"Theorie des projektiven Zusammenhangs *n*-dimensionaler Räume," Math. Ann. **106** (1932), 400-454.

Theory of projective connections on *n*-dimensional spaces.

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§ 1.

Introduction.

1. The relations between the various projective differential geometries of Cartan [1-3], Schouten [1-3], J.M. Thomas [1], T. Y. Thomas [1-6], Veblen [1-7], and Weyl [1-2] have recently been clarified in two works of Schouten and Golab [1]). It has thus been demonstrated that these various geometries can be regarded as special cases of a more general theory of projective connections under a certain assumption (cf. § 14, **49**). Nonetheless, the theory can hardly be regarded as complete; for one thing, it has many unsatisfying aspects from a purely formal standpoint.

First, the asymmetry of the indices for projective quantities is an impediment, i.e., the special role of the index 0 as opposed to the indices 1, …, *n*. Second, it is disturbing that in general the covariant derivative is not associated with a covariant differential. Indeed, the impossibility of defining such a well-defined affine displacement without restrictions has already been proved by Schouten and Golab [2]), but the deeper basis for that fact is still not entirely obvious. Third, it is not entirely clear whether one is necessarily led to the associated admission of densities. Fourth, the frequent appearance of the factor $\Delta^{\frac{1}{n+1}}$ is particularly astounding [3]). One does not completely see whether the exponent must be precisely $\frac{1}{n+1}$, much less whether Δ is a determinant of degree *n* and not (*n*+1). Fifth, up till now there is no theory for the induction of a projective connection in an embedded space.

2. The goal of the present work is to seek to eliminate these "beauty marks" from the theory by way of a different type of representation. The key to making this possible is given by the first of the aforementioned remarks: the special role of the index 0. This

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[†] Translated by D.H. Delphenich.

[1] J. A. Schouten and St. Golab, Über projective Übertrügungen und Ableitungen, I. Math. Zeitscr. **32** (1930), pp. 192-214, II Annali di Mat. (4) **8** (1931), pp. 141-157. (The first of these articles will be denoted by Sch and G.)

[2] Loc. cit. [1]), pp. 207.

[3] Cf., e.g., Sch. and G. § 2; Veblen [4] (u^M); Veblen and J.M. Thomas (θ); T.Y. Thomas [1], $\left(-\frac{1}{n+1}\frac{\partial}{\partial \overline{x}^{\sigma}}\log\Delta\right)$ etc.

very same difficulty has been encountered previously in ordinary projective geometry; there, one succeeded in removing it by the introduction of *homogeneous coordinates.*

To be sure, homogeneous coordinates were also introduced into projective differential geometry by Cartan, but always only in *linear spaces* that were associated with each point of space, never in *the space itself* [4]). (By contrast, Weyl has even recently replaced the homogeneous coordinates in linear spaces with inhomogeneous ones.)

However, it will now be shown that the introduction of *homogeneous ur-variables* casts a thoroughly illuminating light on some of the phenomena that were just mentioned.

For example, one immediately sees that a type of "density" can appear. If v^ν is a point of the associated linear space then the numbers that determine it are generally only defined up to a common factor, i.e., they are homogeneous functions (e.g., of degree $\mathfrak{r}$) of the homogeneous ur-variables x^ν (cf. § 3, 7) If one then replaces the x^ν with ρx^ν then v^ν takes on the factor $\rho^{\mathfrak{r}}$. If one regards the replacement $x^\nu \rightarrow \rho x^\nu$ as a coordinate transformation (which is usually not recommended), and Δ is the determinant of the transformation then one has $\rho = \Delta^{\frac{1}{n+1}}$, and $v^{\mathfrak{r}}$ is therefore a "density" [5]) of weight $\mathfrak{r} \geq \frac{1}{n+1}$. The densities of weight $\frac{1}{n+1}$ are therefore, to a certain extent, quantities of first degree (on this, cf., however, § 14).

Furthermore, the lack of a covariant differential in the general case becomes self-explanatory. There also "exists" no *ordinary* differential: the dx^ν are *not* homogeneous functions of the x^ν (cf. § 3, **9**c). The basis for the lack of a covariant differential is then the same as the basis for the lack of a point difference in ordinary projective geometry.

However, this shows that one can achieve the existence of a covariant differential by a specialization of the displacement (§ 7, **23**) and that this can even come about by means of a *path-preserving change* of the displacement (§ 12, **38**, **40**).

The theory of geodetic lines (§ 10), of path-preserving transformations (§ 12), and of ${}^{m+1}P$ in ${}^{n+1}P$ is then effortlessly carried out. Then one sees that a projective connection is uniquely determined for a given curve by – *inter alia* – the requirement that the quantities Q_μ that are defined in § 6, **17** must vanish (§ 12, **43**). However, the vanishing of the scalar $Q_\mu x^\mu + 1$ is crucial for the existence of the covariant differential (§ 7). This existence is then inseparable from the unique determination of the projective connection, as Schouten and Golab [6]) have already proved.

3. A crucial element of this entire style of representation was the wish that the essential features of ordinary projective geometry should be preserved to the greatest extent possible, and that projective differential geometry should not degenerate into a thinly-veiled affine geometry. The relationships with inhomogeneous coordinates (§ 13,

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[4] Except for *Euclidian* spaces and as *local* coordinates at a single point of a general space. In both cases, one needs to consider only the ordinary projective group L_{n+1}, not, however, our general homogeneous group $\mathfrak{H}_{n+1}$ (§ 2, **6**).

[5] For the analysis of densities, cf. J.A. Schouten and V. Hlavatý, Zur Theorie der allgemeinen linearen Übertragung, Math. Zeitschr. **30** (1929), pp. 414-432. Cf., also footnote ?

[6] Cf. loc. cit.

14) are also established only by means of more reassuring links with the older theory; as usual, they are quite considerable. The geometry of Schouten and Golab is briefly sketched in § 14 on the same basis; we show that it follows from Veblen's theory (and thus all of the older theories), as well as our own.

Furthermore, a standpoint will be principally taken that is as general as possible. Thus, we succeed in establishing not just sufficient, but also necessary conditions for, e.g., the existence of covariant differentials, the map to an infinitesimally close ${}^{n+1}E$, and geodetic lines. An overview of the complete set of conditions that were introduced, along with the most important simplifications in the general formulas, will be given in § 11.

The relationships that are employed draw upon Schouten's Ricci Calculus [7]) in an essential way, in which most of the changes that have, in the course of time, proved to be useful will be incorporated. Some inessential deviations (cf. footnotes ?, ?, ?) are, however, introduced, largely for the purpose of avoiding the vast array of indicial notations, which completely obscure any intuition about the nature of the formulas. As for the form of the representation that is described here, except for the Schouten Ricci calculus, the only older works that had an essential influence on it were those of Cartan and Veblen; indeed, the method of Veblen is closest to the one that is presented here.

4. By the introduction of the homogeneous group $\mathfrak{H}_{n+1}$, which is actually new to the theory, projective differential geometry takes on the character of a *generalization of ordinary projective geometry* to a far greater degree that has been true up till now, in particular, when one does without the existence of a covariant differential and sets $Q_\mu = 0$. Differential geometry may then be immediately discussed in the ${}^{n+1}E$ by means of the Ansatz $\Pi^{\nu}_{\lambda\mu} = 0$; the system of geodetic lines will then be given by the system of straight lines in ${}^{n+1}E$. Therefore, the most important thing is then that one can immediately discuss the differential geometry of an embedded manifold ${}^{m+1}H$, assuming that this is desired. In the event that ${}^{m+1}H$ is given by equations in the x^ν, instead of a parametric representation, the theory is easily altered correspondingly in § 9.

Moreover, the connection with the A_{n+1} [8]) yields the possibility of a generalization of the theory. Just as in ordinary projective geometry, the ${}^{n+1}E$ are easily seen to be the geometry of the lines in an E_{n+1} that includes a fixed point O relative to the subgroup of the affine group of E_{n+1} that leaves this system of lines invariant, and thus one can also regard the ${}^{n+1}P$ as an L_{n+1} in which a system of ∞^n "lines" (i.e., X_1 will be induced under Euclidian translation) through a fixed point O is defined; the group of arbitrary coordinate transformations of the L_{n+1} will be correspondingly replaced by the only group that takes the system of lines to itself and induces a *linear* group (with O as fixed point) on any line. The aforementioned generalization now consists of replacing the "sphere" L_{n+1} with a "ruled surface" L_{n+1}, or even L_{n+m}, i.e., with a space in which a system of ∞^n pairwise distinct lines, E_n, resp., hence, *Euclidian subspaces*, that can be regarded as "points" of new n-dimensional space. The group of L_{m+n} will be correspondingly reduced to those transformations that, first of all, leave the system of ∞^n E_m invariant, and

[7] J.A. Schouten, Der Ricci-Kalkül, Berlin; Julius Springer 1924. Denoted by R.K.

[8] Cf. T.Y. Thomas [4], Sch. and G., § 6, Veblen and Hoffmann, pp. 811, and the works cited there by H. Mandel and J. H. C. Whitehead.

secondly, induce a *linear* transformation in each E_m. A possible generalization of another sort will be briefly indicated in the concluding section.

Finally, we must remark that Veblen's projective differential geometry gives us a quadratic differential form (Veblen [**6**]) quite easily by means of the method followed here. One will probably obtain a new connection and a new glimpse into conformal differential geometry, which I hope to show on a later occasion.

§ 2.

The group.

5. In elementary projective geometry, one ordinarily introduces the (n+1)-ary (= n-dimensional) projective space ${}^{n+1}E$ [9]) in the following way: In an ordinary n-dimensional space E_n, which is given by Cartesian coordinates ξ^i (h, …, l = 1, …, n), one sets:

(1) $$x^\nu = \begin{cases} x^0 & \nu = 0, \\ \xi^i x^0 & \nu = i \end{cases} \qquad (\alpha, \dots, \omega = 0, 1, \dots, n),$$

in which x^0 is an arbitrary variable that is ≥ 0. Any n+1 numbers x^ν that do not all vanish and satisfy the relation (1) will be regarded as the homogeneous coordinates of the point ξ^i. One then completes E_n by means of an "imaginary" hyperplane $x^0 = 0$ and seeks invariants under the group $\mathfrak{L}_{n+1}$ of homogeneous linear coordinate transformations $x^\nu \to x^{\nu'}$ [10]), in which:

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[9] The left-hand upper index indicates the "point value" of the space (cf., P.H. Schoute, Mehrdimensionale Geometrie, Sammlung Schubert, I, pp. 12), i.e., the number of dimensions plus one. We will call a space with a point value of n+1 (n+1)-*ary* (*binary*, *tertiary*, etc.).

[10] In the older theory, coordinate transformations were mostly indicated by changing the kernel symbol ($x^\nu \to y^\nu$) or by attaching a prime to the kernel symbol ($x^\nu \to x^{\nu'}$). However, because we (with J. A. Schouten) will view the quantities as geometrical structures throughout, independently of any coordinate system, it is desirable to always indicate geometrical structure by *the same kernel symbol* and avoid the rather inessential change in the coordinates (numbers that determine them, resp.) defined by changing the *indices*. Each coordinate system will then be associated, firstly, with a series of n+1 *fixed* symbols, and secondly, with a type of notation in which an index of the type in question is regarded as a *variable* that ranges through the associated series of symbols. The fact that the series of symbols must also be different for different coordinate systems thus implies, e.g., that the null component of a vector must be distinct in two different coordinate systems. Consequently, in R.K., the newer method in the later works of J.A. Schouten (since, perhaps, 1928) is still not completely followed (different series of symbols, but only one type of notation and primes on the kernel symbol). In the long run, however, this leads to a much too large number of notations, as in Sch. and G, I:

$$\begin{array}{ll} \iota, \dots, \omega = a_1, \dots, a_n; & \mathrm{I}, \dots, \Omega = A_1, \dots, A_n; \\ h, \dots, m = 1, \dots, n; & H, \dots, M = \dot{1}, \dots, \dot{n}; \\ a, \dots, g = 0, 1, \dots, n; & A, \dots, G = \theta, \dot{1}, \dots, \dot{n}, \end{array}$$

all of which are still in the local E_n. To simplify, we would thus like to indicate *different* coordinate systems *in the same space* by *the same* notation (for the index) and distinguish them from each other by attaching a prime *to the index*, or a point, or underlining *the index*, etc. (e.g., $x^\nu, x^{\nu'}, x^{\nu''}, x^{\dot\nu}, x^{\underline{\nu}}, x^{\underline{\nu}'}$, etc.).

(2) $$x^{\nu'} = A_{\mu}^{\nu'} x^{\mu} \qquad (\alpha', \dots, \omega' = 0', 1', \dots, n')$$

and the $A_{\mu}^{\nu'}$ are any $(n+1)^2$ *constants* with non-vanishing determinant Δ.

One can also proceed in a somewhat different way as follows: One starts with an E_{n+1}, which is given by the Cartesian coordinates x^{ν}, and subjects them to homogeneous linear coordinate transformations of the given group $\mathfrak{L}_{n+1}$. Let E_{n+1}^{*} be the space that results from omitting the origin O from E_{n+1}. Two points $x^{\nu}, \bar{x}^{\nu}$ of E_{n+1}^{*} are collinear with O when and only when:

(3) $$\bar{x}^{\nu} = \rho x^{\nu}, \qquad \rho \geq 0.$$

This relation between x^{ν} and $\bar{x}^{\nu}$ is reflexive, symmetric, and transitive, and is invariant under the group $\mathfrak{L}_{n+1}$; it may then be regarded as an equivalence relation, which we will also refer to as *coincidence*. The sets of coincident points, hence, the lines through O, may be regarded as "points" of a new n-dimensional space that we denote by ${}^{n+1}E$. In order to avoid confusion, we briefly refer to a "point of ${}^{n+1}E$" as a (contravariant) *position*, whereas we reserve the expression (contravariant) *point* for the points of E_{n+1}^{*}. There exist the following relations between the three notions of "point," "position," and "system of n+1 numbers": A system of n+1 numbers that do not all vanish uniquely determines a point, as well as a position, as long as one says which coordinate system they belong to. A given point will thus be represented by different systems of numbers that determine it in different coordinate systems, which relate to each other as in (2); in a single coordinate system, however, different systems of numbers determine different points. However, in a single coordinate system two different systems of numbers can very well determine the same *position*, namely, from (2), when they differ only by the same factor $\rho \geq 0$.

In pure projective geometry, only the concept of *position* has any meaning. Points, along with systems of numbers, serve only to facilitate computation; they may not, however, enter into projective-geometrical *theorems*.

Finally, one may consider *functions* $f(x^{\nu})$ that depend upon only the *ratios* of the x^{ν} up to a factor $\varphi(\rho)$ that depends only upon ρ:

The associated series of symbols is a series of number with *the same* alteration (e.g., $0, 0', 0'', \dot{0}, \underline{0}, \underline{0}'$). We thus use:

$$\left.\begin{array}{l} \alpha, \dots, \omega = 0, 1, \dots, n, \\ \alpha', \dots, \omega' = 0', 1', \dots, n', \\ \underline{\alpha}, \dots, \underline{\omega} = \underline{0}, \underline{1}, \dots, \underline{n}, \text{ etc.} \end{array}\right\} \quad \text{in } {}^{n+1}H,$$

$$\left.\begin{array}{l} h, \dots, l = 1, \dots, n, \\ h', \dots, l' = 1', \dots, n', \text{ etc.} \end{array}\right\} \quad \text{in } X_n,$$

$$\left.\begin{array}{l} a, \dots, g = 1, \dots, \mathfrak{m}, \\ a', \dots, g' = 1', \dots, \mathfrak{m}', \text{ etc.} \end{array}\right\} \quad \text{in an embedded } {}^{n+1}H.$$

(4) $$\overline{f} = f(\rho x^\nu) = \varphi(\rho)\, f(x^\nu).$$

One easily proves that $\varphi(\rho)$ always has the form $\rho^{\mathfrak{r}}$, in which $\mathfrak{r}$ is any *constant* number (the *degree* of $f(x)$): *The functions are homogeneous:*

(5) $$\overline{f} = \rho^{\mathfrak{r}} f.$$

Equation (5) is equivalent to the *Euler homogeneity condition:*

(6) $$x^\mu \partial_\mu f = \mathfrak{r} f, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

or also to:

$$x^\mu \partial_\mu \log f = \mathfrak{r}.$$

If $\mathfrak{r} = 0$, hence, $f = \overline{f}$ is homogeneous of degree zero, then we call f a *function of position*. If f is homogeneous of $\mathfrak{r}^{\text{th}}$ degree then the partial derivatives $\partial_\mu f$ are homogeneous of $(\mathfrak{r} - 1)^{\text{th}}$ degree:

(7) $$\overline{\partial_\mu f} = \frac{\partial f(\rho x^\nu)}{\partial \rho x^\mu} = \rho^{\mathfrak{r}-1} \partial_\mu f,$$

(8) $$x^\lambda \partial_{\lambda\mu} f = (\mathfrak{r} - 1)\, \partial_\mu f; \qquad \partial_{\lambda\mu} = \partial_\lambda . \partial_\mu .$$

6. We would also like to introduce homogeneous coordinates when the n-dimensional space is subjected to the general group $\mathfrak{G}_n$ of all uniquely continuously invertible and sufficiently many times continuously differentiable transformations (in this case, the space will be denoted by X_n). If X_n is given by the ur-variables ξ^i then we further take an arbitrary variable $x_0 \geq 0$ [11]) and introduce the x^ν precisely as before by way of (1). Now, if the ξ^i are subjected to an arbitrary transformation $\xi^i \to \xi^{i'}$ from $\mathfrak{G}_n$, then the ratios of the new homogeneous coordinates $x^{\nu'}$ are functions of the ratios of the x^ν, i.e., the $x^{\nu'}$ are themselves homogeneous functions of null degree of the x^ν, up to a common factor λ. We further assume that the function λ is *homogeneous of first degree* [12]) [13]) in the x^ν (cf., however, § 15). It follows that the $x^{\nu'}$ are also homogeneous of first degree, and, moreover, that they are unique and sufficiently many times differentiable in the neighborhood in question, whereas the functional determinant itself is nowhere vanishing.

