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THE SINGULARITIES OF DIFFERENTIABLE MAPS

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Ever since the classic works of M. Morse, the study of the singularities of a numerical function on a manifold has been the object of a great amount of research. It seems to me that theorem 4 of the article (cf. [4]) by M. Morse is subject to generalization. More generally, I was led to consider the singularities of a map f: $\mathbb{R}^n \to \mathbb{R}^p$ and, more generally, of a map from a manifold V^n to a manifold M^p. In that regard, the definition itself of the singularities of a map, as well as their classification, pose very delicate problems that will be addressed in Chap. I. In Chap. II, one treats "generic" singularities, i.e., singularities that appear for "almost all" maps f: $\mathbb{R}^n \to \mathbb{R}^p$. To that end, I have used the technique that was developed in an earlier article [8]; one thus obtains precise information on the generic dimensions of the critical sets; here, a largely unrecognized phenomenon is brought to light: for a generic map $\mathbb{R}^n \to \mathbb{R}^p$ where $p < n$, the reduction in dimension of relates only to a regular point of the map. The map preserves the dimension of the critical set. Chapter 3 contains the description of the generic singularities for low dimensional target spaces, as well as several existence theorems for critical manifolds. Chapter IV addresses – but does not solve – the question of the stability of generic maps; here, one also finds a list of unsolved problems that are nevertheless quite worthy of interest. Finally, the last chapter treats the homological properties of critical sets; one may see that the outline of a generalization of Morse theory, but I will not pretend to its completeness: Only the theory of characteristic classes was used here, although one may imagine using a homological theory like that of Leray, or an even finer theory like that of the Ljusternik-Schnirelmann "category." However, the scope is so vast and hitherto devoid of applications that it is difficult to state anything but generalities.

FIRST CHAPTER

Here, one considers differentiable maps of class C^r from the Euclidean space $\mathbb{R}^n$ to the Euclidean space $\mathbb{R}^p$; one assumes that $f(0) = 0$. Conforming to the terminology of C. Ehresmann, $\mathbb{R}^n$ will be called the *source* space and $\mathbb{R}^p$ will be called the *target* space. In general, n will be taken to be $\geq p$, and the class r of the map f will be assumed to greater than the larger of the numbers n, p ([1]).

Let x be a point of $\mathbb{R}^n$, and let k be the *rank* of the map f at x; it is convenient to introduce the two differences:

$q = n - k$, which one calls the "*corank at the source*" of x,
$r = p - k$, " " "*corank at the target*" of x.

One lets S_k denote the set of points of $\mathbb{R}^n$ at which the corank r has the value k (strictly); it is clear that S_{k+1} belongs to the adherence of S_k, in such a way that $\overline{S_k}$ constitutes the set of points where the corank r is $\geq k$. In particular, S_0 (if $n \geq p$) constitutes the set of *regular* points of f; it is an open set of $\mathbb{R}^n$.

In the target space $\mathbb{R}^p$, one considers the image sets $Y_k = f(\overline{S_k})$; one calls $Y_1 = f(\overline{S_1})$ the *set of critical values*; one knows that if the map f is of class C^m, where $m \geq n - p + 1$, then this set has measure zero in $\mathbb{R}^p$ [6]. The complement $\mathbb{R}^p - Y_1$ is the set of *regular values.*

For an arbitrary differentiable map, the singularities and the topological structure of the sets S_k (and, *a fortiori*, Y_k) may define a pathological manifold. One nevertheless obtains a structure that is already much simpler if one confines oneself to considering only "generic" maps. The precise definition of a generic map is very delicate; for the moment, we say only that any map may be approached by a generic map (up to an approximation on the derivatives of order r), and that any map that is sufficiently close to a generic map, in the preceding sense, is itself generic.

To any map f: $\mathbb{R}^n \to \mathbb{R}^p$, we associate its graph $G(f)$ in the product space $\mathbb{R}^{n+p}$. Let $y = f(x)$ be the image of a point x of $\mathbb{R}^n$. At the point (x, y) the graph $G(f)$ admits a tangent n–plane T_x. The correspondence $x \to T_x$ defines a map $\bar{f}$ of $\mathbb{R}^n$ into the Grassmannian G_n^p of n-planes through O in $\mathbb{R}^{n+p}$; the map $\bar{f}$ goes by the name of the *derived map* for the map f.

While always assuming (to fix ideas) that $n \geq p$, we let F_r denote the pseudo-manifold of G_n^p that is composed of the n-planes that intersect the $\mathbb{R}^n$ plane $y = 0$ along a linear subspace of dimension $(n - p + r)$.

The F_r are Schubert cycles in G_n^p, and their Schubert symbol is (for the definition, see, for example, [3] or [11]):

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[1] Ultimately, when one says "maps" or "manifolds," one always means maps or manifolds that are differentiable to whatever class is necessary. Any connected component of a manifold is assumed to be paracompact (a denumerable union of compacta).

$$(F_r) \quad \underbrace{(p-r, p-r, \cdots, p-r}_{(n-p+r)} \quad \underbrace{p, \cdots, p, p)}_{(p-r)}.$$

The dimension of F_r is therefore:

$$(p - \text{r})(n - p + \text{r}) + p(p - r) = (p - r)(n + r),$$

and its codimension in G_n^p (which is a manifold of dimension np) is $np - (np - rn + rp - r^2) = r(n - p + r)$. As we verify, these cycles (mod 2) F_r play an important role in the determination of the critical sets S_r; we must also specify their topological nature. At an ordinary point, F_r is a manifold; the singular points of F_r are the ones that belong to $\overline{F}_{r+1}$. It is interesting to specify the nature of the immersion of F_{r+1} into F_r.

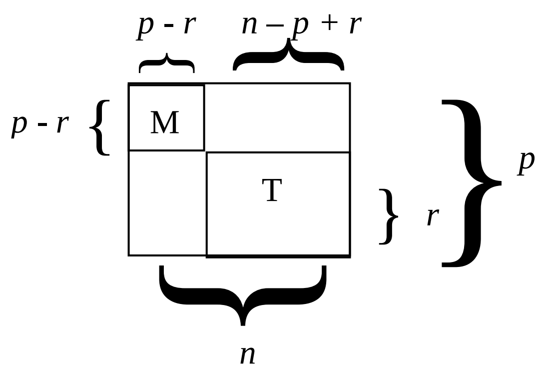
Let u be an ordinary point of F_{r+1}; this n-plane in G_n^p may be generated by n orthogonal vectors, of which $(n - p + r + 1)$ are in $\mathbb{R}^n$ $(y = 0)$, in which they generate a plane Y and $(p - r - 1)$ generate a subspace Z that is orthogonal to Y. A neighborhood of u in F_r that is normal in F_{r+1} is composed of the n-planes such that $(n - p + r)$ of the generating vectors are in the plane Y and $(p - r - 1)$ generate Z. What remains is an n^{th} vector V, which is to be determined, that must be in $\mathbb{R}^p$ and orthogonal to Z; as a result, its locus is a linear space of dimension $p - (p - r - 1) = r + 1$. From this, it results that the normal neighborhood of F_r in F_{r+1} is fibered into Euclidean spaces $\mathbb{R}^{r+1}$ over the set $\mathbb{R}^{n-p+r}$, which is contained in $\mathbb{R}^{n-p+r+1}$, and thus in a sphere S^{n-p+r}; this neighborhood is therefore a fiber of the form $S^{n-p+r} \times S^r \times \mathbb{R}$, a fiber that is trivial, at least homologically. From this, it results that F^{r+1} is embedded in F^r not as a submanifold, but only as a locus of singular points.

Analytical determination of a normal neighborhood of F_r in G_n^p.

Let x_i ($i = 1, 2, \ldots, n$) be coordinates in $\mathbb{R}^n$ and let y_j ($j = 1, 2, \ldots, p$) be coordinates in $\mathbb{R}^p$. An n-plane of G_n^p is defined by a system of p linear equations of the form:

$$y_j = \sum_i a_j^i x_i \ .$$

Suppose that this plane belongs to F_r; this says that the matrix a_j^i has rank at least $(p - r)$. If it is an ordinary point of F_r, moreover, then the matrix a_j^i has rank strictly $(p - r)$. Suppose that the minor M that is formed from the first $(p - r)$ rows and the first $(p - r)$ columns is different from zero. There is a complementary rectangle T in the matrix that is formed from the last p rows and the last $(n - p + r)$ columns. We associate an element of the rectangle T to the minor M and, upon completing M with the row and column that intersect this element, form a minor of order $(p - r + 1)$. We thus obtain $r(n - p + r)$ minors of order $(p - r + 1)$. Upon annulling these minors, one thus writes a system of local equations for F_r in G_n^p.



This remark is useful for the practical determination of F_r in the neighborhood of one of its ordinary points. Similarly, one may obtain the normal neighborhood of F_r in F_{r+1} by writing the quadratic relations between the minors of order $(p - r + 1)$ that appear when the minors of order $(p - r + 2)$ are all null.

The homology of the cycles F_r. – For $r > 1$, each of the pseudo-manifolds F_r have a fundamental cycle mod 2. Indeed, it is more practical to use the cohomology classes that correspond to them by Poincaré-Veblen duality; this is why F_1 corresponds to the Stiefel-Whitney class W_{n-p+1}. In a general fashion, the classes that are dual to the F_r are polynomials in the classes W_i that one calculates explicitly, thanks to the multiplication formulas between Schubert cocycles that were given by S. Chern [1]. Note that for $n - p = 2(k - 1)$, the class that is dual to F_2 is defined to have integer coefficients; it is none other than the Pontrjagin class P_{4k} (cf. Wu [11]). It will be interesting to know the cohomological expression with integer coefficients of the classes that are dual to the F_{2k}, which are defined to have integer coefficients when $(n - p)$ is even.

Critical sets and cycles F_r. – If the rank at a point x of $\mathbb{R}^n$ is reduced to $p - r$ then the tangent plane to the corresponding point (x, y) on the graph $G(f)$ is projected onto $\mathbb{R}^p$ along a $(p - r)$-plane; i.e., the kernel of that projection, which is the intersection of the plane with $\mathbb{R}^n$, has dimension $n - p + r$; therefore, the tangent plane to $G(f)$ at (x, y) belongs to the cycle F_r, and conversely. If one reverts to the definition of the map $\bar{f}$, which is the derived map to f, then one sees that the critical sets S_r are the inverse images under $\bar{f}$ of the cycles F_r of the Grassmannian.

DEFINITION. – A critical point x of S_r will be called *transversally critical,* or, furthermore, *generic,* if the tangent planes to $\bar{f}(\mathbb{R}^n)$ and F_r are in general position at the point $\bar{f}(x)$ of the Grassmannian, which is assumed to be ordinary on F_r.

In chapter II, we will show that any map f may be approached by a map g for which all of the points of S_r are generic. If this hypothesis is realized then the critical sets S_r are true submanifolds of $\mathbb{R}^n$ of codimension $r(n - p + r)$.

Moreover, S_{r+1}, which is situated in the adherence of S_r, admits a normal neighborhood that is homeomorphic to the cone over the product $S^r \times S^{n-p+r}$, which is described in the local immersion of F_{r+1} into F_r.

