"Sur le mouvement des planètes autour du Soleil, d'après la loi électrodynamique de Weber," C. R. Acad. Sci. Paris **75** (1872), 760-763.

On the motion of planets around the Sun according to Weber's electrodynamical law

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Under that law, the force that produces the motion of the planet around the Sun is:

$$F = \frac{fm\mu}{r^2}\left(1 - \frac{1}{h^2}\frac{dr^2}{dt^2} + \frac{2}{h^2}r\frac{d^2r}{dt^2}\right),$$

in which f is the constant of universal attraction, m is the mass of the planet, μ is the sum of that mass and that of the Sun, r is the distance from the planet to the Sun, and h is the velocity by which the attraction propagates in space.

The integration of the equations of motion is accomplished rigorously with the aid of elliptic function. Upon starting with that solution, one can obtain some approximate formulas that will be convenient for the sake of obtaining numerical values. Nonetheless, one will arrive at the goal more rapidly by setting:

$$F = \frac{fm\mu}{r^2} + F_1$$

and regarding F_1 as a perturbing force. Moreover, it will suffice to vary the constants of the elliptical motion.

Here are the equations of the perturbed motion:

$$(1) \qquad \left\{\begin{aligned} \frac{d^2x}{dt^2} + \frac{f\mu x}{r^2} + X &= 0, \\ \frac{d^2y}{dt^2} + \frac{f\mu y}{r^2} + Y &= 0, \\ \frac{d^2z}{dt^2} + \frac{f\mu z}{r^2} + Z &= 0, \end{aligned}\right\} \qquad \text{in which} \qquad \left\{\begin{aligned} X &= \frac{f\mu}{h^2}\frac{x}{r^2}\Omega, \\ Y &= \frac{f\mu}{h^2}\frac{x}{r^2}\Omega, \\ Z &= \frac{f\mu}{h^2}\frac{z}{r^2}\Omega \\ \Omega &= -\frac{dr^2}{dt^2} + 2r\frac{d^2r}{dt^2}. \end{aligned}\right.$$

The equations of elliptic motion are obtained by setting $X = Y = Z = 0$ in equations (1). Suppose that these equations have been integrated and let the elliptic elements be taken to be: a, the semi-major axis, e, the eccentricity, φ, the inclination, θ is the longitude of the node, ϖ is that of the perihelion, and ε is that of the epoch. For the present case, one will have formulas that determine the variation of the constants by taking the well-known formulas and replacing the derivative dR / dp of the perturbing function with respect to an arbitrary element p with $X\frac{dx}{dp}+Y\frac{dy}{dp}+Z\frac{dz}{dp}=R_p$ in them.

Now, one has:

$$R_p = \frac{f\mu}{h^2}\frac{\Omega}{r^3}\left(x\frac{dx}{dp}+y\frac{dy}{dp}+z\frac{dz}{dp}\right) = \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{dp}.$$

Since the expression for the radius vector depends upon only a, e, $\varepsilon - \varpi$, one will have:

$$\frac{dr}{d\varphi} = 0, \qquad \frac{dr}{d\theta} = 0, \qquad \frac{dr}{d\varepsilon} = -\frac{dr}{d\varpi},$$

and as a result:

$$R_\varphi = 0, \qquad R_\theta = 0, \qquad R_\varepsilon = -R_\varpi .$$

One will easily find the following formulas:

$$(2) \qquad \left\{\begin{array}{ll} \dfrac{da}{dt} = -\dfrac{2}{na}R_\varepsilon, & \dfrac{d\theta}{dt} = 0, \\[2ex] \dfrac{de}{dt} = -\dfrac{1-e^2}{na^2 e}R_\varepsilon, & \dfrac{d\varpi}{dt} = -\dfrac{\sqrt{1-e^2}}{na^2 e}R_e, \\[2ex] \dfrac{d\varphi}{dt} = 0, & \dfrac{d\varepsilon}{dt} = \dfrac{2}{na}R_a - \dfrac{\sqrt{1-e^2}}{na^2 e}\left(1-\sqrt{1-e^2}\right)R_e. \end{array}\right.$$

One will remark that these formulas (2) that φ and θ are not altered by the perturbing force, which is obvious *a priori*; however, what is less obvious is that the parameter does not change either. Indeed, one will have:

$$\frac{d[a(1-e^2)]}{dt} = -\frac{2}{na}(1-e^2)\,R_\varepsilon + 2a\,e\,\frac{1-e^2}{na^2 e}R_\varepsilon = 0.$$

In order for us to get some idea of the value of the perturbation, we shall develop those perturbations into series that proceed in sines and cosines of multiples of the mean anomaly ζ and neglect the powers of e that are greater than one.