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[11] This x^0 corresponds to the e^{x^0} of Veblen [**5**]. [**6**], [**7**].

[12] Such a thing is completely different from a *linear* function. The latter has the form $a_\lambda x^\lambda$, in which the a_λ are *constants*; the former can also be given this form (but not uniquely); however, the a_λ are then *arbitrary* functions of the ratios of the x^ν.

[13] The condition that the degree shall equal one can also be omitted, provided that it is ? 0 (otherwise, the $x^{\nu'}$ would be dependent). Cf., also footnotes ?) and ?), along with § 15.

The situation becomes clearer when we choose the second path. The total system of $n+1$ numbers x^ν define an $(n+1)$-dimensional space X_{n+1}. If it is subjected to the group of all uniquely invertible and continuous and sufficiently many times differentiable transformations then neither the notion of "line through the origin" nor the notion of origin itself is invariant. We thus reduce from the group $\mathfrak{G}_n$ to the subgroup $\mathfrak{H}_{n+1}$, which consists of all transformations $x^\nu \to x^{\nu'}$ in $\mathfrak{G}_{n+1}$ for which the $x^{\nu'}$ are *homogeneous of first degree* [14]) in the x^ν:

$$\bar{x}^{\nu'} = x^{\nu'}(\rho x^\mu) = \rho\, x^{\nu'} = \rho x^{\nu'}(x^\mu)\,. \tag{9}$$

We denote an X_{n+1} with this sort of reduced transformation group by H_{n+1}. The (now invariant) lines through the (now invariant) origin of this H_{n+1} can, moreover, be regarded as "points" of a new space that we denote by ${}^{n+1}H$. We once again call them (contravariant) *positions*, and once again reserve the term (contravariant) *position* for the points of H^*_{n+1}, i.e., H_{n+1} without the origin. In order to give a position, one must give, first, one coordinate system, and second, a system of $n+1$ numbers that do not all vanish.

The group $\mathfrak{H}_{n+1}$ includes a subgroup $\mathfrak{H}^*$ that consists of the coordinate transformations in $\mathfrak{H}_{n+1}$ for which $x^{0'} = x^0$, and which is (einstufig) isomorphic with the group $\mathfrak{G}_n$ of transformations of the ξ^i; *the group $\mathfrak{H}_{n+1}$ is then no less "enveloping" than the group $\mathfrak{G}_n$.* The fact that H_{n+1} can be completed by an "imaginary" manifold $x^0 = 0$ is inessential since one must usually restrict oneself to a neighborhood (that is chosen to be arbitrary small) in differential geometry, in such a way that the imaginary manifold can be ignored.

One must observe that the map $x^\nu \to \bar{x}^\nu$, although it is a change in the numbers that determine *the same* point, *is not to be viewed as a coordinate transformation.* Namely, from the degree condition (9), this is possible when and only when *the proportionality factor ρ is a function of position, hence, homogeneous of null degree in the x^ν.* The invariance requirement, upon which the definition of projectors (§ 3) rests, thus relates *only* to the group $\mathfrak{H}_{n+1}$, not, however, to the change of factor (3) when ρ is not homogeneous of null degree.

We now call a differential geometry in ${}_{n+1}H$ *purely projective* when its theorems involve only the notion of position, but not the notion of point. The differential geometry that is introduced and developed in the following sections is not purely projective, but projective in a broader sense, since we are indeed working with homogeneous functions exclusively, but the *degree* of these functions will also be considered, which introduces a non-projective element.

[14] Cf., the previous footnote.

§ 3.

The quantities.

7. The group $\mathfrak{H}_{n+1}$ induces a *linear* group $\mathfrak{L}_{n+1} = \mathfrak{L}_{n+1}(x^\nu)$ at every point x^ν. It is composed of matrices with constant coefficients, which come about when one substitutes the coordinates for the point in question into the functions:

$$A_\mu^{\nu'} = \partial_\mu x^{\nu'}, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{10}$$

The inverse matrix to (10) is:

$$A_{\mu'}^{\nu} = \partial_{\mu'} x^{\nu}, \qquad \partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}}. \tag{11}$$

For the sake of later use, we remark that the $A_\mu^{\nu'}$ (and likewise the $A_{\mu'}^{\nu}$) are homogeneous of null degree, hence, they are pure functions of position:

$$\bar{A}_\mu^{\nu'} = A_\mu^{\nu'}(\rho x^\lambda) = A_\mu^{\nu'}. \tag{12}$$

Due to the Euler homogeneity condition this is equivalent to:

$$x^\rho \partial_\rho A_\mu^{\nu'} = 0. \tag{13}$$

We now associate each point x^ν with an E_{n+1}, in such a way that each coordinate transformation of H_{n+1} from $\mathfrak{H}_{n+1}$ induces a coordinate transformation of E_{n+1} from L_{n+1}, namely, the one that is given by the associated matrix (10). In order to give a point of E_{n+1} one must first give a coordinate system (e.g., x^ν) in H_{n+1}, and secondly a system of n+1 numbers. The coordinate system in H_{n+1} is, in fact, uniquely associated with a coordinate system in E_{n+1}; the given numbers are the coordinates of the points of E_{n+1} relative to this associated coordinate system. If $v^{\nu'}$ are the coordinates of the same point relative to the coordinate system in E_{n+1} that is associated with the coordinate system $x^{\nu'}$ then the $v^{\nu'}$ must go over to the v^ν by means of a transformation of $\mathfrak{L}_{n+1}$:

$$v^{\nu'} = A_\mu^{\nu'} v^\mu; \qquad v^\nu = A_{\mu'}^{\nu} v^{\mu'}. \tag{14}$$

A *point field* in H_{n+1} is composed of a system of n+1 (single-valued, continuous, sufficiently many times differentiable) functions of the coordinates that one of the coordinate systems in H_{n+1} is associated with, in such a way that the functions that are associated with two different coordinate systems go to each other by means of associated coordinate transformations of $\mathfrak{L}_{n+1}$ according to (14). Geometrically, a point field simply means that each point of H_{n+1} is associated with a point of the associated ("local") E_{n+1}.

Now, if $x^\nu, \bar{x}^\nu$ are any two coincident points of H_{n+1} then we identify the two associated E_{n+1} with each other in such a way that we can identify each point of one E_{n+1} with that point of the other E_{n+1} that has the same coordinates. This condition is satisfied for *any* coordinate system whenever it is satisfied for *one* coordinate system, since the $A_\mu^{\nu'}$ have the same values at x^ν and $\bar{x}^\nu$, due to (12). Thus, the local ${}^{n+1}E$ (which we construct out of the E_{n+1} as in § 2, 5) to which the coincident points of H_{n+1} belong will be equivalent and can be identified with each other.

In general, a point field associates a one-dimensional family of points with each *position* in ${}^{n+1}H$, hence, a curve in the local E_{n+1}. In the event that this curve is a straight line through the origin of E_{n+1}, or also contracts to this origin, we call the point field a *position field*. Such a construction thus associates each *position* in ${}^{n+1}H$ with a unique *position* in the associated local ${}^{n+1}E$ or with the origin.

Thus, in order for a point field to be a position field it is necessary and sufficient that the $\bar{v}^\nu \equiv v^\nu(\rho\, x^\nu)$ be proportional to the $v^\nu \equiv v^\nu\,(x^\mu)$. We would like to further assume something rather far-reaching, that (i.e., we restrict ourselves to such positions fields for which) *the proportionality factor is a power of* ρ whose exponent $\mathfrak{r}$ (the *degree* [15]) of v^ν) is constant over the ${}^{n+1}H$, i.e., that the system of numbers are all *homogeneous functions of the same* ($\mathfrak{r}^{\text{th}}$) *degree* in the ur-variables:

$$\bar{v}^\nu = \rho^{\mathfrak{r}}\, v^\nu . \tag{15}$$ [16])

[15] Our notion of degree has nothing to do with the notion of the "degree of an affinor" (R.K., pp. 23), i.e., it is actually the degree (number of indices) of the associated *form*, which is often also called the degree. It appears here for the first in the ordinary *elementary* sense (the degree of the system of numbers when considered to be *functions of the ur-variables*) of differential geometry, in which, up till now, one does not ordinarily bother with the type of functional dependency of the quantities upon the ur-variables (except for differentiability, resp., analyticity, requirements).

[16] One can also regard the association $x^\nu \to \rho x^\nu$ as a coordinate transformation (which is not possible for a general ρ in our theory; cf. § 2, 6, conclusion) and introduce densities of index (weight) $\dfrac{\mathfrak{r}}{n+1}$ instead of quantities of $\mathfrak{r}^{\text{th}}$ degree. Such a transition must then transform under the *unimodular* group (Δ = Det $(v_\lambda x^\nu)$ =) according to (14), although like a "point density" according to:

$$\mathfrak{V}^{\nu'} = \Delta^{\frac{\mathfrak{r}}{n+1}} A_\mu^{\nu'} \mathfrak{V}^\mu . \tag{14a}$$

However, it seems to me that the notion of volume (and thus, the notion of density) possesses a typically non-projective (affine) character to a far greater degree than the notion of degree, in such a way that statement of the theory that is given in the text has merit. One will first arrive at a purely projective theory (cf. § 2, 6, conclusion) when one combines the Ansatz (14a) with the footnote ?) that was mentioned, and defines a position by way of:

$$v^{\nu'} = \mathfrak{r}\, A_\mu^{\nu'}\, v^\mu \tag{14b}$$

with a *completely arbitrary* factor r; the geometry that is thus defined will show a certain agreement with the theory of pseudo-quantities of Schouten and Hlavatý (cf. ?). On this, cf. § 15.

Two position fields $u^{\mathfrak{r}}$, $v^{\mathfrak{r}}$ are *coincident* when and only when they differ by a scalar factor (see below):

$$u^{\nu} = p\, v^{\nu}.$$

In order for this to be true, it is necessary and sufficient that:

$$u^{[\lambda}\, v^{\mu]} = 0.$$

8. Analogously, we define a *covariant position field* w_{μ} of degree $\mathfrak{r}$ by means of:

(16) $$w_{\mu'} = A^{\nu}_{\mu'} w_{\nu}, \qquad w_{\mu} = A^{\nu'}_{\mu} w_{\nu'},$$

(17) $$\bar{w}_{\mu} = \rho^{\mathfrak{r}}\, w_{\mu}.$$

Geometrically, w_{μ} describes a hyperplane (${}^{n}E$) in the local ${}^{n+1}E$. It includes the contravariant position v^{ν} when and only when:

(18) $$v^{\rho}\, w_{\rho} = 0.$$

We will define general quantities that we would like to call *projectors*, corresponding to the affinors of affine geometry [17]), as a system of $(n+1)^{r+s}$ homogeneous functions of $\mathfrak{r}^{\text{th}}$ degree that transform like a product of r covariant and s contravariant points:

(19) $$X_{\lambda'_1 \ldots \lambda'_r}^{\bullet \cdots \bullet\, \nu'_1 \ldots \nu'_s} = A^{\rho_1 \ldots \rho_r\, \nu'_1 \ldots \nu'_s}_{\lambda'_1 \ldots \lambda'_r\, \sigma_1 \ldots \sigma_s} X_{\rho_1 \ldots \rho_r}^{\bullet \cdots \bullet\, \sigma_1 \ldots \sigma_s}$$ [18])

(20) $$\bar{X}_{\lambda_1 \ldots \lambda_r}^{\bullet \cdots \bullet\, \nu_1 \ldots \nu_s} = \rho^{\mathfrak{r}}\, X_{\lambda_1 \ldots \lambda_r}^{\bullet \cdots \bullet\, \nu_1 \ldots \nu_s}.$$

Geometrically, a projector describes some algebraic relationship in the local ${}^{n+1}E$.

As usual, one can also refer a quantity to two or more different coordinate systems. Thus, e.g., in $X_{\lambda'\mu'}^{\cdot\;\cdot\;\nu}$, both of the covariant indices are referred to the system of $x^{\nu'}$ and the contravariant index, to the system of x^{ν}:

(21) $$X_{\lambda'\mu'}^{\cdot\;\cdot\;\nu} = A^{\rho\sigma}_{\lambda'\mu'} X_{\rho\sigma}^{\cdot\;\cdot\;\nu} = A^{\nu}_{\rho'} X_{\lambda'\mu'}^{\cdot\;\cdot\;\nu}.$$

Finally, a *scalar* is a homogeneous function of $\mathfrak{r}^{\text{th}}$ degree that assumes values that are independent of coordinate system at *each* point. A scalar is a *position* function when and

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[17] We use the expression "affine geometry," as opposed to projective (differential) geometry, for the *general* linear displacement IIIA α in the classification scheme of R.K., pp. 75, hence, without regard for the symmetry condition.

[18] For the sake of simplicity, we shall *not* recall the notation for unit-projectors and differential symbols:

$$A^{\nu\sigma}_{\lambda\rho} = A^{\nu}_{\lambda} A^{\sigma}_{\rho}, \qquad B^{\nu\sigma}_{\lambda\rho} = B^{\nu}_{\lambda} B^{\sigma}_{\rho}, \qquad \partial_{\lambda\mu} = \partial_{\lambda}\, \partial_{\mu} = \frac{\partial^2}{\partial x^{\lambda} \partial x^{\mu}}, \qquad \nabla_{\lambda\mu} = \nabla_{\lambda}\, \nabla_{\mu}.$$

only when its degree is zero. Two quantities that differ only by a scalar factor have the same meaning geometrically. A scalar of $\mathfrak{r}^{\text{th}}$ degree may always be brought into the form $p^{\mathfrak{r}}q$ (e.g., $(x^0)^{\mathfrak{r}}\, q$), in which p is an arbitrarily chosen, but fixed, non-vanishing scalar of first degree and q is a scalar of null degree. This was done by Veblen (cf., footnote ?) and ?)).

9. Examples.

a) The *contact position* x^ν. By means of the Euler homogeneity condition:

(22) $$x^\rho A_\rho^{\nu'} = x^\rho \partial_\rho x^{\nu'} = x^{\nu'} ,$$

which is equivalent to (9), $v^\nu = x^\nu$ satisfies equation (14), i.e., the n+1 numbers determine a point of the first degree in E_{n+1} that is associated with the point x^ν in H_{n+1}; we can then think of the point x^ν in H_{n+1} as being identified with the point x^ν in the associated E_{n+1}, just like the associated position in ${}^{n+1}H$ and ${}^{n+1}E$. In affine geometry (cf., last footnote) the x^ν transform nonlinearly; there, the contact position corresponds to the null point (null-vector) in E_{n+1}.

b) The unit projector A_λ^ν. If both of its indices refer to the *same* coordinate system then its values are equal to 1 (0, resp.) whenever both of the indices are equal (different, resp.). However, if one refers the unit projector to two *different* coordinate systems – e.g., $x^\nu, x^{\nu'}$ – then it represents the functional matrix $\partial_\lambda x^{\nu'}$ ($\partial_{\lambda'} x^\nu$, resp.), which we have denoted by the same kernel symbol A all along (cf., footnote ?)). The degree of A_λ^ν is equal to zero (cf. (13)). Geometrically, a quantity of the type $X_\lambda^{\cdot\,\nu}$ always means a single-valued (but not necessarily one-to-one) projective map (collineation) of the local ${}^{n+1}E$ onto itself: an arbitrary position v^ν in ${}^{n+1}E$ will be mapped to the position $X_\lambda^{\cdot\,\nu}\, v^\nu$; A_λ^ν is the *identity* map.

c) The *differentials* of the x^ν do not determine *any* position field. Indeed, if they transform according to (14)

(23) $$dx^{\nu'} = A_\mu^{\nu'}\, dx^\mu ;$$

however, they are *not homogeneous:*

(24) $$\overline{dx}^\nu = d\bar{x}^\nu = \rho\,(dx^\nu + x^\nu\, d\log\rho) .$$

They *do* define a point field (as long as each point in H_{n+1} is uniquely associated with a line element dx^ν, hence, a point in the local E_{n+1}), but the point of E_{n+1} that is associated with the coincident points of the position x^ν *does not* define a line through the origin, but a completely undetermined curve in the plane in E_{n+1} that is spanned by origin and both of the points x^ν and dx^ν. The basis for the non-existence of a covariant differential that was mentioned in the introduction resides in equation (24).

d) On the other hand, any *infinitesimally neighboring position* $y^\nu = x^\nu + dx^\nu$ in ${}^{n+1}H$ uniquely determines, up to quantities of second order, a position in ${}^{n+1}E$ that has the same coordinates relative to any coordinate system in ${}^{n+1}E$, up to quantities of second order, as the position y^ν in ${}^{n+1}H$ relative to the associated coordinate system in ${}^{n+1}H$. One then has (up to quantities of second order!):