Ordinary critical points and exceptional points.

Now consider the map f restricted to S_r, which takes S_r to the image set Y_r. A question presents itself: Does this map have maximal rank, in general?

The response is affirmative, by reason of dimension; indeed, the generic dimension of S_r is $n - r(n - p + r)$, which is less than $p - r$ whenever $r \geq 1$. Now, any tangent plane to S_r is situated in a tangent n-plane to $G(f)$, which is a point of F_r whose projection onto $\mathbb{R}^p$ has dimension $p - r$; the projection of the tangent plane onto S_r is therefore carried out with preservation of the dimension, generically. Such a point will be called "*ordinary* (generic) critical point."

On the contrary, if the tangent to S_r at the point x of S_r is mapped onto $\mathbb{R}^p$ by f with reduction of the rank then one will say that x is an "*exceptional*" critical point. In order to comprehend the origin of these exceptional points it is convenient to refer to the preceding method and define them by intersections; let m be the dimension of S_r, and let q be the dimension of the cycle F_r. F_r may be considered to be the base space of a fiber bundle H, namely, the set of m-planes that are contained in the n-planes that define F_r ($m < n$); as a result, the fiber is the Grassmannian G_m^{n-m}. In each fiber, one may consider the set F_r' of m-planes that project onto the $\mathbb{R}^{p-r}$ that is the intersection of $\mathbb{R}^p$ and the n-plane, with a corank r'; the set of F_r' is obviously invariant under the operation of the structure group for the fiber bundle H in such a way that the set of F_r' comprises a certain cycle Z_r in H. The cycle Z_r is, moreover, a true submanifold of H at its ordinary points. Now, in the base space F_r of H one has a manifold $\bar{f}(S_r)$ of dimension m. Upon associating any point of that manifold with its tangent m-plane one defines a *section* of the fiber bundle H over the manifold $f(\mathrm{S}_r)$; the points of intersection between this section and the cycles Z_r constitute the exceptional critical points upon projecting onto the base space Z_r. Therefore, on each manifold S_r there exists a submanifold (without singularities, generically) Z_r on which f (restricted to S_r) presents a "corank at the target" equal to r'; under the map f: $S_r \to Y_r$ the image $f(Z_r)$ is, in general, a locus of singularities for Y_r; this fact was pointed out for the first time by F. Roger [5]. The set of critical values presents singularities in the target space. As one verifies, these singularities are *stable*, i.e., they persists under a small deformation of the map; they have a special, non-generic, character, moreover. In order to understand this phenomenon, the simplest example is that of the apparent contour of a surface that is projected onto a coordinate plane. One knows that such an apparent contour presents points of regression, in general (example: the apparent contour of a torus when viewed very obliquely with respect to its axis); one verifies that these points of regression are stable. Nevertheless, when considered as a singularity of the projection of a curve onto the plane, the point of regression is an unstable singularity.

In order to arrive at a complete description of a map, it is now necessary to consider the manifolds Z_r of exceptional points. When we iterate the argument made above one verifies that these critical manifolds present submanifolds $W_{r''}$ on which the rank of f (restricted to $Z_{r'}$) is reduced by r'' units, etc. One thus defines "super-exceptional" critical points. Nevertheless, by reason of dimension, one is sure that this process stops, generically or not; indeed, if $p \leq n$ then one may not have super-exceptional points of order $> p$. One ultimately subdivides the source space into a union of manifolds (without singularities, generically) X_i such that the restriction of f to each X_i has maximum rank.

We conclude with several remarks on the subject of exceptional critical manifolds. We first specify the order of the partial derivatives of the map f that intervenes. In the determination of the ordinary critical manifolds S_r only the map $\bar{f}$ is involved, i.e., the first order derivatives; in the determination of the $Z_{r'} \subset S_r$ the tangent plane to $\bar{f}(\mathbb{R}^n)$ is involved, hence, the second order derivatives; in the determination of the super-exceptional manifolds $W_{r''}$, the third derivatives, etc. Moreover, it is interesting to give an idea – which will be only intuitive – of the genesis of the singularities in the critical values that are images of the exceptional critical manifolds. One obtains them by the notion of "ventilating" a singularity by displacing the target space. We explain: suppose that we are given a map f: $\mathbb{R}^n \to \mathbb{R}^p$ that is critical at O; suppose, moreover, that the target space $\mathbb{R}^p$ moves in $\mathbb{R}^{p+m}$, and that this (n-parameter) motion is defined by a map g: $\mathbb{R}^n \to G_p^m$ into the Grassmannian of p-planes in $\mathbb{R}^{p+m}$. Upon assuming that the target space $\mathbb{R}^p$ is carried along by the motion that is defined by g, one defines a map F: $\mathbb{R}^n \to \mathbb{R}^{p+m}$, and at O the image $F(\mathbb{R}^n)$ presents the ventilated singularity of the initial singularity of f: $\mathbb{R}^n \to \mathbb{R}^p$. Example: $n = p = m = 1$. The map f: $\mathbb{R}^1 \to \mathbb{R}^1$ will be defined by $u = t^2$; suppose that the target-line $\mathbb{R}^1$ moves in the plane $\mathbb{R}^2$, and let θ be its polar angle with respect to a fixed direction. Suppose that the map g is defined by $\theta = at$. The "ventilated" singularity will then be defined in $\mathbb{R}^2$ by the equations:

$$x = t^2 \cos at = t^2 + \ldots$$
$$y = t^2 \sin at = t^2 + \ldots,$$

which defines a regression precisely.

The images of the exceptional and super-exceptional critical points present singularities that are all susceptible to this mode of generation. In the important particular case of the critical points of S_1, Z_1, etc., under ventilation, the singularities give singularities of the type $\mathbb{R}^m \to \mathbb{R}^m$; hence, the importance of this type of map.

CHAPTER II

One lets $L(\mathbb{R}^n, \mathbb{R}^p; r)$ denote the set of differentiable maps of class r ($r \geq \text{Sup } n, p$) of $\mathbb{R}^n$ into $\mathbb{R}^p$, endowed with the topology that is defined by the chart on the partial derivatives of order $\leq r$ over any compactum; under these conditions, $L(\mathbb{R}^n, \mathbb{R}^p; r)$ is a complete metric space, and therefore a Baire space. A property (P) of maps f: $\mathbb{R}^n \to \mathbb{R}^p$ that are defined locally at any point of the source space will be called *generic* if the set of f that possess property (P) is a *rare closed* set (closed without interior point), at least at a point of a compactum K of $\mathbb{R}^n$ defined in $L(\mathbb{R}^n, \mathbb{R}^p; r)$.

DEFINITION. – A map f: $\mathbb{R}^n \to \mathbb{R}^p$ will be called "*generic at the source*" if:

1) The derived map $\bar{f}$ of $\mathbb{R}^n$ into the Grassmannian G_n^p is t-regular (in general position) over the Schubert cycles F_r of G_n^p;
2) The section that is defined by a new derivation in the fiber H over each critical manifold $S_r = f^{-1}(F_r)$ intersects the cycle $Z_{r'}$ t-regularly.

The exceptional critical manifolds $Z_{r'}$ are then without singularities. One postulates that the sections (in an appropriate fiber of the Grassmannian) that define the super-exceptional critical manifolds $W_{r'}$ in $Z_{r'}$ are themselves t-regular on the appropriate cycle, and so on.

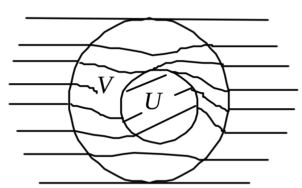
It is clear that the set of functions f: $\mathbb{R}^n \to \mathbb{R}^p$ that are generic at the source define an open subset of $L(\mathbb{R}^n, \mathbb{R}^p; r)$. Indeed, from theorem I.5 of [8], the property of general position (more exactly, of t-regularity) is preserved under a sufficiently small deformation (by approximation of the derivatives); the critical manifolds $S_r, Z_{r'}, W_{r'}$ remain mutually isotopic.

(Note that in theorem I.5 of [8], one assumes that the deformation is valid when it is composed with a homeomorphism of the target space; this is not a useful restriction here, and the theorem is valid under any deformation in $L(\mathbb{R}^n, \mathbb{R}^p; r)$.)

It remains to be proved – and this is more delicate – that any map f may be arbitrarily approached by a map that is "generic at the source." We shall first prove the property for "ordinary" critical manifolds S_r, and then present the proof for exceptional critical manifolds, later on. The principle of the proof is the following: Let f be the given map; assume that the critical set S_r of f is non-vacuous, unless S_r is vacuous for any sufficiently close map, and the property is proved. Therefore, cover S_r by a number (finite or infinite, but in the latter case, denumerable, because of paracompactness) of compacta K_j. Let M_j denote the set of maps in $L(\mathbb{R}^n, \mathbb{R}^p; r)$ that are not t-regular (after derivation) on the Schubert cycle F_r for at least one point of the compactum K_j; it is easy to see that M_j is closed in L. One must show that in a neighborhood of f that is assumed to belong to M_j there are points that do not belong to M_j. One knows that since the rank is strictly equal to ($p - r$) on S_r, one may find a system of ($p - r$) functions $u_1, u_2, \ldots, u_{p-r}$ in a neighborhood of any point x of S_r that are coordinate functions in the target space $\mathbb{R}^p$, and which one may take for coordinate functions in $\mathbb{R}^n$. Under these conditions, in a neighborhood of any $x \in S_r$ there is a chart in which $\mathbb{R}^n$ has the coordinates ($u_1, u_2, \ldots, u_{p-r}, u_{p-r+1}, \ldots, x_n$) and $\mathbb{R}^p$ has the coordinates ($u_1, u_2, \ldots, u_{p-r}, y_{p-r+1}, \ldots, y_p$), and in which the map f is represented by the equations:

$$y_{p-r+j} = h_j(u_k, x_{p-r+m}), \qquad j = 1, 2, \ldots, r-1.$$

The set S_r is then defined by the relations $\frac{\partial y_i}{\partial x_k} = 0$. One assumes that S_r is covered by an atlas of such charts, and that the compacta K_j are sufficiently small that they are each contained in a chart. One considers the set of maps, g, that are sufficiently close to f that they admit the same atlas of distinguished charts as f; it is obviously a Baire space X. One then shows that for any map f there exists an approximation g whose restriction to K_j is, after derivation, t-regular on the cycle F_r . The deformation of f into g happens as follows: for the map f one projects the graph $G(f)$ into the target space parallel to the source space $\mathbb{R}^n$; one obtains g by projecting $G(f)$ along a direction that is slightly oblique with respect to the initial direction; analytically, in the associated chart, this amounts to replacing the functions y with functions y' such that: $y'_j = y_j + \sum m'_j x_i$, in which the coefficients m'_j are assumed to be small. The deformation $f \to g$ is therefore defined only on the open subset U of the associated chart. It is nevertheless clear that one may prolong the deformation of f into g to the exterior of U; this results from classical theorems of Whitney on the prolongation of differentiable maps [9]. One may likewise suppose that this deformation reduces to the identity outside of a neighborhood V that contains U; in the planar case, the figure above gives us a satisfactory idea in spirit with being imbued with excessive formalism.



