -type different from*

one, then, if  $(M(c - \varepsilon), X(c - \varepsilon))$  is simply stratified then  $(M(c + \varepsilon), X(c + \varepsilon))$  is simply stratified, and the (stratified) topological type of  $(M(c + \varepsilon), X(c + \varepsilon))$  is completely defined by the restriction of the map  $q: M(c - \varepsilon) \rightarrow S$  to the “attaching sphere” at the point, 0.

## 2. Critical values of the second species.

Let  $O$  be a critical point of the second species, of the ordinary type that was described above. One sees, in turn, that there exists a stratum of type one in the neighborhood of  $O$  only in the case of transition points of  $X$ -type  $(0, 1)$  or  $(1, 2)$ . These are therefore the only two cases that may lead to a local identification in the leaf space. Now, in these two cases, the examination of the local chart shows that the identifications that were imposed by the strata of type one always bring a “new” stratum of type zero into play. The identification does not work for  $f = c - \varepsilon$ . One may likewise see this upon remarking that for any  $m$  the hypersurface  $f = m$  is projected surjectively onto the space of the  $u_i$ , which is the local quotient space (leaf space). This surjective character is due to the fact that one must solve an equation of third degree in  $(x_r)$ , which always admits a real root, no matter what values are attributed to the other variables. From this, one concludes that crossing a point of the second species of ordinary type does not modify the simply stratified character of the foliation. The same argument is no longer valid for points of more complicated type. For example, for a point of type  $S'S(\Gamma)$ , in the case of a differential system one has an equation of the 4<sup>th</sup> degree in  $(x)$ . There is thus good reason to proceed with a study of this case in particular.

We further clarify the following point: In the ordinary case, where one has solved an equation of third degree in  $(X_r)$ , it is quite possible that the leaf space is not the space  $\mathbb{R}^q$  of variables  $(u_i)$ , but an étalé space over  $\mathbb{R}^q$ . For a given value  $a$ , the set,  $f \leq a$  splits into at most two connected components on any leaf. One may thus obtain – for example, for negative  $a$  – a supplementary leaf  $(\Phi)$  in the leaf space  $F$  that reduces to the origin  $(u_i = 0)$  when  $a$  tends to the critical value  $c = 0$ . Indeed, for  $a > c$  the equation of third degree has only one real root and the local leaf space is isomorphic to  $\mathbb{R}^q$ . From this, it results that the identifications that are true for the leaf space have no importance, because they define only a correspondence between the given leaves of  $M(c - \varepsilon)$  and the leaves of the local foliation  $\Phi_i(F)$ . Now, since these leaves may be continuously deformed to  $O$  when  $\varepsilon$  tends to zero, they have no pre-existing connection with the leaves of  $M(c - \varepsilon)$ . This “argument” is likewise valid if one reverses the sense of the variation of the function  $f$ . One may verify this explicitly for the equation  $f = u + (x)^3 - ux$ .

## 3. Critical values of the third species.

An explicit description of the singularities of the third species seems out of reach. To simplify, we confine ourselves to the case of a differential system  $(M, X)$ . In  $M(a)$ , one has a contact manifold that is formed from connected components  $(C_i)$ . It may happen that  $K(C_i)$  does not encounter  $C_j$ , but that for a value  $a' > a$  these two manifolds are in contact in the boundary  $M(a')$  at a point  $x \in M(a')$ . For a value  $a'' > a'$ , these two

manifolds intersect along a small sphere ( $s$ ). If the  $X$ -type of  $C_j$  is repulsive then this may have the effect of creating new stratifications in the space  $K(\Gamma)$ ; for example, another locus of  $M(a'')$ , a small disk bounded by a sphere  $s'$  in  $K(s)$ . Such critical values, which have only the effect of subdividing the existing strata in a finite manner (one may call them *contact values*), do not affect the simply stratified character of a foliated manifold.

Nevertheless, it may happen that the critical contact values  $c_1, \dots, c_k$  of the type described above accumulate to a "limiting" value  $a$ . The stratification  $K(\Gamma)$  then acquires finer and finer strata, in such a way that one no longer has a simple finite stratification, in general,  $f \leq a$ . In particular, if  $F$  is a compact leaf, and if an element of the fundamental group  $w \in \pi_1(F)$  operates in the holonomy of this leaf as an element of infinite order then the smallest value  $a$  of  $f$  for which one may realize a loop of class  $(w)$  in  $M(a)$  is necessarily an accumulation point for the critical values of the third species. Such a value sometimes corresponds to a critical point of the first species with  $X$ -type equal to one.

