"Über eine gemeinsame Behandlungsweise verschiedener Differentialgeometrien," Math. Zeit. **21** (1924), 254-285.

On a common way of treating various geometries

By

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As a logical consequence of **Klein**'s ideas, nowadays, one treats the differential geometries of geometrically-meaningful groups with invariant formal apparatuses that are specially constructed for these investigations. **Wilczynski** ([1]) and **Fubini** ([2]) have founded projective differential geometry. The corresponding investigation for affine geometry has found a systematic presentation by **Blaschke** and **Reidemeister** ([3]). Such formal apparatuses were also exhibited recently for the conformal ([4]) and **Laguerre** ([5]) groups (especially for the treatment of the theory of surfaces). On the one hand, these investigations confirm the power of **Klein**ian principles to discover the geometrically-important conceptual structures. However, they also yield a broad range of applications for the *tensor* and *Ricci calculus*, and prove its superiority over the other methods of differential geometry that were applied up to now.

Since the exhibition of a formal apparatus for the geometry of surfaces or other manifolds that one might deal with proceeds in the same manner, the search for an invariantly-linked coordinate system that is concomitant to the manifold, and in the presentation of differential equations and integrability conditions naturally stimulates the desire for a common way of treating the various geometries that is aa far-reaching as possible.

In this paper, the question that shall be split off from all the individual investigations as their essential, common component shall not, however, be regarded in its ultimate generality. That sort of abstract and general examination might probably first become fruitful and interesting in connection with deeper group-theoretic questions. In contrast to the general problem, this paper will move into a narrower context. *The following investigation will establish the fact that one can derive the formulas for a large number of geometries by simple specializations of a formal apparatus whose essential components must emerge from a systematic treatment of the projective geometry of ray*

([1]) **Wilczynski**, Transactions of the American Math. Soc. **8** (1907); **9** (1908); **10** (1909), and numerous other papers.

([2]) One finds **Fubini**'s papers summarized, for the most part, in the Rendiconti di Palermo **43** (1919), pp. 2.

([3]) **W. Blaschke**, *Vorlesungen über Differentialgeometrie II.*

([4]) **G. Thomsen**, Abhandlungen aus dem Hamb. math. Sem. **3** (1924), 31-56.

([5]) **W. Blaschke**, Abh. aus dem. Hamb. math. Sem. **3** (1924), 176-194.

systems ([6]). (Cf., the table on pp. 15) **Fubini** ([7]) has treated *the projective geometry of ray systems* that are based upon a quadratic and a bi-quadratic differential form. Here, we shall embark upon a different path that might be more useful in the geometric applications. Its essential Ansätze were also suggested by **Fubini** ([7]). It employs *line coordinates.* A ray $\mathfrak{p}$ of the system will be given as a function of two parameters u^1 and u^2. Corresponding to the number of coordinates, three more rays $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ will be introduced, in addition to $\mathfrak{p}$ and both of its derivatives with respect to the parameters, so the second derivatives of $\mathfrak{p}$ and the first derivatives of $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ will be linear combinations of them (viz., differential equations) ([7a]).

The projective geometry of ray systems is essentially identical to *the higher sphere geometry* (i.e., Lie geometry) *of sphere systems* ([8]) under **Lie**'s line-sphere transformation. The rays $\mathfrak{p}$, $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ correspond to *oriented* spheres. One will arrive at the ten-parameter group of conformal transformations of space when one adds a linear (sphere, resp.) complex. The spheres that belong to a distinguished complex are the points. Thus, the *conformal theory of surfaces* is nothing but the **Lie** geometry of those sphere systems whose spheres all belong to one and the same linear complex. One will arrive at **Laguerre** geometry by adjoining a sphere. When one considers the sphere systems whose spheres all contact a fixed sphere, one will be dealing with the *surface theory of the Laguerre group.* When one considers a second envelope that does not degenerate into a sphere in connection with such a sphere system, one will implicitly adjoin a sphere to it.

One arranges the formulas of conformal geometry in such a way that **Laguerre** geometry emerges as a special case.

The presentation of the analytical apparatus for the projective geometry of ray systems shows that one does not need to establish the dependency of the rays $\mathfrak{p}$, $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ upon the system completely in the slightest in order to be able to present differential equations and integrability conditions in a final form that is as simple as can be achieved. One can let them be arbitrary and variable, to a certain extent, without affecting the formal character of these equations. The ultimate foundation manifests itself in further relationships that emerge between the tensors that appear. When one now treats the projective geometry of ray systems without establishing $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ completely, one can arrive at further geometries by specializing its formulas. For example, one can employ the arbitrariness in the choice of those rays to demand that one of them should continually lie in the same fixed plane for all positions of the system. In that way, a plane will be adjoined implicitly, and one will arrive at the *affine geometry of ray systems*. The requirement that is imposed upon the position of one of the rays will manifest itself in a relationship that emerges between the tensors that appear in the differential equations. Conversely, one can propose that relationship between the tensors as a requirement; it will then follow from it that the ray will lie in a fixed plane and that one will arrive at

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([6]) (viz., ray congruences)

([7]) **Fubini**, Rom. acc. Lincei **27** (1918), **28** (1919).

([7a]) **Fubini** introduced three linear complexes instead of the three rays that are connected with them in a simple way.

([8]) Viz., sphere congruences.

affine geometry. The latter path is the one that shall always be followed in the sequel. (See the table on page 15.)

One will obtain non-Euclidian (elliptic, as well as hyperbolic) geometry from conformal geometry when one adds a fixed sphere, and the usual Euclidian geometry of motion by adjoining a point. One can then arrive at the *non-Euclidian theory of surfaces* from the **Lie** geometry of a sphere system that belongs to a fixed linear complex by the above process of implicit adjunction when one adjoins a sphere that is outside of the complex, and a *Euclidian* theory when one adjoins a sphere that is in the complex.

One further arrives at the *conformal geometry of sphere systems* in such a way when one adjoins a linear complex that does not belong to the spheres of the system. One will arrive at the ***Laguerre** geometry of sphere systems* by adjoining a sphere that is not contacted by the spheres of the system. Now, a sphere that is linked invariantly with a surface and contacts it plays an important role in the theory of surfaces, namely, the *central sphere* ([9]). For the better part of all problems in the conformal theory of surfaces, it is even preferable to base them upon the study of the *systems of central spheres* that are linked with the surface. **W. Blaschke** has discovered a similar situation in **Laguerre**'s theory of surfaces, in which the *middle sphere* and the *middle sphere system* have considerable significance. In order to then arrive at the theory of surfaces that corresponds to the conformal and **Laguerre** geometries of sphere systems, one must impose the condition that a sphere system consists of the central spheres (middle spheres, resp.) of an envelope. If one consider the spheres that all contact a fixed sphere (go through a fixed point, resp.) in the conformal geometry of sphere systems then one will arrive at the *non-Euclidian (Euclidian, resp.) theory of surfaces in plane coordinates;* the spheres of the systems will then be the *tangential planes* in the two geometries.

If one considers one sheet of a ray system whose second sheet is a second-order surface projective-geometrically in connection with the ray system then one will be compelled to define a *non-Euclidian theory of surfaces* by the implicit adjunction of an F_2, and in fact, to *the line coordinates of one family of isotropic tangents to the surface.*

Finally, the projective and affine geometry of ray systems will give one a new perspective into a *projective and affine theory of surfaces in line coordinates.* Just as one is forced into a conformal theory of surfaces from the study of systems of central spheres whose envelope is the surface, one can be forced into a projective and affine theory of surfaces by the study of a system of invariant surface tangents. For example, in the projective geometry of ray systems, it is easy to pose the condition that the rays that coincide with a families of *Darboux tangents* ([10]) to the one focal sheet [viz., ray systems whose rays are asymptotes ([11]) to a surface] must be excluded from a general theory ([11a]). A surface is then (except for exceptional cases) determined, up to projective transformations, by two quadratic forms and one invariant. Along with that possibility, there is another one whose pursuit will perhaps prove to be more convenient.

A corresponding path would lead to an affine theory of surfaces.

([9]) And indeed the central sphere is characterized among all of the spheres that contact the surface by the fact that the surface intersects it along a curve that has two perpendicular tangents at the contact point.

([10]) Cf., e.g., **Blaschke**, *Vorlesungen über Differentialgeometrie II*, § 42.

([11]) Asymptotes = tangents to the asymptotic lines.

([11a]) The author will devote a subsequent paper to these ray systems and a projective theory of surfaces that is based upon their study.

The general theory offers an opportunity to solve problems that belong to different geometries in a common way. In addition, it gives a heuristic means for discovering the important geometric concepts in the geometries that have been researched only slightly up to now.

A complete geometric utilization of the formal apparatus and the relationships between the various geometries must be reserved for a later work. In many respects, the following developments are only a program, for the moment, while in other respects, they are already more extensive depending upon which results are already present.

Now and then, special geometric results will be sprinkled about, and without proof, to some extent.

Herr **W. Blaschke** has supported my work by his advice and suggestions so many times that I must take this occasion to emphasize that fact explicitly.

§ 1.

Basic concepts.

In what follows, Greek indices (μ, ν, σ, ρ, etc.) shall run from 1 to 6, while Latin ones (i, k, l, …) shall run from 1 to 2. Doubly-appearing indices will be summed over.

In order to simultaneously introduce the basic concepts and facts of projective line geometry and **Lie**'s sphere geometry, anywhere that those of the former first appear, the ones of the latter that correspond to them will be placed in parentheses. The spheres of **Lie**'s geometry will always be understood to be *oriented spheres.* The six line (sphere, resp.) coordinates $\mathfrak{p}^\rho$, which cannot all vanish at once, are coupled by a homogeneous, quadratic equation. In line geometry, one usually employs *normal coordinates*, for which, that equation will take the form:

$$\mathfrak{p}^1\mathfrak{p}^4 + \mathfrak{p}^2\mathfrak{p}^5 + \mathfrak{p}^3\mathfrak{p}^6 = 0. \tag{1a}$$

[In **Lie**'s geometry, one employs coordinates that make the equation assume the normal form:

$$(\mathfrak{p}^1)^2 + (\mathfrak{p}^2)^2 + (\mathfrak{p}^3)^2 + (\mathfrak{p}^4)^2 - (\mathfrak{p}^5)^2 - (\mathfrak{p}^6)^2 = 0.] \tag{1b}$$

General projective line coordinates shall be introduced here that are any sort of linear functions of the "normal coordinates." Equation (1) will then assume the general form:

$$a_{\mu\nu}\,\mathfrak{p}^\mu\,\mathfrak{p}^\nu = 0, \tag{2}$$

in which the $a_{\mu\nu}$ are any numbers whose determinant satisfies:

$$|\,a_{\mu\nu}\,| = \Delta \neq 0. \tag{3}$$

Should those general line (sphere, resp.) coordinates be real, the form (2) would need to have the same index of inertia ([12]) as the forms (1*a*) [(1*b*, resp.], whose index is, however, 3 (2, resp.)

Under projective maps (higher sphere transformations of space, resp.) the $\mathfrak{p}^\rho$ will be subjected to the homogeneous linear transformations $\mathfrak{p}^\rho = c^\rho{}_\sigma \mathfrak{p}^{\sigma*}$ that leave equation (2) invariant. Since the coordinates are homogeneous, the expressions of an invariant character must remain unchanged under not only these transformations, but also the *renormalizations* of coordinates:

(4) $$\mathfrak{p}^\rho = \lambda\,(\mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^3, \mathfrak{p}^4, \mathfrak{p}^5, \mathfrak{p}^6) \cdot \mathfrak{p}^{\rho*},$$

where λ is an arbitrary function of its argument.