(25) $$y^{\nu'} = x^{\nu'} + dx^{\nu'} = A_\mu^{\nu'} x^\mu + A_\mu^{\nu'} dx^\mu = A_\mu^{\nu'} y^\mu$$

and:

(26) $$\bar{y}^\nu = \bar{x}^\nu + d\bar{x}^\nu = \rho x^\nu + \rho\, dx^\nu + x^\nu\, d\rho = (\rho + d\rho)\, y^\nu.$$

The position y^ν in ${}^{n+1}H$ can therefore be identified with the position y^ν in the local ${}^{n+1}H$ (that is associated with the position x^ν in ${}^{n+1}H$), i.e., an infinitesimal neighborhood of the contact point x^ν in ${}^{n+1}H$ will be uniquely embedded in the local ${}^{n+1}E$, up to quantities of second order. Since y^ν is then embedded in the ${}^{n+1}E$ belongs to x^ν, along with the ${}^{n+1}E$ that belongs to y^ν, there thus exists a link (of first order) between the various local ${}^{n+1}E$. We thus point out that the differentials dx^ν are *not* determined by the being given the *positions* x^ν and y^ν; this is the case only when each *point* x^ν (y^ν, resp.) is distinguished. Nonetheless, the position y^ν *is* determined by being given the *position* x^ν and the differentials dx^ν; on the contrary, *any* position on the line through x^ν and y^ν may be obtained by a particular choice of the point that represents x^ν.

e) On the other hand, this *connecting line* is indeed determined, and therefore the "*bi-position*" $J^{\lambda\mu} = x^{[\lambda}y^{\mu]} = x^{[\lambda}dx^{\mu]}$:

(27) $$J^{\lambda'\mu'} = A_{\rho\sigma}^{\lambda'\mu'} J^{\rho\sigma},$$

(28) $$\bar{J}^{\lambda\mu} = \rho\,(\rho + d\rho)\, J^{\lambda\mu}.$$

Therefore, there are *no line elements* in projective differential geometry, but only *directions*, i.e., lines through the contact point and an infinitesimally neighboring point in the local ${}^{n+1}E$.

f) A (contravariant) (n+1)-*point* $J^{\nu_0\nu_1\ldots\nu_n}$ is a projector whose values are $+J$, $-J$, 0, whenever the indices ν_0, ν_1, …, ν_n represent an even (odd, resp.) permutation of the numbers 0, 1, …, n (contain two equal numbers, resp.). Therefore, J is any function ("density") that takes on the factor Δ under a coordinate transformation. One obtains an (n+1)-point as an alternating product of any n+1 linearly independent points. If J is a homogeneous function then the (n+1)-point also determines an (n+1)-*position*. If J is (in a certain coordinate system) equal to one then (for this coordinate system) $J^{\nu_0\nu_1\ldots\nu_n}$ is the *unit* (n+1)-*point* (-*position*, resp.) [19]). If one were to take J to be equal to one in *any* coordinate system then $J^{\nu_0\nu_1\dots\nu_n}$ would no longer be a projector, but a *density*; however, the latter (which we will not usually use) is not uniquely determined, without further assumptions.

g) A *covariant* (n+1)-*point* (-*position*, resp.) is defined analogously. One can choose $\pm\frac{1}{J}$, 0 for its values, in which J has the meaning described in f). If the co- and contravariant (n+1)-points are associated with each other in that way then the computations in §§ 3, 10, and 13 yield the frequently-used relations:

$$J^{\nu_0\nu_1\dots\nu_r\sigma_{r+1}\dots\sigma_n} J_{\lambda_0\lambda_1\dots\lambda_r\sigma_{r+1}\dots\sigma_n} = (n-r)!\,(r+1)!\,A^{[\nu_0\dots\nu_r]}_{[\lambda_0\dots\lambda_r]}, \tag{29}$$

and, in particular:

$$J^{\nu_0\dots\nu_n} J_{\lambda_0\dots\lambda_n} = (n-r)!\,A^{[\nu_0\dots\nu_r]}_{[\lambda_0\dots\lambda_r]}, \tag{30}$$

$$J^{\nu\rho_1\dots\rho_n} J_{\lambda\rho_1\dots\rho_n} = n!\,A^{\nu}_{\lambda}, \tag{31}$$

$$J^{\rho_0\rho_1\dots\rho_n} J_{\rho_0\rho_1\dots\rho_n} = (n-1)! \tag{32}$$

§ 4.

The ${}^{m+1}H$ in ${}^{n+1}H$.

10. Let an H_{m+1} ($m < n$) be given in H_{n+1} in such a way that the x^ν are given as *homogeneous functions of the first degree* [20]) of the m+1 homogeneous parameters x^a (a, …, g = 0, 1, …, m); let these parameters be subject to the transformation of the group $\mathfrak{H}_{m+1}$ that corresponds to the group $\mathfrak{H}_{n+1}$ that was introduced above. At the same time, one is then given a ${}^{m+1}H$ in ${}^{n+1}H$. Furthermore, let the local spaces E_{m+1} and ${}^{m+1}E$ be introduced, as above. As usual, there exist the quantities:

$$B^\nu_a = \partial_a x^\nu, \qquad \partial_a = \frac{\partial}{\partial x^a}, \tag{33}$$

which associate each (contravariant) point (position, resp.) v^a in the local ${}^{m+1}E$ with a unique point (position, resp.):

$$v^\nu = B^\nu_a v^a \tag{34}$$

in the local ${}^{n+1}E$, which we can think of being identified with v^a, and for this reason we shall denote them with the kernel symbol v. Thus, the local ${}^{m+1}E$ also appears to be a

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[19] In R.K., pp. 42, the non-vanishing values of the unit n-vector were chosen to be equal to $\frac{1}{n!}$, instead of = 1. We have changed the factor, since it is more customary to choose the unit volume of a *parallelotope* to be a *simplex* with side = 1. However, formulas (29) to (32) will then become a little less simple.

[20] The condition that the degree = 1 can also be omitted here. Cf., footnote ?).

manifold that is *embedded in* ${}^{n+1}E$. In particular, by means of the Euler homogeneity condition (6), one has:

(35) $$x^a B_a^\nu = x^a \partial_a x^\nu = x^\nu ,$$

such that, in particular, the point x^a in ${}^{m+1}E$ will be identified with the point x^ν in ${}^{n+1}E$. Thus, there are four points (positions, resp.) x that are identified with each other and lie in ${}^{n+1}H$, ${}^{m+1}H$, ${}^{n+1}E$, and ${}^{m+1}E$, respectively. Correspondingly, we have immediately denoted the parameter in ${}^{m+1}H$ by x^a (instead of, e.g., u^a).

Each hyperplane (hence, each ${}^nE$) w_λ in the local ${}^{n+1}E$ is associated with a hyperplane (hence, an ${}^mE$):

(36) $$w'_a = B_a^\lambda w_\lambda$$

in the local ${}^{m+1}E$. Geometrically, w'_a is the intersection of w_λ with ${}^{m+1}E$. Objects on ${}^{m+1}E$ that are induced from objects in ${}^{n+1}E$ (or conversely) by intersection or projection, which might not be uniquely determined, will be denoted by attaching a prime to the kernel symbol.

For the sake of later use, we remark that B_a^ν is homogeneous of null degree (in the x^a):

(37) $$x^b \partial_b B_a^\nu = x^b \partial_{ab} x^\nu = 0.$$

11. If one introduces a quantity $J'^{c_0\cdots c_m}$ in ${}^{m+1}H$ that is analogous to the one in **9.** f) then the contact ${}^{m+1}E$ in ${}^{m+1}H$ will also be represented by the $(m+1)$-point:

(38) $$J'^{\nu_0\cdots\nu_m} = B_{c_0\cdots c_m}^{\nu_0\cdots\nu_m} J'^{c_0\cdots c_m} = (m+1)!\ B_{[0\ldots m]}^{\nu_0\cdots\nu_m} ,$$

or by the covariant $(n-m)$-point:

(39) $$t_{\lambda_{m+1}\cdots\lambda_n} = \frac{1}{(m+1)!} J_{\lambda_0\cdots\lambda_n} J'^{\lambda_0\cdots\lambda_m} .$$

Thus, one has:

(40) $$J'^{[\nu_0\cdots\nu_m} B_c^{\rho]} = 0, \qquad t_{\lambda_{m+1}\cdots\lambda_n} B_a^{\lambda_p} = 0 \qquad (m+1 \leq p \leq n).$$

Geometrically, (40) represents the condition for the points of ${}^{m+1}E$ to be incident with the ${}^{m+1}E$ that is represented in ${}^nE$-coordinates. In general, one can represent the tangential ${}^{m+1}E$ in mixed coordinates by way of:

(41) $$t_{\lambda_r\cdots\lambda_n}^{\cdots\cdots\ c_r\cdots c_m} = \frac{1}{r!} J_{\lambda_0\cdots\lambda_n} B_{c_0\cdots c_{r-1}}^{\lambda_0\cdots\lambda_{r-1}} J'^{c_0\cdots c_m} \qquad (0 \leq r \leq m+1).$$

12. The case $m = 0$ is trivial. Therefore, since the x^ν must be *homogeneous functions of first degree* in the single coordinate x^0, the equations for ${}^1H$ ultimately read like:

(42) $$x^\nu = a^\nu x^0,$$

in which the a^ν are *constants;* ^1H is therefore a single position.

For $m = 1$, ^2H is a "binary" manifold, hence, a curve. Correspondingly, the tangent through the bi-point will be represented by:

$$J'^{\lambda\mu} = B^{\lambda\mu}_{ab} J'^{ab} = 2B^{\lambda\mu}_{[01]} . \tag{43}$$

For $m = n - 1$ the tangential hyperplanes ^nE or hypersurfaces ^nH will be given, analogously to (39), (41), by:

$$t_\lambda = \frac{1}{n!} J_{\nu_0\ldots\nu_{n-1}\lambda} B^{\nu_0\ldots\nu_{n-1}}_{a_0\ldots a_{n-1}} J'^{a_0\ldots a_{n-1}} = = (n+1)\ B^{[0\ldots n-1}_{[0\ldots n-1]} A^{n]}_\lambda , \tag{44}$$

or by:

$$t_{\lambda\mu}^{\cdot\ \cdot\ c} = \frac{1}{(n-1)!} J_{\nu_0\ldots\nu_{n-2}\lambda\mu} B^{\nu_0\ldots\nu_{n-2}}_{a_0\ldots a_{n-2}} J'^{a_0\ldots a_{n-2}c} = = n(n+1)\ B^{[0\ldots n-2}_{[0\ldots n-2]} A^{n-1 n]}_{[\lambda\ \ \mu]} B^c_{n-1} , \tag{45}$$

etc. The incidence condition (40) reads like:

$$t_\lambda B^\lambda_a = 0, \tag{46}$$

or:

$$t_{\lambda\mu}^{\cdot\ \cdot\ c} B^{\lambda\mu}_{ab} = 0, \tag{47}$$

etc.

13. ^{m+1}H is *taut* if an ^{n-m}E is given at every position in the local ^{n+1}E that has no points in common with the tangential ^{m+1}E. Let this ^{n-m}E be represented by the $(n - m)$-point $n^{\nu_1\cdots\nu_{n-m}}$; let it be normalized such that:

$$t_{\lambda_1\cdots\lambda_{n-m}} n^{\nu_1\cdots\nu_{n-m}} = (n-m)!\ ^{21}). \tag{48}$$

If one then sets:

$$B^c_\lambda = \frac{1}{(n-m)!} t_{\lambda\mu_1\cdots\mu_{n-m}}^{\cdot\ \cdot\ \ \ \ \cdot\ c} n^{\mu_1\cdots\mu_{n-m}} = \tag{49}$$

$$= \frac{1}{m!(n-m)!} J_{\lambda_0\cdots\lambda_n} J'^{c_0\cdots c_{m-1}c} B^{\lambda_0\cdots\lambda_{m-1}}_{c_0\cdots c_{m-1}} A^{\lambda_m}_\lambda n^{\lambda_{m+1}\cdots\lambda_n} =$$

$$= (m+1)(n-m)\ B^{|01\cdots m-1[c|}_{[01\cdots m\text{-}1m]} A^m_\lambda n^{m+1\cdots n]} ,$$

then, due to (30), (31), (39), (48), one has:

[21] Cf., the "first normalization condition" (193) R.K., pp. 157; consistent with footnote ?), the factor in R.K is chosen differently.

(50) $$B_{a\lambda}^{\lambda c} = B_a^{\lambda} B_{\lambda}^c = B_a^c = \partial_a x^c = \begin{cases} 1, & c = a, \\ 0, & c \neq a. \end{cases}$$

As in affine geometry, one can also associate any contravariant position v^ν in ${}^{n+1}E$ with its projection onto ${}^{n-m}E$”:

(51) $$v'^c = B_\lambda^c v^\nu,$$

and any covariant position (i.e., ${}^mE$) w_a in ${}^{m+1}E$ can be associated with its “contact hyperplane (i.e., ${}^nE$) with ${}^{n-m}E$”:

(52) $$w_\lambda = B_\lambda^a w_a,$$

which is not uniquely possible without tautness.

§ 5.

Projective connections.

14. A projective connection in ${}^{n+1}H$ can be given in four ways:

A. By defining a covariant derivative.

B. By defining a covariant differential.

C. By defining a map from any ${}^{n+1}E$ to the infinitesimally close ones (“displacement”).

D. By defining a system of curves (geodetic lines; “paths”).

The four corresponding types of definition of an *affine* connection are equivalent to each other and will all be governed by a system of n^3 functions Γ_{ij}^k with well-known transformation laws. We will see that here in *projective* geometry, as well, the four types of definition will be governed by a corresponding system of functions, together with a covariant point, but they are *by no means* equivalent to each other: Namely, the corresponding functions $\Pi_{\lambda\mu}^\nu$ *cannot* be chosen arbitrarily, but they must satisfy certain conditions that are different for *A*, *B*, and *D*.

15. We would now like introduce a corresponding system of $(n + 1)^3$ functions $\Pi_{\lambda\mu}^\nu$ in ${}^{n+1}H$, with a corresponding law of transformation:

(53) $$\Pi_{\lambda'\mu'}^{\nu'} = A_{\lambda'\mu'\nu}^{\lambda\mu\nu'} \Pi_{\lambda\mu}^\nu + A_\rho^{\nu'} \partial_{\lambda'} A_\mu^\rho ,$$

without concern for which method of definition of a projective connection we have chosen.

We would further like to assume that the $\Pi_{\lambda\mu}^\nu$ are functions of position, hence, homogeneous functions, e.g., of $\mathfrak{k}^{\text{th}}$ degree.

16. As in affine geometry there exist:
the *torsion quantities:*

(54) $$S_{\lambda\mu}^{\cdot\,\cdot\,\nu} = \Pi^{\nu}_{[\lambda\mu]}$$

and the *curvature quantities:*

(55) $$N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = -2\partial_{[\omega}\Pi^{\nu}_{|\lambda|\mu]} - 2\Pi^{\nu}_{\rho|\omega}\Pi^{\rho}_{|\lambda|\mu]},$$

which are homogeneous of $\mathfrak{k}^{\text{th}}$ (($\mathfrak{k}$ + 1)$^{\text{th}}$, resp.) degree.

In contrast to affine geometry, however, there exist two quantities [22]) of null order (and ($\mathfrak{k}$ + 1)$^{\text{th}}$ degree):

(56) $$P_{\lambda}^{\cdot\,\nu} = \Pi^{\nu}_{\lambda\mu}x^{\mu},$$

(57) $$Q^{\nu}_{\cdot\,\mu} = \Pi^{\nu}_{\lambda\mu}x^{\lambda} = P_{\mu}^{\cdot\,\nu} + 2S_{\lambda\mu}^{\cdot\,\cdot\,\nu}x^{\lambda},$$

as writing out the transformation formulas, by the use of (13), yields immediately.

§ 6.

Covariant derivatives.

17. As usual, we would like to establish the covariant derivative of a quantity by the following three conditions:

A. The covariant derivative of a quantity is itself a quantity (in particular, it is therefore homogeneous).

B. The difference between the covariant derivative and the ordinary (partial) derivative of a quantity is a homogeneous linear function of the values of the quantity.

C. The covariant derivative of products and contractions of arbitrary quantities satisfies the Leibniz rule for differentiation.

From the second condition, it follows that the most general form for a covariant derivative of a scalar of $\mathfrak{r}^{\text{th}}$ degree must be:

(58) $$\nabla_{\mu}\, q = \partial_{\mu}\, q + \overset{\mathfrak{r}}{Q}_{\mu}\, q.$$

Thus, $\overset{\mathfrak{r}}{Q}_{\mu}$ can certainly be of degree $\mathfrak{r}$, but it can no longer be dependent upon q itself. If one defines $\nabla_{\mu}\, q^{\mathfrak{r}}$ according to (58) then condition C yields: $\nabla_{\mu}\, q^{\mathfrak{r}} = \mathfrak{r}q^{\mathfrak{r}-1}\nabla_{\mu}\, q$; hence, when q is a scalar of null degree: $\overset{0}{Q}_{\mu} = 0$. Furthermore, if the degree $\mathfrak{r}$ of q is arbitrary and p is a scalar of first degree then C, when applied to $\nabla_{\mu}\, q\, p^{-\mathfrak{r}}$, and together with $\overset{0}{Q}_{\mu} =$ 0, yields:

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[22] These two quantities roughly correspond to the A^{c}_{a0} (A^{c}_{0a}, resp.) in Sch. and G., etc. Cf., § 14.

$$\overset{\mathfrak{r}}{Q}_\mu = \mathfrak{r}\, Q_\mu ,$$

in which we have written Q_μ, in stead of $\overset{1}{Q}_\mu$.

