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On the equations of motion of non-holonomic material systems

By

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1. Imagine a system that is initially subject to some constraints *that are expressible by relations in finite terms between the coordinates of the various points.* Let the number of independent parameters $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ that fix the position of the system be $k + p$ when one takes those constraints into account. If one assumes that the constraints are independent of time then one will have:

$$x_0 = f(t, q_1, q_2, \ldots, q_k, \ldots, q_{k+p}), \qquad y_0 = \varphi(\ldots), \qquad z_0 = \omega(\ldots) \tag{1}$$

for the coordinates of an arbitrary point of the system.

One will get a virtual displacement that is compatible with those constraints at the moment t by varying $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ by $\delta q_1, \delta q_2, \ldots, \delta q_k, \ldots, \delta q_{k+p}$, resp.; one will get:

$$\delta x_0 = \sum_{\alpha=1}^{k} \frac{\partial x_0}{\partial q_\alpha} \delta q_\alpha + \sum_{i=1}^{p} \frac{\partial x_0}{\partial q_{k+i}} \delta q_{k+i}, \qquad \delta y_0 = \ldots, \qquad \delta z_0 = \ldots \tag{2}$$

However, now suppose that one adds some new constraints to the preceding ones that depend upon time and are expressible by *p differential relations between the parameters* $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ of the form:

$$dq_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha} dq_\alpha + a_i\, dt \qquad (i = 1, 2, \ldots, p), \tag{3}$$

in which the coefficients $a_{i\alpha}$, a_i can depend upon time t and some parameters $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$.

For a virtual displacement that is compatible with those constraints, one will have:

$$\delta q_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha} \delta q_\alpha \qquad (i = 1, 2, \ldots, p). \tag{4}$$

One will get a virtual displacement that is compatible with the two kinds of constraints at the moment t by introducing the values of δq_{k+i} , …, δq_{k+p} into (2); one will have:

$$\text{(5)} \qquad \delta x = \sum_{\alpha=1}^{k}\left(\frac{\partial x_0}{\partial q_\alpha}+\sum_{i=1}^{p} a_{i\alpha}\frac{\partial x_0}{\partial q_{k+i}}\right)\delta q_\alpha\,, \qquad \delta y = \ldots, \qquad \delta z = \ldots$$

Therefore, in order to get the most general virtual displacement that is compatible with the constraints that exist at the instant t, it will suffice to subject the k parameters q_1, q_2, …, q_k to arbitrary variations δq_1, δq_2, …, δq_k . The system considered will then possess k degree of freedom and its coordinates will be $k + p$ in number.

The general equation of dynamics is:

$$\sum m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z),$$

in which x'', y'', z'' are the second derivatives of the coordinates (1) of an arbitrary point of the system with respect to time, while taking into account equations (3), and X, Y, Z are the projections of any of the forces.

That equations must be true for all displacements (5) that are compatible with the constraints; it then decomposes into the following k equations:

$$\text{(6)} \qquad \sum m\left[x''\left(\frac{\partial x_0}{\partial q_\alpha}+\sum_{i=1}^{p} a_{i\alpha}\frac{\partial x_0}{\partial q_{\alpha+i}}\right)+y''(\cdots)+z''(\cdots)\right] = Q_\alpha \qquad (a = 1, 2, \ldots, k),$$

in which Q_α is the coefficients of the δq_α in the expression for the sum of the virtual works that are done by applied forces.