It would be convenient for us to shall carry out that construction of invariant expressions in two steps:

I. The construction of invariants under the linear transformations of $\mathfrak{p}^\rho$ that leave the form $a_{\mu\nu}\,\mathfrak{p}^\mu\,\mathfrak{p}^\nu$ absolutely invariant (*viz.*, *semi-invariants*).

II. The construction of expressions that are invariant under renormalizations (4), in addition.

Obviously, the manifold of transformations in I and II is identical with the transformations above.

The rays (spheres, resp.) that satisfy a linear equation:

(5) $$\mathfrak{q}_\nu\,\mathfrak{p}^\nu = 0$$

belong to a *linear complex*. Upon introducing new quantities $\mathfrak{q}^\mu$, we can write that equation in the form:

(6) $$a_{\mu\nu}\,\mathfrak{q}^\mu\,\mathfrak{p}^\nu = 0.$$

The quantities $\mathfrak{q}^\mu$ are the *coordinates of the linear complex*. For two linear complexes ζ^μ, η^μ, the expression:

(7) $$(\zeta, \eta) = a_{\mu\nu}\,\zeta^\mu \eta^\nu,$$

which we would like to refer to as their *scalar product*, is semi-invariant. The determinant $|\,\zeta_{(1)}, \zeta_{(2)}, \zeta_{(3)}, \zeta_{(4)}, \zeta_{(5)}, \zeta_{(6)}\,|$ of six linear complexes $\zeta^\mu_{(1)}, \ldots, \zeta^\mu_{(6)}$ is a semi-invariant. From the multiplication law for determinants, it can be expressed in terms of the scalar products of the complexes:

([12]) Index of inertia = number of negative signs in the aggregate of complete squares that one can put the form into by means of real transformations.

(8) $$\Delta \cdot |\ \zeta_{(1)},\ \zeta_{(2)},\ \zeta_{(3)},\ \zeta_{(4)},\ \zeta_{(5)},\ \zeta_{(6)}\ |^2 = |\ (\zeta_{(\rho)},\ \zeta_{(\sigma)})\ |\ .$$

Since the complex coordinates transformation like line coordinates, we can regard the rays as special complexes that satisfy equation (2) $\{(\mathfrak{p}\ \mathfrak{p}) = 0\}$. In fact, all rays $\mathfrak{p}$ that satisfy an equation (5) $(\mathfrak{q}\ \mathfrak{p}) = 0$ (so they belong to the complex $\mathfrak{q}$) will be a fixed line whose coordinates are precisely the $\mathfrak{q}^{\mu}$ in the case where $(\mathfrak{q}\ \mathfrak{q}) = 0$. In what follows, we will then often use the term "complexes" to collectively mean "proper" complexes and the term "lines" (spheres, resp.) to mean "degenerate" complexes.

All lines $\mathfrak{q}$ whose coordinates have the form:

(9) $$\mathfrak{p}^{\rho} = \alpha\ \mathfrak{a}^{\rho} + \beta\ \mathfrak{b}^{\rho},$$

in which $\mathfrak{a}$ and $\mathfrak{b}$ are two fixed intersecting lines:

$$[(\mathfrak{a}\ \mathfrak{a}) = (\mathfrak{b}\ \mathfrak{b}) = (\mathfrak{a}\ \mathfrak{b}) = 0],$$

and α and β are two parameters that do not vanish at the same time, define a *pencil*.

We will write equations that have the same type as (9) symbolically in the form $\mathfrak{p} = \alpha\ \mathfrak{a} + \beta\ \mathfrak{b}$ by omitting the coordinate indices.

Four rays that belong to a pencil:

$$\mathfrak{p}_{(\rho)} = \alpha_{(\rho)}\ \mathfrak{a} + \beta_{(\rho)}\ \mathfrak{b} \qquad (\sigma = 1, 2, 3, 4)$$

will have an invariant, namely, the *double ratio:*

(10) $$D = \frac{(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3\beta_4 - \alpha_4\beta_3)}{(\alpha_1\beta_4 - \alpha_4\beta_1)(\alpha_3\beta_2 - \alpha_2\beta_3)}.$$

[We will refer to the corresponding invariant of four spheres of a pencil as the *angle ratio*, in order to avoid later confusion.] We refer to the expression:

(11) $$\varphi = \frac{i}{2}\log D \qquad (i = \sqrt{-1})$$

as the *logarithmic double (angle, resp.) ratio.* The formula:

(12) $$\cot\varphi = i\,\frac{1+D}{1-D}$$

is true for φ.

Four elements of a pencil have a *harmonic double (angle, resp.) ratio* when one has:

(13) $$D = -1, \qquad \varphi = \frac{\pi}{2}.$$

§ 2.

General theory of ray and sphere systems.

We assume that the ray $\mathfrak{p}$ of the system is given as a function of two parameters u^1 and u^2. Since we will also be sometimes forced to use complex geometry, we will assume that the functions are analytic. For most of the results, one can make do with fewer assumptions. For the individual quantities that depend upon the functions $\mathfrak{p}^\rho (u^1, u^2)$ and their derivatives, we shall introduce the usual notations of tensor analysis in regard to their behavior under parameter transformations. Two quantities define a *contravariant vector* when they transform like the differentials du^i. If $s_i\, du^i$ is invariant then the s_i will define a *covariant vector*, and so forth. If we set $\partial\mathfrak{p} : \partial u^i = \mathfrak{p}_i$ then the corresponding coordinates of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ will define a covariant vector. If we deal with only quantities that depend upon the ray system then we can write the transformations (4) in the form:

(14) $$\mathfrak{p} = \lambda\,(u^1, u^2) \cdot \mathfrak{p}^*.$$

One has:

(15) $$(\mathfrak{p}\ \mathfrak{p}) = (\mathfrak{p}\ \mathfrak{p}_i) = 0$$

identically in the u^i. Furthermore:

(16) $$(d\mathfrak{p}\ d\mathfrak{p}) = (\mathfrak{p}_i\ \mathfrak{p}_k)\, du^i\, du^k = g_{ik}\, du^i\, du^k$$

is a differential form that is invariant, up to a multiplicative factor. If we consider a ruled surface (channel surface, resp.) $\mathfrak{q}(t)$ that goes through $\mathfrak{p}$, where t is a parameter, in the neighborhood of $\mathfrak{p}$: $[\mathfrak{q}(t) = \mathfrak{p} + t\, d\mathfrak{p} + d^2\mathfrak{p} \ldots]$, and if we consider quadratic terms in t then we will recognize from the fact that $(\mathfrak{p}\ \mathfrak{q}) = 0$ or $(\mathfrak{p}\ d^2\mathfrak{p}) = -(d\mathfrak{p}\ d\mathfrak{p}) = 0$, that the null family of (16) consists of the *torses (contact family, resp.)* of the system. We exclude the case of $g = 0$ ($g = |\, g_{ik}\, |$), for which the two families of torses coincide. As is known, these systems consist of one of the two families of asymptotic tangents (curvature spheres, resp.) of a surface for which one sheet coincides with the focal surface (envelope, resp.) ([13]). We introduce the form g_{ik} as a fundamental form, to which we will relate covariant derivatives, as well as the raising and lowering of indices, from now on, by employing the tensor g^{ik} that is reciprocal to g_{ik}. For the time being, we shall not concern ourselves with the fact that g_{ik} is not absolutely invariant. Later, we will

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([13]) Cf., footnote ([11]).

"normalize" g_{ik} and then use the normalized form as the fundamental form. From **Ricci**'s lemma ([14]), one has:

(17) $$g_{ikr} = 0,$$

and from that, one easily derives the fact that ([14a]):

(18) $$(\mathfrak{p}_l\, \mathfrak{p}_{ik}) = 0.$$

(New indices that are appended mean covariant derivatives.) For the six linearly-independent *basic complexes* ([15]), we introduce $\mathfrak{p}$, $\mathfrak{p}_1$, and $\mathfrak{p}_2$, as well as three rays [spheres, resp.] $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$, that are coupled by the conditions that are given in the following table. The scalar products that the complexes define with each other are calculated:

(19)

	$\mathfrak{p}$	$\mathfrak{p}_k$	$\mathfrak{v}$	$\bar{\mathfrak{v}}$	$\mathfrak{z}$
$\mathfrak{p}$	0	0	0	0	–1
$\mathfrak{p}_i$	0	g_{ik}	0	0	0
$\mathfrak{v}$	0	0	0	1	0
$\bar{\mathfrak{v}}$	0	0	1	0	0
$\mathfrak{z}$	–1	0	0	0	0

From the determinant law, we easily derive the relation:

(20) $$\Delta \cdot |\, \mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{v}, \bar{\mathfrak{v}}, \mathfrak{z}\, |^2 = g.$$

Since $g \neq 0$, the basic complexes are linearly-independent. $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ are not established completely by the conditions that appear in the table.

The totality of rays $\mathfrak{t}$ [$(\mathfrak{t}\ \mathfrak{t}) = 0$] that satisfy the equations:

(21) $$(\mathfrak{t}\ \mathfrak{p}) = (\mathfrak{t}\ \mathfrak{p}_i) = 0$$

fill up the two pencils of tangents [contacting spheres, resp.] to the sheets in the surface elements to the envelope that belong to the system ray. If one then introduces the parameters u, v of the torses then $\alpha\,\mathfrak{p} + \beta\,\mathfrak{p}_u$ and $\gamma\,\mathfrak{p} + \delta\,\mathfrak{p}_v$ will be the tangent pencils of the two torses at the associated focal points ([16]).

Since $(\mathfrak{v}\bar{\mathfrak{v}}) \neq 0$, $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ are tangents to different sheets. Moreover, both are still arbitrary with one degree of freedom. If one ultimately establishes $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ then, from

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([14]) Cf., e.g., **Blaschke**, *Differentialgeometrie II*, § 56.
([14a]) Cf., the paper cited in ([4]), formula (28).
([15]) Which can naturally also be rays.
([16]) The torses cut the enveloping sheet at its edges of regression.