Hence, (58) takes the form:

(59) $$\boxed{\nabla_\mu = \partial_\mu q + \mathfrak{r}\, Q_\mu q} .$$

Since $\partial_\mu\, q$ is homogeneous of $(\mathsf{r} - 1)^{\text{th}}$ degree and $\nabla_\mu\, q$ must be *homogeneous*, Q_μ must be *homogeneous of degree* –1. Since $\nabla_\mu\, q$, like $\partial_\mu\, q$, transforms like a covariant point, Q_μ must also be a *covariant point*.

18. Condition *B* then gives the most general form for the covariant derivative of a point of degree $\mathfrak{r}$:

(60) $$\nabla_\mu\, v^\nu = \partial_\mu\, v^\nu + \overset{\mathfrak{r}}{\Pi}{}^{\nu}_{\lambda\mu} v^\lambda .$$

If one substitutes $v^\nu = p^{\mathsf{r}}\, u^\nu$ in this, in which p is an arbitrary scalar of first degree, then u^ν is a point of null degree, and one finds:

(61) $$\overset{\mathsf{r}}{\Pi}{}^{\nu}_{\lambda\mu} = \Pi^{\nu}_{\lambda\mu} + \mathsf{r}\, A^{\nu}_{\lambda}\, Q_\mu ,$$

when we write $\Pi^{\nu}_{\lambda\mu}$, instead of $\overset{0}{\Pi}{}^{\nu}_{\lambda\mu}$. If one substitutes (61) in (60) then this equation takes the form:

(62) $$\boxed{\nabla_\mu v^\mu = \partial_\mu v^\mu + \Pi^{\nu}_{\lambda\mu} v^\lambda + \mathfrak{r}\, Q_\mu v^\nu.} .$$

The condition that $\nabla_\mu\, v^\nu$ must be a *quantity* yields the well-known transformation laws (53) for the $\Pi^{\nu}_{\lambda\mu}$. Since the $\partial_\mu\, v^\nu$ are homogeneous of $(\mathfrak{r} - 1)^{\text{th}}$ degree, the $\Pi^{\nu}_{\lambda\mu}$ (like the Q_μ) are homogeneous of $(-1)^{\text{th}}$ degree:

I [23]) $$\boxed{x^\rho \partial_\rho\, \Pi^{\nu}_{\lambda\mu} = -\,\Pi^{\nu}_{\lambda\mu} ; \qquad x^\lambda \partial_\lambda\, Q_\mu = -\, Q_\mu}$$

The quantities $P^{\nu}_{\cdot\lambda}$ and $Q^{\cdot\nu}_{\lambda}$ that were introduced in § 5, **16** thus have degree 0. The torsion quantities and curvature quantities have degrees –1 and –2, resp.

Thus, we have:

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[23]) The numbering by Roman numerals relates to § 11.

Theorem 1. *In order for a system of functions* $\Pi^{\nu}_{\lambda\mu}$ *with the transformation law* (53), *together with a covariant point* Q_μ, *to define a covariant derivative, it is necessary and sufficient that the* $\Pi^{\nu}_{\lambda\mu}$ *and the* Q_μ *are homogeneous of degree* -1 [24]).

19. For a covariant point w_λ of $\mathfrak{r}^{\text{th}}$ degree, application of the Leibniz rule to $\nabla_\mu w_\rho v^\rho$ yields:

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda - \overset{-\mathfrak{r}}{\Pi}{}^{\nu}_{\lambda\mu} w_\nu , \tag{63}$$

$$\boxed{\nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Pi^{\nu}_{\lambda\mu} w_\nu + \mathfrak{r}\, Q_\mu w_\lambda} \tag{64}$$

For a general quantity of degree $\mathfrak{r}$ one finds, e.g.:

$$\boxed{\begin{aligned}\nabla_\mu X^{\cdots\cdot \nu_1\cdots\nu_s}_{\lambda_1\cdots\lambda_r} &= \partial_\mu X^{\cdots\cdot \nu_1\cdots\nu_s}_{\lambda_1\cdots\lambda_r} + \sum_{i=1}^{s} \Pi^{\nu_i}_{\rho\mu} X^{\cdots\cdot \nu_1\cdots\nu_{i-1}\rho\nu_{i+1}\cdots\nu_i}_{\lambda_1\cdots\lambda_r} \\ &\quad - \sum_{i=1}^{r} \Pi^{\rho}_{\lambda_i\mu} X^{\cdots\cdots\cdots\cdot \nu_1\cdots\nu_s}_{\lambda_1\cdots\lambda_{i-1}\rho\lambda_{i+1}\cdots\lambda_r} + \mathfrak{r}\, Q_\mu X^{\cdots\cdot \nu_1\cdots\nu_s}_{\lambda_1\cdots\lambda_r} .\end{aligned}} \tag{65}$$

We will denote a space in which a covariant derivative is defined by means of (59), (62), (64), (65) *and satisfies condition I by* ${}^{n+1}P$.

We should point out that Q_μ itself transforms like a covariant point, although this is *not* the case for $\Pi^{\rho}_{\rho\mu}$. Unlike in the analysis of densities one therefore does not set $Q_\mu = \Pi^{\rho}_{\rho\mu}$ [25]). On the other hand, the equation:

$$\boxed{Q_\mu = 0} \tag{IV}$$

is *completely* invariant, i.e., one can define a covariant derivative for which the $\overset{\mathfrak{r}}{\Pi}{}^{\nu}_{\lambda\mu}$ are *independent* of $\mathfrak{r}$. E.g., the operator:

$$\overset{0}{\nabla}_\mu = \nabla_\mu - \mathfrak{r}\, Q_\mu , \tag{66}$$

which agrees with the operator ∇_μ in affine geometry, because of (59), (62), (64), (65), is such a differential operator that is independent of $\mathfrak{r}$. However, we would like to *not* make

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[24]) Whereas, for us, the degree of a quantity is reduced by one under covariant differentiation, for Veblen the "weight" of a quantity is invariant under covariant derivative. On this, cf., § 14, **49**.

[25]) Except when one restricts oneself to unimodular (or at least constant modular) transformations and, as in footnote ?), introduces "densities," which is what was done in most of the older presentations.

the assumption IV, since we will see in § 7 that it excludes the existence of a covariant differential [26]).

20. Covariant differentiation of (59) and alternation yields:

$$\nabla_{[\omega\mu]}\, q = T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho q + \mathfrak{r}\, U_{\omega\mu}\, q, \tag{67}$$

in which we have set:

$$T_{\omega\mu}^{\cdot\cdot\,\nu} = S_{\omega\mu}^{\cdot\cdot\,\nu} + A_{[\omega}^{\nu} Q_{\mu]}\,, \tag{68}$$

$$U_{\omega\mu} = \nabla_{[\omega}\, Q_{\mu]} - T_{\omega\mu}^{\cdot\cdot\,\rho} Q_\rho\,. \tag{69}$$

Covariant differentiation of (62), (64), and alternation yields:

$$\begin{aligned}\nabla_{[\omega\mu]}\, v^\nu &= -\frac{1}{2}\overset{\mathfrak{r}}{N}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} v^\lambda + T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho v^\nu = \\ &= -\frac{1}{2} N_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} v^\lambda + T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho v^\nu + \mathfrak{r}\, U_{\omega\mu} v^\nu\end{aligned} \tag{70}$$

$$\begin{aligned}\nabla_{[\omega\mu]}\, w_\lambda &= +\frac{1}{2}\overset{-\mathfrak{r}}{N}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} w_\nu + T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho w_\lambda = \\ &= +\frac{1}{2} N_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} w_\nu + T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho w_\lambda + \mathfrak{r}\, U_{\omega\mu} w_\lambda\,.\end{aligned} \tag{71}$$

Formulas (67), (70), (71) differ from the corresponding formulas of affine geometry [27]) solely by the facts that, firstly, $S_{\omega\mu}^{\cdot\cdot\,\nu}$ is replaced by $T_{\omega\mu}^{\cdot\cdot\,\nu}$, and secondly, that the term in $U_{\omega\nu}$ appears, which has degree $\mathfrak{r}$ and is proportional to differentiated quantities, as the following relation, which is a result of (66), yields:

$$\overset{0}{\nabla}_{[\omega\mu]} - S_{\omega\mu}^{\cdot\cdot\,\rho}\overset{0}{\nabla}_\rho = =_{[\omega\mu]} - T_{\omega\mu}^{\cdot\cdot\,\rho}\nabla_\rho - \mathfrak{r}\, U_{\omega\mu}\,. \tag{72}$$

21. In addition to the *first identity* [28]), which is trivial, the curvature quantities satisfy the *second identity* [29]):

$$\begin{aligned}N_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} &= 2\nabla_{[\omega} S_{\mu\lambda]}^{\cdot\cdot\,\nu} + 4 S_{[\omega\mu}^{\cdot\cdot\,\rho} S_{\omega]\rho}^{\cdot\cdot\,\nu} + 2 Q_{[\omega} S_{\mu\lambda]}^{\cdot\cdot\,\nu} = \\ &= 2\nabla_{[\omega} T_{\mu\lambda]}^{\cdot\cdot\,\nu} + 4 T_{[\omega\mu}^{\cdot\cdot\,\rho} T_{\omega]\rho}^{\cdot\cdot\,\nu} + 2 U_{[\omega\mu} A_{\lambda]}^{\nu}\,,\end{aligned} \tag{73}$$

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[26]) For this reason, assumption IV, in its essential features, may be said to belong to a true *projective* differential operator. Without it, in any position-like ${}^{n+1}E$ a covariant point Q_μ, hence, an ${}^{n}E$, would be distinguished, which one can regard as at an "infinite distance" in ${}^{n+1}E$; geometry would then take on a certain affine character again in the small.

[27]) Cf., R.K., pp. 85.

[28]) Cf., R.K., pp. 87.

[29]) Cf., R.K., pp. 88.

which takes on the conventional form $N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = 0$ for the symmetric case, as well as for the quasi-symmetric case (cf., below **22**) when $U_{\omega\nu} = 0$, moreover. For the *Bianchi identity* [30]), one finds:

(74) $$\nabla_{[\kappa} N_{\omega\mu]\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = -2T_{\kappa\omega}^{\cdot\,\cdot\,\rho} N_{\mu]\rho\lambda}^{\cdot\,\cdot\,\cdot\,\nu} .$$

Furthermore, there exist the following two identities, which do not occur in affine geometry:

(75) $$N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} x^{\lambda} = -2\nabla_{[\omega} P_{\mu]}^{\cdot\,\nu} + 4P_{\rho}^{\cdot\,\nu} T_{\omega\mu}^{\cdot\,\cdot\,\rho} + 2x^{\lambda} \nabla_{[\omega} S_{\mu]\lambda}^{\cdot\,\cdot\,\nu}$$

(76) $$N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} x^{\omega} = -\nabla_{\mu} P_{\lambda}^{\cdot\,\nu} .$$

Finally, we note the following relations:

(77) $$\nabla_{\lambda} x^{\nu} = A_{\lambda}^{\nu} + Q_{\cdot\lambda}^{\nu} + Q_{\kappa} x^{\nu} = QA_{\lambda}^{\nu} + P_{\lambda}^{\cdot\,\nu} - 2T_{\lambda\mu}^{\cdot\,\cdot\,\nu} x^{\mu} ,$$

(78) $$x^{\lambda} \nabla_{\lambda} v^{\nu} = + P_{\lambda}^{\cdot\,\nu} v^{\lambda} + \mathfrak{r}\, Q\, v^{\nu} ,$$

(79) $$x^{\lambda} \nabla_{\lambda} w_{\mu} = - P_{\lambda}^{\cdot\,\rho} w_{\rho} + \mathfrak{r}\, Q\, w_{\mu} ,$$

in which we have set:

(80) $$Q = 1 + x^{\rho} Q_{\rho} .$$

Equation (77) follows from (62) by means of the identity $\partial_{\lambda} x^{\nu} = A_{\lambda}^{\nu}$ by applying (57) and (68); (78) and (79) follow from (62) ((64), resp.) by means of (56) and the Euler homogeneity condition.

22. We call the projective connection *quasi-symmetric* when we have:

Vβ $$\boxed{T_{\omega\mu}^{\cdot\,\cdot\,\nu} = 0.}$$

The condition of quasi-symmetry says that the "curvature" $\nabla_{[\omega\mu]}\, q$, i.e., the rotation of the gradient of a scalar q of null degree, vanishes. Quasi-symmetry expresses an essential property of projective connections by the symmetry condition:

Vα $$\boxed{S_{\omega\mu}^{\cdot\,\cdot\,\nu} = 0.}$$

The additional term $A_{[\omega}^{\nu} Q_{\mu]}$ in (68) originates in the fact that the gradient $\nabla_{\mu}\, q$ of a scalar of null degree is not of null degree, but has degree –1.

Theorem 2. *The condition:*

VI $$\boxed{U_{\mu\nu} = 0}$$

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[30]) Cf., R.K., pp. 91.

is necessary and sufficient for Q_μ *to be a gradient* [31]).

In fact, due to (67), (69) the integrability conditions for the equation:

$$\text{IV}\alpha \qquad \boxed{Q_\mu = \nabla_\mu \log g}$$

now read:

$$(\mathfrak{r} - 1)\, U_{\omega\mu} = 0\,,$$

if $\mathfrak{r}$ is the degree of g. However, if $\mathfrak{r} \neq 1$ then, due to (59), IVα would take the form: $\partial_\mu \log g = 0$, hence, g = constant, which is not possible for a scalar of first degree, except for the trivial case (which is usually excluded) in which g vanishes [32]). From IVa, and contraction with x^μ, by means of (59), (80) it follows that:

$$Q = \frac{1}{1-\mathfrak{r}} = \text{const.} \tag{81}$$

and:

$$Q_\mu = Q\, \partial_\mu \log g = \partial_\mu \log g^Q. \tag{82}$$

§ 7.

The covariant differential.

23. In general, there exists no covariant differential. If one sets:

$$\delta v^\nu = dx^\mu\, \nabla_\mu\, v^\nu \tag{83}$$

then the δv^ν indeed transform like the values of a covariant point, but they are not homogeneous: one has (cf., (24), (78)):

$$\overline{\delta v^\nu} = \rho^{\mathfrak{r}}\, \{\delta v^\nu + (P_\lambda^{\cdot\nu} v^\lambda + \mathfrak{r}\, Q\, v^\nu)\, d \log \rho\}. \tag{84}$$

A covariant differential obviously exists when and only when the coefficient of $d \log \rho$ vanishes, hence, for all of the points of a *given* degree $\mathfrak{r}$, when $P_\lambda^{\cdot\nu} v^\lambda + \mathfrak{r}\, Q\, v^\nu = 0$; hence, one has:

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[31]) The theory of Sch. and G. belongs to the special case $U_{\omega\mu} = 0$ (cf., § 14, **49**). Since the older presentation may be regarded as a special case of the one in Sch. and G., and, moreover, the newer theory of Veblen that was mentioned in § 14, **47** is the same special case, in either event, we have $U_{\omega\mu} = 0$.

[32]) If g is homogeneous of $\mathfrak{r}^{\text{th}}$ degree and $\mathfrak{r} \neq 0$ then $\log g$ is not homogeneous, and $\nabla_\mu \log g$ does not actually exist. However, we use the expression as an abbreviation for $\frac{1}{g}\nabla_\mu g$, and also speak of "gradients," accordingly.

IIa $$\boxed{P_{\kappa}^{\cdot\nu} = PA_{\kappa}^{\nu}}$$

and:

(85) $$\frac{P}{Q} = \text{const.} = -\mathfrak{r}\,.$$

Theorem 3. *There exists a covariant differential for quantities of each degree when and only when both of the following conditions:*

IIb $$\boxed{P_{\kappa}^{\cdot\nu} = 0}$$

IV*b* $$\boxed{Q = 0}$$

are satisfied.

Under the assumption IIa, (77), (78) take the form:

(86) $$\nabla_{\lambda} x^{\nu} = (P + Q) A_{\lambda}^{\nu} - 2T_{\lambda\mu}^{\cdot\cdot\nu} x^{\mu}\,,$$

(87) $$x^{\lambda} \nabla_{\lambda} v^{\nu} = (P + \mathfrak{r}\, Q)\, v^{\nu}\,,$$

and under the assumptions IIb, IVβ, they take the form:

(88) $$\nabla_{\lambda} x^{\nu} = -2T_{\lambda\mu}^{\cdot\cdot\nu} x^{\mu}\,,$$

(89) $$x^{\lambda} \nabla_{\lambda} v^{\nu} = 0\,.$$

These relations will be used in the calculations often.

§ 8.

Position displacements.

24. A displacement is essentially a means of equating quantities that exist at one position x^ν in a space with the quantities that exist at an infinitesimally close position $x^\nu + dx^\nu$ [33]) Assuming that one has a means of equating ("simultaneously measuring") quantities that are defined to have the same type at the *same* position, which is the case in projective differential geometry for quantities of the same type *and degree*, one must then define what it means for two quantities of the same type to be called "equal" at two neighboring points; i.e., the quantities at x^ν shall be mapped to the quantities at y^ν. However, in order for this to be the case it suffices to map the two ${}^{n+1}E$ to each other in a one-to-one way [34]).