We first address Ω, which contains the term $r\frac{d^2r}{dt^2}$; now, we have:

$$r\frac{d^2r}{dt^2} = -\left(\frac{dr}{dt^2}\right)^2 + x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}+\frac{dx^2+dy^2+dz^2}{dt^2},$$

or even, with an approximation that is entirely satisfactory:

$$r\frac{d^2r}{dt^2} = -\frac{dr^2}{dt^2} - \frac{f\mu}{r} + f\mu\left(\frac{2}{r} - \frac{1}{a}\right);$$

it will then result that:

$$\frac{\Omega}{r^2} = 2f\mu\left(\frac{1}{r^3} - \frac{1}{ar^2}\right) - \frac{3}{r^3}\frac{dr^2}{dt^2},$$

which is an expression that is developed as follows:

$$\frac{\Omega}{r^2} = n^2 e\left[2\cos\zeta + \frac{e}{2}(1 + 11\cos 2\zeta)\right] + \ldots$$

One will then have:

$$R_a = \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{da} = \frac{2f\mu}{h^2}n^2 e\cos\zeta + \ldots,$$

$$R_e = \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{de} = -\frac{f\mu}{h^2}n^2 ae\left[1 + \cos 2\zeta + \frac{3e}{4}(3\cos\zeta + 5\cos 3\zeta)\right]n^2 e\cos\zeta + \ldots,$$

$$R_\varepsilon = \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{d\varepsilon} = \frac{f\mu}{h^2}n^2 ae\sin 2\zeta + \ldots,$$

and upon neglecting e^2, as always, one will deduce that:

$$\frac{da}{dt} = 0, \qquad \frac{d\theta}{dt} = 0,$$

$$\frac{de}{dt} = -\frac{f\mu}{h^2}\frac{ne}{a}\sin 2\zeta, \qquad \frac{d\varpi}{dt} = \frac{f\mu}{h^2}\frac{n}{a}\left[1 + \cos 2\zeta + \frac{3e}{4}(3\cos\zeta + 5\cos 3\zeta)\right],$$

$$\frac{d\varphi}{dt} = 0, \qquad \frac{d\varepsilon}{dt} = \frac{4f\mu}{h^2}\frac{ne}{a}\cos\zeta,$$

so upon integrating those equations:

$$\delta\alpha = 0, \qquad \delta\theta = 0,$$

$$\delta e = \frac{f\mu}{h^2}\frac{e}{2a}\cos 2\zeta, \qquad \delta\varpi = \frac{f\mu}{h^2}\frac{n}{a}t + \frac{f\mu}{h^2}\frac{1}{a}\left[\tfrac{1}{2}\sin 2\zeta + \frac{9e}{4}\sin\zeta + \frac{5}{4}e\sin 3\zeta\right],$$

$$\delta\varpi = 0, \qquad \delta\varepsilon = \frac{4f\mu}{h^2}\frac{e}{a}\sin\zeta.$$

We will then see that the perturbations of the elements are zero or periodic, with the exception of those of $\delta\varpi$, which contain a secular part. Later, we shall confirm that the periodic parts are entirely negligible under the various hypotheses that one can make on the value of h, in such a way that we will arrive at the following conclusion:

Under Weber's law, the elements will remain the same as under Newton's law. Only the longitude of perihelion will be found to have increased by $\frac{f\mu}{h^2}\frac{n}{a}$, which is a quantity that will get larger as the planet gets closer to the Sun.

Consider the case of Mercury. Upon taking the mean solar day to be the unit of time, and the semi-major axis of the orbit of the Earth to be the unit of distance, one will find that:

$$\delta\varpi = \frac{(1.05160)}{h^2}\ t.$$

If we assume that h has the same value as in Weber's experiments on electricity, namely, $h = 439450 \times 10^6$, with seconds and millimeters for units, then we will first have:

$$\log h = 2.40805 \qquad \text{and} \qquad \delta\varpi = (\bar{4}.23550)\,t$$

per century, with our units, and then find that:

$$\delta\varpi = +\ 6.28'';$$

for Venus, one will have only:

$$\delta\varpi = +\ 1.32''.$$

If one supposes that h is equal to the speed of propagation of light then one will have:

$$\log h = 2.23948$$

and then

For Mercury and one century…………. $\delta\varpi = +\ 13.65''$,
For Venus and one century……………. $\delta\varpi = +\ 2.86''$.

In order to show that these periodic terms are negligible, it will suffice to take the biggest of them, which is $\delta\varpi$, namely, $\frac{f\mu}{2ah^2}\sin 2\zeta$; one will find that its coefficient does not reach 0.003″.