(19), $\mathfrak{z}$ will also be determined completely. Conversely, if one proposes that the ray $\mathfrak{z}$ should satisfy the conditions:

(22) $$(\mathfrak{p}\ \mathfrak{z}) = -1, \qquad (\mathfrak{p}_i\ \mathfrak{z}) = 0$$

then $\mathfrak{v}$ and $\overline{\mathfrak{v}}$ will be determined by that. If one normalizes $\mathfrak{p}$ then $\mathfrak{z}$ will also be normalized, since $(\mathfrak{p}\ \mathfrak{z}) = -1$. However, the normalization of $\mathfrak{v}$ and $\overline{\mathfrak{v}}$ is still not established by $(\mathfrak{v}\overline{\mathfrak{v}}) = 1$; it is still possible by a renormalization of the form:

(23) $$\mathfrak{v} = \lambda\,\mathfrak{v}^*, \qquad \overline{\mathfrak{v}} = \frac{1}{\lambda}\overline{\mathfrak{v}}^*,$$

in which λ is an arbitrary function of u^1 and u^2. We exhibit the following table of scalar products of the second covariant derivatives $\mathfrak{p}_{ik}$ with the basic complexes:

(24)

	$\mathfrak{p}$	$\mathfrak{p}_l$	$\mathfrak{v}$	$\overline{\mathfrak{v}}$	$\mathfrak{z}$
$\mathfrak{p}_{ik}$	$-g_{ik}$	0	c_{ik}	$\overline{c}_{ik}$	$-a_{ik}$

The last three tensors are defined by this. In addition, we introduce the skew-symmetric tensor ([17]):

(25) $$e_{ik} = \sqrt{\Delta}\cdot|\,\mathfrak{p}, \mathfrak{p}_i, \mathfrak{p}_k, \mathfrak{v}, \overline{\mathfrak{v}}, \mathfrak{z}\,|$$

whose components are:

(26) $$e_{11} = 0, \qquad e_{12} = \pm\sqrt{g}, \qquad e_{21} = \mp\sqrt{g}, \qquad e_{22} = 0.$$

We can choose one of the signs – perhaps the upper one ([18]). For the tensor e^{ik}, one will have:

$$e^{11} = 0, \qquad e^{12} = \frac{1}{\sqrt{g}}, \qquad e^{21} = -\frac{1}{\sqrt{g}}, \qquad e^{22} = 0.$$

One further has the equations:

(27) $$e_{ir}\,e^{kr} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k, \end{cases}$$

(28) $$\tfrac{1}{2}e_{ik}\,e^{ik} = \tfrac{1}{2}\,g_{ik}\,g^{ik} = 1.$$

We finally exhibit the following table of scalar products:

([17]) e_{ik} is only invariant under parameter substitutions with the same sense.

([18]) The presence of a double sign is connected with the arbitrary permutability of $\mathfrak{v}$ and $\overline{\mathfrak{v}}$.

(29)

	$\mathfrak{p}$	$\mathfrak{p}_i$	$\mathfrak{v}$	$\overline{\mathfrak{v}}$	$\mathfrak{z}$
$\mathfrak{v}_l$	0	$-c_{il}$	0	$-n_l$	$-w_l$
$\overline{\mathfrak{v}}_l$	0	$-\overline{c}_{il}$	$+n_l$	0	$-\overline{w}_l$
$\mathfrak{z}_l$	0	$+a_{il}$	$+w_l$	$+\overline{w}_l$	0

The vectors n_l, w_l; $\overline{w}_l$ will be defined here, while the remaining relations can be derived from (19) by differentiation when one makes note of (24). We can now exhibit the differential equations:

(30) $$\mathfrak{p}_{ik} = a_{ik}\mathfrak{p} + \overline{c}_{ik}\mathfrak{v} + c_{ik}\overline{\mathfrak{v}} + g_{ik}\mathfrak{z},$$

(31) $$\mathfrak{v}_l = w_l\mathfrak{p} - c_l^g\mathfrak{p}_g - n_l\mathfrak{v},$$

(32) $$\dot{\mathfrak{v}}_l = \overline{w}_l\mathfrak{p} - \overline{c}_l^{\,g}\mathfrak{p}_g - n_l\overline{\mathfrak{v}},$$

(33) $$\mathfrak{z}_l = a_l^g\mathfrak{p}_g + \overline{w}_l\mathfrak{v} + w_l\overline{\mathfrak{v}},$$

whose validity one can verify by scalar multiplication with the "basic complexes."

We can write the integrability conditions for the system with the help of the tensor e^{ik} invariantly as follows ([19]):

(34) $$e^{ik}\,\mathfrak{p}_{ikl} = K\,e_{si}\,\mathfrak{p}^s, \qquad e^{ik}\,\mathfrak{v}_{ik} = 0, \qquad e^{ik}\,\overline{\mathfrak{v}}_{ik} = 0, \qquad e^{ik}\,\mathfrak{z}_{ik} = 0.$$

Here, K means the **Gauss**ian curvature of the form g_{ik}. If one expresses the derivatives of the basic complexes in the conditions (34) with respect to the latter with the help of the differential equations (30) to (33) and the formulas that are obtained from them by differentiating and ultimately sets the individual factors in them equal to zero then one will be led to seven essential relations between the tensors g_{ik}, c_{ik}, $\overline{c}_{ik}$, a_{ik}, w_i, $\overline{w}_i$, and n_i:

(35) 1) $$K = e^{ik}e^{pq}c_{ip}\overline{c}_{kq} - a_i^i,$$

(36) 2) $$e^{kl}(w_{kl} + w_k n_l + c_l^g a_{sk}) = 0,$$

(37) 3) $$e^{kl}(\overline{w}_{kl} - \overline{w}_k n_l + \overline{c}_l^{\,g} a_{sk}) = 0,$$

(38) 4) $$e^{kl}(c_{ikl} + c_{ik}n_l + g_{ik}w_l) = 0,$$

(39) 5) $$e^{kl}(\overline{c}_{ikl} - \overline{c}_{ik}n_l + g_{ik}\overline{w}_l) = 0,$$

(40) 6) $$e^{kl}(a_{ikl} + c_{ik}\overline{w}_l + \overline{c}_{ik}w_l) = 0,$$

(41) 7) $$e^{kl}(n_{kl} + g^{rs}c_{kr}\overline{c}_{ls}) = 0.$$

([19]) These equations emerge from the usual, non-invariantly-written integrability conditions $\frac{\partial^3\mathfrak{p}}{\partial u^i\partial u^k\partial u^l} = \frac{\partial^3\mathfrak{p}}{\partial u^i\partial u^l\partial u^k} = \frac{\partial^3\mathfrak{p}}{\partial u^i\partial u^k\partial u^l}$, $\frac{\partial^2\mathfrak{p}}{\partial u^i\partial u^k} = \frac{\partial^2\mathfrak{p}}{\partial u^k\partial u^i}$, etc. Cf., on this, **W. Blaschke**, *Differentialgeometrie II*, pp. 151-152, formulas (102), (105^*).

These equations first take on an invariant character when first of all, the arbitrariness of two degrees of freedom that exists in $\mathfrak{v}$, $\overline{\mathfrak{v}}$, and $\dot{\mathfrak{z}}$ is resolved by a two suitable additional equations, and second of all, the normalization of $\mathfrak{p}$ and the normalizations of $\mathfrak{v}$, $\overline{\mathfrak{v}}$ [cf., (23] are established by two further equations. Since we will carry out the definition of $\mathfrak{v}$, $\overline{\mathfrak{v}}$, and $\mathfrak{z}$, as well as the normalization of the individual geometries that are depicted in the table in various way, in the following paragraph, we would like to derive some formulas without imposing those four additional requirements, even though they would simplify the formulas for the projective geometry of ray systems.

§ 3.

Invariant theory of three quadratic forms. Asymptotic lines and curvature lines.

Three quadratic forms g_{ik}, c_{ik}, $\overline{c}_{ik}$ have five simultaneous invariants whose representatives we can write as ([20]):

$$(42)\qquad \begin{cases} \tfrac{1}{2} g^{ik} c_{ik} = h, & \dfrac{c}{g} = k, \\ \tfrac{1}{2} g^{ik} \overline{c}_{ik} = \overline{h}, & \dfrac{\overline{c}}{g} = \overline{k}, \\ \tfrac{1}{2} e^{lk} g^{pq} c_{lp} \overline{c}_{kq} = d, & (i = \sqrt{-1}), \end{cases}$$

upon defining g_{ik} to be the fundamental form. c and $\overline{c}$ are the determinants of the forms c_{ik} and $\overline{c}_{ik}$, resp. One easily proves the formulas ([21]) from ([22]):

$$(43)\qquad \frac{c}{g} \equiv \tfrac{1}{2}\, e^{ik} e^{pq}\, c_{ip}\, c_{kp}\,,$$

$$(44)\qquad g^{rs}\, c_{ir}\, c_{ks} = 2h\, c_{ik} - k\, g_{ik}\,.$$

One obtains analogous equations by switching c_{ik} with $\overline{c}_{ik}$. There is a further *Jacobi form* for any two of the forms. The three **Jacobi** forms are the following ones:

$$(45)\qquad p_{ik} = \tfrac{1}{2}(e_{ir} c_k^r + e_{kr} c_i^r)\,,$$

([20]) h and k are the two simultaneous invariants of g_{ik} and c_{ik}, and similarly, $\overline{h}$ and $\overline{k}$ are those of g_{ik} and $\overline{c}_{ik}$.

([21]) At best, by reverting to the components in the parameters of the torses $g_{11} = g_{22} = 0$.

([22]) This formula is the same as the one in the motion-geometric theory of surfaces that expresses the form of the spherical image in terms of the two fundamental forms.

(46) $$\overline{p}_{ik} = \tfrac{1}{2}(e_{ir}\overline{c}_k^r + e_{kr}\overline{c}_i^r),$$

(47) $$f_{ik} = \tfrac{i}{2}e_{rs}(c_i^r\,\overline{c}_k^s - c_k^s\,\overline{c}_i^r).$$

One has the following equations:

(48) $$g^{ik}\,p_{ik} = 0, \qquad g^{ik}\,\overline{p}_{ik} = 0,$$

(49) $$c^{ik}\,p_{ik} = 0, \qquad \overline{c}^{ik}\,\overline{p}_{ik} = 0,$$

(50) $$\frac{i}{2}e^{ik}g^{pq}p_{ip}\overline{p}_{kq} = d,$$

(51) $$g^{ik}\,f_{ik} = d.$$

The quantity:

$$j = \frac{1}{2}e^{ik}e^{pq}c_{ip}\overline{c}_{kq}$$

that enters into the integrability condition (35) is connected with the invariants (42) in the following way:

(52) $$(j + h\overline{h})^2 = (h^2 - k)(\overline{h}^2 - \overline{k}) + d^2.$$

We know that 𝔭 and 𝔳 are two non-coincident rays of the pencil of tangents to a sheet of the envelope, namely, the "first" one. If we exclude the case in which 𝔳 coincides precisely with one of the asymptotes (curvature spheres, resp.) of the sheet then we can give the two asymptotes of the first sheet in the form 𝔱 = 𝔭 + γ𝔳, where γ is a scalar. The asymptotes have the property that they will still be tangents to the sheet at a neighboring surface element, so from (21) one must have (𝔱 d𝔭$_i$) = 0, (𝔭 d𝔭$_i$) + γ (𝔳 d𝔭$_i$) = 0, in addition to (𝔱 𝔱) = 0, (𝔱 𝔭) = (𝔱 𝔭$_i$) = 0, or one must have:

(53) $$(-g_{ik} + \gamma c_{ik})\,du^k = 0$$

for the direction of the asymptotic lines $du^1 : du^2$. Should the two equations (53) have a solution g, then $g_{ik}\,du^k$ would have to be proportional to $c_{ik}\,du^k$. That condition can be written in the form:

(54) $$\tfrac{1}{2}e^{rs}(g_{ir}\,c_{ks} - g_{ks}\,c_{ir})\,du^i\,du^k = 0.$$

From (45), we get:

(55) $$p_{ik}\,du^i\,du^k = 0$$

as the *differential equation of the asymptotic lines (curvature lines, resp.)* of the first sheet. For the second sheet, we get, analogously:

$$\bar{p}_{ik}\ du^i\ du^k = 0.$$

From (48), (49), one will have $p_{11} = p_{22} = g_{12} = c_{12} = 0$ by introducing the asymptotic lines of the first sheet as parameter curves, and from (53), one will get the value c_{11} / g_{11} for γ when $du^1 = 0$, $du^2 = 0$, but $\gamma = g_{22} / c_{22}$ for $du^1 = 0$, $du^2 = 1$.