In fact, such a thing also exists in affine geometry, and indeed there is an *affine map* between neighboring E_n; it is given by the requirement of "*covariant constancy*," i.e., the vanishing of the covariant differential. For example, the vector v^ν is considered to be "equal" to the vector:

$$\tilde{v}^\nu = v^\nu + dv^\nu = v^\nu - v^\lambda \Gamma^\nu_{\lambda\mu} \, dx^\mu \tag{90}$$

at y^ν.

We now like to define a displacement in projective differential geometry, as well, and indeed, by means of a *projective* map between neighboring ${}^{n+1}E$. Furthermore, we would *not* like to derive such a thing by means of the covariant differential that was introduced in § 7 by means of the requirement of covariant constancy, but independently of that, so we will consider the covariant derivative of the most general map, and, from that, derive the conditions for the $\Gamma^\nu_{\lambda\mu}$. These two paths recommend themselves since they allow us to proceed in a purely geometric way (up to degree considerations), and because the starting point is general, since we will only assume a map of the ${}^{n+1}E$, but not the E_{n+1}. A displacement for arbitrary projectors will then follow from the displacement of the position.

25. Therefore, let there be given a *projective map* of the ${}^{n+1}E$ at a position x^ν to the ${}^{n+1}E$ at a neighboring point $y^\nu = x^\nu + dx^\nu$. It shall satisfy the following conditions:

P.1. As $dx^\nu \to 0$, it goes to the identity continuously. By that, we shall mean the following: If v^ν is a position in the ${}^{n+1}E$ at x^ν and $\tilde{v}^\nu$ is its image in the ${}^{n+1}E$ at y^ν then the *ratios* of the coordinates of $\tilde{v}^\nu$ shall converge continuously to the ratios of the coordinates of v^ν whenever y^ν converges to x^ν. Thus, the coordinates in both of the ${}^{n+1}E$ must be considered relative to two linear coordinate systems, which are both associated with the *same* coordinate system in the ${}^{n+1}H$.

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[33]) Cf., D. van Dantzig, Die Wiederholung des Michelson-Versuchs and Relativitätstheorie, Math. Annalen **96** (1926), pp. 261-283, in particular, § 7, Metrik and Physik.

[34]) Cf., J.A. Schouten [**3**], E. Cartan [**2**].

P.2. Coincident positions will be mapped to coincident positions. Therefore, if $u^{[\lambda} v^{\mu]} = 0$ then one also has $\tilde{u}^{[\lambda}\tilde{v}^{\mu]} = 0$.

P.3. The degree of a position is preserved by the map.

This map then yields a map of the *scalars* at x^ν to the *scalars* at y^ν. Namely, if u^ν, v^ν are two coincident points at x^ν, so $u^\nu = pv^\nu$, then, because of *P*.2, one can define $\tilde{p}$ by $\tilde{u}^\nu = \tilde{p}\tilde{v}^\nu$. We would like to make the following assumptions about this scalar map:

S.1. As $dx^\nu \to 0$, it goes to the identity continuously.

S.2. Sums and products of two scalars go to sums and products of scalars (this follows from *P*.2).

S.3. The degree of a scalar is preserved by the map (this follows from *P*.3).

Therefore, we must consider that for scalars of *null* degree a scalar field associates a *number* with each position of ${}^{n+1}H$; a scalar field of higher degree maps the points at a position onto the number continuum in a well-defined way. Only for scalars of *null* degree can we then infer, from the well-known theorem that the field of real numbers admits no continuous automorphism other than the identity, that $\tilde{p} = p$. If we also allow complex numbers for function values then we must have $\tilde{p} = p$ for scalars of null degree, since the only non-identical automorphism that the field of complex numbers admits (namely, the one that takes any number to its conjugate) is not continuously reachable from the identity.

From *S*.2, one then proceeds in the well-known way to infer that for scalars of the same degree, $\dfrac{\tilde{p}}{p}$ must be independent of p, i.e., that $\tilde{p}$ must have the form:

$$\tilde{p} = \overset{\mathfrak{r}}{\Theta}\, p\,. \tag{91}$$

Therefore, $\overset{\mathfrak{r}}{\Theta}\, p$ is a function in the $2(n + 1)$ arguments x^ν, y^ν, or briefly, a *two-point function.* Furthermore, it follows from *S*.2 in a well-known way that:

$$\overset{\mathfrak{r}}{\Theta} = \Theta^{\mathfrak{r}}, \tag{92}$$

in which we have written Θ, instead of $\overset{1}{\Theta}$; (92) is valid at least for rational degrees, so we shall restrict ourselves to such numbers. If we develop Θ as a power series in dx^ν with coefficients that depend only upon x^ν, and no longer on y^ν, and we truncate the development at the terms of second order, then this shows that Θ must have the form:

$$\Theta = 1 - Q_\mu\, dx^\mu\,, \tag{93}$$

since *S*.1 implies that the first term must equal one. The truncation of the development is justified *since we call two displacements "equal" when their effects differ only by second* *order quantities*. All of the following equations are therefore valid only up to second order quantities, and the maps are uniquely defined only up to second order quantities.

Now, if p is any scalar of $\mathfrak{r}^{\text{th}}$ degree then, on account of $S.2$:

$$\tilde{p} = \Theta^{\mathfrak{r}}\, p = (1 - Q_\mu\, dx^\mu)^{\mathfrak{r}}\, p = (1 - \mathfrak{r}\, Q_\mu\, dx^\mu)\, p \tag{94}$$

is likewise a scalar of $\mathfrak{r}^{\text{th}}$ degree, but at y^ν. However, since y^ν goes to $(\rho + d\rho)\, y^\nu$ as $x^\nu \to \rho\, x^\nu$ (cf., § 2, **9**d), $\tilde{p}$ includes the factor $(\rho + d\rho)^{\mathfrak{r}} = \rho^{\mathfrak{r}}\,(1 + \rho\, d \log \rho)$. Hence, one has:

$$\begin{aligned} \bar{\Theta}^{\mathfrak{r}} &= \Theta^{\mathfrak{r}}\,(1 + \mathfrak{r}\, d \log \rho) = \\ &= (1 - \mathfrak{r}\, Q_\mu\, dx^\mu)\,(1 + \mathfrak{r}\, d \log \rho) = \\ &= 1 - \mathfrak{r}\, Q_\mu\, dx^\mu + \mathfrak{r}\, d \log \rho\,. \end{aligned} \tag{95}$$

Equating this with:

$$\bar{\Theta}^{\mathfrak{r}} = 1 - \mathfrak{r}\, \rho\, \bar{Q}_\mu\,(dx^\mu + x^\mu\, d \log \rho)$$

yields $\bar{Q}_\mu = \rho^{-1} Q_\mu$, hence, the homogeneity conditions I for Q_μ (cf., § 6, **18**), as well as the condition IVβ (cf., § 7, **23**).

26. We now go on to the subject of projective maps of positions of $\mathfrak{r}^{\text{th}}$ degree. Such a thing is well-known to be given by:

$$\tilde{v}^\nu = \overset{\mathfrak{r}}{\Theta}{}^\nu_\lambda\, v^\lambda\,, \tag{96}$$

in which the $\overset{\mathfrak{r}}{\Theta}{}^\nu_\lambda$ are two-point functions. Since $\tilde{v}^\nu$ is a position at y^ν, its values transform by means of the values of the $A^{\nu'}_\mu$ at y^ν, i.e., by means of $A^{\nu'}_\mu + dA^{\nu'}_\mu$. From this, the transformation formulas for the $\tilde{v}^\nu$ become:

$$\tilde{v}^{\nu'} = (A^{\nu'}_\mu + dA^{\nu'}_\mu)\,\tilde{v}^\mu\,,$$

and for the $\overset{\mathfrak{r}}{\Theta}{}^\nu_\lambda$ they become:

$$\overset{\mathfrak{r}}{\Theta}{}^{\nu'}_{\lambda'} = (A^\mu_\rho + dA^{\nu'}_\rho)\,\overset{\mathfrak{r}}{\Theta}{}^\rho_\sigma A^\sigma_{\lambda'}\,. \tag{97}$$

If $\tilde{v}^\mu$ is an arbitrary position of $\mathfrak{r}^{\text{th}}$ degree and p is an arbitrary scalar of $\mathfrak{r}^{\text{th}}$ degree then the equation $v^\nu = p\, u^\nu$ defines u^ν to be a position of null degree that is coincident with v^ν. On account of $\tilde{v}^\nu = \tilde{p}\tilde{u}^\nu$ and (94), one then has:

$$\overset{\mathfrak{r}}{\Theta}{}^\nu_\lambda = \Theta^{\mathfrak{r}}\; \Theta^\nu_\lambda\,, \tag{98}$$

in which we have again omitted the upper index 0 in $\overset{0}{\Theta}{}^{\nu}_{\lambda}$. We again develop the Θ^{ν}_{λ} in dx^{ν} with coefficients that depend only upon x^{ν}, and truncate the development at terms of second order. Geometrically, the terms that are independent of dx^{ν} must represent the identity map; hence, *up to a factor T*, they must equal the unit projector A^{ν}_{μ}. For the terms that are linear in dx^{ν}, we write $-T\ \Pi^{\nu}_{\lambda\mu}\ dx^{\mu}$; hence, we have:

$$\Theta^{\nu}_{\lambda} = T\,(A^{\nu}_{\mu} - \Pi^{\nu}_{\lambda\mu}\ dx^{\mu})\,. \tag{99}$$

Equating the infinitesimal terms in (97) yields the well-known transformation laws (53) of the $\Pi^{\nu}_{\lambda\mu}$. If u^{ν} is any position of null degree then, from P. 3, this must also be true for $\tilde{u}^{\nu}$, and it follows that the Θ^{ν}_{λ} are also homogeneous of null degree:

$$\bar{\Theta}^{\nu}_{\lambda} = \Theta^{\nu}_{\lambda}\,. \tag{100}$$

Substitution in (99) yields:

$$\bar{T}\,(A^{\mu}_{\lambda} - \rho\bar{\Pi}^{\nu}_{\lambda\mu}dx^{\mu} - \bar{\Pi}^{\nu}_{\lambda\mu}x^{\mu}d\rho) = T\,(A^{\mu}_{\lambda} - \Pi^{\nu}_{\lambda\mu}dx^{\mu})\,. \tag{101}$$

Equating the finite terms yields:

$$\bar{T} = T, \tag{102}$$

i.e., T is a scalar of null degree. Equating the infinitesimal terms, which do not depend upon $d\rho$, yields the homogeneity condition I, and the remaining terms yield condition IIb.

Substituting (98), (92), (93), (99) in (96) yields, for a position of $\mathfrak{r}^{\text{th}}$ degree:

$$\tilde{v}^{\nu} = T\,(1 - Q_{\mu}\,dx^{\mu})^{\mathfrak{r}}\,(v^{\nu} - \Pi^{\nu}_{\lambda\mu}\,v^{\lambda}\,dx^{\mu}), \tag{103}$$

which we can also write as:

$$\tilde{v}^{\nu} = T\,(v^{\nu} - \Pi^{\nu}_{\lambda\mu}\,v^{\lambda}\,dx^{\mu} - \mathfrak{r}\,Q_{\mu}\,dx^{\mu}). \tag{104}$$

The image point $\tilde{v}^{\nu}$ is then identical with the one that is obtained by covariant constancy, up to a scalar factor T.

If the position displacement in an ${}^{n+1}E$ likewise determines a point-displacement in the E_{n+1} then obviously one must have $T = 1$. The displacement remains projective-geometric in the event that $\tilde{v}^{\nu}$ is changed by an arbitrary scalar factor of null degree. If one decomposes it into a factor that is finite and dependent only on x^{ν} and a factor that is infinitesimal and deviates from 1 then the first factor causes a change in T, whereas the second can likewise cause a change in Q_{μ} in the form of an increase of $\Pi^{\nu}_{\lambda\mu}$ by a product of A_λ^ν with a covariant point. Only the sum of these last two changes is determined by the change in $\tilde{v}^\nu$. Thus, we have:

Theorem 4. *Conditions* I, IIb, *and* IVβ *for the existence of the covariant differential are also necessary and sufficient for the existence of a position displacement. The* $\Pi_{\lambda\mu}^\nu$ *are determined by way of the geometric map of the neighboring* ${}^{n+1}E$, *up to a multiplicity of* A_λ^ν, *whereas the* Q_μ *remain completely undetermined* (naturally while preserving the condition).

27. Covariant constancy of the contact point means that x^ν is mapped to y^ν, i.e., that $\tilde{x}^{[\lambda} y^{\mu]} = 0$. From (96), (104), we now have:

$$\tilde{x}^\nu = T\,\{x^\nu(1 - Q_\mu\, dx^\mu) - Q^\nu_{\cdot\,\mu} dx^\mu\}. \tag{105}$$

Therefore, up to a factor T and quantities of second order, $\tilde{x}^{[\lambda} y^{\mu]}$ is equal to:

$$x^{[\lambda}\, dx^{\mu]} - Q^{\lambda}_{\cdot\,\rho} y^{[\mu}\, dx^{\rho]} = x^{[\lambda}\,(Q^{\mu]}_{\cdot\,\rho} + A^{\mu]}_{\rho})\, dx^\rho.$$

Thus, the contact point is covariantly constant under a displacement in an *arbitrary* direction when and only when $Q^\nu_{\cdot\,\rho} + A^\nu_\rho$ is proportional to x^ν, i.e., when a covariant point q_ρ exists such that:

$$Q^\nu_{\cdot\,\mu} = x^\nu q_\mu - A^\nu_\lambda. \tag{106}$$

Now, since:

$$Q^\nu_{\cdot\,\mu} x^\mu = \Pi^\nu_{\lambda\mu} x^\lambda x^\mu = P_\mu^{\cdot\,\nu} x^\lambda, \tag{107}$$

vanishes, on account of IIb, contracting (106) with x^μ yields:

$$q = q_\mu\, x^\mu = 1\,.$$

Conditions (106), (108) are also sufficient for the covariant constancy of the contact point. From (57), IIb they are consistent with the symmetry condition Vα; they are consistent with the quasi-symmetry condition Vβ when and only when:

$$q_\mu = -\,Q_\mu\,. \tag{109}$$

From (105), equation (109) is (under the assumptions IIb, IV*b*, (106), (108) and for T = 1) necessary and sufficient for the covariant constancy of the contact *point* ($\tilde{x}^\nu = y^\nu$). In order for this to be the case, quasi-symmetry is sufficient, but not necessary.

On account of § 2, **9**d, the contact point $x^\nu = y^\nu - dx^\nu$ also lies in the local ${}^{n+1}E$ at y^ν; we say that it is *invariant* under the displacement when the position x^ν in the ${}^{n+1}E$ at x^ν is mapped to the position $y^\nu - dx^\nu$ in ${}^{n+1}E$ at y^ν. The condition for this reads $\tilde{x}^{[\lambda}y^{\mu]} = 0$. From (105), this is the case when and only when $Q^{\nu}_{\cdot\mu}$ has the form:

(110) $$Q^{\nu}_{\cdot\mu} = x^\nu q_\mu \;,$$

which, from (107), IIb, leads to:

(111) $$q = x^\nu q_\mu = 0 \;.$$

This condition is inconsistent with quasi-symmetry, but for:

(112) $$q_\mu = 0$$

it is consistent with symmetry. The case of invariance of the contact point was previously established by J. A. Schouten [35]). From (105), (109) is necessary for the invariance of contact *point*, which is impossible, due to the inconsistency of IVβ with (109), (111).

Conditions (106), (108), (110), (111) can be summarized by:

III $$\boxed{Q^{\nu}_{\cdot\mu} = x^\nu q_\mu + (P - x^\rho q_\rho) A^\nu_\mu,}$$

in which $P = 0$, with the extra conditions:

IIIα $$\boxed{x^\rho q_\rho = 1}$$

for covariant constancy, and:

IIIβ $$\boxed{x^\rho q_\rho = 0}$$

for the invariance of the contact point, resp.

§ 9.

The ${}^{m+1}P$ in the ${}^{n+1}P$.

28. Let there be given any system of functions Π'^{c}_{ab}, Q'_a on a ${}^{m+1}H$ in ${}^{m+1}P$ that satisfy the homogeneity conditions I′ that correspond to I. There then exists the curvature projector:

(113) $$\Pi_{ab}^{\cdot\,\cdot\,\nu} = \partial_a B^\nu_b + B^{\lambda\mu}_{ab}\Pi^\nu_{\mu\lambda} - B^\nu_c \Pi'^{c}_{ba},$$

which is a quantity of $(-1)^{\text{th}}$ degree in the x^ν and satisfies the identities:

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[35]) Cf., J. A. Schouten [**3**].