If we denote the left-hand side of equations (6) by P_α then one will have:

$$\begin{aligned} P_\alpha = {} & \frac{d}{dt}\sum m\left[x'\left(\frac{\partial x_0}{\partial q_\alpha}+\sum_{i=1}^{p} a_{i\alpha}\frac{\partial x_0}{\partial q_{\alpha+i}}\right)+y'(\cdots)+z'(\cdots)\right] \\ & - \sum m\left\{x'\left[\left(\frac{\partial x_0}{\partial q_\alpha}\right)+\frac{d}{dt}\left(\sum_{i=1}^{p} a_{i\alpha}\frac{\partial x_0}{\partial q_{\alpha+i}}\right)\right]+y'(\cdots)+z'(\cdots)\right\}. \end{aligned}$$

One infers from (1) and (3) that:

$$\text{(7)} \qquad x'_0 = \frac{\partial x_0}{\partial t}+\sum_{\alpha=1}^{k}\frac{\partial x_0}{\partial q_\alpha}q'_\alpha+\sum_{i=1}^{p}\frac{\partial x_0}{\partial q_{k+i}}q'_{k+i}\,, \quad y'_0 = \ldots, \; z'_0 = \ldots,$$

$$\text{(8)} \qquad q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\,q'_\alpha + a_i \qquad (i = 1, 2, \ldots, p).$$

One will then have:

$$\frac{\partial x'}{\partial q'_\alpha} = \frac{\partial x_0}{\partial q_\alpha} + \sum_{i=1}^{p} a_{i\alpha}\frac{\partial x_0}{\partial q_{k+i}}, \qquad \frac{\partial y'}{\partial q'_\alpha} = \ldots, \qquad \frac{\partial z'}{\partial q'_\alpha} = \ldots,$$

and the expression for P_α will take the form:

$$\begin{aligned} P_\alpha = {} & \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q'_\alpha} + y'\frac{\partial y'}{\partial q'_\alpha} + z'\frac{\partial z'}{\partial q'_\alpha}\right) \\ & - \sum m\left[x'\frac{d}{dt}\left(\frac{\partial x_0}{\partial q'_\alpha}\right) + y'\frac{d}{dt}\left(\frac{\partial y_0}{\partial q'_\alpha}\right) + z'\frac{d}{dt}\left(\frac{\partial z_0}{\partial q'_\alpha}\right)\right] \\ & - \sum m\left[x'\frac{d}{dt}\left(\sum a_{i\alpha}\frac{\partial x_0}{\partial q'_\alpha}\right) + y'\frac{d}{dt}\left(\sum a_{i\alpha}\frac{\partial y_0}{\partial q'_\alpha}\right) + z'\frac{d}{dt}\left(\sum a_{i\alpha}\frac{\partial z_0}{\partial q'_\alpha}\right)\right]. \end{aligned}$$

It is easy to see from equations (1), (7), and (8) that one has:

$$\frac{d}{dt}\left(\frac{\partial x_0}{\partial q_\alpha}\right) = \frac{\partial x'}{\partial q_\alpha} + \sum_{i=1}^{p}\frac{\partial x_0}{\partial q_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}, \qquad \frac{d}{dt}\left(\frac{\partial y_0}{\partial q_\alpha}\right) = \ldots, \qquad \frac{d}{dt}\left(\frac{\partial z_0}{\partial q_\alpha}\right) = \ldots,$$

$$\frac{\partial x'_0}{\partial q'_{k+i}} = \frac{\partial x_0}{\partial q_{k+i}}, \qquad \frac{\partial y'_0}{\partial q'_{k+i}} = \frac{\partial y_0}{\partial q_{k+i}}, \qquad \frac{\partial z'_0}{\partial q'_{k+i}} = \frac{\partial z_0}{\partial q_{k+i}} \qquad (i = 1, 2, \ldots, p),$$

$$\frac{\partial q_{k+i}}{\partial q'_{k+i}} = a_{i\alpha} \qquad (i = 1, 2, \ldots, p),$$

and the function P_α will take the following form:

$$\begin{aligned} P_\alpha = {} & \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q'_\alpha} + y'\frac{\partial y'}{\partial q'_\alpha} + z'\frac{\partial z'}{\partial q'_\alpha}\right) - \sum m\left(x'\frac{\partial x'}{\partial q_\alpha} + y'\frac{\partial y'}{\partial q_\alpha} + z'\frac{\partial z'}{\partial q_\alpha}\right) \\ & + \sum m\left[x'\left(\sum_{i=1}^{p}\frac{\partial x'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}\right) + y'\left(\sum_{i=1}^{p}\frac{\partial y'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}\right) + z'\left(\sum_{i=1}^{p}\frac{\partial z'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}\right)\right] \\ & - \sum m\left[x'\frac{d}{dt}\left(\sum_{i=1}^{p}\frac{\partial x'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right) + y'\frac{d}{dt}\left(\sum_{i=1}^{p}\frac{\partial y'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right) + z'\frac{d}{dt}\left(\sum_{i=1}^{p}\frac{\partial z'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right)\right]. \end{aligned}$$