If we determine the quantities r and r' by:

$$(56) \qquad h = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{r'}\right), \qquad k = \frac{1}{rr'}$$

then by an application of the special parameters, one will have directly:

$$r = g_{11} : c_{11}\,, \qquad r' = g_{22} : c_{22}\,.$$

If we introduce analogous quantities $\bar{r}$, $\bar{r}'$ for the second sheet then *the four asymptotes of the sheet* will be given by:

$$(56a) \qquad \mathfrak{p} + r\,\mathfrak{v}, \qquad \mathfrak{p} + r'\mathfrak{v}, \qquad \mathfrak{p} + \bar{r}\,\bar{\mathfrak{v}}, \qquad \mathfrak{p} + \bar{r}'\bar{\mathfrak{v}}\,.$$

If we take the two rays $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ and the infinitely-close rays $\mathfrak{v} + d\mathfrak{v}$, $\bar{\mathfrak{v}} + d\bar{\mathfrak{v}}$ that belong to a well-defined direction of advance then, from (19), (29), all four of them will cut $\mathfrak{p}$. We ask: When is there yet another infinitely-close ray of the system $\mathfrak{p} + \delta\mathfrak{p}$ that likewise cuts all four rays? Obviously, one must then have:

$$(56b) \qquad \begin{aligned} (\delta\mathfrak{p}\ d\mathfrak{v}) &= -\,c_{ik}\ \delta u^i\ du^k = 0, \\ (\delta\mathfrak{p}\ d\bar{\mathfrak{v}}) &= -\,\bar{c}_{ik}\ \delta u^i\ du^k = 0. \end{aligned}$$

Should both equations be fulfilled for δu^i that do not vanish simultaneously, then $c_{ik}\ du^k$ would have to be proportional to $\bar{c}_{ik}\ du^k$. We can write that condition in the form:

$$\tfrac{1}{2}e^{rs}(c_{ir}\bar{c}_{ks} - c_{ks}\bar{c}_{ir})\ du^i\ du^k = 0,$$

just as we did above with the corresponding condition for the asymptotic (curvature, resp.) lines [cf., (54)].

From (47), the rays $\mathfrak{v}$, $\mathfrak{v} + d\mathfrak{v}$, $\bar{\mathfrak{v}}$, $\bar{\mathfrak{v}} + d\bar{\mathfrak{v}}$ will then be cut by $\mathfrak{p}$ and yet another neighboring ray of the system $\mathfrak{p} + d\mathfrak{p}$ only when we advance the form $f_{ik}\ du^i\ du^k$ in one of the null directions. The null families of that form depend upon how one defines $\mathfrak{v}$ and $\bar{\mathfrak{v}}$. We would then like to refer to them as the *principal families* that belong to the systems of rays $\mathfrak{v}$ and $\bar{\mathfrak{v}}$, and to the associated curves on the sheet as *principal curves.* If we regard equations (56b) as linear equations in du^1 and du^2 then we will see that the solutions for

du^i that do not vanish simultaneously must imply that the direction δ is that of the second principal family.

Four directions of advance $du_1 : dv_1$, $du_2 : dv_2$, $du_3 : dv_3$, $du_4 : dv_4$ in the ray system have an invariant, namely, the *double ratio:*

(57) $$D = \frac{(du_1 dv_2 - du_2 dv_1)(du_3 dv_4 - du_4 dv_3)}{(du_1 dv_4 - du_4 dv_1)(du_3 dv_2 - du_2 dv_3)}.$$

It is equal to the double ratio of the four tangents to one of the sheets that belong to the direction of advance. One will get:

(58) $$\cot \varphi = i \, \frac{e^{ik} e^{pq} r_{ip} s_{kq}}{\sqrt{e^{ik} e^{pq} r_{ip} r_{kq}} \cdot \sqrt{e^{ik} e^{pq} s_{ip} s_{kq}}}$$

for the logarithmic double (angle, resp.) ratio φ of the directions of advance that belong to the null families (lines, resp.) of the forms $r_{ik}\, du^i\, du^k = 0$ and $s_{ik}\, du^i\, du^k = 0$. It follows from this that:

(58) $$e^{ik}\, e^{pq}\, r_{ip}\, s_{kq} = 0$$

is the *condition for harmonic position* of the null lines that belong to r_{ik} and s_{ik} . Since $e^{is}\, e^{kr}\, p_{sr} = - p^{ik}$, from (48), (49), the null lines of the forms g_{ik} and c_{ik} on the first sheet will be harmonic to the asymptotic lines p_{ik} (have harmonic angle ratios to the lines of curvature, resp.). The null lines of g_{ik} and $\bar{c}_{ik}$ are harmonic to the asymptotic lines $\bar{p}_{ik}$ on the second sheet. All of these curves then define a *conjugate net.* From (50), one easily recognizes that $d = 0$ is the necessary and sufficient condition for the asymptotic lines (lines of curvature, resp.) to correspond on the sheets. That equation will then characterize the *W-ray (W-sphere, resp.) systems.*

A relation:

(59) $$\alpha g_{ik} + \beta p_{ik} + \gamma \bar{p}_{ik} = 0 \qquad (\alpha, \beta, \gamma \text{ scalar})$$

can exist only for *W*-ray systems, as one recognizes upon multiplying by g^{ik}. For $d \neq 0$, we can then make an Ansatz for the symmetric tensor a_{ik} that is linear in these three tensors and whose coefficients are invariants. From equations (35), (36), (37), we can then calculate these invariants (and thus, the a_{ik}) from the remaining tensors that enter into the integrability conditions. If we multiply (38) and (39) by e^{is} then we will get:

(60) $$w_s = e_{si}\, e^{kl}\, c^i_{kl} + e_{si}\, e^{kl}\, c^i_k n_l ,$$

(61) $$\bar{w}_s = e_{si}\, e^{kl}\, \bar{c}^i_{kl} - e_{si}\, e^{kl}\, \bar{c}^i_k n_l .$$

For $d \neq 0$, we can then calculate a_{ik} , w_l, and $\bar{w}_l$ from the tensors g_{ik} , c_{ik} , $\bar{c}_{ik}$, n_l, and their derivatives. We would now like to exhibit the differential equation of the **Darboux** curves on the sheets. (The analogous curves in **Lie** geometry are called *cyclide curves*.) According to **E. Čech**, they can be characterized as follows: If one draws the four asymptotes of the surface of one of the two families through four consecutive points of a **Darboux** curve then all four of the asymptotes that go through the starting points will cut the other family. One easily sees from this that $(\mathfrak{p} + r\,\mathfrak{v}, \mathfrak{p} + r'\mathfrak{v}) = 0$, $(\mathfrak{p} + r\,\mathfrak{v}, d\langle\mathfrak{p} + r'\mathfrak{v}\rangle) = 0$, $(\mathfrak{p} + r\,\mathfrak{v}, d^2\langle\mathfrak{p} + r'\mathfrak{v}\rangle) = 0$ is fulfilled identically for any direction of advance. *The differential equation for Darboux curves* is then:

$$(\mathfrak{p} + r\,\mathfrak{v}, d^3\langle\mathfrak{p} + r'\mathfrak{v}\rangle) = s_{ikl}\, du^i\, du^k\, du^l = 0. \tag{62}$$

One can also switch r and r' in this. Analogous equations are true for the other sheet.

The formulas that were derived here are applicable to all of the geometries that will be treated in what follows with no further analysis. We will now turn to the special cases. Through the relations between the tensors that were given in the table, one will come to the geometries that will be discussed as special cases of the general theory. The rays (spheres, resp.) $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ will also be established each time by these relations between the tensors. For example, one sets $\frac{1}{2}c_i^i = h = 0$, $\frac{1}{2}\bar{c}_i^i = \bar{h} = 0$ for the projective geometry of ray systems. However, that means that, from (24), one will have $(\mathfrak{v}, g^{ik}\,\mathfrak{p}_{ik}) = 0$, $(\bar{\mathfrak{v}}, g^{ik}\,\mathfrak{p}_{ik}) = 0$, and the still-missing defining equations for the rays $\mathfrak{v}$, $\bar{\mathfrak{v}}$ will be given by that, while $\mathfrak{z}$ will also be established.

The more detailed explanation of this will be given in the paragraphs that describe the individual cases.

To abbreviate, one sets:

$$\begin{cases} m_i = w_i + c_i^s n_s, \\ \bar{m}_i = \bar{w}_i - \bar{c}_i^s n_s. \end{cases} \tag{63}$$

Table of the various geometries

Projective geometry of ray systems, § 4	$e_i^i = \bar{c}_i^i = 0$
Lie's geometry of sphere systems, § 4	
Conformal surface theory (point coordinates), § 5	$e_{ik} = \bar{c}_{ik}$ $\quad e_i^i = 0$ $w_l = \bar{w}_l$ $\quad n_l = 0$
Laguerre surface theory (plane coordinates), § 5	$\bar{e}_{ik} = 0$ $\quad e_i^i = 0$ $\bar{w}_l = 0$ $\quad n_l = 0$

Conformal geometry of sphere systems, § 6	$a_{ik} = -\frac{1}{2} g_{ik}, \quad w_l = \overline{w}_l = 0$
Laguerre geometry of sphere systems, § 6	$a_{ik} = 0, \quad w_l = \overline{w}_l = 0$
Conformal surface theory (central sphere systems), § 7	$a_{ik} = -\frac{1}{2} g_{ik}, \quad w_l = \overline{w}_l = 0, \quad e_i^i = 0$
Laguerre surface theory (middle sphere systems), § 7	$a_{ik} = 0, \quad w_l = \overline{w}_l = 0, \quad e_i^i = 0$
Non-Euclidian surface theory (point coordinates), § 8	$e_{ik} = \overline{c}_{ik} \quad a_{ik} = -\frac{1}{2} g_{ik}$ $n_l = 0 \quad w_l = \overline{w}_l = 0$
Euclidian (ordinary motion-geometric) surface theory (point coordinates), § 8	$e_{ik} = \overline{c}_{ik} \quad a_{ik} = 0$ $n_l = 0 \quad w_l = \overline{w}_l = 0$
Non-Euclidian surface theory (plane coordinates), § 8	$e_{ik} = \overline{c}_{ik} \quad a_{ik} = -\frac{1}{2} g_{ik}$ $n_l = 0 \quad w_l = \overline{w}_l = 0$
Euclidian surface theory (plane coordinates), § 8	$\overline{c}_{ik} = 0 \quad a_{ik} = -\frac{1}{2} g_{ik}$ $n_l = 0 \quad w_l = \overline{w}_l = 0$
Non-Euclidian surface theory (line coordinates), § 4	$c_i^i = \overline{c}_i^i = 0, \quad \overline{m}_i = 0$ ([23])
Affine geometry of ray systems, § 4	$g^{rs} a_{ir} a_{ks} + w_i \overline{w}_k + w_k \overline{w}_i = 0$
Projective surface theory (Appendix)	$e_i^i = \overline{e}_i^i = 0$ For that, one must have $m^i m_i = 0$ ([23]). Other conditions besides that one can come into question. (Cf., Appendix.)
Surface theory of higher sphere geometry (Appendix)	

([23]) m_i and $\overline{m}_i$ are explained in (63).