### General remarks and open problems.

Let  $(M, X)$  be a foliated manifold, and let  $f$  be a "generic" function on  $(M, X)$  that admits zero for its minimum value. One has seen that for sufficiently small  $c$ , the manifold  $(M(c), X(c))$  is simply stratified. This situation persists whenever one crosses only critical points of the first species and  $X$ -type different from one, "ordinary values of the second species, and contact values of the third species. This situation ceases, in general, whenever one crosses a limit value, which is an accumulation value for critical values of the third species. One will observe that the definition of a simply stratified foliation involves a generic function ( $f$ ). Indeed, this is probably an intrinsic notion; one may conjecture that any foliated manifold with boundary (the foliation being "generic" on the boundary) such that all of the leaves are compact and simply connected is simply stratified. On this subject, compare the stability theorem (B, II, 16) of G. Reeb in [3].

It will be likewise good to specify a generic form (or forms) for the appearance of recurrence in a foliation of given codimension. One will therefore have – and this is the generally accepted conjecture – that is the appearance of recurrence is subordinate to the existence of a compact leaf whose holonomy has infinite order (at least in the "generic" case).

Finally, it remains to be known whether the notion of stratification loses all interest when one passes to a limit value and non-compact leaves and recurrence appears. The known examples of "structurally stable configurations" (homoclinic points, etc.) suggest that there exists an infinite stratification in the same case, whose global topological type is invariant under very small perturbations. There is likewise also good reason to know how the topological type of  $(M(c), X(c))$  varies as a function of the parameter  $c$ . Maybe it is not unreasonable to think that this type varies only on a dense disconnected null perfect set of values of  $(c)$ .

### Applications to certain differential systems.

First of all, observe that the preceding theory extends to the case of manifolds with boundary  $(M, X)$ . One then supposes that the foliation  $(X)$  is generic on the

boundary  $\partial M$ . One then takes a function  $f$  that is likewise generic on the boundary. As before, one has the notion of a simply stratified manifold.

### Structural stability of gradient vector fields.

Let  $f$  be a function that possesses only a finite number of non-degenerate critical points  $m_j$  on  $M$ . One supposes that the nappes of the gradients that are attached to these critical points intersect transversally, for a given Riemannian metric. Let  $X = \text{grad } f$  be this field. One removes a ball of sufficiently small radius  $r$  around each point  $m_j$  from  $M$ , and one applies the preceding theory to the manifold with boundary  $N = M - \bigcup_i D_i$ , which is foliated by  $(X)$ . The chosen function is the given function  $f$ . The set  $(G)$  reduces to certain contact values  $(i)$  of  $X$  in the spheres  $\partial D_j$  and their saturations, which are neighborhoods of the nappes of the gradient for small  $r$ . As a result, like the latter, these diverse sets intersect transversally, and their union constitutes a stratified set. The quotient  $S$  is represented by certain open sets of spheres  $D$ , and the equivalence relation  $(X)$  is induced by a stratified map  $q: N \rightarrow S$ . Since any leaf here is simply connected, one has a simple stratification whose topological type is, moreover, independent of the radius  $r$  if  $r$  is sufficiently small. It results from this that the field  $(X)$  is structurally stable in the manifold  $(N)$ . Since the connecting homeomorphisms that express the structural stability leave the spheres  $D_j$  (globally) invariant, one may possibly extend these homeomorphisms to any manifold  $M$  by letting  $r$  go to zero.

### BIBLIOGRAPHY

- [1] R. THOM, Les singularités des applications différentiables, *Annales de l'Institut Fourier*, 1950.
  - [2] R. THOM, La stabilité topologique des applications polynomiales, *L'Enseignement Mathématique*, t. VIII, 1-2, 1962.
  - [3] G. REEB, Sur les espaces fibrés et les Variétés Feuilletées, *Actualités scientifiques et Industrielles*, Hermann, Paris, no. 1183.
-