In that expression, the last sum transforms into:

$$\frac{d}{dt}\sum m\left[x'\left(\sum_{i=1}^{p}\frac{\partial x'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right) + y'\left(\sum_{i=1}^{p}\frac{\partial y'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right) + z'\left(\sum_{i=1}^{p}\frac{\partial z'_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right)\right]$$

$$-\sum m\left[x''\left(\sum_{i=1}^{p}\frac{\partial x_0''}{\partial q_{k+i}''}\frac{\partial q_{k+i}''}{\partial q_\alpha''}\right)+y''\left(\sum_{i=1}^{p}\frac{\partial y_0''}{\partial q_{k+i}''}\frac{\partial q_{k+i}''}{\partial q_\alpha''}\right)+z''\left(\sum_{i=1}^{p}\frac{\partial z_0''}{\partial q_{k+i}''}\frac{\partial q_{k+i}''}{\partial q_\alpha''}\right)\right],$$

since equations (7) and (8) give:

$$\frac{\partial x_0'}{\partial q_{k+i}'}=\frac{\partial x_0''}{\partial q_{k+i}''},\qquad \frac{\partial y_0'}{\partial q_{k+i}'}=\frac{\partial y_0''}{\partial q_{k+i}''},\qquad \frac{\partial z_0'}{\partial q_{k+i}'}=\frac{\partial z_0''}{\partial q_{k+i}''}\quad (i=1,2,\ldots,p),$$

$$\frac{\partial q_{k+i}'}{\partial q_\alpha'}=\frac{\partial q_{k+i}''}{\partial q_\alpha''}\qquad (i=1,2,\ldots,p).$$

Hence, for the equations of motion (6), one will have:

$$(6')\quad\left\{\begin{aligned}&\frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q_\alpha'}+y'\frac{\partial y'}{\partial q_\alpha'}+z'\frac{\partial z'}{\partial q_\alpha'}\right)-\sum m\left(x'\frac{\partial x'}{\partial q_\alpha}+y'\frac{\partial y'}{\partial q_\alpha}+z'\frac{\partial z'}{\partial q_\alpha}\right)\\&\quad+\sum m\left[x'\left(\sum_{i=1}^{p}\frac{\partial x_0'}{\partial q_{k+i}'}\frac{\partial q_{k+i}'}{\partial q_\alpha}\right)+\cdots+\cdots\right]\\&\quad-\frac{d}{dt}\sum m\left[x'\left(\sum_{i=1}^{p}\frac{\partial x_0'}{\partial q_{k+i}'}\frac{\partial q_{k+i}'}{\partial q_\alpha'}\right)+\cdots+\cdots\right]\\&\quad+\sum m\left[x''\left(\sum_{i=1}^{p}\frac{\partial x_0''}{\partial q_{k+i}''}\frac{\partial q_{k+i}''}{\partial q_\alpha''}\right)+\cdots+\cdots\right]=Q_\alpha\qquad(\alpha=1,2,\ldots,k),\end{aligned}\right.$$

in which:

$$(6'')\quad\left\{\begin{aligned}&\frac{d}{dt}\sum m\left(x'\frac{\partial x_0'}{\partial q_\alpha'}+y'\frac{\partial y_0'}{\partial q_\alpha'}+z'\frac{\partial z_0'}{\partial q_\alpha'}\right)-\sum m\left(x'\frac{\partial x_0'}{\partial q_\alpha}+y'\frac{\partial y_0'}{\partial q_\alpha}+z'\frac{\partial z_0'}{\partial q_\alpha}\right)\\&\quad+\sum m\left[x''\left(\sum_{i=1}^{p}\frac{\partial x_0''}{\partial q_{k+i}''}\frac{\partial q_{k+i}''}{\partial q_\alpha''}\right)+\cdots+\cdots\right]=Q_\alpha\qquad(\alpha=1,2,\ldots,k),\end{aligned}\right.$$