(114) $$\Pi_{ab}^{\cdot\cdot\nu}\, x^a = P_\mu^{\cdot\nu} B_b^\mu - B_c^\nu P_b'^{\cdot c},$$

(115) $$\Pi_{ab}^{\cdot\cdot\nu}\, x^b = Q_{\cdot\lambda}^{\nu} B_a^\lambda - B_c^\nu Q'^{c}_{\cdot a}.$$

If we denote covariant differentiation with respect to Π'^c_{ab}, Q'_a by V' then we have identically for the covariant (contravariant, resp.) point $w_a = B_a^\nu w_\nu$ ($v^\nu = B_c^\nu v^c$, resp.) in the local ${}^{m+1}E$:

(116) $$B_{ba}^{\mu\lambda}\nabla_\mu w_\lambda - \nabla'_b w_a = -\Pi_{ba}^{\cdot\cdot\nu}\, w_\nu,$$

(117) $$B_b^\mu \nabla_\mu v^\nu - B_c^\nu \nabla'_b v^c = +\Pi_{ba}^{\cdot\cdot\nu}\, v_a,$$

when, as we would like to do, we define the Q'_a by means of:

(118) $$Q'_a = B_a^\lambda Q_\lambda.$$

29. If the ${}^{m+1}H$ is taut (cf., § 4, **13**) then, from (118) and:

(119) $$\Pi'^{c}_{ab} = B_{ab\nu}^{\lambda\mu c}\Pi_{\mu\lambda}^\nu + B_\nu^c \partial_a B_b^\nu,$$

induces a system Π'^c_{ab}, Q'_a on the ${}^{m+1}H$. Then, one has:

(120) $$P'^{\cdot c}_{a} = B_{a\nu}^{\lambda c} P_\lambda^{\cdot\nu}, \qquad Q'^{\,c}_{\cdot b} = B_{\nu b}^{c\lambda} Q_{\cdot\mu}^{\nu};$$

hence, (114), (115) turn into:

(121) $$\begin{aligned} H_{ab}^{\cdot\nu} x^a &= (A_\sigma^\nu - B_\sigma^\nu) Q_{\cdot\rho}^{\sigma} B_b^\rho, \\ H_{ab}^{\cdot\nu} x^b &= (A_\sigma^\nu - B_\sigma^\nu) P_\rho^{\cdot\sigma} B_b^\rho. \end{aligned}$$

In this case there also exist the two curvature quantities:

(122) $$L_{a\cdot\nu}^{\cdot c} = B_{ab}^{\lambda\mu}\,\nabla_\lambda B_\nu^\mu,$$

and we have, as usual:

(123) $$H_{ab}^{\cdot\cdot\nu} = B_{ab}^{\lambda\mu}\,\nabla_\lambda B_\mu^\nu,$$

(124) $$H_{ab}^{\cdot\cdot\nu} B_\nu^c = 0, \qquad L_{a\cdot\nu}^{\cdot c} B_b^\nu = 0.$$

The equations of Gauss, Codazzi, Rizzi may be stated quite easily; however, we will not go into that here.

§ 10.

Geodetic lines.

30. The equation for geodetic lines does *not* have the usual form:

(125) $$\frac{d^2x^\nu}{dt^2} + \Pi^\nu_{\lambda\mu}\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt} - \alpha\frac{dx^\nu}{dt} = 0\,,$$

since a curve is a binary manifold (^2H). Instead of (125), one obtains:

(126) $$H_{ab}^{\cdot\cdot\nu} = \partial_a B^\nu_b + B^{\lambda\mu}_{ab}\Pi^\nu_{\mu\lambda} - B^\nu_c \Pi'^{\,c}_{ba} = 0 \qquad (a, b, \ldots, = 0, 1).$$

One easily sees the analogy with (125) when one writes (126) in the form:

$$\frac{\partial^2 x^\nu}{\partial x^a \partial x^b} + \Pi^\nu_{\lambda\mu}\frac{\partial x^\lambda}{\partial x^a}\frac{\partial x^\mu}{\partial x^b} - \Pi'^c_{ab}\frac{\partial x^\lambda}{\partial x^c} = 0\,,$$

and replaces α with $\Pi'^{\,1}_{11}$ in (125). The elimination of the Π'^c_{ab} from (126) yields (cf., (43)):

(127) $$H_{ba}^{\cdot\cdot[\nu} J'^{\rho\sigma]} = (\partial_a B^{[\nu}_b + B^{\lambda\mu}_{ab}\Pi^{[\nu}_{\mu\lambda})J'^{\rho\sigma]} = 0\,.$$

From (37), contracting with x^b yields the necessary condition:

(128) $$B^\lambda_a P_\lambda^{\cdot[\nu} J'^{\rho\sigma]} = 0\,.$$

If v^ν is an arbitrary position in $J'^{\rho\sigma}$ that is $\neq x^\nu$ then $J'^{\rho\sigma}$ is proportional to $x^{[\rho}\, v^{\sigma]}$. If one contracts (128) with v^a then one sees that $v^\lambda P_\lambda^{\cdot\nu}$ must be linearly dependent on v^ν and x^ν:

(129) $$P_\lambda^{\cdot\nu} v^\lambda = \alpha\, v^\nu + \beta\, x^\nu\,.$$

It follows that β must depend on v^ν: $\beta = p_\lambda\, v^\lambda$, and since this must be true for arbitrary v^ν, $P_\lambda^{\cdot\nu}$ must then have the form:

(130) $$P_\lambda^{\cdot\nu} = p_\lambda\, x^\nu + \alpha\, A^\nu_\lambda\,.$$

Contracting this with x^λ yields $\alpha = P - p_\rho\, x^\rho$, in which:

(131) $$P_\lambda^{\cdot\nu} x^\lambda = \Pi^\nu_{\lambda\mu} x^\lambda x^\mu = P\, x^\nu,$$

hence:

II $$\boxed{P_\lambda^{\cdot\nu} = p_\lambda x^\nu + (P - p_\rho x^\rho)A^\nu_\lambda.}$$

Likewise, contraction of (127) with x^a yields the necessary condition:

III $$\boxed{Q^\nu_{\cdot\mu} = x^\nu q_\mu + (P - x^\rho q_\rho)A^\nu_\mu.}$$

If one brings (127) into the form:

(132) $$(B_a^\lambda \nabla_\lambda B_{(b)}^{[\nu}) J'^{\rho\sigma]} = 0 ,$$

in which the parentheses around the index b means that this index has been "turned off" [36]), i.e., that B_b^ν is to be differentiated like a contravariant point, then one sees that the integrability conditions for (127) read like:

(133) $$J'^{ac} B_c^\mu \nabla_\mu B_{(a)}^\lambda J'^{\rho\sigma} \nabla_\lambda B_{(b)}^\nu = 0 .$$

A small calculation then yields that (133) is satisfied, on the basis of (127), II, and III. Hence, we have:

Theorem 5. *Conditions* II, III *are necessary and sufficient for a geodetic line to go through any position in* ^{n+1}P *in any direction*.

31. Let $v^\nu = B_b^\nu v^a$ be an arbitrary position of first degree that is $\neq x^\nu$ that determines a line (hence, ^2E) in the ^{n+1}E at x^ν by way of the B_b^ν. Then, one has, for a particular k of first degree:

(134) $$x^{[\lambda} v^{\mu]} = \frac{1}{2} k^2 J'^{\lambda\mu} .$$

Since, from II, III, the system of equations (127) or (132) becomes an identity when one contracts it with x^a or x^b, it means the same thing as the equation:

(135) $$v^a v^b H_{ab}^{\cdot\cdot[\nu} J'^{\rho\sigma]} = 0 ,$$

which, since the index b in (132) is turned off and $B_c^{[\nu} J'^{\rho\sigma]}$ vanishes, can also be brought into the form:

(136) $$v^\lambda (\nabla_\lambda v^{[\nu}) J'^{\rho\sigma]} = 0 .$$

That is, $v^\lambda \nabla_\lambda v^\nu$ must be a linear combination of x^ν and v^ν:

(137) $$v^\lambda \nabla_\lambda v^\nu = \alpha x^\nu + \beta v^\nu .$$

Hence: *The line in* ^{n+1}E *that is determined by* x^ν *and* v^ν (*i.e., the tangent to the geodetic line*) *goes to itself under covariant differentiation in its own direction*. If one then replaces v^ν with a certain linear combination of x^ν and v^ν then a brief calculation by means of (149), (150), shows that α and (or) β can be taken to vanish. (137) is then equivalent with:

(138) $$v^\lambda \nabla_\lambda v^\nu = 0 .$$

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[36]) Cf., J. A. Schouten, On infinitesimal deformations of V_m in V_n, Amsterdam, Proceedings Kon. Akad. **31** (1928), pp. 208-218.

Instead of this, one can also introduce the weaker condition:

(139) $$v^\lambda \nabla_\lambda v^\nu = \beta v^\nu ,$$

or, equivalently:

$$(v^\lambda \nabla_\lambda v^{[\mu}) v^{\nu]} = 0 ,$$

which is invariant to a change in v^ν, up to an *arbitrary* factor of arbitrary degree. We would like to call a position field v^ν that satisfies the condition (139) a *geodetic position field.*

32. One must still point out that geodetic lines are generally *not* projective-Euclidian, i.e., the $\Pi'^{\,c}_{ab}$ cannot be taken to vanish by a judicious choice of the parameter x^a (although this is indeed the case with α and β in (137)!). In order for the ${}^2P$ to be projective-Euclidian it is necessary and sufficient that the $S'^{\,\cdot\,\cdot\,c}_{ab}$, $P'^{\,\cdot\,c}_{a}$, $Q'^{\,c}_{\cdot b}$, and $N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc}$ do not vanish. The condition $N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} = 0$ is, however, equivalent with:

(140) $$J'^{ab} N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} = 0 .$$

On the other hand, it follows from (126) that:

(141) $$B^{\omega\mu\lambda}_{abc} N'^{\,\cdot\,\cdot\,\cdot\,\nu}_{\omega\mu\lambda} = B^{\nu}_{d} N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} .$$

If one contracts (141) with $J'^{ab} x^c$ and $J'^{ab} v^c$ and regards v^ν as an *arbitrary* position in the ${}^{n+1}E$ at x^ν then one finds, by means of (134), that (141) is equivalent with the two conditions:

(142) $$x^\omega x^\lambda N'^{\,\cdot\,\cdot\,\cdot\,\nu}_{\omega\mu\lambda} = 0 ,$$

(143) $$x^\omega N'^{\,\cdot\,\cdot\,\cdot\,\nu}_{\omega[\mu\lambda]} = 0 .$$

These conditions together are thus *necessary and sufficient for us to have* $N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} = 0$. From $P'^{\,\cdot\,c}_{a} = 0$, by means of (126), it follows that:

$$B^\lambda_a P^{\cdot\,\nu}_\lambda = B^\lambda_a p_\lambda x^\nu + (P - p) B^\lambda_a = 0 ;$$

hence, by contracting with v^a and considering that $v^{[\lambda} x^{\mu]} \neq 0$ and v^ν is arbitrary, we have:

(143a) $$p_\lambda = 0, \qquad P = 0 .$$

Likewise, it follows from $Q'^{\,c}_{\cdot b} = 0$ that:

(143b) $$q_\lambda = 0 .$$

Due to (126), $S'^{\,\cdot\,\cdot\,c}_{ab} = 0$ is equivalent with $J'^{ab} B^{\lambda\mu}_{ab} S'^{\,\cdot\,\cdot\,\nu}_{\lambda\mu} = 0$; this condition is, however, a result of (143a), (143b). From (76), it follows that (142), (143) are also satisfied, on the basis of (143a), (143b). Thus, we have:

Theorem 6. *In order for all geodetic lines to be projective-Euclidian it is necessary and sufficient that* $P^{\cdot\nu}_{\lambda} = Q^{\nu}_{\cdot\lambda} = 0$.

Finally, we would like to give the condition $N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} = 0$ another form *without assuming* $S'^{\,\cdot\,\cdot\,c}_{ab} = 0$, $P'^{\,\cdot\,c}_{a} = 0$, $Q'^{\,c}_{\cdot b} = 0$. Thus, we prove that:

Theorem 6a. *The necessary and sufficient condition that* $N'^{\,\cdot\,\cdot\,\cdot\,d}_{abc} = 0$ *for any geodetic line reads like:*

(144) $$\nabla_\mu P = 0$$

together with either:

(145) $$\nabla_{(\lambda}\, p_{\mu)} + p_{(\lambda}\, Q_{\mu)} + p_{(\lambda}\, q_{\mu)} = 0$$

and:

(146) $$q \equiv x^\rho q_\rho = P + 1$$

or (instead of (145), (146)):

(147) $$p_\lambda = 0\,.$$

The latter case is equivalent to IIa with constant P (from (144)); in this case, (145) is also satisfied, but not (146). In the former case, from (146), one also has that q, and, from (145) (contracting (145) with x^λ and applying (144), (146), (77), (78), II, III), also p, is constant. In the quasi-symmetric case, from II, III, (68), (57) equivalent to:

IVγ $$Q_\mu = p_\mu - q_\mu$$

and (145) turns into:

(148) $$\nabla_{(\lambda}\, p_{\mu)} + p_\mu\, p_\mu = 0\,.$$

Proof of 6a. From II, III, the identities (77), (78), (79) turn into:

(149) $$\nabla_\lambda x^\nu = (Q_\lambda + q_\lambda)\, x^\nu + (P - q + 1)\, A^\mu_\lambda\,,$$

(150) $$x^\lambda \nabla_\lambda v^\nu = (P - p + \mathfrak{r}\, Q)\, v^\nu + p_\lambda\, v^\lambda x^\mu,$$

(151) $$x^\lambda \nabla_\lambda w_\mu = (P - p + \mathfrak{r}\, Q)\, w_\mu - x^\rho w_\rho \cdot p_\mu\,.$$

In particular, one also has:

(152) $$x^\lambda \nabla_\lambda p_\mu = -\,(P + Q)\, p_\mu\,.$$

Now, from (76), (142) is equivalent with $x^\lambda\, \nabla_\lambda P^{\cdot\nu}_{\lambda} = 0$. From II, III, (149), (152) this yields:

(153) $$x^\nu\, \{\nabla_\mu P - (P - q + 1) p_\mu\} + p(P - q + 1)\, A^\mu_\lambda = 0\,.$$

Contracting with an arbitrary non-vanishing covariant point, which we denote by x^ν, shows that both of the terms must each vanish:

$$\nabla_\mu P - p_\mu (P - q + 1) = 0 \,, \tag{154}$$
$$p\,(P - q + 1) = 0 \,. \tag{155}$$

Likewise, (143) is equivalent $\nabla_{[\lambda} P_{\mu]}^{\cdot\,\nu} = 0$; furthermore, substituting this in II, III, (149), (152) yields an equation whose terms must both vanish individually. The first term gives (145), and the second one yields:

$$\nabla_\mu P - x^\rho \nabla_\mu p_\rho - p(Q_\mu + p_\mu) = 0 \,. \tag{156}$$

From (152), (154), (145), the latter equation satisfied; likewise (155), such that (145), (154) together are necessary and sufficient. From (155), it follows that either (146) is true, and thus (144), conditions that, along with (145), are sufficient, or that we have:

$$P - q + 1 \geq 0 \,, \tag{157}$$

hence, $p = 0$, and therefore, from (156), (149):

$$\nabla_\mu P = x^\rho \nabla_\mu p_\rho = - p_\mu (P - p + 1) \,.$$

Together with (154), this again yields (144), hence, from (157), also (147), from which, everything else follows. However, with that we have proved theorem 6a.

§ 11.

Overview.

33. Little by little, we have now introduced the following conditions:

I.	$x^\omega \partial_\omega \Pi^\nu_{\lambda\mu} = - \Pi^\nu_{\lambda\mu}$;	$x^\omega \partial_\omega Q_\mu = - Q_\mu$;	
II.	$P_\lambda^{\cdot\,\nu} = p_\lambda x^\nu + (P - p) A^\nu_\lambda$;	$p = p_\rho x^\rho$;	$(P_\lambda^{\cdot\,\nu} = \Pi^\nu_{\lambda\mu} x^\mu)$.
IIa.	$P_\lambda^{\cdot\,\nu} = P A^\nu_\lambda$, or	$p_\lambda = 0$.	
IIb.	$P_\lambda^{\cdot\,\nu} = 0$, or	$p_\lambda = 0$, $P = 0$.	
III.	$Q^\nu_{\cdot\,\mu} = x^\nu q_\mu + (P - q)\, A^\nu_\lambda$,	$q = x^\rho q_\rho$;	$(Q^\nu_{\cdot\,\mu} = \Pi^\nu_{\lambda\mu} x^\lambda)$.
IIIα.	$Q^\nu_{\cdot\,\mu} = x^\nu q_\mu + A^\nu_\lambda$, or	$P - q + 1 = 0$.	
IIIβ.	$Q^\nu_{\cdot\,\mu} = x^\nu q_\mu$, or	$P - q = 0$.	
IV.	$Q_\mu = 0$.		
IVα.	$Q_\mu = \nabla_\mu g = \partial_\mu \log g^Q$;	$Q = \dfrac{1}{1-\mathfrak{r}}$ = const.,	$\mathfrak{r}$ = degree of g .
IVβ.	$Q = 0$;	$(Q = x^\rho Q_\rho + 1)$.	

IVγ. $Q_\mu = p_\mu - q_\mu$.
IVδ. $Q_\mu = - q_\mu$.
Vα. $S_{\lambda\mu}^{\cdot\cdot\nu} = 0$; $\quad (S_{\lambda\mu}^{\cdot\cdot\nu} = \Pi^\nu_{[\lambda\mu]})$.
Vβ. $T_{\lambda\mu}^{\cdot\cdot\nu} = 0$; $\quad (T_{\lambda\mu}^{\cdot\cdot\nu} = S_{\lambda\mu}^{\cdot\cdot\nu} + A^\nu_{[\lambda}Q_{\mu]})$.
VI. $U_{\omega\mu} = 0$; $\quad (U_{\omega\mu} = \nabla_{[\omega}Q_{\mu]} - T_{\lambda\mu}^{\cdot\cdot\rho}Q_\rho)$.