Affine surface theory (Appendix)	$g^{rs}\, a_{ir}\, a_{ks} + w_i \bar{w}_k + w_k \bar{w}_i = 0$ This implies yet another condition. (Cf. Appendix.)

§ 4.

Ultimate formulas for the projective geometry of ray systems.

We set $\sigma = \frac{1}{2} g^{ik} \dot{\mathfrak{p}}_{ik}$. One can also write the requirements $c_i^i = h = 0$ in the form $(\mathfrak{v}\sigma) = (\bar{\mathfrak{v}}\sigma) = 0$ then, and $\mathfrak{v}$, $\dot{\bar{\mathfrak{v}}}$, and $\mathfrak{z}$ can then be established by them. One obtains the following expression for $\mathfrak{z}$: $\mathfrak{z} = \sigma + \frac{1}{2}(\sigma\sigma)\,\mathfrak{p}$.

We fix the normalization of $\mathfrak{p}$, $\mathfrak{v}$, and $\bar{\mathfrak{v}}$ by the requirements:

(64) $$c = \bar{c} = -g.$$

In this, we exclude the cases $c = 0$ or $d = 0$, for which, one of the sheets would become a torse (surface with nothing but umbilic points, resp.; in real spheres or planes). One easily recognizes that one obtains the normalized values $\hat{\mathfrak{p}}$, $\hat{\mathfrak{v}}$, $\hat{\bar{\mathfrak{v}}}$ from the un-normalized ones by the normalization (64) in the following way ([24]):

(65) $$\hat{\mathfrak{p}} = \mathfrak{p}\cdot\sqrt[4]{\frac{c\cdot d}{g^2}}, \qquad \hat{\mathfrak{v}} = \mathfrak{v}\cdot\sqrt[4]{\frac{d}{c}}, \qquad \hat{\bar{\mathfrak{v}}} = \bar{\mathfrak{v}}\cdot\sqrt[4]{\frac{c}{d}}.$$

Likewise, the normalized values of g_{ik}, c_{ik}, $\bar{c}_{ik}$, a_{ik}, etc., can be expressed in terms of the un-normalized ones. In the following, all of these quantities will be assumed to have been normalized, without stating that explicitly. *Since* $h = \bar{h} = 0$, $k = \bar{k} = -1$, *one will now get the asymptotes in the form:*

(66) $$\mathfrak{p} \pm \mathfrak{v},\ \mathfrak{p} \pm \bar{\mathfrak{v}}.$$

From (10), one now easily calculates that $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ are the tangents to the sheet that are conjugate to $\mathfrak{p}$ (the contacting spheres that have harmonic angle ratios with the curvature spheres, resp.). Likewise, since $c_i^i = 0$, one easily deduces from (58) that the null lines of the form c_{ik} yield the pair of conjugate directions on the first sheet, which lies harmonically to the pair of null lines of g_{ik}.

([24]) One easily sees which triples of values one must take for the roots in the three expressions simultaneously. Different possibilities exist.

The normalized quantities $\hat{\mathfrak{p}}$, $\hat{\mathfrak{v}}$, $\hat{\bar{\mathfrak{v}}}$, $\hat{g}_{ik}$, $\hat{c}_{ik}$, $\hat{\bar{c}}_{ik}$ depend upon derivatives of $\mathfrak{p}$ up to the second, $\hat{\mathfrak{z}}$, $\hat{w}_i$, $\hat{\bar{w}}_i$, $\hat{n}_i$, $\hat{\mathfrak{p}}_i$ depend upon derivatives up to the third, and $\hat{a}_{ik}$, $\hat{\mathfrak{p}}_{ik}$, upon derivatives up to fourth order.

With the chosen normalization, one will have the following *orthogonality conditions* between the tensors g_{ik}, c_{ik}, p_{ik}, e_{ik}, which are very important in calculations:

(66) $$\tfrac{1}{2}g^{ik}\,g_{ik} = \tfrac{1}{2}c^{ik}\,c_{ik} = \tfrac{1}{2}p^{ik}\,p_{ik} = \tfrac{1}{2}e^{ik}\,e_{ik} = 1,$$

(67) $$g^{ik}\,c_{ik} = g^{ik}\,p_{ik} = c^{ik}\,p_{ik} = e^{ik}\,g_{ik} = e^{ik}\,c_{ik} = e^{ik}\,p_{ik} = 0.$$

Furthermore:

(68) $$c^{ir}\,c_{rk} = p^{ir}\,p_{rk} = e^{ir}\,e_{rk} = g^{ir}\,g_{rk} = g_k^i,$$

(69) $$e_{ir}\,c_k^r = e_{ik}, \qquad e_{ri}\,p_k^r = c_{ik}, \qquad p_{ir}\,c_k^r = e_{ik}.$$

One obtains analogous equations for the other sheet.

One derives from this that:

(70) $$(\mathfrak{v}_i\,\mathfrak{v}_k) = (\bar{\mathfrak{v}}_i\,\bar{\mathfrak{v}}_k) = g_{ik}.$$

The torses of the *ray systems* $\mathfrak{v}$ and $\bar{\mathfrak{v}}$, which are *conjugate to* $\mathfrak{p}$, then cut out the same conjugate net from the corresponding sheet as the torses of $\mathfrak{p}$.

Since $h = \bar{h} = 0$, $k = \bar{k} = -1$, one easily recognizes from (42) that a ray system has only one projective invariant of order two – namely, d ([25]). It depends in a simple way upon the logarithmic double ratio φ of the four directions of advance of the asymptotic lines on the sheets: One has: $\cot\varphi = -d$.

One further has the equation: $f_{ik} = \tfrac{1}{2}d\cdot g_{ik}$. The principal families then coincide with the torses for $d \neq 0$. For the W-ray systems, any curve on the sheet will be a principal curve.

We would now like to further calculate the *differential equation of the Darboux curves (cyclide curves, resp.)* in connection with (62). Since $r = +1$, $r' = -1$, $(\mathfrak{p}\;\mathfrak{p}_{ikl}) = (\mathfrak{v}\;\mathfrak{v}_{ikl}) = 0$, we get:

$$s_{ikl} = (\mathfrak{p}\;\mathfrak{v}_{ikl}) - (\mathfrak{v}\;\mathfrak{p}_{ikl}).$$

With the use of (69), equations (60), (61) take on the simpler form:

(71) $$w_s = c_{sl}^l + c_s^l n_l,$$

(72) $$\bar{w}_s = \bar{c}_{sl}^l - \bar{c}_s^l n_l.$$

([25]) The **Waelsch** invariant , Wiener Ber. **100** (1891), pp. 158.

With the help of (71), one easily sees the identity:

(73) $$c_{ikl} = p_{ik}\ p_l^r c_{rs}^l .$$

With the help of this relation and with the use of the abbreviation (63), one will obtain:

(74) $$s_{ikl} = m_l\, g_{ik} + m_s\, c_{ik} ,$$

and analogously:

(75) $$\bar{s}_{ikl} = \bar{m}_l\, g_{ik} + \bar{c}_l^s\, \bar{m}_s\, \bar{c}_{ik} .$$

There are six independent projective invariants of a ray system that depend upon third derivatives. If we introduce the abbreviations:

(76) $$\begin{cases} J = m^i m_i, & S = p^{ik} m_i\, m_k, \quad T = c^{ik} m_i\, m_k, \\ \bar{J} = \bar{m}^i \bar{m}_i, & \bar{S} = \bar{p}^{ik} \bar{m}_i\, \bar{m}_k, \quad \bar{T} = \bar{c}^{ik} \bar{m}_i\, \bar{m}_k, \end{cases}$$

then we will have ([26]):

$$J^2 = S^2 + T^2, \qquad \bar{J}^2 = \bar{S}^2 + \bar{T}^2 .$$

If we further set:

(77) $$A = c^{ik}\, n_i\, n_k , \qquad B = \bar{c}^{ik}\, n_i\, n_k$$

then we can regard, say, S, T, $\bar{S}$, $\bar{T}$, A, and B as the representatives of the six invariants.

One has $J = \frac{1}{2} S^{ikl}\, S_{ikl}$.

$J = 0$ ($\bar{J} = 0$, resp.) is the condition for one of the sheets to be a *ruled surface (channel surface, resp.)*, while $m_i = 0$ ($\bar{m}_i = 0$, resp.) is the condition for one of them to be a *second-order surface (Dupin cyclide, resp.)*. The study of the latter ray systems is identical with the *non-Euclidian surface theory* of sheets that do not degenerate into an F_2. If, say, $\bar{m}_i = 0$ then the rays $\mathfrak{p}$ will be the *isotropic surface tangents to one family* (tangents to the absolute surface, resp.). The null lines of g_{ik} will be the minimal lines of that family and the curves that are conjugate to them.

The coefficients of the form g_{ik} can be calculated as $c_i^i = 0$, $\bar{c}_i^i = 0$, $- g = c = \bar{c}$ ([26a]). The double sign in the last quadratic equation can be fixed by a special choice of normalization. Upon following through on formula (61), the proof of the following theorem will no longer be difficult:

A ray system with disjoint focal sheets that is not a W-ray system will be determined by the two quadratic forms c_{ik} and $\bar{c}_{ik}$ and the linear form n_i up to projective transformations. (A corresponding statement is true for the **Lie** geometry of sphere systems.)

([26]) Proof is by introducing special parameters!

([26a]) If the normalization condition $- g = c = \bar{c}$ is not needed then one can employ other ones in place of it for the calculation of g_{ik} .

A *Lie*-F_2 *(Lie cyclide, resp.)* is coupled with a surface in a projectively-invariant way. If one draws three consecutive asymptotes to the other family along an asymptotic lines then the **Lie**-F_2 will be the F_2 that is drawn through these three lines. If one introduces the asymptotic parameters of one of the sheets of our ray systems then one of the families of generators σ of its **Lie**-F_2 will be given by:

$$(\sigma\,\sigma) = 0, \qquad (\sigma, \mathfrak{p} + \mathfrak{v}) = 0, \qquad (\sigma, \mathfrak{p}_v + \mathfrak{v}_v) = 0, \qquad (\sigma, \mathfrak{p}_{vv} + \mathfrak{v}_{vv}) = 0.$$

One can find a representation for the **Lie**-F_2 in general parameters from this.

§ 5.

On the conformal and Laguerre theories of surfaces.