This being the case, the graph of the map g in U admits a tangent plane at any point (x_i) that is defined by the system of equations:

$u_i = u_i$: $$dy'_{p-r+j} = \sum_k \frac{\partial y'_{p-r+j}}{\partial u_k} du_k + \sum_i \frac{\partial y'_{p-r+j}}{\partial x'_{p-r+i}} dx_{p-r+i},$$

namely:

$$dy'_{p-r+j} = \sum_k \frac{\partial h_j}{\partial u_k} du_k + \sum_i \left(\frac{\partial h_j}{\partial x_{p-r+i}} + m_j^i \right) dx_{p-r+i} .$$

The set S_r for the map g is therefore defined by the equations: $\frac{\partial h_j}{\partial x_{p-r+i}} + m_j^i = 0$, which amounts to looking at the inverse image of F_r by the derived map $\bar{g}$ if one remarks that minors relative to the square ($u_i = u_i$) form a system of transverse coordinates for the Schubert cycle F_r.

The map $G: \mathbb{R}^n \to \mathbb{R}^{r(n-p+r)}$ that is defined by the equations $m_j^i = -\frac{\partial h_j}{\partial x_{p-r+i}}$ admits regular values that are close as one wants to the origin. If (m_j^i) denote the coordinates of such a value then this signifies that the corresponding map g is t-regular, when derived, on the Schubert cycle F_r that one would like to obtain precisely. We therefore see that the subspaces M_j in the Baire space X are rare; therefore, their union is meager in X, and admits no interior point. This shows that in any neighborhood of any f there are g that are t-regular on F_r after derivation, and this confirms the stated generic property. We shall elaborate on several consequences of this result.

THEOREM 1. – *Any map* $f: \mathbb{R}^n \to \mathbb{R}^p$ *may be approached arbitrarily closely* (*relative to the partial derivatives of order* $\leq r$) *by a map g whose derived map* $\bar{g}$ *is t-regular on the Schubert cycles* F_r; as a result, the points of $\mathbb{R}^n$ where g has corank at the target strictly r define true submanifolds S_r.

The codimension of S_r in $\mathbb{R}^n$ is equal to $r(n - p + r)$.

COROLLARY. – If $n < r(n - p + r)$ then the set S_r is "generically" vacuous.

If $n \geq p$, moreover, then this shows that for $n < r^2$ there is no stable singularity of corank at the target $\geq r$. Examples of this are:

Stable critical points of corank 2. – These appear only for $n = p = 4$ (maps $\mathbb{R}^4 \to \mathbb{R}^4$); by contrast, the maps $\mathbb{R}^5 \to \mathbb{R}^4$ do not admit stable critical points of corank 2.

The first case of a critical point of corank 3 appears for the maps $\mathbb{R}^9 \to \mathbb{R}^9$, and the first case of a critical point of corank r appears for maps $\mathbb{R}^{r^2} \to \mathbb{R}^{r^2}$. In chapter III we verify that these singularities essentially exist and that they are "generic" (stable under any small deformation).

The formula that gives the codimension of S_r is itself valid for $p < n$. One may remark that if r is the corank at the target then $n - p + r$ is the corank at the source; one therefore has the very mnemonic formula:

THEOREM 2. – *The generic codimension of the critical set of a map is equal to the product of the corank at the source with the corank at the target.*

From this, one deduces a very curious consequence concerning how the dimensions of the critical sets S_r compare for the maps $\mathbb{R}^n \to \mathbb{R}^{n-k}$ and $\mathbb{R}^n \to \mathbb{R}^{n+k}$. For each of these cases, the formula gives:

for $\mathbb{R}^n \to \mathbb{R}^{n-k}$, codimension of $S_{r'} = r(k + r)$,

for $\mathbb{R}^n \to \mathbb{R}^{n+k}$, codimension of $S_{r+k} = (k + r)r$.

For example, for $n = 2$, $k = 1$ the critical sets of the maps $\mathbb{R}^2 \to \mathbb{R}^1$ and $\mathbb{R}^2 \to \mathbb{R}^3$ have the same dimension, namely zero. In the first case, one has the critical points of a function, and in the second case, the cuspoidal points of an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.

By way of example, we give a table below of the dimensions of the critical manifolds of a map $\mathbb{R}^5 \to \mathbb{R}^n$; r always denotes the corank at the target and – denotes a strictly negative dimension, hence, the set is generically void.

$r =$	0	1	2	3	4	5
$n = 1$	5	0	–	–	–	–
$n = 2$	5	1	–	–	–	–
$n = 3$	5	2	–	–	–	–
$n = 4$	5	3	–	–	–	–
$n = 5$	5	4	1	–	–	–
$n = 6$	–	5	3	–	–	–
$n = 7$	–	–	5	2	–	–
$n = 8$	–	–	–	5	1	–
$n = 9$	–	–	–	–	5	0
$n = 10$	–	–	–	–	–	5

The symmetry therefore exhibits a curious phenomenon that we may call "Whitney duality." We return to this later on in the context of the global properties of critical sets. In a more general fashion, one notes that if n is fixed then it is for $p = n$ that the maps $\mathbb{R}^n \to \mathbb{R}^p$ present the greatest diversity from the standpoint of critical sets and corank.

Remarks on the topological structure of the critical set.

The set of points of a map f where the corank at the target is $\geq r$ is nothing but the adherence $\overline{S_r}$; $\overline{S_r}$ obviously contains S_{r+1}. Since S_{r+1} is the inverse image of the Schubert cycle F_{r+1} under the derived map $\overline{f}$, which t-regular on F_{r+1}, one deduces that the normal neighborhood of S_{r+1} in S_r is homeomorphic to a neighborhood of F_{r+1} in F_r; e.g., recall the cone over the product of the spheres $S^{n-p+r} \times S^r$ at a generic point. From this, it results that the closed set $\overline{S_r}$ is not, in general, a true submanifold (unless $\overline{S_{r+1}}$ is generically vacuous); $\overline{S_r}$ is nevertheless a pseudo-manifold (for $r \geq 1$), in such a way that one may speak of the fundamental cycle of $\overline{S_r}$; this cycle has integer coefficients if the pseudo-manifold $\overline{S_r}$ is orientable, modulo 2, in general.

The exceptional critical singularities.

We shall describe the logic in the most banal case of an exceptional critical point, viz., a point of S_1 at which the restriction of f has corank (at the source) equal to 1. In a neighborhood U of such a point (as at any generic point of S_1) there exists a system of coordinates $(u_1, u_2, \ldots, u_{p-1}, x_p, \ldots, x_n)$ in which the map f is expressed by the equations:

$$U_i = u_i; \qquad y = \varphi(u_j, x_i) + \ldots,$$

where φ is a quadratic form that is not identically null (indeed, the hypothesis of t-regularity of φ on F_1 implies that φ is non-null). The critical set S_1 is defined in U by the equations: $\dfrac{\partial y}{\partial x_i} = 0$, $i = p, p + 1, \ldots, n$, and its tangent plane at the point (x_i, u_j) is defined by:

$$\sum_j x_j \frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_j u_k \frac{\partial^2 y}{\partial x_r \partial u_k} = 0\,.$$

This plane is therefore defined in the product $\mathbb{R}^{p-1} \times \mathbb{R}^n$ by the matrix:

$$\left|\begin{array}{ccc} & u_i & x_j \\ U_i & \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} & 0 \\ \dfrac{\partial y}{\partial x_i} & \dfrac{\partial^2 y}{\partial x_i \partial u_k} & \dfrac{\partial^2 y}{\partial x_i \partial x_j} \end{array}\right|.$$

(One does not include the row that pertains to y, since this row is a linear combination of the preceding ones on S_1.)

With these conditions, the exceptional critical manifold Z_1 is defined in $\mathbb{R}^n$ by the nullity of the determinant:

$$D = \left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right|.$$

In fact, D may be considered as a coordinate that is transverse to the cycle Z in H (where Z is defined by $D = 0$ in H). Suppose that the image of a neighborhood of 0 in S_1 under the (twice) derived map in H is all integers in the neighborhood of a generic point $z \in Z$; this must say that on the proposed neighborhood the map f when restricted to S_1, has rank at least $(p - 2)$, or, furthermore, that on the proposed (possibly restricted) neighborhood, one may find a minor of the matrix M of order $(p - 2)$ that is not annulled in that neighborhood. Suppose (to fix ideas) that it is the relative minor of the first element (1st row, 1st column) of the matrix $M = \left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right|$, namely, $Q_1(x, u)$ for this minor. We then deform the map f into g, where g is defined on the neighborhood considered by the equations: $U_i = u_i$; $Y = y + a(x_p)^2$. The value $D(g)$ of the determinant D, for the map f will be: $D(g) = D(f) + 2aQ_1(x, u)$.

The relation $D(g) = 0$ may be written $a = - 1/2D(f) / Q_1(x, u)$. In the neighborhood of 0 considered, this relation defines a map of S_1 onto $\mathbb{R}^1$. Take a regular value $a = c$ of this function for a, which is sufficiently close to 0 as one prefers. The map of S_1 into H that is associated with g and defined by $Y = y + cx_p^2$ is therefore t-regular on the cycle Z because the transverse coordinate D has maximum rank on S_1.

As in the case of the addition of linear functions, one shows that for a sufficiently small c the deformation of f into g may be prolonged by a deformation that reduces to the identity outside a sufficiently large neighborhood.

It remains to show that any map f is of rank $\geq p - 2$ "almost everywhere" on S_1; indeed, if all of the minors of order $p - 2$ are null in a neighborhood of 0 then one may first suppose that at least one of the minors of order $p - 3$ is non-null in a – possibly restricted – neighborhood. With these conditions, by adding a conveniently chosen quadratic form at x to y, one may obtain that the rank is everywhere $\geq p - 2$ on this neighborhood, except possibly on the points of a submanifold, and so on for the lesser ranks. The proof is completed, as before, by "combining the pieces." It is nevertheless important to say that this does not generalize without difficulty to exceptional critical manifolds of corank at the source > 1. Indeed, as we shall verify at the end of chapter 3, when one forms the intersection with a cycle $\underline{F}_r$ of a Grassmannian that defines the exceptional manifold, one is led to consider the coefficients of a matrix $\underline{T}$. In general, these coefficients contain second derivatives of the map f; now, due to the "integrability" relations $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ it is possible that the rank of this matrix is reduced. Nevertheless, due to the linear character of these relations, the image space is a linear subspace of the space of vectors that are transverse to $\underline{F}_r$, in such a way that one again speak of the t-regular map as $\underline{F}_r$, considered as embedded in this image linear subspace. In particular, it results from this that the codimension of an exceptional critical manifold generally smaller than the one that is given by the formula of theorem 2. (See the example at the end of chapter 3.) With this restriction, the principle of the last proof remains valid.

Case of the super-exceptional critical manifolds.