because one deduces from equations (7) and (8) that:

$$(9)\quad\left\{\begin{aligned}&\frac{\partial x'}{\partial q_\alpha'}=\frac{\partial x_0'}{\partial q_\alpha'}+\sum_{i=1}^{p}\frac{\partial x_0'}{\partial q_{k+i}'}\frac{\partial q_{k+i}'}{\partial q_\alpha'},\quad\frac{\partial y'}{\partial q_\alpha'}=\ldots,\quad\frac{\partial z'}{\partial q_\alpha'}=\ldots\quad(\alpha=1,2,\ldots,k)\\&\frac{\partial x'}{\partial q_\alpha}=\frac{\partial x_0'}{\partial q_\alpha}+\sum_{i=1}^{p}\frac{\partial x_0'}{\partial q_{k+i}'}\frac{\partial q_{k+i}'}{\partial q_\alpha},\quad\frac{\partial y'}{\partial q_\alpha}=\ldots,\quad\frac{\partial z'}{\partial q_\alpha}=\ldots\quad(\alpha=1,2,\ldots,k).\end{aligned}\right.$$

Let T_0 and S_0 denote one-half the *vis viva* and one-half the energy of acceleration of the system, when they are calculated by taking into account the finite constraints that are imposed upon the system:

$$2\,T_0 = \sum m\,(x_0'^2 + y_0'^2 + z_0'^2)\,, \qquad\qquad 2\,S_0 = \sum m\,(x_0''^2 + y_0''^2 + z_0''^2)\,,$$

in such a way that T_0 will be a function of $t, q_1, \ldots, q_k, \ldots, q_{k+p}, q_1', \ldots, q_k', \ldots, q_{k+p}'$, in which the quantities $q_{k+1}', \ldots, q_{k+p}'$ are independent variables, and S_0 will be a function of $t, q_1, \ldots, q_k, \ldots, q_{k+p}, q_1', \ldots, q_k', \ldots, q_{k+p}', q_1'', \ldots, q_k'', \ldots, q_{k+p}''$, in which the quantities $q_{k+1}'', \ldots, q_{k+p}''$ are independent variables.

On the other hand, let T and S denote the analogous quantities when one also takes the differential constraints that are given by equation (8) into account:

$$2T = \sum m\,(x'^2 + y'^2 + z'^2)\,, \qquad\qquad 2S = \sum m\,(x''^2 + y''^2 + z''^2)\,.$$

Finally, let T_1 denote the function T_0, when it is considered to be a function of only $q_{k+1}', \ldots, q_{k+p}'$, and let S_1 denote the function S_0, when it is considered to be a function of only $q_{k+1}'', \ldots, q_{k+p}''$, while, of course, not forgetting that $q_{k+1}', \ldots, q_{k+p}'$ are determined for T_1, and $q_{k+1}'', \ldots, q_{k+p}''$ are determined for S_1 by equations (8).

Upon remarking that $x', y', z', x'', y'', z''$ are deduced from $x_0', y_0', z_0', x_0'', y_0'', z_0''$, while taking the relations (8) into account, but that by virtue of equations (9), the same thing is not true for $\dfrac{\partial x'}{\partial q_\alpha'}, \dfrac{\partial x'}{\partial q_\alpha}, \ldots$, one can then put the equations of motion (6′) or (6″) for non-holonomic systems into the following form:

$$\frac{d}{dt}\frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} + \frac{\partial T_1}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial T_1}{\partial q_\alpha'} + \frac{\partial S_1}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k), \tag{10}$$

or

$$\frac{d}{dt}\frac{\partial T_0}{\partial q_\alpha'} - \frac{\partial T_0}{\partial q_\alpha} + \frac{\partial S_1}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k). \tag{11}$$

I have already deduced roughly those results in the same manner in the article "Sur les équations du mouvement des systèmes matériels non holonomes," that was printed in the *Journal de Mathématiques pures et appliquées*, 1920.