We link directly to what was just done in the previous paragraph and now appeal to the terminology of **Lie**'s sphere geometry. We consider the special sphere systems for which:

(78) $$\varepsilon\, c_{ik} = \overline{c}_{ik}, \qquad \varepsilon\, w_l = \overline{w}_l, \qquad n_l = 0.$$

Due to (42), one then has $d = 0$. We are thus dealing with W-sphere systems. It then follows from (31) and (32) that $\varepsilon\, \mathfrak{v}_l - \overline{\mathfrak{v}}_l$ = const., $\varepsilon\, \mathfrak{v} - \overline{\mathfrak{v}}$ = const. If we introduce the notation $\mathfrak{q} = \varepsilon\, \mathfrak{v} - \overline{\mathfrak{v}}$ then we will have $(\mathfrak{p}\ \mathfrak{q}) = 0$. The spheres $\mathfrak{p}$ then belong to the constant linear complex $\mathfrak{q}$. $\mathfrak{q}$ can also degenerate into a sphere, namely, for $(\mathfrak{q}\ \mathfrak{q}) = -2\varepsilon = 0$; all spheres $\mathfrak{p}$ then contact the fixed sphere $\mathfrak{q}$. In the Introduction, it was shown that one will arrive at conformal surface theory for $\varepsilon \neq 0$. We now turn to that case! The spheres $\mathfrak{p}$ that belong to the complex are the points of the surface. If we introduce a normal coordinate system [cf., § 1, (1b)] such that $\mathfrak{q}$ has the coordinates $(0, 0, 0, 0, 0, \sqrt{2\varepsilon})$ then we will have $\mathfrak{p}_6 = 0$. The first five coordinates of $\mathfrak{p}$ are then its *five penta-spherical point coordinates.* If we introduce the notation:

(79) $$[\zeta\,\eta] = \zeta^1\eta^1 + \zeta^2\eta^2 + \zeta^3\eta^3 + \zeta^4\eta^4 - \zeta^5\eta^5$$

then $[\mathfrak{p}\ \mathfrak{p}] = 0$. One will have $[r\ r] \neq 0$ for the coordinates of a sphere $\mathfrak{r}$ that does not belong to the complex $\mathfrak{q}$ – i.e., $(\mathfrak{r}\ \mathfrak{q}) \neq 0$, $(r\ r) = 0$. The first five coordinates of r are now the *penta-spherical sphere coordinates.* In that way, a sphere with given penta-spherical coordinates $r_1, r_2, \ldots, r_5$ will always correspond to two spheres in higher sphere geometry, namely, the ones with the sixth coordinates $r_6 = +\sqrt{[\mathfrak{r}\,\mathfrak{r}]}$ and $r_6 = -\sqrt{[\mathfrak{r}\,\mathfrak{r}]}$.

Such a pair of spheres will be referred to as simply "a sphere" in conformal geometry. (The difference between "oriented" spheres goes away.) In place of a linear complex σ $[(\sigma\,\sigma) \neq 0]$, we will have a sphere with penta-spherical coordinates $\sigma_1, \sigma_2, \ldots, \sigma_5$ in the conformal geometry of a sphere. The two spheres with the sixth coordinates σ_6 =

$\pm\sqrt{[\sigma\sigma]}$ are the spheres that are contained in the pencil of linear complexes $\alpha\,\sigma + \beta\,\mathfrak{q}$ (α, β are scalars). They then correspond to (σ_1, σ_2, …, σ_5).

Due to the fact that we identify spheres that differ by only the sign of σ_6, the two sheets of the envelope of our special sphere system will also coincide ([27]). $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ then, in fact, coincide in their first five coordinates. $(\mathfrak{v}\ \mathfrak{v}) = 0$ implies: $\mathfrak{v}_6 = \pm\sqrt{[\mathfrak{v}\mathfrak{v}]}$, and since $[\bar{\mathfrak{v}}\,\bar{\mathfrak{v}}] = [\mathfrak{v}\ \mathfrak{v}]$, we will get $\bar{\mathfrak{v}}_6 = \pm\sqrt{[\mathfrak{v}\mathfrak{v}]}$. Thus, $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ will be identified with each other in conformal geometry, and therefore, with the sheet, as well. We write out only five coordinates in the formulas of conformal surface theory and can then set $\mathfrak{v} = \bar{\mathfrak{v}}$.

Two null spheres (i.e., points) are always contained in a pencil of spheres. *The angle between two spheres is the logarithmic angle ratio that they make with the null spheres in the pencil that they define.*

We obtain the same equations for establishing $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$, as well as determining their normalizations that were employed in § 3.

The form g_{ik} goes to that of the minimal lines. *The angle between two directions that emanate from a point of the surface is the logarithmic angle ratio that those directions male with the two isotropic surface directions.* (58) implies that the angle φ of the null lines of the form $r_{ik}\,du^i\,du^k$ will satisfy:

(80) $$\cot\varphi = \frac{i}{2}\frac{r_i^i\sqrt{g}}{\sqrt{r}} \qquad (r = |\,r_{ik}\,|).$$

The sphere $\eta = \mathfrak{v} + \varepsilon\,\bar{\mathfrak{v}}$ is the central sphere of the surface that contacts the surface and defines a harmonic angle ratio with the curvature spheres, in conjunction with the null sphere of the point of the surface. Among the spheres of that pencil, it is characterized by the fact that it intersects the surface along a curve that has two perpendicular tangents at the point of contact.

The directions of those distinguished tangents are the directions of the null lines of the form $(\eta\ \mathfrak{p}_{ik}) = (1 + \varepsilon^2)\ c_{ik}$, namely, the angle bisectors of the lines of curvature, which we would like to call the *intersecting tangent curves* of the surface ([27a]). From (31), (32), one will have:

(81) $$(\eta_i\ \eta_k) = g_{ik}\cdot[1 + \varepsilon^2]^2.$$

The contact families of the system of central spheres that are linked with the surface then coincide with the minimal lines. Essentially four integrability conditions remain, namely: (35), (36), (38), (40).

At this point, let us mention the following notions, which are important in the advanced conformal theory of surfaces:

([27]) Naturally, this "coincidence" of the sheets does not have $g = 0$ as a consequence. (Cf., § 3)
([27a]) Confer the more detailed treatment in the paper cited in ([4]) for this and what follows.

1. The *transverse sphere* $d\hat{\mathfrak{p}}$ that belongs to a line element $\mathfrak{p} \to \mathfrak{p} + d\mathfrak{p}$ ([28]). (The caret suggests that one takes the differential of the normalized $\mathfrak{p}$.) Geometrically, we arrive at it in the following way: The two neighboring curvature spheres of one of the two families at $\mathfrak{p}$ and $\mathfrak{p} + d\mathfrak{p}$ intersect in a circle. We draw the sphere through that circle that is perpendicular to the curvature sphere at $\mathfrak{p}$. That sphere intersects the corresponding curvature spheres that are constructed from the other family in a circle that is perpendicular to the surface at $\mathfrak{p}$, namely, the *transverse circle* to the line element. Now, the transverse sphere is the one that goes through $\mathfrak{p}$, is perpendicular to the line element, and contains the transverse circle.

2. The *normal circle* of a surface that is perpendicular to it at a point of the surface and is the circle of intersection of all transverse spheres to it.

3. The *principal point* $\mathfrak{z}$, at which the normal circle pierces the central sphere a second time.

4. *The conformal-geodetic lines.* Just as the usual geodetic lines can be characterized by the fact that their curvature circles cut the surface normal a second time, in addition to the point of the surface, the conformal-geodetic lines are characterized by the fact that their curvature circles cut the normal circle twice. These curves are the extremals of the variational problem $\delta \int \sqrt{g_{ik} du^i du^k} = 0$, where g_{ik} is the normalized form, as in § 4.

If one would consider only the conformal theory of surfaces then one would have to set $\varepsilon = 1$. However, if one first leaves ε arbitrary then one can arrive at the formulas of the **Laguerre** theory of surfaces as special cases of those of the conformal theory, corresponding to the discussion in the Introduction, when one sets $\varepsilon = 0$. $\mathfrak{q} = -\bar{\mathfrak{v}}$ = const. will then be a sphere in **Lie** geometry. The spheres that contact that fixed sphere are the *"oriented" planes* of **Laguerre** geometry. A system of planes is then given by $\mathfrak{p}$ (u^1, u^2), and we will have the surface theory of the non-degenerate sheet in plane coordinates. By introducing suitable normal coordinates, we will get the coordinate values (0, 0, 0, 1, 0, 1) for $\mathfrak{q}$, so $\mathfrak{p}_4 = \mathfrak{p}_6$. Since $(\mathfrak{p}\ \mathfrak{p}) = 0$, $\mathfrak{p}$ will be determined from $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_5$ by establishing such a distinguished coordinate system. Those four quantities are then the homogeneous *Laguerre plane coordinates* that were employed by **Blaschke**. If we normalize the spheres r that do not contact $\mathfrak{q}$ by $(r\ \mathfrak{q})$ = const. then we can introduce r_1, r_2, r_3, r_5 as inhomogeneous *Laguerre sphere coordinates.*

By choosing that coordinate system, we can drop the fourth and sixth coordinates from the formulas and set $\mathfrak{q} = -\bar{\mathfrak{v}} = 0$. Since $\bar{c} = 0$, only the one normalization condition $c = -g$ is permissible. However, that condition will then suffice to establish all

([28]) The interpretation of the transverse sphere that is given in the paper that was cited in ([4]) is not correct.

quantities up to an arbitrary constant factor that is applied to $\bar{\mathfrak{v}}$. That indeterminacy corresponds to similar behavior in the geometry of motion, where the most important quantities are also invariant only up to a constant factor that depends upon the choice of units. The main ideas of conformal surface theory go to the corresponding ones in **Laguerre** theory for $\varepsilon = 0$. For example, the central sphere will become **Blaschke**'s *middle sphere.*

The formulas of **Laguerre** surface theory thus-obtained are, however, not identical with those of **Blaschke** with no further assumptions. Later on, we will come back to some other facets of conformal **Laguerre** surface theory and obtain **Blaschke**'s formulas.

§ 6.

On the conformal and Laguerre geometries of sphere systems.

In § 4, we deduced the equations that fixed $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ and the equations for determining the normalization in the conformal theory of surfaces directly in § 4. Here, in contrast to that, we will appeal to the general formulas of § 3. $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$ will then be undetermined, at first. It was shown above that $\mathfrak{z}$ satisfied the equations:

(82) $$(\mathfrak{z}\ \mathfrak{z}) = 0, \qquad (\mathfrak{z}\ \mathfrak{p}) = -1, \qquad (\mathfrak{z}\ \mathfrak{p}_i) = 0,$$

but can still be chosen freely, to a certain extent. $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ will then be determined (up to normalization) upon fixing $\mathfrak{z}$.

We now demand: The complex $\mathfrak{z} + \frac{1}{2}\delta\mathfrak{p}$, where δ is a given constant, shall continually be equal to a given fixed complex ζ. [If we set:

(83) $$\mathfrak{z} + \tfrac{1}{2}\delta\mathfrak{p} = \zeta$$

then ζ must naturally be normalized in such a way that $(\zeta\ \zeta) = -\delta$.] That demand can be fulfilled when equations (82) are consistent with that demand. We normalize $\mathfrak{p}$ by $(\mathfrak{p}\ \zeta) = -1$. We will then have $(\mathfrak{p}_i\ \zeta) = 0$. However, for that distinguished normalization, equations (82) will then follow from (83).

It follows from (83) that $\mathfrak{z}_i = -\frac{1}{2}\delta\cdot\mathfrak{p}_i$, or, from (33):

(84) $$a_{ik} = -\tfrac{1}{2}\delta\cdot g_{ik}\,, \qquad w_l = \bar{w}_l = 0.$$

Instead of (83), we can also demand (84) as a way of fixing $\mathfrak{z}$, $\mathfrak{v}$, $\bar{\mathfrak{v}}$.

If $\delta \neq 0$ then $\mathfrak{z}$ will be a "proper" complex, and since we have adjoined a complex, we will be forced into conformal geometry. Since $(\mathfrak{p}\ \zeta) \neq 0$, the $\mathfrak{p}$ are proper (i.e., not degenerating to a point) spheres, in the sense of conformal geometry ([29]). We will then be led to the conformal geometry of sphere systems by equations (84). One likewise arrives at **Laguerre**'s geometry of sphere systems for the case of $\delta = 0$, when one adjoins a sphere to ζ.