In the case of a super-exceptional manifold of order k, one is led to consider a *flag manifold* of order k (i.e., a system of k-planes that are each contained in the following one), and a subset H in such a manifold that is defined by conditions on the dimension of the projection onto the factor $\mathbb{R}^p$, and finally, a certain cycle Z in the submanifold H. In a neighborhood of a generic point of Z, the immersion of Z into H is isomorphic to that of a Schubert cycle in a Grassmannian, in such a way that in a neighborhood of such a point, a system of normal coordinates of Z in H may be given by the $r(n - p + r)$ minors that appear in a matrix. What appear in the rows of this matrix are the coefficients of the tangent plane to the critical manifold of order $(k - 1)$ that one maps into H. These coefficients linearly contain the derivatives of order k of the map f.

In an associated local chart, one will be led to add homogenous forms of degree k at x to the variables y by taking their variable coefficients to be the ones that appear, after derivation, in the $r(n - p + r)$ minors that give the local equations of Z in H. Here again, one chooses the values of these coefficients so that the map obtained in H is t-regular on the cycle Z, possibly taking into account linear relations between the coefficients that give us the integrability relations $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. The previous combination of the pieces then applies without other modifications. I envision returning to this point in a later publication.

CHAPTER III

GENERIC FORM OF SINGULARITIES FOR LOW DIMENSIONS

Here, we will be occupied with the singularities that are presented by generic maps of $\mathbb{R}^n$ into $\mathbb{R}^p$ when p is small.

1. $p = 1$. **Singularities of a function.** – Let $f: \mathbb{R}^n \to \mathbb{R}^1$ be a function and let $G(f)$ be its graph in $\mathbb{R}^{n+1}$. One considers the derived map of $\mathbb{R}^n$ into G_n^1. The Schubert cycle F_1 reduces to a point of G_n^1 (the horizontal n-plane). A system of normal coordinates for F_1 is given by the coefficients of the equation for the tangent plane, hence, by the partial derivatives $\frac{\partial f}{\partial x_i}$. From theorem 2, for an f that is generic at the source the critical set S_1 reduces to a certain number of points. At each of these points, the derivatives $\frac{\partial f}{\partial x_i}$ are linear forms in the local coordinates, and, as a result of the regularity of F_1 *these linear forms are independent.* This says that the local development of f about each critical point begins with a non-degenerate quadratic form. Therefore:

THEOREM 3. – *Any differentiable function on $\mathbb{R}^n$ may be approached as close as one wants* (*as well as its partial derivatives up to an arbitrary large, but finite, order*)*, by a function g whose only singularities are non-degenerate quadratic critical points.*

[Cf. M. Morse, The Calculus of Variations in the Large. *Colloque*, Publ. XVIII, pp. 178, th. 8.7.]

2. $p = 2$. **Singularities of a map into the plane**. – There exists only one non-vacuous generic critical set; its dimension is 1, and, since S_2 is vacuous, it is a true submanifold (a curve in $\mathbb{R}^n$). There are exceptional critical points that are defined as $S_1(S_1)$. It is easy to see the corresponding singularity in the target plane; it is obtained by "ventilating" the singularity $\mathbb{R}^1 \to \mathbb{R}^1$, which is defined by $u = t^2$. As we saw in chapter I, when it is composed with a map of degree 1 of $\mathbb{R}^1$ into $G_1^1 = S^1$ it gives a *regression of ordinary type.* One easily obtains the local equations of the singularities. One may give $\mathbb{R}^n$ a system of local coordinates of the form $\mathbb{R}^n (u, x_2, x_n) \to \mathbb{R}^n (u, v)$ in which v is a quadratic form φ in u and x_i. The critical curve in $\mathbb{R}^n$ is defined by $\frac{\partial v}{\partial x_i} = 0$, and its tangent vector at 0 is defined by $\frac{\partial \varphi}{\partial x_i} = 0$. In the present case, where the rank of the form $\varphi(0, x_i)$ is reduced, one may assume that v admits a development that begins with:

$$v = ux_1 + x_3^2 + \ldots + (x_n)^2,$$

and the critical curve is then locally parameterized by the variable x_2. Its image under the map is composed of $u = (x_2)^2$ by a "ventilation" of the angle kx_2. If we set $x_2 = t$ then this gives $u = t^2$, $v = kt^3 + \ldots$, for the image of the critical curve. Moreover, one directly recovers this result by writing the term in $(x_2)^3$ of the development of v. In the case $\mathbb{R}^2 \to \mathbb{R}^2$, the singularity is represented by the equation: $z = x^3 - xy$, in which f sends the plane (x, y) onto the plane (y, z). We shall return to this in a more general case, moreover.

3. $p = 3$. – Here again, one has only one critical set S_1 on S_1 that is a true surface; one has exceptional curves $S_1(S_1)$ and super-exceptional points $S_1(S_1(S_1)))$. In the target space, the exceptional curves have images that are curves of regression of surface of critical values. In general, a map $\mathbb{R}^2 \to \mathbb{R}^3$ presents points of corank at the target equal to 2; they are cuspoidal points. Here we see that *the critical value surface does not, in general, present cuspoidal points.* Therefore, a critical value set presents stable singularities other than the generic singularities, and, moreover, certain generic singularities for the dimensions considered may not present themselves. The local equations for the map $\mathbb{R}^3 \to \mathbb{R}^3$ in the neighborhood of a super-exceptional point that is taken to be the origin may be written in the form:

$$X = x, \qquad Y = y, \qquad Z = zy + z^2x - z^4.$$

The critical surface is defined by $\dfrac{\partial Z}{\partial z} = y + z^2x - 4z^3 = 0$, and its image in the XY-plane, which is defined by the parametric representation in x at z: $X = x$, $Y = -2zx + 4z^3$, presents a singularity in the form of an exceptional critical point $\mathbb{R}^2 \to \mathbb{R}^3$. The third coordinate:

$$Z = z(-2zx + 4z^3) + z^3x - z^4 = -z^2x + 3z^4,$$

defines the "ventilation" of this singularity in $\mathbb{R}^3$.

For the sake of completeness, we also mention the generic singularity of a map $\mathbb{R}^2 \to \mathbb{R}^3$. It amounts to the set S_2, which is composed of isolated points. In a neighborhood of such a point, the map is represented by the equations: $x = u$, $y = uv$, $z = v^2$; it amounts to the classical cuspoidal point of a surface. One knows that such a point is the extremity of a curve segment of the self-intersection $u = 0$ whose image is the z-axis. The manifold of self-intersection therefore presents a singularity of corank 1 at this point.

4. $p = 4$. – The case $\mathbb{R}^3 \to \mathbb{R}^4$ presents a unique singularity, which is an S_2 that is a curve of cuspoidal points (topological product of the singularity $\mathbb{R}^3 \to \mathbb{R}^4$ with $\mathbb{R}$.); this curve may, moreover, itself present exceptional critical points that may give rise to regressions.

The interesting case is presented by the maps $\mathbb{R}^4 \to \mathbb{R}^4$; S_1 is then composed of a 3-dimensional manifold that may itself present exceptional and super-exceptional critical manifolds that one may easily specify. However, one has, moreover, an S_2 that is composed of isolated points that are singular points for $\overline{S_1}$. A system of local equations in the neighborhood of a point of S_2 is given by:

$$x = u, \qquad y = v, \qquad z = w^2 + t^2, \qquad Z = uw + vt.$$

Indeed, if one forms the coefficient matrix of the tangent plane to the graph then one finds:

$$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & w & t \\ 0 & & u & v \end{bmatrix}$$

and, as a result, the derived map $\overline{f}$ is regular on the Schubert cycle F_2 with the normal coordinates of F_2 given by the elements of the lower right-hand submatrix.

For the maps $\mathbb{R}^5 \to \mathbb{R}^4$, S_2 is vacuous; likewise for $\mathbb{R}^n \to \mathbb{R}^4$, $n > 5$. We conclude these particular considerations and begin the study of general singularities.

Critical manifolds S_r. *Local generic form. Existence.*

Up to the present, we have not shown the actual existence of non-vacuous critical submanifolds S_r when their generic codimension allows this; i.e., when $n \geq r(n - p + r)$. Their existence is not obvious; indeed, if we are given a map g: $\mathbb{R}^n \to G_n^p$ then it is, in general, impossible to "integrate" this map, i.e., to find a map f: $\mathbb{R}^n \to \mathbb{R}^p$ such that g is its derivative. g must satisfy integrability conditions that are expressed, for example, by the symmetry of certain coefficients. This is why the existence of a map f whose derivative $\overline{f}$ is t-regular on the Schubert cycle F_r necessitates a proof. If we are given a system $(u_1, u_2, \ldots, u_{p-r}, x_{p-r+1}, \ldots, x_n)$ of coordinates in $\mathbb{R}^n$ then we consider the map into $\mathbb{R}^p$ with coordinates $(U_1, U_2, \ldots, U_{p-r}, Y_{p-r+1}, \ldots, Y_p)$ that is defined by the equations:

$$\begin{aligned}
&U_m = u_m, \qquad m = 1, 2, \ldots, p - r;\\
&Y_{p-r+1} = \sum_i (x_{p-r+1})^2 ;\\
&Y_{p-r+2} = U_1 x_{p-r+1} + U_2 x_{p-r+2} + \ldots + U_{n-p} x_n ;\\
&Y_{p-r+3} = U_{n-p+r+\mathrm{I}} x_{p-r+1} + U_{n-p+r+2} x_{p-r+2} + \ldots
\end{aligned}$$

and so on, until we have exhausted all of the variables U_i. However, this may not happen because there are $(p - r)$ variables U_j, and one disposes of $(r - 1)$ quadratic forms Y_{p-r+j}, $j > 1$, which each contain $(n - p + r)$ coefficients, and one has: $(p - r) > (r - p + r)$ of them such that $n > r(n - p + r)$, as one initially assumed.

The critical set S_r of the map f is then a plane defined by the linear forms: $\dfrac{\partial Y_{p-r+j}}{\partial x_{n-p-r+k}} =$ 0, forms that are $r(n - p + r)$ in number. If one specifies these forms at x_j and u_k then one confirms that the set of these forms at u_k and x_j has maximum rank, which is equal to $r(n - p + r)$. From this, one deduces that the derived map $\overline{f}$ is t-regular on the Schubert cycle F_r.

Therefore, all of the critical sets S_r with a non-negative generic dimension are essentially presented, and may be realized by algebraic maps that are expressed locally thanks to the quadratic forms. It remains to be known – and this is a problem that we defer – whether any generic point of S_r admits a local chart in which the map f has the form above. We treat the problem only in the case of the generic points of S_1.

Case of points of corank at the target equal to 1 ($n \geq p$) (cf. [5]).

At such a point, there exists a local chart (u_1, u_{p-1}, x) in which the map into $\mathbb{R}^p$ (U_1, U_2, …, U_{p-1}, y) takes the form: $U_j = u_j$, $j = 1, 2, \ldots, p - 1$; $y = \varphi(u, x_{p-1+i}) + \ldots$, in which φ is a quadratic form in u and x; by a change of variables that is a homeomorphism of the source space tangent to the identity at 0, one may assume that y is reduced to the just the quadratic form φ, and that the terms of higher order have disappeared [4]. Moreover, one may add an arbitrary quadratic form in the (u_i) to y. This amounts to performing a homeomorphism of the target space $\mathbb{R}^p$ that is tangent to the identity at the origin. The condition of t-regularity of the derived map $\overline{f}$ may be expressed by the fact that the $(n - p + 1)$ linear forms $\dfrac{\partial y}{\partial x_{n-p+i}}$ are linearly independent. Therefore, the rank of the form $\varphi(u_j, x)$ may not be (generically) reduced by more than $(n - p + 1)$. By subtracting an arbitrary quadratic form in u_i the reduced form $\varphi(0, x_i)$ that is obtained for $u_i = 0$ has rank at least $n - 2(p - 1)$.