2. P. Woronetz ([1]) has presented the equations of motion for non-holonomic systems in another form. Upon introducing our notations for the functions and parameters, the equations that he obtained can be written in the following fashion:

$$\frac{d}{dt}\frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} - \sum_{i=1}^{p} a_{i\alpha}\frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p}\left[\frac{\partial T}{\partial q_{k+i}'}\right]\left(\sum_{j=1}^{k} A_{\alpha j}^{(i)} + A_\alpha^{(i)}\right) = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k), \tag{12}$$

([1]) "Über die Bewegung eines starren Körper, der ohne Gleitung auf belibiger Fläche rollt," Math. Ann. **70** (1910).

with

$$q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha} q'_\alpha + a_i \qquad (i = 1, 2, \ldots, p),$$

(13)
$$\begin{cases} A_{\alpha j}^{(i)} = \left(\dfrac{\partial a_{i\alpha}}{\partial q_j} + \displaystyle\sum_{\mu=1}^{p} a_{\mu j} \dfrac{\partial a_{i\alpha}}{\partial q_{k+\mu}} \right) - \left(\dfrac{\partial a_{ij}}{\partial q_\alpha} + \displaystyle\sum_{\mu=1}^{p} a_{\mu\alpha} \dfrac{\partial a_{ij}}{\partial q_{k+\mu}} \right), \\ A_{\alpha}^{(i)} = \left(\dfrac{\partial a_{i\alpha}}{\partial t} + \displaystyle\sum_{\mu=1}^{p} a_{\mu} \dfrac{\partial a_{i\alpha}}{\partial q_{k+\mu}} \right) - \left(\dfrac{\partial a_{i}}{\partial q_\alpha} + \displaystyle\sum_{\mu=1}^{p} a_{\mu\alpha} \dfrac{\partial a_{i}}{\partial q_{k+\mu}} \right) \end{cases}$$

$$(\alpha \text{ and } j = 1, 2, \ldots, k; \;\; i = 1, 2, \ldots, p),$$

$\left[\dfrac{\partial T_0}{\partial q'_{k+1}} \right]$ denotes the value that $\dfrac{\partial T_0}{\partial q'_{k+1}}$ takes when one replaces the variables q'_{k+1} in it by their values in (8).

I now propose to show the manner by which our equations (10) lead to Woronetz's equations (12).

Indeed, the third term on the left-hand side of equation (10) is equal to:

(14)
$$\frac{\partial T_1}{\partial q_\alpha} = \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+1}} \frac{\partial q'_{k+1}}{\partial q_\alpha} = \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+1}} \left(\sum_{j=1}^{k} \frac{\partial a_{ij}}{\partial q_\alpha} q'_j + \frac{\partial a_i}{\partial q_\alpha} \right).$$

For the fourth term, one will get:

(15)
$$-\frac{d}{dt} \frac{\partial T_1}{\partial q'_\alpha} = -\frac{d}{dt} \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+1}} a_{i\alpha} \, .$$

For the fifth, one will have:

$$\frac{\partial S_1}{\partial q''_\alpha} = \sum_{i=1}^{p} \frac{\partial S_0}{\partial q''_{k+1}} a_{i\alpha} \, ,$$

or even, as is easy to verify:

$$\frac{\partial S_0}{\partial q''_{k+i}} = \frac{d}{dt} \frac{\partial T_0}{\partial q'_{k+1}} - \frac{\partial T_0}{\partial q_{k+1}} \qquad (i = 1, 2, \ldots, p),$$

so one will finally get:

$$\frac{\partial S_1}{\partial q''_\alpha} = \frac{d}{dt} \left(\sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+1}} a_{i\alpha} \right) - \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+1}} \frac{da_{i\alpha}}{dt} - \sum_{i=1}^{p} \frac{\partial T_0}{\partial q_{k+1}} a_{i\alpha}$$

for that term. We shall transform the right-hand side of that formula. We have:

$$\frac{da_{i\alpha}}{dt} = \frac{\partial a_{i\alpha}}{\partial t} + \sum_{j=1}^{k} \frac{\partial a_{i\alpha}}{\partial q_j} q'_j + \sum_{\mu=1}^{p} \frac{\partial a_{i\alpha}}{\partial q_{k+\mu}} \left(\sum_{j=1}^{k} a_{\mu j} q'_j + a_\mu \right)$$

$$= \frac{\partial a_{i\alpha}}{\partial t} + \sum_{j=1}^{k} \frac{\partial a_{i\alpha}}{\partial q_j} q'_j + \sum_{j=1}^{k} q'_j \sum_{\mu=1}^{k} a_{\mu j} \frac{\partial a_{i\alpha}}{\partial q_{k+\mu}} + \sum_{\mu=1}^{p} a_\mu \frac{\partial a_{i\alpha}}{\partial q_{k+\mu}};$$

on the other hand, since:

$$\frac{\partial T}{\partial q_{k+\mu}} = \frac{\partial T_0}{\partial q_{k+i}} + \sum_{\mu=1}^{p} \frac{\partial T_0}{\partial q'_{k+\mu}} \frac{\partial q'_{k+\mu}}{\partial q_{k+i}},$$

one will have:

$$\sum_{i=1}^{p} \frac{\partial T_0}{\partial q_{k+i}} a_{i\alpha} = \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p} a_{i\alpha} \sum_{\mu=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \frac{\partial q'_{k+i}}{\partial q_{k+i}}$$

$$= \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \sum_{\mu=1}^{p} a_{\mu\alpha} \frac{\partial q'_{k+i}}{\partial q_{k+i}}$$

$$= \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p} \frac{\partial T}{\partial q'_{k+i}} \sum_{\mu=1}^{p} a_{\mu\alpha} \left(\sum_{j=1}^{k} \frac{\partial a_{ij}}{\partial q'_{k+\mu}} q'_j + \frac{\partial a_i}{\partial q'_{k+\mu}} \right)$$

$$= \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \sum_{j=1}^{p} q'_j \sum_{\mu=1}^{p} a_{\mu\alpha} \frac{\partial a_{ij}}{\partial q'_{k+\mu}} - \sum_{j=1}^{k} \frac{\partial T_0}{\partial q'_{k+i}} \sum_{\mu=1}^{p} a_{\mu\alpha} \frac{\partial a_i}{\partial q_{k+\mu}}.$$

The expression for $\partial S_1 / \partial q''_\alpha$ will then take the form:

(16)
$$\frac{\partial S_1}{\partial q''_\alpha} = \frac{d}{dt} \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} a_{i\alpha} - \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}}$$

$$- \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \left(\frac{\partial a_{i\alpha}}{\partial t} + \sum_{j=1}^{k} \frac{\partial a_{i\alpha}}{\partial q_j} q'_j + \sum_{j=1}^{k} a_\mu \frac{\partial a_{i\alpha}}{\partial q_j} + \sum_{j=1}^{k} q'_j \sum_{\mu=1}^{p} a_{\mu j} \frac{\partial a_{i\alpha}}{\partial q_{k+\mu}} \right)$$

$$+ \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \left(\sum_{j=1}^{k} q'_j \sum_{\mu=1}^{p} a_{\mu\alpha} \frac{\partial a_{ij}}{\partial q_{k+\mu}} + \sum_{j=1}^{k} a_{\mu\alpha} \frac{\partial a_i}{\partial q_{k+\mu}} \right).$$

Upon adding corresponding sides of formulas (14), (15), and (16), one will get:

$$\frac{\partial T}{\partial q_\alpha} - \frac{d}{dt} \frac{\partial T_0}{\partial q'_\alpha} + \frac{\partial S_1}{\partial q''_\alpha} = - \sum_{i=1}^{p} a_{i\alpha} \frac{\partial T}{\partial q_{k+i}} - \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \left(\sum_{j=1}^{k} A^{(i)}_{\alpha j} q'_j + A^{(i)}_\alpha \right).$$

These are Woronetz's equations precisely.