If we once more set $\zeta = (0, 0, 0, 0, 0, \sqrt{\delta})$ for the case $\delta \neq 0$ of conformal geometry then we can drop the sixth coordinate, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_5$ will be the penta-spherical sphere coordinates, and one will then set $\zeta \equiv 0$, so $\mathfrak{z} \equiv -\frac{1}{2}\delta\mathfrak{p}$.

Since $(\mathfrak{v}\ \zeta) = (\bar{\mathfrak{v}}\ \zeta) = 0$, $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ will be points, and in fact, from (19), they will be the *envelope points of the sphere system*. The normalization of the sphere $\mathfrak{p}$ is fixed by $(\mathfrak{p}\ \zeta) = -1$ ([29a]). We will best fix the normalization of $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ in the general case by the requirement:

(85) $$c = \bar{c}\ .$$

However, in special cases, it is often more convenient to employ other normalizations.

Since $(\mathfrak{p}\ \mathfrak{v}) = (\mathfrak{p}\ d\mathfrak{v}) = 0$, the tangent directions to the intersection curve of the first sheet with the sphere $\mathfrak{p}$ of the system are given by $c_{ik}\ du^i\ du^k = (\mathfrak{p}\ d^2\mathfrak{v}) = 0$. We would like to refer to the null lines of c_{ik} and the corresponding form $\bar{c}_{ik}$ as the *intersection tangent curves of the sphere system* on the sheets.

In connection with the argument in § 3 [formula (56*b*), *et seq.*], we can now make the following statement about a null line of the form f_{ik} : It must go through the sphere $\mathfrak{p}$, as well as a neighboring sphere $\mathfrak{p} + \delta\mathfrak{p}$ through the points $\mathfrak{v}$, $\bar{\mathfrak{v}}$ of the envelope, as well as the ones that neighbor them $\mathfrak{v} + d\mathfrak{v}$, $\bar{\mathfrak{v}} + d\bar{\mathfrak{v}}$ in the direction of one of the principal families. However, the spheres $\mathfrak{p}$ and $\mathfrak{p} + \delta\mathfrak{p}$ intersect along a circle. The four points must then lie on a circle, and we will thus be led to the well-known **Darboux** definition of the *principal curves of a congruence of spheres.*

With the help of § 3 (42) and the relation $k = \bar{k}$ that follows from (85), one easily sees that there are four conformal differential invariants of a congruence of spheres that depend upon derivatives up to order two. Geometric interpretations of four representatives of them will be simple with the help of § 3 (58), § 5 (80).

The minimal lines of the sheets are the null lines of the forms:

(86) $$g^{rs}\, c_{ir}\, c_{ks} \qquad \text{and} \qquad g^{rs}\, \bar{c}_{ir}\, \bar{c}_{ks}\ .$$

The *congruence circle* of the system that is perpendicular to both sheets at the envelop points $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ is represented by:

([29]) They do not belong to the fixed complex.

([29a]) The normalization of ζ is the fixed uniquely by $(\zeta\ \zeta) = -\delta$ for a given δ.

(87) $$\sigma(t) = \tfrac{1}{2}t^2\mathfrak{v} + \frac{t}{\sqrt{\delta}}\mathfrak{p} + \overline{\mathfrak{v}}\,,$$

where the point σ is given as a function of t [since $[\mathfrak{p}\ \mathfrak{p}] = \delta$, $[\sigma\,\sigma] = 0$ (29b)].

The theory of W-sphere systems [$d = 0$, cf., (50)], on whose sheets the lines of curvature correspond, in conjunction with the consideration of the system $\sigma(u^1, u^2, t)$ of congruence circles, will make it possible to derive *Ribacour's theory of normal systems of circles* quite easily. If $\mathfrak{p}(u^1, u^2)$ is a sphere system that refers to the corresponding lines of curvature of the sheet and has $d = 0$, and if $\mathfrak{v}$ and $\overline{\mathfrak{v}}$ are the associated points of the envelope that a *triply-orthogonal system* of surfaces u^1 = const., u^2 = const, t = const. will be given by $\sigma(u^1, u^2, t)$.

Sphere systems whose sheets are related to each other in an angle-preserving way are special W-sphere systems.

For $\delta = 0$, one arrives at considerations that are analogous to those of **Laguerre**'s geometry of sphere systems. $\mathfrak{v}$ and $\overline{\mathfrak{v}}$ are then the *tangential planes* of the sheet. The normalization of $\mathfrak{p}$ is fixed by $(\mathfrak{p}\ \zeta) = -1$ in this case only up to a constant factor since ζ is not normalized.

§ 7.

Central and middle sphere systems. More on the conformal and Laguerre theories of surfaces.

In the conformal (**Laguerre**, resp.) geometry of sphere systems, the simplest invariant variational problem $\delta \iint \sqrt{g}\ du\,dv = 0$ will lead to the differential equations $c_i^i = 0$ and $\overline{c}_i^i = 0$ for the extremals.

Now, $c_i^i = 0$ is the necessary and sufficient condition for one (the first, resp.) sheet of a sphere system to consist of the central spheres (middle spheres, resp.) of it (29c).

In fact, the angle ratio D between the two curvature spheres $(\mathfrak{p} + r\,\mathfrak{v})$ and $(\mathfrak{p} + r'\mathfrak{v})$, on the one hand, and the null sphere (tangential plane, resp.) $\mathfrak{v}$ of the envelope point and the sphere $\mathfrak{p}$ of the system, on the other, is calculated to be r' / r. The condition for a harmonic angle ratio is:

$$D = \frac{r'}{r} = -1, \text{ or } \qquad r + r' = 0, \qquad \text{or } h = \tfrac{1}{2}c_i^i = 0,$$

(29b) The meaning of the square bracket is again the one that one derives from (79).
(29c) The expressions in parentheses are the ones that apply to **Laguerre** geometry.

with which, the proof is complete.

Now, it can be very fruitful to study the conformal (**Laguerre**, resp.) geometric properties of surfaces in terms of the properties of their central (middle, resp.) sphere systems. One sees from (44) that for $c_i^i = 0$, the tensors $g^{rs}\, c_{ir}\, c_{ks}$ and g_{ik} of the minimal lines of the first sheet and the contact family, resp., are proportional. Instead of (85), we now choose the normalization that makes $g^{rs}\, c_{ir}\, c_{ks} = g_{ik}$. From (44), we will then have $k = -1$, and we will easily see from this that the surface point $\mathfrak{v}$ is normalized in entirely the same way (except for an inessential factor) as was done in § 5 in the conformal (**Laguerre**, resp.) theory of surfaces "external to the surface." With that, it will no longer be difficult to exhibit the connection with the formulas of § 5. For **Laguerre** geometry, we will now be led directly to **Blaschke**'s formulas. The intersecting tangent curves c_{ik} of the sphere system now go to the intersecting tangent curves of the surface in the sense of § 5.

The extremals of the variational problem above will yield the sphere systems that are the systems of central (middle, resp.) spheres for both sheets. Naturally, the sheets of such systems are entirely special surfaces. They will also be the extremals of the simplest invariant variational problem for surfaces. For conformal geometry, they will be *conformal minimal surfaces,* while for **Laguerre** geometry, they will be the *L-minimal surfaces.* According to **Blaschke**, the latter interesting class of surfaces can be represented in a manner that is completely free of integrals.

In the conformal geometry of the surfaces with distinguished sheets, the sphere systems with $c_i^i = 0$, $d = 0$ will yield the *isothermals*, while for **Laguerre** geometry, they will yield the *surfaces for which the spherical images of the lines of curvature define a system of isotherms on the sphere* ([30]).

§ 8.

Non-Euclidian and Euclidian theories of surfaces in point and plane coordinates.

We continue the investigations of the previous paragraph directly and set, not only (84), but:

(88) $$c_{ik} = \bar{c}_{ik}, \qquad\qquad n_l = 0.$$

Analogous to § 4, we then obtain: $\varepsilon\,\mathfrak{v} - \bar{\mathfrak{v}} = \mathfrak{q} = \text{const.}$ We then once more consider the sphere system that belongs to a linear complex, but this time from the standpoint of conformal geometry, instead of **Lie** geometry.

We can give our equations yet another interpretation. From § 4, we will arrive at the conformal theory of surfaces by the conditions $c_{ik} = \bar{c}_{ik}$, $w_l = \bar{w}_l$, $n_l = 0$. In the same way as in § 5, the constancy of $\mathfrak{z} + \frac{1}{2}\delta\mathfrak{p}$ will be required by $a_{ik} = -\frac{1}{2}\delta \cdot g_{ik}$, $w_l = \bar{w}_l = 0$, and with that, one firstly adjoins a linear complex, and secondly completes the definitions of $\mathfrak{v}$, $\bar{\mathfrak{v}}$, and $\mathfrak{z}$. However, from § 4, we can make that linear complex correspond to a

([30]) See the paper of **W. Blaschke** that was cited in ([5]).

sphere in conformal geometry that will degenerate to a point for $\delta = 0$, so for $\delta = 0$, $\zeta = \mathfrak{z}$ will be a sphere in **Lie** geometry. We will rise to conformal geometry by the adjunction of the complex $\mathfrak{q} = \mathfrak{v} = \bar{\mathfrak{v}}$, so $(\zeta\,\mathfrak{q}) = 0$, and ζ will thus belong to that complex, and will then be a point. However, by adjoining a sphere, one will arrive at conformal non-Euclidian geometry, while the adjunction of a point will give conformal Euclidian geometry. We will thus be led to the non-Euclidian theory of surfaces by our conditions (84), (88) in the case of $\delta \neq 0$, and to the Euclidian theory in the case of $\delta = 0$.

We next once more set $\mathfrak{q} = \mathfrak{v} - \bar{\mathfrak{v}} = (0, 0, 0, 0, 0, \sqrt{2})$. The first five coordinates are then penta-spherical. We then introduce penta-spherical coordinates in the case $\delta \neq 0$ such that $z = (0, 0, 0, 0, \sqrt{\delta})$ ([31]). Since:

$$[\mathfrak{p}\ \mathfrak{p}] = 0, \qquad [\mathfrak{p}\ \zeta] = -1, \qquad \mathfrak{p}_5 = \frac{1}{\sqrt{\delta}},$$

we will then have:

$$\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2 + \mathfrak{p}_4^2 = \frac{1}{\delta}$$

for the surface point $\mathfrak{p}$. If one sets $\delta = 1$ then $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_4$ will be the *Weierstrass coordinates* of the points of the surface ([32]). $\xi = \mathfrak{v} + \bar{\mathfrak{v}}$ is the *tangential plane.*

For $\delta = 0$, one can given the values (0, 0, 0, 1, 1) to the penta-spherical coordinates of the point $\zeta = \mathfrak{z}$. Corresponding to the arbitrariness in the unit of measurement, the normalization of $\mathfrak{p}$ is established by $(\mathfrak{p}\ \zeta) = -1$ only up to the constant factor that is applied to ζ. $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ are the *Cartesian coordinates* of the point of the surface. $\xi = \frac{\sqrt{2}}{2}(\mathfrak{v} + \bar{\mathfrak{v}})$ is the *tangential plane*. ξ_1, ξ_2, ξ_3 are the *direction cosines of the surface normal*. $C_{ik} = \sqrt{2} \cdot c_{ik}$ is the form of the *asymptotic lines* $[(\xi\, d^2\mathfrak{p}) = C_{ik}\, du^i\, du^k\ !]$, $\sqrt{2} \cdot h$ and $2k$ are the *mean* and *Gaussian curvatures*, resp. By dropping the last three equations, one will get from (30), (31) that:

$$\mathfrak{p}_{ik} = + C_{ik}\,\zeta \qquad (\textit{Gauss equations}),$$
$$\xi_l = - C_l^s\,\mathfrak{p}_s \qquad (\textit{Weingarten equations}).$$

What will remain are two integrability conditions, namely, (35) and (38):

$$K = 2k \qquad (\textit{Gauss's Theorema egregium}),$$
$$e^{kl}\, c_{ikl} = 0 \qquad (\textit{Codazzi equations}).$$

([31]) Since $(\zeta\,\zeta) = -\delta$, $(\zeta\,\mathfrak{q}) = 0$, one will have $[\zeta\,\zeta] = -\delta$.