It is clear that the index of the quadratic form restricted to the plane $u = 0$, $\varphi(0, x_i)$ is an invariant. One may interpret this in the following manner: suppose that the point $x = 0$ is an ordinary critical point. The map $\overline{f}$ has maximum rank on the critical manifold at x, and, as a result, at $y = f(x)$, the image of the critical manifold is a regularly embedded $(p - 1)$-dimensional submanifold of $\mathbb{R}^p$. Consider a line $\mathbb{R}^1$, which is transverse to that manifold at y and let Q be its inverse image under f. In general, it is a submanifold Y and the restriction of f to Y defines a function $f: Y \to \mathbb{R}^1$. The index of this function, which presents a critical point at x in Y, is the index of the form $\varphi(0, y)$. One calls this the "*transverse index*" of the indicated point x.

If one wants to analytically express the condition for a critical point to not be exceptional, then one must express that the map $\dot{f}$ on the tangent plane to S_1 has maximum rank, therefore its kernel is null. Now, in the space of vectors tangent to $\mathbb{R}^n$ at O, $\dot{f}$ has a kernel composed of the $(n - p + 1)$-plane $u_1 = u_2 = \ldots = u_{p-1} = 0$, and the tangent plane to S_1, which is defined by $\dfrac{\partial y}{\partial x_j} = 0$, is nothing but the plane that is conjugate to the kernel of f with respect to the cone of second degree that is defined by $\varphi(u, x) = 0$. Now, a plane and its conjugate may admit a common line only if the plane is tangent to the cone along this line. This amounts to saying that the hyperquadric that is defined projectively by the equation $\varphi(0, x) = 0$ admits a double point. The exceptional critical points are therefore characterized by the fact that the quadric $\varphi(0, x_i) = 0$ presents multiple points. On the contrary, at a generic point, this quadric has no singularity, and conversely.

Thus, it is easy to determine the topological structure at a point that is not exceptional whose corank at the target is equal to 1. Indeed, one may assume that the variables u_i (therefore U_i) are chosen in such a fashion that the tangent plane to the critical manifold in the target space $\mathbb{R}^p$ is defined by $Y = 0$. This imposes the condition that the tangent plane to S_1 at O must be the plane that is defined by $x_{n-p+i} = 0$. Under these conditions, Y takes the form:

$$Y = \sum a_j^i (x_j)(x_i)\,,$$

in which the terms in $u_i x_j$ are zero.

One then sees that the singularity is nothing but the topological product of the map Y: $\mathbb{R}^{n-p+1} \to \mathbb{R}^1$ that presents a critical point at the origin with a space, (that of the $u_i = U_i$).

Generic exceptional critical points.

Let O be such a point. At the point O the critical manifold S_1 admits a tangent $(p - 1)$-plane, which one assumes to be parameterized by the coordinates $u_1, u_2, \ldots, u_{p-2}, v$. One supposes that the functions u_i are chosen in such a fashion that they have total rank $(p - 2)$ on this tangent plane, and the kernel of the prolonged map $\dot{f}$ is the line $u_i = 0$ parameterized by the function, v. Under these conditions, the restricted map f on S_1 may be defined by the formulas:

$$U_i = u_i, \qquad i = 1, 2, \ldots, p - 2; \qquad V = 3v^2.$$

We complete the system of coordinates of $\mathbb{R}^n$ with:

$$(u_1, u_2, \ldots, u_{p-2}, w, v, x_{p+1}, \ldots, x_n),$$

and those of $\mathbb{R}^p$ with $(U_1, U_2, \ldots, U_{p-2}, W, Y)$. One then embeds S_1 into $\mathbb{R}^n$ by setting: $w = V = 3v^2$, and since the map f: $\mathbb{R}^n \to \mathbb{R}^p$ is defined by $U_i = u_i$, $W = w$, Y will begin with a quadratic form in (u_i, v, x_j). However, the critical set S_1 is then defined by the equations $\dfrac{\partial Y}{\partial x_i} = 0$, $\dfrac{\partial Y}{\partial v} = 0$. Since this set must be identified with S_1, which is defined by $x_j = 0$, and $w - 3v^2 = 0$, one sees that $Y = wv - v^3 + \sum (x_j)^2$, in which $\sum (x_j)^2$ is a quadratic form of rank strictly $n - p$. In this reduced form, one sees that the singularity is the product of the singularity $\mathbb{R}^{n-p+2} \to \mathbb{R}^2$ that is associated with a regression point of the apparent contour curve with a space $\mathbb{R}^{p-2}$ (that of the $u_i = U_i$).

One easily verifies that the transversal index on that reduced form at an ordinary critical point that is close to an exceptional point varies by a unit when this point traverses the manifold of exceptional points.

Several results on the exceptional singularities.

The explicit description of "all" the generic singularities of the differentiable maps obviously appears to be a quasi-chimerical task. On this subject, one may at least pose the prejudicial question of their existence. One has already seen that all of the critical manifolds S_r of corank at the target r exist when the dimensional conditions of theorem 2 are satisfied. In turn, one may demand that the same be true for the exceptional singularities. The response is affirmative when one is dealing with super-exceptional singularities of finite order k, provided that the k singularities are "banal," i.e., of corank 1. It is negative, in general, for singularities of corank greater than 1. We show:

The existence of super-exceptional singularities of arbitrary order such that all of the components of the singularity are of corank (*at the target*) *equal to* 1.

This amounts to constructing a map f: $\mathbb{R}^n \to \mathbb{R}^p$ such that if S_{j+1} denotes the critical manifold of S_j (which is assumed to pass through the origin O) then the point O is a critical point of corank at the target equal to 1, generically, for all restrictions $f|S_j$, $j = 1, 2, k - 1$, and proceeding by recurrence on k. We assume known a map g: $\mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$ such that O is a critical point of corank 1 for all the successive critical manifolds (which are $k - 1$ in number). Since the map f is of corank 1, one assumes that one may find $(p - 2)$ functions u_1, u_p, …, u_{p-1} that are preserved by the map. Upon completing these functions u_i with a $(p - 1)^{\text{th}}$ function v one obtains a system of coordinates in the source space in which the map f into $\mathbb{R}^{p-1}$ with coordinates (U_1, U_{p-2}, W) is expressed by the equations: $U_i = u_i$, $W = h(u_i, v)$; h is a polynomial of degree k. One then immerses the source space $\mathbb{R}^{p-1}$ into a space $\mathbb{R}^n$ of coordinates (u_i, w, v, x_{n-p+1}, …, x_n) by setting: $x_{n-p+1} = 0$; $w = h(u_i, v)$. One then sets:

$$Y = wv - \int_0^v h(u_i, v)dv + \sum (x_{n-p+j})^2 \ . \tag{P}$$

One immediately verifies that the map F:

$$\mathbb{R}^n (u_i, w, v, x_{n-p+j}) \to \mathbb{R}^p (U_i, W, Y)$$

is defined by the equations: $U_i = u_i$, $i = 1, 2, \ldots, p - 2$; $W = w$; Y = the polynomial (P) that answers to the desired conditions. One sees that the polynomial that serves to define Y is then of degree $k + 1$. Indeed, in the case of S_1 ($k = 1$), it suffices to define Y as a quadratic form.

Inexistence (*in general*) *of an exceptional singularity of corank* 2.

Consider a map f: $\mathbb{R}^n \to \mathbb{R}^5$, $n \geq 5$. The critical manifold S_1 is of dimension 4. At each point $x \in S_1$ the plane that is tangent to S_1 is mapped into $\mathbb{R}^5$ with a rank equal to 4. One may thus see whether the manifold S_1 presents exceptional singularities of type $S_2(S_1)$ of corank 2, which will thus be isolated points. For $n = 5$, it is obvious that such a singularity is impossible. Indeed, at a generic point of S_1, the kernel of the map f has dimension 1 in the space of tangent vectors (corank 1). As a result, an $\mathbb{R}^1$ may not be mapped onto an $\mathbb{R}^2$ by *f*. For n = 6, this argument is no longer valid. Take a system of coordinates u, v, x, y, z, t in $\mathbb{R}^6$ such that the 4 functions u, v, x, y define a map of maximum rank on $\mathbb{R}^4$. Then take a system of coordinates (U, V, X, Y, Z) in $\mathbb{R}^5$ such that $U = u$, $V = v$, $X = x$, $Y = y$. The critical set S_1 is then defined by the equations $\frac{\partial Z}{\partial z} = 0$ and $\frac{\partial Z}{\partial t} = 0$. The tangent plane must contain the kernel of *f*; one may suppose that it is the plane $u = v = 0$. One obtains this situation by setting:

$$Z = uz + vt + g(u, v, x, y, z, t),$$

in which g contains terms of at least third order. A parametric representation of S_1 in $\mathbb{R}^6$ is given by the 4 variables (x, y, z, t) and the equations: $ux = -\frac{\partial g}{\partial x}$, $v = -\frac{\partial g}{\partial t}$. As a result, the tangent plane to S_1 at the point (x, y, z, t) will be defined by the matrix:

$$\begin{array}{c|c} & \begin{matrix} \underline{x} \quad & \underline{y} \quad & \underline{z} \quad & \underline{t} \end{matrix} \\ \begin{matrix} X \\ Y \\ U \\[12pt] V \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\partial^2 g}{\partial z \partial x} & \frac{\partial^2 g}{\partial z \partial y} & \frac{\partial^2 g}{\partial z^2} & \frac{\partial^2 g}{\partial z \partial t} \\ \frac{\partial^2 g}{\partial t \partial x} & \frac{\partial^2 g}{\partial t \partial y} & \frac{\partial^2 g}{\partial t \partial z} & \frac{\partial^2 g}{\partial t^2} \end{bmatrix} \end{array}.$$

The intersection of the image of S_1 under the twice-derived map with the Schubert cycle $F_2 \subset G_2^2$ will thus be defined by the equations: $\frac{\partial^2 g}{\partial z^2} = \frac{\partial^2 g}{\partial z \partial t} = \frac{\partial^2 g}{\partial t^2} = 0$. They are three in number, instead of the four that normally appear. This proves that because of the integrability condition $\frac{\partial^2 g}{\partial z \partial t} = \frac{\partial^2 g}{\partial t \partial z}$ the intersection of $\overline{f}(S_1)$ with the cycle F_2 (or its extension to a fiber bundle such as *H*, pp 5) is *never t-regular*. The critical set $S_1(S_2)$ will exist, but it will be of dimension 1 instead of zero. In the case $n > 6$ one has, apparently, an analogous conclusion. Here, we only draw attention to this phenomenon, which merits a more concerted study. In any case, it emphasizes the difference between the nature of the singularities of critical manifolds and the critical singularities, period – a difference that we have pointed out previously.