Thus, IIa is a specialization of II and IIb is a specialization of IIa; IIIα and IIIβ are inconsistent specializations of III. III follows from II and Vα or Vβ. IV is a specialization of IVα, as well as IVγ, but it is inconsistent with IVγ. IVγ follows from II, III, and Vβ; follows from IVγ and IIa, or also from IIa, III, and Vβ. IVα and VI are equivalent.

34. Condition I is necessary and sufficient for the existence of a covariant derivative.

Conditions I, IIb, IVβ are necessary and sufficient for the existence of a covariant differential, as well as a position displacement for arbitrary quantities of arbitrary degree.

Conditions II, II, III are necessary and sufficient for the existence of a geodetic line through each position in each direction.

Condition IIIα (IIIβ, resp.) states (assuming I, IIb, IVβ) the covariant constancy (invariance, resp.) of the contact point.

If a taut ${}^{m+1}H$ in ${}^{n+1}P$ induces a projective connection by means of (119) then (118), (120), shows that each of the conditions I, …, VI is valid in ${}^{m+1}P$, as long as it is valid in ${}^{n+1}P$. Hence:

Theorem 7. *Conditions* I, …, VI *are all invariant under the embedding of a manifold.*

In the next section, it will shown that conditions IIb and IVβ can be satisfied under the assumption of I, II, III, as long as either Vα or Vβ can be satisfied under path-preserving changes of the projective connection.

35. If we understand the term *directionless derivative* to mean the effect of the operator $x^\omega\partial_\omega$ then the *stronger* requirement that the directionless derivative of a quantity vanishes is equivalent to conditions IIb and IVβ; on the other hand, the *weaker* requirement that the directionless derivative of any quantity be proportional to this quantity is equivalent to condition IIa. This last requirement has a simple geometric meaning: In the H_{n+1} that arises from ${}^{n+1}H$ by way of § 2, **6** the $\Pi^\nu_{\lambda\mu}$ determine an affine (cf., ?)) displacement L_{n+1}. If one now identifies the various local E_{n+1} that belong to the points of the same "ray" (= line through O) with each other the one obtains an "osculating E_{n+1}." If one now considers a vector field such that the various points of a ray are merely Euclidian parallel vectors – hence, vectors with proportional components – then one has a position field in our sense. The aforementioned weak requirement now says that these vectors are also parallel in the sense of displacement, and indeed in the weak sense, that the vector that results from the displacement of a vector of the field that originates at a point of the same ray is only proportional (but not necessarily *equal*) to the vector of the field that belongs to the new point.

The covariant derivative of the contact point determines a projectivity (collinearity) $\nabla_\lambda x^\nu$ in ${}^{n+1}E$. The stronger requirement that they are undetermined (vanish) is equivalent to IIIα and IVδ. The weaker requirement that when this projective map takes ${}^{n+1}E$ to itself it is geometrically the identity, and thus that $=_\lambda x^\nu$ is proportional to A_λ^ν, is equivalent to III and IVδ. When interpreted in H_{n+1} the weaker requirement states that a vector that has the same direction as the ray preserves its direction in the Euclidian sense under displacement. The still weaker condition that $P_\lambda^{\cdot\nu} x^\lambda = Q^\nu_{\cdot\mu} x^\mu$ be proportional to x^ν says only that the rays are geodetic lines in L_{n+1}. One could further demand that the rays be geodetically parallel; this would be equivalent to IIIβ.

Both strong conditions together yield IIb, IIIα, IVβ, IVδ; both weak conditions together yield IIa, III, and IVδ, hence, the unrestricted existence of geodetic lines.

§ 12.

Path-preserving changes of connection.

36. A general change in a projective connection will be given by a quantity $X_{\lambda\mu}^{\cdot\cdot\nu}$ and a covariant point Y_μ that are both of degree –1:

(158) $$\overset{*}{\Pi}{}^{\nu}_{\lambda\mu} = \Pi^\nu_{\lambda\mu} + X_{\lambda\mu}^{\cdot\cdot\nu} ; \qquad \overset{*}{Q}_\mu = Q_\mu + Y_\mu .$$

We assume that a system of geodetic lines exists in ${}^{n+1}P$, such that conditions II, III are satisfied. The requirement that these conditions remain satisfied under the change (130) reads like:

(159) $$X_{\lambda\mu}^{\cdot\cdot\nu} x^\mu = u_\lambda x^\nu + (X - u_\rho x^\rho) A_\lambda^\nu ,$$

(160) $$X_{\lambda\mu}^{\cdot\cdot\nu} x^\lambda = v_\mu x^\nu + (X - v_\rho x^\rho) A_\mu^\nu ,$$

for a certain choice of the covariant point:

(161) $$u_\lambda = \overset{*}{p}_\lambda - p_\lambda ,$$

(162) $$v_\lambda = \overset{*}{q}_\lambda - q_\lambda ,$$

and the scalar:

(163) $$X = P^* - P .$$

In order for the new projective connection to have the same paths as the old one it is necessary and sufficient that equation (127) remain invariant under the change (158); hence:

(164) $$B_{ab}^{\lambda\mu} X_{\lambda\mu}^{\cdot\cdot[\nu} J'^{\rho\sigma]} = 0 .$$

Contraction by x^a (x^b, resp.) yields equations that are satisfied as a result of (159), (160). Hence, (164) is equivalent with the equation that arises from it by contraction with $v^a\, v^b$, in which $v^a \neq x^a$, but is otherwise arbitrary. Therefore, we must have:

(165) $$v^a\, v^b\, X_{\lambda\mu}^{\cdot\,\cdot\,\nu} = \lambda\, x^\nu + \mu\, v^\nu .$$

However, since such an equation must be valid for an *arbitrary* point v^ν (indeed, it must give geodetic lines in *every* direction!), λ (μ, resp.) must be a homogeneous quadratic (homogeneous linear, resp.) form in v^ν:

(166) $$\lambda = Z_{\lambda\mu}\, v^\lambda\, v^\mu, \qquad Z_{\lambda\mu} = Z_{\mu\lambda}; \qquad \mu = z_\rho\, v^\rho .$$

The necessary and sufficient condition for the change to be path-preserving then reads like:

(167) $$X_{(\lambda\mu)}^{\cdot\,\cdot\,\nu} = Z_{\lambda\mu}\, x^\nu + 2\, z_{(\lambda}\, A_{\mu)}^\nu .$$

37. The alternating part of $X_{\lambda\mu}^{\cdot\,\cdot\,\nu}$ is nowhere to be found in the condition equation (167). If one now sets:

(168) $$X_{[\lambda\mu]}^{\cdot\,\cdot\,\nu} = -S_{\lambda\mu}^{\cdot\,\cdot\,\nu}, \qquad X_{(\lambda\mu)}^{\cdot\,\cdot\,\nu} = 0 ,$$

then, from II, III, conditions (159), (160), (167) are satisfied, it becomes:

(169) $$\overset{*}{S}{}_{\lambda\mu}^{\cdot\,\cdot\,\nu} = 0 .$$

However, if one sets, instead of (168):

(170) $$X_{[\lambda\mu]}^{\cdot\,\cdot\,\nu} = -T_{\lambda\mu}^{\cdot\,\cdot\,\nu}, \qquad X_{(\lambda\mu)}^{\cdot\,\cdot\,\nu} = 0 ,$$

then the conditions will be likewise satisfied, and one will have:

(171) $$\overset{*}{T}{}_{\lambda\mu}^{\cdot\,\cdot\,\nu} = 0 .$$

Hence: *one can always arrange, by a path-preserving change of connection, that a projective connection be either symmetric or quasi-symmetric, as one desires.*

38. Since Y_μ appears nowhere in the condition equations, one can arrange that $\overset{*}{Q}_\mu$ be equal to an arbitrarily chosen covariant point of degree –1. In particular, one can arrange that $\overset{*}{Q}_\mu$ be a gradient, or vanish completely, or also that $Q^* = 0$.

39. We would further like to leave the alternating part of $\Pi_{\lambda\mu}^\nu$, as well as Q_μ unchanged. One then has:

(172) $$X_{\lambda\mu}^{\cdot\,\cdot\,\nu} = X_{(\lambda\mu)}^{\cdot\,\cdot\,\nu} = Z_{\lambda\mu}\, x^\nu + 2\, z_{(\lambda}\, A_{\mu)}^\nu .$$

Substituting of (172) in (159), (160) shows that the latter conditions are satisfied, on the basis of (172). We have:

(173) $$u_\lambda = z_\lambda + Z_{\lambda\mu}\, x^\nu ,$$

(174) $$v_\mu = z_\lambda + Z_{\lambda\mu}\, x^\lambda ,$$

(175) $$X = 2z_\lambda\, x^\lambda + Z_{\lambda\mu}\, x^\lambda x^\mu .$$

In the event that $\Pi^\nu_{\lambda\mu}$ has already been changed as in **37** or **38**, we omit the *; we set:

(176) $$z_\lambda = -\, p_\lambda ; \qquad Z_{\lambda\mu} = 0 ,$$

which makes:

(177) $$p^*{}_\lambda = 0 .$$

Condition IIa will also be satisfied then.

40. We further leave p_λ unchanged; instead of (173), one now has:

(178) $$z_\lambda = -\, Z_{\lambda\mu}\, x^\mu ,$$

Substitution in (141) yields:

(179) $$X_{\lambda\mu}^{\cdot\cdot\nu} = Z_{\lambda\mu}\, x^\nu - 2\, x^\rho\, Z_{\rho(\lambda}\, A^\nu_{\mu)} .$$

We again omit the star and set:

(180) $$X = -\, Z_{\lambda\mu}\, x^\lambda x^\mu = -\, P .$$

If one leaves the $Z_{\lambda\mu}$ otherwise arbitrary then one has:

(181) $$P^* = 0 .$$

Condition IIb is then also satisfied. Hence, from theorem 3, one has:

Theorem 8. *The existence of a covariant differential may be achieved by means of path-preserving changes of connection.*

41. We further leave $P_\lambda^{\cdot\nu}$ unchanged. (175) then turns into:

(182) $$Z_{\lambda\mu}\, x^\lambda x^\mu = 0 .$$

We again omit the star. If one now wishes that the displacement be uniquely determined then this can happen only under restricted assumptions. From the transformation laws of $\Pi^\nu_{\lambda\mu}$ (53), it follows that:

(183) $$\Pi^{\rho'}_{\rho'\mu'} = A^\nu_{\mu'}\Pi^\sigma_{\sigma\mu} - \partial_{\mu'} \log\Delta , \qquad \Delta = \mathrm{Det}(A^{\nu'}_\lambda) .$$

If one then sets:

(184) $$z_\lambda = -Z_{\lambda\mu}\, x^\mu = -\frac{1}{n}\Pi^{\rho}_{\rho\mu},$$

in the given coordinate system, and if one chooses the $Z_{\lambda\mu}$ to be otherwise arbitrary then, since $P = 0$, condition (182) is satisfied, and we have, in the given coordinate system:

(185) $$\Pi^{*\rho}_{\rho\mu} = 0;$$

in any other coordinate system, $\Pi^{*\rho'}_{\rho'\mu'}$ will be the gradient of a density.

Hence, one will also have:

(186) $$N^{*\cdot\cdot\cdot\rho}_{\omega\lambda\rho} = 0\,,$$

i.e., the displacement is "volume-preserving" [37]). This is again independent of the coordinate system.

From (73), the Ricci tensor then becomes symmetric in the event that the displacement is made symmetric by means of **37**, as we would like to assume:

(187) $$N^{*\cdot\cdot\cdot\rho}_{\rho\lambda\mu} = N^{*\cdot\cdot\cdot\rho}_{\rho\mu\lambda}\,.$$

42. We again omit the star and would like to further leave $\Pi^{\rho}_{\rho\mu}$ unchanged. As a result, we must have:

(188) $$z_\lambda = -Z_{\lambda\mu}\, x^\mu = 0\,.$$

Under the assumption of IIb, III, we have:

(189) $$N^{*\cdot\cdot\cdot\rho}_{\rho\mu\lambda} - N^{\cdot\cdot\cdot\rho}_{\rho\mu\lambda} = - =_\rho Z_{\mu\lambda}\, x^\rho + 2T^{\cdot\cdot\sigma}_{\rho\mu}\, x^\rho Z_{\lambda\sigma} = -(n-1)(1-q)\, Z_{\lambda\mu}\,.$$

Thus, if $n \geq 1$ and $q \geq 1$ then, from (187), one can set:

(190) $$Z_{\lambda\mu} = \frac{1}{(n-1)(1-q)}\, N^{\cdot\cdot\cdot\rho}_{\rho\mu\lambda}\,,$$

and we have:

(191) $$N^{*\cdot\cdot\cdot\rho}_{\rho\mu\lambda} = 0\,.$$

Thus, the $\Pi^{\nu}_{\lambda\mu}$ are uniquely established (e.g.) by conditions I, IIb, Vα, (186), (191).

43. The Q_μ are not uniquely determined in this way; in the event that a covariant differential exists, they obey only the condition $Q = 0$. Thus, n parameters remain

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[37]) The notion of "volume" is affine, not projective, in character. However, condition (186) corresponds to the condition for volume preservation in the affine case.

undetermined, i.e., exactly as many as in the projective change of an affine displacement. If one would like to determine the Q_μ uniquely then one can set $Q_\mu = 0$, which does not, however, agree with $Q = 0$.

In summation, we have:

Theorem 9. *For a given system of paths, a projective displacement is determined uniquely, up to the indeterminacy of Q_μ, by the following requirements: Existence of a covariant differential, symmetry, "volume preservation," vanishing of the Ricci tensor* [38]).

Theorem 10. *If we forego the existence of a covariant differential then a projective connection will be uniquely determined for a given system of paths by the conditions:* I, IIb, IV, *symmetry* (= *quasi-symmetry*), *"volume preservation," and the vanishing of the Ricci tensor.*

§ 13.

Inhomogeneous coordinates.

44. In order for the projective connection to be derivable by means of one of the known methods from a linear displacement the existence of geodetic lines is necessary in any case. Therefore, let conditions I, II, III be assumed. Let a geodetic position field v^ν be given:

(139) $$v^\lambda \nabla_\lambda v^\nu = \beta v^\nu.$$

This condition is equivalent to:

(192) $$v^a \nabla'_a v^c = \beta v^c,$$

in which ∇' denotes the differential operator on the geodetic line. If one then sets:

(193) $$w_a = \frac{1}{J'_{cd} x^c v^d} \cdot J'_{ab} v^b,$$

in which denotes J'_{ab} the covariant unit bi-point of the geodetic line (relative to an arbitrary coordinate system), then a brief calculation shows that the rotation of w_a vanishes:

(194) $$\partial_{[a} w_{b]} = 0,$$

which then implies that w_a is a gradient field:

(195) $$w_a = \partial_a \log w.$$

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[38]) The symmetry condition Vα cannot be replaced with Vβ here, since then, from IIb, III, IVβ, IVα, IIIα would also be valid, such that the substitution (190) would be impossible.

From the fact that $x^a\, w_a = 1$, w *is* a homogeneous function of first degree, for which, moreover:

(196) $$v^a\, \partial_a w = w^\lambda\, \partial_\lambda\, v = 0\,.$$

Condition (139) ((192), resp.), as well as the definition of w, remain invariant under a change of v^ν by an arbitrary factor.

45. Furthermore, let t be any arbitrary, but chosen once and for all, non-constant homogeneous function of null degree along the path. Thus, one has $v^a \partial_a\, t \neq 0$, since otherwise the fact that $x^a \partial_a\, t = 0$ in general would make $\partial_a\, t = 0$; hence, t would be constant. We normalize v^a by means of the condition:

(197) $$v^a \partial_a\, t = 1\,.$$

The degree of v^a will then be = 1 .

If f is then any other homogeneous function *of null degree* on the path, so $f = f(t)$, then:

$$v^a\, \partial_a\, f(t) = v^a\, \frac{df}{dt}\, \partial_a\, t\,,$$

hence, from (197):

(198) $$v^a\, \partial_a\, f = \frac{df}{dt}\,.$$

We further set:

(199) $$\xi^\nu = \frac{v^\nu}{w}\,.$$

The ξ^ν are therefore $n + 1$ homogeneous functions of null degree in the x^ν that can be regarded as (over-specified) coordinates in the ${}^{n+1}H$ that is associated with X_n. Then (198), (196) yields:

(200) $$\frac{d\xi^\nu}{dt} = \frac{v^\nu}{w}$$

and (139) gives:

(201) $$\frac{v^\lambda}{w}\,\nabla_\lambda \frac{v^\nu}{w} = \frac{\gamma}{w}\cdot\frac{v^\nu}{w}\,, \qquad \gamma = \beta - v^\lambda\, \nabla_\lambda \log w = \beta - Q_\lambda\, v^\lambda\,.$$

If one substitutes (200) in (201) then one finds, from (198), the following equations for geodetic lines:

(202) $$\frac{d^2\xi^\nu}{dt^2} + w\Pi^\nu_{\lambda\mu}\,\frac{d\xi^\lambda}{dt}\,\frac{d\xi^\mu}{dt} = \gamma\,\frac{d\xi^\nu}{dt}\,.$$

46. From (202), one can easily derive the equation for geodetic lines in homogeneous coordinates. By means of a coordinate transformation one can, for the time being, deduce that:

(203) $$x^0 = w.$$

From (199), (200), one then has:

(204) $$\xi^0 = 1, \qquad x^0 = 0 .$$

Furthermore, one can perform a path-preserving change that makes the displacement symmetric (§ 12, **37**) and, moreover, that:

(205) $$\Pi^0_{\lambda\mu} = 0 .$$

In fact, if one takes a covariant point z_λ of degree –1 that is arbitrary, up to the condition:

(206) $$z_\lambda \, x^\lambda = 0 ,$$

and one sets:

$$Z_{\lambda\mu} = -\frac{1}{x^0}\Pi^0_{\lambda\mu} + \frac{2}{x^0} A^0_{(\lambda} z_{\mu)} \; ,$$

then one will have (cf., (158), (172)) $\Pi^{*0}_{\lambda\mu} = 0$, hence, also $P^* = 0$ and $p^*_\lambda = q^*_\lambda = \dfrac{p^*}{x^0} A^0_\lambda$.