([32]) At least, for the elliptical case.

We can then arrive at the non-Euclidian and Euclidian theories of surfaces from yet another direction.

If we consider sphere systems whose spheres are all orthogonal to a fixed sphere then we will be forced into the study of a non-Euclidian surface theory of one of the sheets. By considering sphere systems whose spheres all go through a fixed point, we will similarly arrive at Euclidian surface theory. Since the spheres of the systems are orthogonal to the fixed sphere (go through the fixed point, resp.) they will be the planes of non-Euclidian (Euclidian, resp.) geometry, and indeed they will be the tangential planes to the sheet in question.

These formulas are then written in *plane coordinates.* The intersection tangent curves of the system go to the asymptotic lines of the sheet. The form g_{ik} is the *form of the line element on the unit sphere in the spherical image.*

In the context of the theory of congruences of central spheres, one can now derive theorems that represent relations between conformal, non-Euclidian, and Euclidian geometry, such as perhaps the following one:

The conformally-minimal surfaces that are, at the same time, isothermal surfaces are the minimal surfaces of non-Euclidian and Euclidian geometry and the ones that are conformally-related to them.

If the congruence of circles of a congruence of central spheres all go through a fixed point then the distinguished sheet will be a surface of fixed (Euclidian) mean curvature or one that is conformally-related to it.

§ 9.

Affine geometry of ray systems.

We now employ the terminology of the projective geometry of ray systems. We demand that:

(89) $$g^{rs}\, a_{ir}\, a_{ks} + w_i \bar{w}_k + w_k \bar{w}_i = 0.$$

One calculates from (33) that the expression on the left is equal to $(\mathfrak{z}_i\ \mathfrak{z}_k)$. However, $(\mathfrak{z}_i\ \mathfrak{z}_k) = 0$ is the necessary and sufficient condition for all rays of the system $\mathfrak{z}\ (u^1, u^2)$ to lie in a fixed plane. If we choose parameters such that $(\mathfrak{z}_1\ \mathfrak{z}_1) = (\mathfrak{z}_2\ \mathfrak{z}_2) = 0$ then the rays $\mathfrak{z} + \mathfrak{z}_1\, du^1$ and $\mathfrak{z} + \mathfrak{z}_2\, du^2$ must intersect and give $(\mathfrak{z}_1\ \mathfrak{z}_2) = 0$. For that reason, the condition is necessary; however, it is also sufficient, since all further scalar products $(\mathfrak{z}\ \mathfrak{z}_{ik})$, $(\mathfrak{z}\ \mathfrak{z}_{ikl})$, etc., will vanish. If we adjoin the plane that is defined by the ray $\mathfrak{z}$ then we can now given the following twist to the requirement (89): $\mathfrak{z}$ shall continually lie in a given fixed plane.

If $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ $[(\mathfrak{a}\ \mathfrak{a}) = (\mathfrak{b}\ \mathfrak{b}) = (\mathfrak{c}\ \mathfrak{c}) = (\mathfrak{a}\ \mathfrak{b}) = (\mathfrak{b}\ \mathfrak{c}) = (\mathfrak{a}\ \mathfrak{c}) = 0$, resp.] are three rays that lie in general position in a fixed plane then the rays $\mathfrak{p}$ of the system can be given a distinguished normalization relative to that plane that is well-defined up to a constant factor by the demand that $e^{rs} \mid \mathfrak{p}, \mathfrak{p}_r, \mathfrak{p}_r, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \mid = 1$. Namely, if one replaces the rays $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ with other ones that lie in the plane then they will be linear combinations of the latter, while the normalization will change by only a constant factor. If we think of $\mathfrak{p}$ as having been normalized in that way relative to the given plane then we can normalize $\mathfrak{z}$ by the demand that $(\mathfrak{z}\ \mathfrak{p}) = -1$. $(\mathfrak{z}\ \mathfrak{p}_i) = 0$ are then two conditions that will determine the position of $\mathfrak{z}$ in the fixed plane. Since the normalized $\mathfrak{p}$ depends upon first-order derivatives, $\mathfrak{z}$ will be of second order, due to $(\mathfrak{z}\ \mathfrak{p}_i) = 0$.

We can now introduce a normal coordinate system [cf., § 1 (1*a*)] such that $\mathfrak{z}$ takes on the coordinates: $(0, 0, 0, z_4, z_5, z_6)$, so from (1*a*), we will have $(\mathfrak{z}\ \mathfrak{z}) = 0$, and all $\mathfrak{z}$ will lie in the plane that is defined by the three lines (1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0). If we restrict ourselves to distinguished coordinate systems such that the only possible transformations of the line coordinate $\mathfrak{p}$ are the ones that transform the first coordinates $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ amongst themselves then any line whose first three coordinates are equal to zero must also possess that property under the transformation. *The first three line coordinates then define a vector.* Now, the ratios $\mathfrak{p}_1 : \mathfrak{p}_2 : \mathfrak{p}_3$ give the *direction* of the ray, so all rays with the same ratios of the first three coordinates will be intersected by that pencil of planes in the fixed (i.e., "infinitely-distant") plane ([33]).

If we take the ray $\mathfrak{p}$ of the system with its distinguished normalization then the first three coordinates of the vector must acquire an invariant meaning. That vector must have the direction of $\mathfrak{p}$ and depend upon first-order derivatives. Now, the vector that connects the two focal points has those properties. Since there are no affine invariants of a ray system that depend upon only first-order derivatives, $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = p$ will then be the vector that connects the focal points, up to an inessential constant factor. One can gain the insight from this that our coordinates essentially coincide with the ones that **W. Krause** ([34]) employed for the treatment of the affine geometry of ray systems. $(\mathfrak{z}\ \mathfrak{p}) = -1$, $(\mathfrak{z}\ \mathfrak{p}_i) = 0$ implies that the last three components of $\mathfrak{z}$ define a contravariant vector z. One has:

(90)
$$z = -\frac{p_1 \times p_2}{\mid p, p_1, p_2 \mid},$$

where $p_1 \times p_2$ is the vectorial product, and $\mid p, p_1, p_2 \mid$ the determinant of the vectors. If we displace all of the vectors p from a fixed point then we will obtain a surface $p(u^1, u^2)$:

([33]) The bilinear form that belongs to (1*a*) is:

$$(\zeta\,\eta) = \zeta_1\,\eta_4 + \zeta_2\,\eta_5 + \zeta_3\,\eta_6 + \zeta_4\,\eta_1 + \zeta_5\,\eta_2 + \zeta_6\,\eta_3\,,$$

and only ζ_1, ζ_2, ζ_3 enter into the condition for the intersection of a line ζ with an infinitely distant line η [$\eta_1 = \eta_2 = \eta_3 = 0$], namely : $(\zeta\,\eta) = 0$.

([34]) **W. Krause**. Dissertation, Hamburg 1922 (Abstract). The complete work is found at the Hamburgischen Staats- und Universitätsbibliothek and the preussischen Staatsbibliothek (typescript).

namely, the affine curvature image of the ray system. Now, as we see in (90), $\mathfrak{z}$ *is the line of intersection of the tangential plane of the curvature image with the fixed plane.* One also sees from this that $\mathfrak{v}$ and $\bar{\mathfrak{v}}$ are the tangents to the focal sheet that are parallel to that plane.

Similar to what we did in the conformal geometry of sphere congruences, one concludes here that there are four affine differential invariants of a second-order ray system. As was true there, the *principal curves* $f_{ik}\, du^i\, du^k = 0$ of the system will also play an important role here. The calculation of the *affine normals* to the sheet will involve no complications.

Appendix.

On the projective and affine theories of surfaces. (Higher sphere-geometric theory of surfaces.)

Looking back upon the remarks that were made in the Introduction, we would now like to present the condition *for a ray* $\mathfrak{p}$ *to coincide with a Darboux tangent to one sheet* in the context of the projective geometry of ray systems that was developed in § 4. If one introduces torses as families of parameters then one will have $g_{11} = g_{22} = 0$, and since $\mathfrak{p}$ contacts the curves on the sheet that correspond to one of the torses, from (74), one must have, say, $s_{111} = 0$. A tangent to the curve $dv = 0$ is then a **Darboux** tangent. From (74), and since $c_{11} \neq 0$, $c_1^1 = 0$, $c_1^2 \neq 0$ ([35]), $s_{111} = 0$ then implies the condition that $m_2 = 0$. However, that can be written invariantly as: $J = m^i\, m_i = 0$ ([cf., (76)].

The null lines of the form g_{ik} on the first sheet are now the one **Darboux** tangent and its conjugate. If we exclude the case in which the surface of the first sheet is an F_2 ($m_i = 0$) ([36]) then we can show that $S \neq 0$. [Cf., (76)] Now, due to the two equations $J = 0$, $S = p^{ik}\, m_i\, m_k$, from (76), one can calculate the form m_i in terms of the forms g_{ik}, c_{ik}, $\bar{c}_{ik}$, and S from (76), and also the form n_i from (71), (63). Since one can calculate the coefficients of g_{ik} from the coefficients of c_{ik} and $\bar{c}_{ik}$ as in § 4, we can now make the following statement in connection with the theorem that was stated in the conclusion to § 4: In general ([37]), a surface will be determined by being given the forms c_{ik}, $\bar{c}_{ik}$, and the invariant S, up to projective transformations.

Imposing the same condition in the affine geometry of ray systems would lead to an affine theory of surfaces.

However, it is possible that other conditions might be more convenient. One can let the null lines of the form g_{ik} coincide with any invariant conjugate net of curves on the surface of the sheet. The tangents to that curve net then define the ray systems $\mathfrak{p}$ and $\mathfrak{v}$

([35]) Cf., (66), (67).

([36]) Cf., § 4.

([37]) Namely, if the surface has disjoint focal sheets, if it is not developable and is not a F_2, and if the three ray systems that are defined by the systems of **Darboux** tangents are not all three of them W-ray systems.

([38]). An apparent difficulty in such investigations must be emphasized, namely, that in projective geometry, a purely line-geometric treatment of the theory of surfaces on the basis of the duality principle might be the most appropriate one, and that theorems on the surface theory of the sheet are already relatively simple to obtain in the general theory of ray systems ([39]).

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([38]) In projective geometry, one can imagine the null lines of the third differential form that **Fubini** introduced, and the affine lines of curvature in affine geometry.

([39]) However, one will get the simplest line-geometric way of treating the projective and affine theories of surfaces from the study of the ray systems that consist of the asymptotic tangents to one of the two families, which can be extracted from the theory that was given here, according to § 2. Cf., footnote ([11a]).