CHAPTER IV

GLOBAL PROPERTIES OF GENERIC MAPS

Recall that a property (P) of maps $f: \mathbb{R}^n \to \mathbb{R}^p$ that are defined locally at any point of the source space is called "generic" if the set of f that do not possess property (P) at one or more points of a compactum K in $\mathbb{R}^n$ form a *rare closed* subset (closed, without interior) in $L(\mathbb{R}^n, \mathbb{R}^p; r)$.

Passage from the local to the global.

If V^n and M^p denote two manifolds of dimension n, p, respectively then the space of maps of class r from V^n to M^p, $L(V^n, M^p; r)$, is well defined and is a Baire space. One must show that if (P) is generic then the set of maps $f: V^n \to M^p$ that do not possess property (P) at one or more points V^n is a *meager* subset of $L(V^n, M^p; r)$.

Let K_j be a compactum of V^n and let (D_j) be the set of f that do not possess property (P) at one or more points of K_j. One must show that (D_j) is *rare* in $L(V^n, M^p; r)$. Indeed:

i) (D_j) *is closed*. This is true because if g belongs to the complement of (D_j) then g possesses property (P) at any point of K_j. The same must be true for any g that is sufficiently close to g since (P) is generic.

ii) (D_j) *has no interior point*. Let g be a point of (D_j). Choose an atlas U_j on V^n and an atlas W^k on M^p, such that the image of an open subset U_i under g is contained in an open subset W_j of W^k. Let (G) denote an open neighborhood of g in $L(V, M)$ that is sufficiently small that if $f \in (G)$ then the image $f(U_j)$ is also contained in W_j. As a result, there exists a canonical map h of (G) into $L(U_i, W_j)$ that is defined by the restriction of the map to (U_i); h is obviously an open map for the Baire space topology that is defined on (G) and $L(U_i, W_j)$. Set $K_{ji} = U_i \cap K_j$, and let Z_{ij} denote the set of $f \in (G)$ that do not possess (P) at one or more points of K_{ji}. Z_{ij} is a rare set in $L(U_i, W_j)$. As a result, its inverse image $h^{-1}(Z_{ij})$ in (G) is likewise rare in (G). The union (over the variable i) of the $h^{-1}(Z_{ij})$ in (G) obviously gives the intersection $(G) \cap D_j$. It is therefore a meager set in (G), hence it has no interior point, and, as a result, there is a $g' \in (G)$ in the neighborhood of g that does not belong to (D_j).

If one then covers V^n by an at most denumerable infinitude of compacta then one will deduce that the set of $f \in L(V^n, M^p)$, which do not possess property (P) at a point of V^n is represented by $\bigcup_i (D_j)$. Hence, by a "meager" subset (*rare,* if V^n is compact; there are only a finite number of K_j).

As an application of this theorem, we cite:

THEOREM 4. – *Any real numerical function on a manifold may be approached arbitrarily closely* (*as well as its derivatives up to order r*) *by a function that possesses only non-degenerate quadratic critical points.*

It is clear that one may state an analogous theorem for any system of generic singularities $\mathbb{R}^n \to \mathbb{R}^p$ once one has obtained an exhaustive description of these singularities. For example:

THEOREM 5. – *Any map f of a manifold V^n onto the plane $\mathbb{R}^2$ may be approached by a map g that has only a critical curve of corank at the target equal to* 1, *and is without singularities in V^n. By projection onto $\mathbb{R}^2$, the critical curve will possess, other than self-intersection points, ordinary regression points* (*images of the exceptional critical points*).

Some generalities on generic maps.

We first give several definitions.

Differentiable set. – A subset E of is $\mathbb{R}^k$ called *differentiable of class r* if at any point x of E there exists a neighborhood U_x of x in $\mathbb{R}^k$ and an ideal (with a finite basis) of the ring of differentiable functions of class r in U_x such that $U_x \cap E$ is the set of points that annul all of the functions of the ideal. One may assume that E is defined by a "coherent system of ideals" relative to an atlas U_j that covers E. [In fact, any closed set is differentiable in the preceding sense. In the sequel, one assumes, moreover, that E is a *manifold with singularities*: On the basis for the ideal, the functions have partial derivatives of non-null finite order at any point of E.]

Differentiably isotopic sets. – Two differentiable sets E, E' that are both defined by a coherent system of ideals in the same open subset U of $\mathbb{R}^k$ will be called differentiably isotopic if there exists a homeomorphism h: $U \to U$ that reduces to the identity on the frontier of U, and transforms the system of ideals relative to E into a system relative to E'.

Locally isotopic sets. – Two differentiable sets E, E' will be called locally isotopic if there exists an atlas U_i on a common neighborhood U such that the intersections $E \cap U_i$ and $E' \cap U_i$ are differentiably isotopic in each U_i (the homeomorphisms h^i of U_i do not necessarily reduce to the identity on the frontier of U_i).

Continuously isotopic sets. – Let E be a differentiable set that depends continuously on the parameter t; one says that E remains *continuously isotopic to itself* if for every value of t there exists a homeomorphism h_t of U that transforms E into E_t. There is a similar definition for locally continuously isotopic sets. In that regard, we have the theorem:

THEOREM 6. – Two *locally continuously isotopic* differentiable sets are *globally continuously isotopic*.

Proof. – One forms the product $\mathbb{R}^k \times \mathbb{R}$ in which the factor $\mathbb{R}$ is parameterized by t. In each product $U_i \times \mathbb{R}$ the homeomorphism $h^i(t)$ allows us to define a direction D_y at any point y on which one may assume that t has maximum rank. To any point $y \in U_i \times \mathbb{R}$, one associates a vector $V_i(y)$ that is non-null at any point of $U_i \times \mathbb{R}$ and parallel to D_y. Let $(c_i(y), y \in U)$ be a partition of unity subordinate to the atlas U_i. One associates the vector $Z = \sum c_i(y) V_i(y)$ to any point (y, t). The vector field thus defined has rank 1 everywhere on the t-axis, and its integration gives a system of trajectories that defines the desired global homeomorphism $h(t)$. Moreover, one may assume that the complement to U is covered by an element X of the atlas, on which one takes a field that is parallel to the t-axis. With these conditions, the homeomorphism $h(t)$ so obtained reduces to the identity outside of U.

A fundamental lemma. – Suppose that the set E is defined as the inverse image of a submanifold by a t-regular map of the space $\mathbb{R}^k$ that contains E. E is then itself a submanifold. If one deforms the map f sufficiently little then the set E remains continuously isotopic to itself.

It suffices to show that E remains locally continuously isotopic to itself. The proof has been given in [8] (Isotopy Th., I.5). This result is then generalized to the differentiable sets that contain singular points. It suffices to assume that each of these manifolds of singular points is itself defined as the inverse image of a submanifold by a t-regular map, so the extension of a manifold of singular points in a singular manifold to which it adheres is itself defined by a t-regular map. If these conditions are satisfied then for a sufficiently small deformation of the map E remains continuously isotopic to itself.

One may define a map f: $\mathbb{R}^n \to \mathbb{R}^p$ to be generic at the source as follows: f is *generic at the source* if each of the critical manifolds of f (simply critical S_r exceptional and super-exceptional critical) may be defined as the inverse image of the submanifold by a t-regular map (which is canonically defined by "derivation," starting with f).

By this, one intends, for example, either that the critical manifolds S_r are defined as the inverse images of Schubert cycles F_r by the derived map $\bar{f}$, which is assumed to be t-regular on F_r, or that the exceptional manifolds are themselves also inverse image of certain cycles of folded manifolds, etc. As far as this is concerned, one must note that there is reason to take into account the phenomenon that was pointed out at the end of the preceding chapter, namely, that the derived maps are not arbitrary. This may have then the effect of reducing the codimension of the cycle that defines an exceptional manifold.
If these conditions are satisfied then any sufficiently small deformation of the map f will define a continuous isotopy of the set S of critical manifolds (simple, exceptional, super-exceptional) into the source space $\mathbb{R}^n$.

Now consider the set $f(S)$ of critical manifolds. One may think that it results from the preceding fact that for any sufficiently small deformation of f the set $f(S)$ remains continuously isotopic to itself in the target space $\mathbb{R}^p$. This is true only for "almost all" maps. Indeed, if we are given a differentiable map $E \to \mathbb{R}^n$ and a map f:$E \to \mathbb{R}^m$ then any f may be approached by a map g that is “generic at the target,” which enjoys the following characteristic property: for any sufficiently small deformation of g the set $g(E)$ remains isotopic to itself. (We simply accept this result, since the proof seems to be difficult without some prior analysis of the generic singularities.) In the case of $f(S)$, the situation is then further complicated by the fact that the generic singularities of the critical manifolds are of a special type.

With these conditions, if a map f: $V^n \to M^p$ is generic at the source and at the target then one may conjecture the following: to any map f' that is sufficiently close to the f, one may associate a homeomorphism h of V^n and a homeomorphism j of M^p such that $f'_0 h = j_0 f$. The (non-unique) homeomorphisms h and j are close to the identity, depend continuously on f', and reduce to the identity if f' tends to f. We shall establish this fact completely in the case of maps of V^n into $\mathbb{R}^1$ (functions).

A function on V^n is generic at the source if and only if it possesses only non-degenerate quadratic critical points. Moreover, it will be *generic at the target* if the corresponding critical values are all distinct and finite in number. Let q_i be the critical points of f and let $c_i = f(q_i)$ be the corresponding critical values, which are all assumed to be distinct. One assumes that we are given a deformation $f(t)$ of f (by approximating the derivatives of order, $r \geq 2$) such that the critical values $f(t, q_i(t))$ remain distinct. One may consider this deformation to be a map F of $V^n \times I$ into $\mathbb{R}^n \times I$, such that F maps (V^n, t) into $(\mathbb{R}, t)$. In the target space to F – viz., $\mathbb{R}^2$ – the trajectories of the critical values $f(t, q_i(t))$ are given, and one may admit that they cut the lines t = constant transversally. Since these trajectories remain distinct, one may define a homeomorphism $h_i(\mathbb{R}) \to \mathbb{R}$ such that $h_i(q_i) = f(t, q_i(t))$. One lets G denote the system of trajectories that are defined by $x \to (h_i(x), t)$, for t variable. Each critical point q_i admits a fixed neighborhood U_i in V^n (U_i disjoint) such that $q_i(t)$ stays continuously isotopic to itself in U_i for t variable. Let $k(t)$ be a homeomorphism of $U = \bigcup_i U_i$ onto itself such that $k(t, q_i) = q_i(t)$, and which reduces to the identity on the frontier of U. This homeomorphism $k(t)$, when completed with the identity outside of U, permits us to define a system of trajectories (K), and the differential dt is not annulled on the product $V^n \times I$ or on any trajectory of (K). Let k denote a non-null vector that is tangent at each point of $V^n \times I$ to the curve (K) that passes through that point, and is a continuous function of that point (as well as its derivatives). Let (x, t) be a point of $V^n \times y$, and let (y, t) be its image in $\mathbb{R}^2$ under F. If x is (x, t) critical for $f(t)$ then the trajectory of (x, t), which is $(q_i(t), t)$, is projected onto $\mathbb{R}^2$ along a (G). Otherwise, consider the tangent plane at (x, t) (which necessarily exists, since F is regular on (G)) to the inverse image $F^{-1}(G)$ of the trajectory (G) that issues from the point (y, t). Let k' be the projection along the direction t = constant that is orthogonal to k in this plane. k' is never null because its projection on the t-axis is non-null. The integration of the vector field k' gives rise to a system of trajectories (K') that must replace the initial system (K). However, under F the trajectories (K') project onto the trajectories (G). As a result, they define a homeomorphism $k'(t)$ of V^n onto itself such that: $f(t) \circ k'(t) = h_i \circ f$, which is precisely what we had to construct.