One sees that the complementary term that one must add to the left-hand side of the Lagrange equation in order to then equate it to Q_α in such a fashion as to obtain the equation of motion for non-holonomic systems is presented by Woronetz in a form that is more complicated than ours, and above all, extremely difficult to remember.

We further remark that our equations (10) and (11) can be applied advantageously, above all, to problems in the rolling of solid bodies, because in order to find the function S_1, one does not need to calculate the function S_0 or S entirely, which is very difficult. In the case considered, the function S_1 reduces to just the energy of acceleration of the whole mass, which is assumed to be concentrated at the center of gravity, and its calculation will present no difficulty. The functions T, T_1, T_0 are calculated very simply once one has calculated the function T_0 . ([2])

3. In § 7 of his paper, Woronetz gave a formula for the case of non-holonomic systems that is analogous to Hamilton's integral while stating the following theorem:

"The equations of motion of a non-holonomic systems can be obtained easily in the form (12) in the following manner: Let $q_1, \ldots, q_{k+p}$ denote the coordinates of a material system, let T_0 denote its kinetic energy, and let Q'_s be the generalized force that corresponds to the coordinate q_s . Suppose that the system satisfies the conditions:

$$q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha} q'_\alpha + a_i \qquad (i = 1, 2, \ldots, p).$$

If upon appealing to those equations, one expresses the kinetic energy of the system and the generalized impulses that correspond to the dependent velocities $q'_{k+1}, \ldots, q'_{k+p}$ as functions of time t, the coordinates $q_1, q_2, \ldots, q_{k+p}$, and the independent velocities $q'_1, \ldots, q'_k$:

$$T_0 = T\,(t, q_1, \ldots, q_{k+p}, q'_1, \ldots, q'_k),$$

$$\left[\frac{\partial T_0}{\partial q'_{k+i}}\right] = k_i\,(t, q_1, \ldots, q_{k+p}, q'_1, \ldots, q'_k) \qquad (i = 1, 2, \ldots, p)$$

then one will have the formula:

$$\int_{t_0}^{t_1} [\delta T + \sum_{j=1}^{k+p} Q'_s\,\delta q_s + \sum_{i=1}^{p} k_i\,\delta(q'_{k+1} - \sum_{\alpha=1}^{k} a_{i\alpha}\,q'_\alpha - a_i)]\,dt = 0 \tag{23}$$

for all variations $\delta q_1, \ldots, \delta q_k$ that annul the moments t_0 and t_1 . The variations $\delta q_{k+1}, \ldots, \delta q_{k+p}$ are defined by the equations:

$$\delta q_{k+1} = \sum_{\alpha=1}^{k} a_{i\alpha}\,\delta q'_\alpha \qquad (i = 1, 2, \ldots, p),$$

([2]) See no. 3 of my cited article in the Journal de Mathématique, where I studied the rolling of a hoop on a horizontal plane.

and all of the differences $\delta q'_\alpha - \frac{d}{dt}\delta q_s$ (s = 1, 2, …, k + p) must be equal to zero."

It is easy to see that Woronetz's integral (23) is identical to the integral:

(24) $$\int_{t_0}^{t_1} [\delta(T - T_1 + T_1^0) + \sum_{\alpha=1}^{k} Q_\alpha \, \delta q_\alpha] dt \,,$$

in which T_1^0 represents the function T_0, when it is considered to be a function of the independent variables $q'_{k+1}, \ldots, q'_{k+p}$, and consequently T_1 is nothing but the function T_1^0, when one takes equations (8) into account.

When one equates the integral (24) to zero, one will get our equations (10). Therefore, our integral is simpler than that of Woronetz.

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