Some problems. –

1. Prove the preceding theorem (stability of generic maps) in full generality, without having to resort to a description of the singularities.

2. Say that a differential set $E \subset V^n$ is *algebroid* if it is isotopic (continuously or not) to a set E' that is defined by a coherent system of ideals of algebraic functions (of $\mathbb{R}^n$ or the algebraic variety V^n). From the preceding theorem, it results that the set of critical manifolds of a generic map into the source space, and the set of critical values in the target space are algebroid because one may approach f by an algebraic map. Problem: Is a set E that is locally algebroid also globally algebroid([2])?

3. One has seen that if the differentiable sets E, E' are locally continuously isotopic then they are globally so. What happens to this statement if one drops the hypothesis of "continuity?"

[2] In all of the examples that were cited in chapter III, a super-exceptional generic singularity of order k always admitted a local algebraic representation in which only polynomials of degree at most $(k+1)$ appeared. There are good reasons for showing whether this is the general phenomenon.

CHAPTER IV

HOMOLOGICAL PROPERTIES OF CRITICAL MANIFOLDS

The formula of theorem 2 that gives the generic codimension of the critical set S_r shows that this difference depends only on the difference $(n - p)$ between the source and target spaces, with the corank r fixed. From this, it results that any simple critical singularity of a map $\mathbb{R}^n \to \mathbb{R}^p$ gives rise, upon topologically multiplying the source and target by a factor $\mathbb{R}$ to a simple singularity of a map $\mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$, which one calls the "*suspension*" of the initial singularity. One easily proves that the suspension of a generic singularity is again a generic singularity and that the dimension of the critical set for the suspended singularity is equal to that of the critical set of the initial singularity plus one. These considerations generalize to the exceptional and super-exceptional singularities, but the proof is too lengthy to describe here. Of course, not all of the generic singularities of maps $\mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$ are suspensions. It is easy to see the necessary and sufficient condition for this to be the case: a map $f: \mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$ such that $f(0) = 0$ is a *suspension* at 0 if and only if the critical set of the minimum dimension that is associated with the singularity has dimension ≥ 1. By this, we mean that if one forms the sequence of simple, exceptional, and super-exceptional critical points in a neighborhood of O, then the manifold with the lowest dimension has dimension 1. Under these conditions, one may find a function t in a neighborhood of V that has maximum rank at any point on each of these manifolds. This entails that the critical sets that are defined by the hypersurfaces t = constant in the intersection of the critical manifolds are "continuously isotopic" for variable t, which shows that the map f is the suspension of a map g, which is the restriction of f to the hypersurface t = constant. From this, it results that if a singularity of a map $f: \mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$ is not the suspension of map from $\mathbb{R}^n$ to $\mathbb{R}^p$ then the critical set of minimum dimension for f has dimension 0; hence, it reduces to the origin O.

Homotopy invariance of the critical cycles.

Let f, g, be two maps of the same manifold V^n into a manifold M^p. Suppose that f and g are *homotopic* and generic. One may then find a generic map F from $V^n \times I$ into $M^p \times I$ such that $F(V, t)$ is in (M, t) and $f = F|(V, 0)$, $g = F|(V, 1)$. One may, moreover, assume that F coincides with f (g, resp.), for values of t that are close to 0 (1, resp.). With these conditions, form the critical sets of the map F. For example, the set S_r of F is a "pseudo-manifold" with boundary whose boundary is composed of critical sets S_r of f and g. These boundary manifolds are pseudo-manifolds themselves; hence, they are cycles mod 2. It may possible be the case that some S_r does not have a representative in $(V, 0)$ and $(V, 1)$. It is then the case that the corresponding singularity is not a suspension, and, as a result, the corresponding singular points are isolated points. The local neighborhoods of these points in the sets S_j of F for $j < r$ are always cyclic (they are quadratic cones), in such a way that the existence of these points does not contradict the statement that any $\overline{S_j}$ is a cycle modulo its boundaries in $(V, 0)$ and $(V, 1)$. This permits us to state:

THEOREM 7. – *The critical pseudo-manifolds S_r of corank r of two homotopic maps f, g, of V^n into M^p define cycles mod* 2 *that are homologous.*

REMARK. – In certain cases, one may substitute homology with integer coefficients for homology mod 2. This will be the case, notably, for the critical cycles that are dual to the Pontrjagin characteristic classes in an orientable manifold.

It is clear that the preceding result may be extended to exceptional and super-exceptional manifolds. Nevertheless, one encounters the following difficulty here: one does not know the neighborhood of the isolated singular points of the map F into the exceptional critical sets. Meanwhile, there is little to doubt that they are all cyclic neighborhoods. The proof may be carried out explicitly in the case of exceptional singularities of corank one for small dimensions for which there are no exceptional singularities of corank ≥ 2, namely, $p < 4$.

Suppose that the critical set S_r exists (for f, g, as well as for F), and that for these three maps the critical set S_{r+1} has a strictly negative critical dimension. With these conditions, S_r is a submanifold with boundary in $V^n \times I$ whose boundary consists of the sets S_r for f and g. Therefore, in this case one sees that the critical cycles may be realized by true submanifolds and that these submanifolds are L-equivalent in the sense of [8] (hence, they are cobordant).

An important particular case. – *The target space is a Euclidean space* $\mathbb{R}^k$. Suppose that we are given a map f of a compact manifold V^n into a Euclidean space $\mathbb{R}^k$. The homotopy classes of the critical sets of this map do not depend on the map f, since all of the maps of V^n into $\mathbb{R}^k$ are homotopic.

One may suppose that V^n is regularly embedded in a space $\mathbb{R}^{k+m}$ in such a fashion that the map f is given by the projection of $\mathbb{R}^{k+m}$ onto the k-plane $\mathbb{R}^k$ of the first k coordinates. With these conditions, the critical sets S_r are inverse images of Schubert cycles F_r of the Grassmannian G_n^{m+k-n} under $\overline{f}$. The homology classes of the cycles S_r may therefore be calculated. Recall that:

1. The cohomology class that is dual to S_1 for a map of V^n into $\mathbb{R}^k$, $p < n$, is the Stiefel-Whitney class W^{n-p+1}.

2. The (integer) cohomology class that is dual to S_2 for a map $V^n \to \mathbb{R}^k$ is the Pontrjagin class P^{4k} (its mod 2 reduction is $(W^{2k})^2$).

Apropos of property 1, it is interesting to remark that for a map g: $V^n \to \mathbb{R}^{n+k}$ the critical set of corank at the source 1 (which has the same dimension as the critical set at the target 1, for a map of V^n into $\mathbb{R}^{n+k}$) has a homology class that is dual to the class $\overline{W_{k+1}}$, with the notations of Whitney.

COROLLARY. – *An invariance theorem for the characteristic classes*:

Suppose we are given two manifolds V, V' of the same dimension, n, and we assume that following:

1. V and V' have the same homotopy type.

2. There exists manifold with boundary Q that admits V and V' for its boundary, and which has the same homotopy type as the product $V \times I$ (or furthermore: such that V and V' are retracts).

With these conditions, the map of Q into a Euclidean space of the form $\mathbb{R}^k \times I$ (V is mapped into $\mathbb{R}^k \times 0$, and V' is mapped into $\mathbb{R}^k \times 1$) permits us to state:

V and V' have characteristic classes (Stiefel-Whitney and Pontrjagin) that correspond to each other under the isomorphism of homotopy types.

Of course, this result says nothing about the Steifel-Whitney classes, which one knows are determined by the homology type of V^n. However, the result is very interesting for the Pontrjagin classes. Indeed, one knows that there exist manifolds of the same homotopy type whose Pontrjagin classes do not correspond. (They are fibrations of spheres S^3 over S^4; cf [2]).

Homology class of the critical cycle of map f: $V^n \to M^p$.

Let $\overline{S_1}$ be the critical set of corank at the target ≥ 1 of a map f of V^n into M^p ($p < n$). f has maximum rank on the complement to $\overline{S_1}$, and, as a result, the space of tangent vectors to V^n decomposes at every point of $V^n - \overline{S_1}$ into a direct sum of two vector spaces: the kernel $\mathbb{R}^{n-p}$ of the extended map $\dot{f}$, and a transverse subspace $\mathbb{R}^k$, which is mapped onto the tangent space of vectors to the target space isomorphically under f. From this, it results that on $V^n - \overline{S_1}$ one may write the multiplication relation between the "Stiefel-Whitney polynomials" that says that the space of tangent vectors at any point of $V^n - \overline{S_1}$ is the sum (in the sense of Whitney) of the two preceding fibers. Since the fiber space with fiber $\mathbb{R}^p$ is nothing but the fiber space that is induced by the tangent spaces to the target under the map, one may deduce that on $V^n - \overline{S_1}$ the image (under the injection onto $V^n - \overline{S_1}$) of the Stiefel-Whitney polynomial $\sum W_i t^i$ of V^n is "divisible" by the polynomial $\sum f^*(U_j)t^j$, which is the image of the Stiefel-Whitney polynomial $\sum U_j t^j$ of M^p under f. Now, if one formally carries out the division (for increasing powers) of $\sum W_i t^i$ by $\sum f^*(U_j)t^j$, and one stops the division at the term of the quotient in t^{n-p} then one generally obtains a remaining polynomial of the form: $\sum t^{n-p+i} c_{n-p+i}$, $1 \leq i \leq n$.

From this, it results that the classes c_{n-p+i}, $i = 1, \ldots, p$ must have a null image in $V^n - \overline{S_1}$. In particular, since $\overline{S_1}$ has dimension $p - 1$ only one cohomology class is annulled by the injection homomorphism:

$$H^{n-p+1}(V^n) \to H^{n-p+1}(V^n - \overline{S_1}),$$

namely, the dual class of the fundamental cycle in $H^{p-1}(\overline{S_1})$ under Poincaré-Veblen duality. One thus has the theorem:

THEOREM 8. – *For any map f of a manifold V^n into a manifold M^p ($n \geq p$) the homology class of the cycle of the critical points of corank at the target* 1 *is dual to the coefficient of the term of degree $n - p + 1$ that appears in the remainder of the division of the Stiefel-Whitney polynomial of V^n by the image of the Stiefel-Whitney polynomial of M^p under f.*

One will remark that all of the classes c_{n-p+i} have null images in the complement to $\overline{S_1}$. One may interpret this result as follows: the homology classes that are dual to the c_{n-p+i} have representative cycles whose support is contained in $\overline{S_1}$.

I know of no practical procedure for the determination of the critical sets S_r of f of corank $r > 1$. One may describe the theoretical procedure as follows: one forms the product $P = V^n \times M^p$, and then the graph $G(f)$ of the map f in P. P is the base for a fiber bundle H whose fiber is the Grassmannian G_n^p of p-planes that issue from the given point x of P. In each fiber, one has a Schubert cycle $F_r(x)$ and the union of all the $F_r(x)$ when x varies over P forms a cycle Z. Furthermore, if one associates to any $x \in G(f)$, the n-plane that is tangent to $G(f)$ at x then one defines a canonical section $G'(f)$ of H over $G(f)$. The problem then comes down to forming the intersection of the cycles Z and $G'(f)$ in H. If we project onto $G(f)$ then it will give the critical set S_r. Here, the essential difficulty resides in the determination of the homology class of Z, a difficulty that seems to be not in the least bit insurmountable, moreover.

Homology classes of the exceptional critical cycles.

If we are given a map $f: V^n \to \mathbb{R}^p$ where $p \leq n$ then the critical set S_r is a pseudo-manifold of dimension $p - 1$. One has an exceptional critical manifold X_1 on S_1 whose corank at the source is equal to 1. One proposes to determine the homology class mod 2 of X_1 in V^n, a class that is independent of (f), as we saw above.

We first treat the case $p = 2$, which is relatively simpler. One then knows that the critical manifold S_1 is a curve whose image in the plane $\mathbb{R}^2$ (the apparent contour of the manifold) presents a certain number of ordinary regression points, which are images of exceptional critical points.

Let q be the number of these points. One assumes that the manifold is compact, n-dimensional, and embedded in a Euclidean space $\mathbb{R}^2 \times \mathbb{R}^N$, and that the map f is the projection of the first two coordinates (x, y) onto the plane.

Let F_1 be the Schubert cycle of G_n^{N+2} that is formed from n-planes that project onto $\mathbb{R}^2$ along an $\mathbb{R}^1$. By associating any element y of F_1 with its projection in $\mathbb{R}^2$, one defines a canonical map G of F_1 into the Grassmannian G_1^1 of *unoriented* lines in the plane. Since S_1 is the inverse image of the intersection by F_1 of the image of V^n by the derived map, one may consider that S_1 is embedded in F_1. Now, since S_1 is the union of circles, it may be given an orientation, which is arbitrary, moreover. To each point of the apparent contour $F(S_1)$ we associate its oriented tangent. One thus defines a map h of S_1 into the Grassmannian $\hat{G}_1^1$ of *oriented* lines in the plane. This map is continuous at any point of S_1, except at q exceptional critical points where it is discontinuous (it passes from the oriented line to the line oriented in the reverse sense). Since every continuous map of a circle into $\hat{G}_1^1$ has an even degree on G_1^1 by identification, one deduces that the degree of the map h: $S_1 \to G_1^1$ is congruent to q, modulo 2.

Now, it is easy to calculate the degree of the map h. It suffices (when calculating mod 2) to compute the number of points where the apparent contour $F(S_1)$ admits a tangent that is parallel to a given direction; for example, the direction $y = 0$. It then amounts to the same thing as computing the number of critical points of the function y on V^n, a number that is equal, mod 2, to the Euler characteristic $\chi(V^n)$ (or furthermore, to the Stiefel-Whitney number $W_n(V^n)$).

As a result, we have proved:

THEOREM 9. – *The number of regression points that are generically presented by the apparent contour of a compact manifold V^n by projecting onto a coordinate plane is congruent mod* 2 *to the Euler characteristic of V.*

We have used the following (perhaps well-known?) property, in passing: *if a system of closed curves in the plane possesses only self-intersections and regression points as singularities then its class* (*the number of tangents that issue from a given point*) *is congruent, mod* 2, *to the number of regression points.* This is a very curious real form of Plhcker's theorem.

Now return to the general case of a map f of V^n into $\mathbb{R}^p$, which we assume is obtained by projecting V^n, which is embedded in $\mathbb{R}^{p+N}$. Here again, the critical set S_1 will be considered as embedded in the Schubert cycle F_1 (which is assumed to be reduced to just its ordinary points).

To simplify, one will assume that S_1 has no points of corank > 1, in such a way that it is a true submanifold of dimension $p - 1$ that is dual to Stiefel-Whitney class W^{n-p+1}. Here again, one has a canonical map of F_1 onto the Grassmannian G_{p-1}^1, of unoriented $(p-1)$-planes, i.e., the projective space $PR(p-1)$. Let h: $S_1 \to G_{p-1}^1$ be the induced map on the critical manifold, which is considered to be embedded in F_1. One must determine the obstruction to the lifting of the map h to a map h' of S_1 into the Grassmannian of oriented

p-planes, which has the same dimension as $\hat{G}^1_{p-1}$. Let Z_1 in S_1 be the p–2-dimensional cycle that is dual to the class W_1 of the tangent structure to the manifold S_1. This must say that the complement $S_1 - Z_1$ is an orientable (p–1)-manifold. Moreover, let X_1 be the exceptional critical manifold of S_1. The embedding in $\mathbb{R}^N$ has maximum rank on $S_1 - Z_1$. From this, it results that on the complement $S_1 - X_1 - Z_1$ one may associate any point of the image $f(S_1)$ with its *oriented* tangent (p–1)-plane. Therefore, on $S_1 - X_1 - Z_1$ the map h is the projection of a map h' into the Grassmannian of oriented (p–1)-planes. On $X_1 + Z_1$, the map h' extends by "jumping from leaf to leaf," i.e., by reversing the orientation of the (p–1)-plane when one traverses the (p–2)-cycle $X_1 + Z_1$. From this, it results that the obstruction to the existence of the map h' is given by the cohomology class that is dual to the cycle $X_1 + Z_1$, namely, $W_1(S) + D$, where D denotes the class that is dual to the exceptional critical cycle X_1. Now, one may easily calculate this obstruction: let u be the characteristic class of the lifting to two leaves $\hat{G}^1_{P_1} \to \hat{G}^1_{P_1}$. The obstruction to the lift of the map h to $\hat{G}^1_{p-1}$ is $h^*(u) = W_1(S) + D$. Now, in $G^1_{p-1} = PR(p-1)$, u is the class that is dual to the linear hyperplane $PR(p-2)$. One then is down to considering the set of (p–1)-planes in G^1_{p-1} that contain a fixed direction (A). Let (J) be the set of n-planes in the initial Grassmannian G^N_n that project onto $\mathbb{R}^p$ along a (p–1)-plane that contains (A). If H_A denotes the hyperplane in $\mathbb{R}^p$ that is orthogonal to (A) then J may be defined as the set of n-planes in G^N_n whose projection onto H_A is a (p–2)-plane. As a result, J is a Schubert cycle F_1 of G^N_n, and its dual cohomology class is the Stiefel-Whitney class W_{n-p+2}. Therefore, let (T) be a tubular neighborhood of (S_1) in V^n, and let:

$$\varphi^*: H^{p-1-k}(S) \cong H^{n-k}(V^n)$$

be the attached isomorphism (φ^*) (whose definition and notation is in [7]). The cycle $J \cap S_1$ admits the obstruction $h^*(u)$ as its dual class in (S_1), and the class W_{n-p+2} as its dual in T. Therefore:

$$W_{n-p+2} = \varphi^*(h(u)) = \varphi^*(W_1(S_1) + D).$$

Let i: $S_1 \to V^n$ be the injection of S_1 into V^n and let W_1^ν be the Stiefel-Whitney class of dimension 1 in the fiber space of vectors that are normal to S_1 in V^n. The Whitney duality formula gives: $W_1(S_1) = i^*(W_1) + W_1^\nu$. Now, formulas (8) and (32) of [7] give: $\varphi^*(i^*(W_1)) = W_1 \cdot \varphi^*(1) = W_1 \cdot W_{n-p+1}$ and $\varphi^*(W_1) = Sq^1\varphi^*(1) = \mathrm{S}q^1 W_{n-p+1} = W_1 \cdot W_{n-p+1} + (n-p+2)W_{n-p+2}$, from the formulas of Wu [12].

What finally remains is: $\varphi^*(D) = (n-p+1)\, W_{n-p+2}$. Therefore:

THEOREM 10. – *The homology class mod* 2 *of an exceptional critical cycle* X_1 *is null in* S_1 *if the codimension* $(n-p+1)$ *of* S_1 *in* V^n *is even. If* $(n-p+1)$ *is odd then the class* X_1 *is dual to the Stiefel-Whitney class* W_{n-p+2} *in* V^n.

Remark. – If the critical manifolds S_1 and X_1 possess singularities of corank > 1 then the preceding proof is still valid if we consider the relative homologies of S_1 and X_1, modulo the singular sets, which changes nothing about the fundamental class of X_1.

The nullity mod 2 of the cycle X_1 in the in the case where the codimension $(n - p + 1)$ is even is, moreover, easily proved at x, thanks to the notion of the *transverse index* of a singularity of corank 1. One knows that when one crosses the cycle X_1 in S_1 the transverse index varies by unity. One must, moreover, remark that since X_1 is not, in general, an orientable neighborhood in S_1 there is no reason to distinguish the index q from the index $c - q$, if c is the codimension. Now, if c is even then by likewise identifying the complementary dimensions any change of a unit in the index leads to a change of class for the index (an index may not be transformed into its complement). As a result, if one considers a loop L in S_1 then the total number of index changes, hence, the exceptional critical points on L, is necessarily even, since the index must finally coincide with the initial index or its complement. Hence, it results that X is homologous to 0 mod 2 in S_1. If the codimension c is odd: $c = 2k + 1$ then the two indices $(k, k + 1)$ and $(k + 1, k)$ are identified, in such a way that the number of exceptional points on L may be odd. One may state, moreover:

If the Stiefel-Whitney class W_{n-p+2} of a manifold V^n is non-null, and if $n - p$ is odd then the critical manifold of a map of V^n into $\mathbb{R}^p$ always possesses points whose index is $(n - p)/2$.

This is true because the change of index from $(n - p)/2$ to its complement must happen at least once for every loop L whose intersection number with X in S_1 is equal to 1.

One may continue the study further, and notably study the homology of super-exceptional critical cycles. We point out this result in passing:

THEOREM 11. – *The number of super-exceptional critical points that are generically presented by a map of a compact three-dimensional manifold V^3 into $\mathbb{R}^k$ is even.*

This results immediately from the fact that any three-dimensional manifold V^3 is the boundary of a four-dimensional manifold with boundary Q^4. The super-exceptional critical points in V^3 are then the boundary of a super-exceptional critical curve for an extension of f to Q^4.

In conclusion, there is no reason to doubt that the study of the local and global properties of the singularities of differentiable maps opens an extremely rich domain to research. At some point, it may be necessary to make some attempt to distinguish the problems and the methods that might be interesting for the neighboring disciplines - notably, Differential Geometry and Algebraic Geometry.

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