Let both of these changes be carried out. From (202), the zero equation is then satisfied identically, on account of (200), (204). If one then lets the indices i, j, k, … range through the numbers 1, …, n then (202) gives the well-known equation for geodetic lines:

(207) $$\frac{d^2\xi^k}{dt^2} + \Gamma^k_{ij}\frac{d\xi^i}{dt}\frac{d\xi^j}{dt} = \gamma\frac{d\xi^k}{dt} ,$$

in which we have set:

(208) $$\Gamma^k_{ij} = x^0\,\Pi^k_{ij} .$$

The Γ^k_{ij} depend only upon the x^ν, but not on x^0, since the Π^k_{ij} are of degree –1 in the x^ν. For the remaining $\Pi^\nu_{\lambda\mu}$, one then easily finds by means of (56):

(208) $$\begin{cases} \Pi^k_{ij} = \dfrac{1}{x^0}\Gamma^k_{ij}, \\[2ex] \Pi^k_{i0} = \Pi^k_{0i} = -\dfrac{1}{x^0}(pA^k_i + \Gamma^k_{ij}\xi^j), \\[2ex] \Pi^k_{00} = \dfrac{1}{x^0}(2p\xi^k + \Gamma^k_{ij}\xi^i\xi^j). \end{cases}$$

We have thus proved:

Theorem 10. *In the event that* ${}^{n+1}P$ *possesses geodetic lines there is* (in the ${}^{n+1}H$ that is attached to X_n) *also an* A_n *with the same geodetic lines.*

§ 14.

Relationships with the older theory.

47. We would now like to briefly discuss how our theory connects with the older theories, in particular, that of Schouten and Golab. For that reason, we replace the homogeneous coordinates x^ν in the ${}^{n+1}H$ with "Veblen coordinates" $x^{\underline{\nu}}$.

Namely, we set:

$$x^{\underline{\rho}} = \log x^0, \qquad x^{\underline{i}} = x^i = \frac{x^i}{x^0}, \quad ^{39})$$
$$\underline{\kappa}, \underline{\lambda}, \ldots, \underline{\omega} = \underline{0}, \underline{1}, \ldots, \underline{n}, \qquad \underline{h}, \underline{i}, \ldots, = \underline{1}, \ldots, \underline{n}, \tag{210}$$

and we briefly denote the transformation (210) by $\mathfrak{S}$ and an arbitrary transformation of $\mathfrak{H}_{n+1}$ by $\mathfrak{T}$, then the transformation $\mathfrak{S}^{-1}\mathfrak{T}\mathfrak{S}$ will be represented by:

$$x^{\underline{0}'} = x^{\underline{0}} + \varphi(x^{\underline{i}}), \qquad x^{\underline{k}'} = x^{\underline{\nu}}(x^{\underline{i}}). \tag{211}$$

Under the transformation to Veblen coordinates, the group $\mathfrak{H}_{n+1}$ thus goes to a group that is conjugate to it, a fortiori, an isomorphic group $\mathfrak{F}_{n+1} = \mathfrak{S}^{-1}\mathfrak{H}_{n+1}\mathfrak{S}$ that is represented by (211). This is, however, precisely the group that Veblen established [40]). The associated functional matrix is:

$$A_{\underline{\lambda}}^{\underline{\nu}'} = \begin{pmatrix} 1 & \partial_{\underline{i}}\varphi \\ 0 & \partial_{\underline{i}} x^{\underline{k}'} \end{pmatrix}. \quad ^{41}) \tag{212}$$

The Veblen "projective tensor" is therefore identical with our quantities.

In order to represent the relationships between our projective connections with the older theories, we would like to refer to only the methods of Schouten and Golab, since these methods subsume most of the other ones.

Sch. and G. introduce no $(n + 1)^{\text{th}}$ coordinates, but restrict themselves to the coordinates ξ^k. Thus, our group $\mathfrak{H}_{n+1}$ ($\mathfrak{F}_{n+1}$, resp.) has no precise analog in their theory. On the other hand, a matrix $A_{\underline{\lambda}}^{\underline{\nu}'}$ appears [42]). In our notation, it is:

$$A_{\underline{\lambda}}^{\underline{\nu}'} = \begin{pmatrix} 1 & -\frac{\mathfrak{n}}{c}\partial_{\underline{i}} \log \Delta_0 \\ 0 & \partial_{\underline{i}} x^{\underline{k}'} \end{pmatrix}, \tag{213}$$

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[39]) The associated functional matrix will be given by (218), (219).

[40]) Veblen [**5**], pp. 144.

[41]) Veblen [**5**], pp. 145.

[42]) Which is denoted by E_A^c by them (loc. cit., pp. 200).

in which $\mathfrak{n} = \dfrac{1}{n+1}$ and Δ_0 is the functional determinant of the transformation $x^{\underline{k}} \to x^{\underline{k}'}$ (hence, not $x^{\underline{\nu}} \to x^{\underline{\nu}'}$!). It therefore differs from (212) by the fact that the arbitrary function $\varphi(\xi^k)$ must be replaced with the special choice $-\dfrac{\mathfrak{n}}{c}\log\Delta_0$. Therefore, instead of the group $\mathfrak{H}_{n+1}$, a subgroup appears in Sch. and G.:

$$x^{\underline{0}'} = x^{\underline{0}} - \frac{\mathfrak{n}}{c}\log\Delta_0\,, \qquad x^{\underline{k}'} = x^{\underline{\nu}}(x^{\underline{i}})\,. \tag{214}$$

There exists the following relation between the determinants Δ and Δ_0:

$$\Delta = \Delta_0^{1-\frac{1}{c}}\,. \tag{215}$$

If we convert (214) to our coordinate system by means of (215) then this relation gives:

$$x^{\underline{0}'} = x^0 \Delta_0^{-\frac{\mathfrak{n}}{c}} = x^0 \Delta^{\frac{1}{(1-c)(n+1)}}\,. \tag{216}$$

The group that was established in the work of Sch. and G. is therefore a subgroup of our group $\mathfrak{H}_{n+1}$; *it is defined by relation* (216).

If we write $\underline{\mathfrak{x}}$ instead of x^0 then (216) turns into:

$$\underline{\mathfrak{x}}' = \underline{\mathfrak{x}}\;\Delta^{\mathfrak{c}}, \qquad \mathfrak{c} = \frac{1}{(1-c)(n+1)}\,. \tag{217}$$

Instead of regarding this equation as a defining condition for a subgroup, we can also treat it as the *transformation equation for a scalar density of weight* $\mathfrak{c}$ *and degree* 1. When we adjoin this scalar density we can then continue to work with our group H_{n+1} (its conjugate group $\mathfrak{F}_{n+1}$).

48. In order to correlate the projective connection of Sch. and G. with our own one, we must convert the defining equation for the former [43]) into homogeneous coordinates. In the calculations, we use the following functional matrix that goes with (210):

$$A_\mu^{\underline{\nu}} = \begin{pmatrix} \dfrac{1}{\mathfrak{x}} & 0 \\ -\dfrac{1}{\mathfrak{x}^2}x^k & \dfrac{1}{\mathfrak{x}}A_i^k \end{pmatrix}, \tag{218}$$

or its inverse:

[43]) Loc. cit., pp. 209.

(219) $$A_{\underline{\lambda}}^{\nu} = \begin{pmatrix} \mathfrak{r} & 0 \\ x^k & \mathfrak{r}A_i^k \end{pmatrix},$$

respectively. Elimination of the components of the affine connection gives that the defining condition (in our notation) is equivalent to:

(220) $$\Pi_{\underline{k}\,\underline{i}}^{k} = \Pi_{\underline{i}\,\underline{k}}^{k} = 0, \qquad \Pi_{\underline{\lambda}\,\underline{0}}^{\nu} = \Pi_{\underline{0}\,\underline{\lambda}}^{\nu} = c\,A_{\underline{\lambda}}^{\nu}, \quad P_{\underline{\nu}\,\underline{i}\,\underline{k}}^{\cdot\cdot\cdot\underline{\nu}} = 0, \qquad \Pi_{\underline{k}\,\underline{i}}^{\underline{0}} = \Pi_{\underline{i}\,\underline{k}}^{\underline{0}}.$$

Converting the $\Pi_{\underline{\lambda}\underline{\mu}}^{\nu}$ by means of (218), (219), and the transformation equation (53) gives that these equations are equivalent to $S_{\lambda\mu}^{\cdot\cdot\nu} = 0$, and:

(221) $$\Pi_{\rho i}^{\rho} = 0\,, \qquad \Pi_{\rho 0}^{\rho} = \frac{1}{x^0}(c-1)(n+1)\,,$$

(222) $$P_{\lambda}^{\cdot\nu} = (c-1)\,A_{\lambda}^{\nu}\,,$$

(223) $$N_{\rho\mu\lambda}^{\cdot\cdot\cdot\rho} = 0\,.$$

The first of these equations has no invariant meaning, however, it yields the invariant condition:

(224) $$N_{\omega\mu\rho}^{\cdot\cdot\cdot\rho} = 0$$

that is "volume-preserving" (cf., [44])) since it can be written as $\Pi_{\rho\lambda}^{\rho} = \partial_\lambda \log \mathfrak{r}^{(c-1)(n+1)}$, i.e., it states that $\Pi_{\rho\lambda}^{\rho}$ is a gradient. The second equation shows that condition IIa is satisfied. From symmetry, III will also be satisfied then, i.e., the equation of geodetic lines is integrable without restriction. Finally, the third condition states the vanishing of the Ricci tensor (which is symmetric, from (224)). When c = 1, and only when, the weak condition IIa turns into the strong condition IIb; this is, however, precisely the case that the (older) Veblen theory singles out.

49. We must point out the remarks of Sch. and G. that their theory subsumes Veblen's theory only in its older form (which is no longer discussed here; Veblen [**4**]), but *not*, however, the more recent and more general theory that was mentioned in **47** (Veblen [**5**]), which first appeared as a discussion in the work of Sch. and G. The latter is based on a group that is isomorphic to ours, whereas Sch. and G. restricted themselves to a proper subgroup. Furthermore, Veblen makes essentially fewer far-reaching assumptions than Sch. and G about the $\Pi_{\underline{\lambda}\underline{\mu}}^{\nu}$ in his general theory. In fact, he assumes only the symmetry of $\Pi_{\underline{\lambda}\underline{\mu}}^{\nu}$, together with the condition $\Pi_{\underline{\lambda}\underline{0}}^{\nu} = A_{\underline{\lambda}}^{\nu}$. From (220), (222), the latter is, however, not equivalent to $P_{\lambda}^{\cdot\nu} = 0$. On the other hand, his $\overset{\mathrm{r}}{\Pi}{}_{\lambda\mu}^{\nu}$ is independent of r, i.e., one has $Q_\mu = 0$. Thus, one has:

[44]) Loc. cit., pp. 209.

Theorem 11. *The generalized Veblen theory is equivalent to the special case of our theory that is singled out by conditions* IIb, IV, *and* V, *as well as* $\mathrm{V}\alpha = \mathrm{V}\beta$. *In particular,* III *is therefore also satisfied, such that the equation of geodetic lines is integrable without restriction* (*it is also projective-Euclidian*). *From* IV, *a covariant differential exists in Veblen's theory only for points of degree* 0. [45])

The fact that, for Veblen, the "weight" of a quantity does not change under differentiation, whereas the degree is lowered by one that way, implies that the Veblen operator is $\partial_{\underline{i}} = x^0\,\partial_i$, and thus preserves the degree; the operator $\partial_{\underline{0}}$ is simply the Euler homogeneity operator $x^\lambda\,\partial_\lambda$. Analogous statements are true for the covariant derivative.

50. In conclusion, we would like to briefly present the relationships between our *homogeneous functions* with the *projective densities* of Sch. and G.

Since every projector density can be written as the product of a projector and a power of an arbitrary scalar density, in order to define the projective derivative of an arbitrary projector density by means of the Leibniz rule for the differentiation of a product, it suffices to define the covariant derivative of a single scalar density (whose degree we can choose arbitrarily). We thus choose the $\mathfrak{x}$ that was introduced above. The expression $\frac{1}{\mathfrak{x}}\nabla_\mu \mathfrak{x}$ may then be taken to be an arbitrary covariant point of degree –1 (weight = 0!). We thus choose our Q_μ. We are then in precise agreement with the theory of Sch. and G. Namely, if we make the usual Ansatz for densities $\mathfrak{p}$ of weight $\mathfrak{k}$ and null degree:

$$\nabla_\mu\,\mathfrak{p} = \partial_\mu\,\mathfrak{p} + k\Pi^\rho_{\rho\mu}\,\mathfrak{p} \tag{225}$$

then this implies that for an arbitrary density $\mathfrak{p}$ of weight $\mathfrak{k}$ and degree $\mathfrak{r}$, one has:

$$\nabla_\mu\,\mathfrak{p} = \partial_\mu\,\mathfrak{p} + k\Pi^\rho_{\rho\mu}\,\mathfrak{p} + \mathfrak{r}\,Q_\mu\,\mathfrak{p}, \tag{226}$$

and, in particular, for $\mathfrak{p} = \mathfrak{x}$, from $\mathfrak{x} = 1$ and $\mathfrak{k} = \mathfrak{c}$, one has:

$$\nabla_\mu\,\mathfrak{x} = \partial_\mu\,\mathfrak{x} + \mathfrak{c}\,\Pi^\rho_{\rho\mu}\,\mathfrak{x} + Q_\mu\,\mathfrak{x}\,. \tag{227}$$

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[45]) In a dissertation that appeared in May 1931, The representation of projective spaces, Ann. of Math. **32** (1931), pp. 327-360, a student of Veblen, J.H.C. Whitehead, obtained some results that have many points of contact with our own. He has replaced Veblen's e^{x^0} with x^0 (cf. ?)), and is thus dealing with homogeneous coordinates in the X_n, indeed, only with normal coordinates at a single point, such that our group H_{n+1} does not appear in his work. His projective connection, like that of Veblen, is characterized by IIb, IV, $\mathrm{V}\alpha = \mathrm{V}\beta$.

If one calculates this only in a coordinate system in which $\mathfrak{x} = x^0$ [46]) then one will have $\partial_\mu \mathfrak{x} = A_\mu^0$. On the other hand, from (221) one has $\Pi_{\rho\mu}^{\rho} = -\frac{1}{\mathfrak{a}} A_\mu^0$; hence, we find:

$$\nabla_\mu \mathfrak{x} = Q_\mu \mathfrak{x}. \tag{228}$$

Finally, we point out that the projectors are in one-to-one correspondence with the projective densities of null degree (as they ultimately occur in Sch. and G.), so if we associate, e.g., a point density $\mathfrak{v}^\nu$ of weight $\mathfrak{x}$ and degree 0 with the point:

$$v^\nu = \frac{\mathfrak{v}^\nu}{\mathfrak{a}^{\mathfrak{r}}} \tag{229}$$

of weight 0 and degree $-\frac{\mathfrak{k}}{\mathfrak{c}}$ then we see that when one singles out an arbitrary scalar density of degree 1 and *arbitrary* weight $\mathfrak{c}$ (from which, one calculates $c = 1 - \frac{1}{(n+1)\mathfrak{c}}$) the *density theory* of Sch. and G. is also completely contained in our theory. In the theory of Sch. and G., one comes upon certain "disadvantages" since one must adjoin only a scalar density with a single value there, whereas one needs n + 1 values for a covariant point. One thus sacrifices some generality. One can then work, not only without singling out a Q_μ (which can probably vanish, as well), but ultimately with projectors and projector densities of *null degree*, whereas one can also operate *with* the Q_μ in (226) with *arbitrary* densities *without* singling out an $\mathfrak{x}$; however, equating (228) with $\nabla\alpha$ yields, moreover, that, from theorem 2, the theory that is obtained in this way is singled out by the special case $U_{\omega\mu} = 0$ precisely, as was already remarked in footnote ?).

In summary, we can therefore say:

Theorem 12. *The projective differential geometry of Sch. and G.* (and thus, from theorem 11, all of the older theories, as well) *may be considered to be a special case of our own. It arises by adjoining a scalar density* $\mathfrak{x}$ *of degree* 1 *and arbitrary weight* c *and by the following specializing conditions:*

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[46]) Such a thing always exists. If $\mathfrak{x} \neq 0$ is given as an *arbitrary* function of first degree in an arbitrary coordinate system and $\mathfrak{c}$ is an *arbitrary* number $\neq 0$ then there is always a homogeneous solution of null degree λ of the differential equation $\lambda + x^0\,\partial_0\lambda = \lambda^{\frac{1}{\mathfrak{c}}}\left(\frac{x^0}{\mathfrak{x}}\right)^{\frac{1}{\mathfrak{c}}}$. If one then sets $x^{0'} = \lambda\, x^0$, $x^{i'} = x^i$ then the transformation $x^\nu \to x^{\nu'}$ belongs to $\mathfrak{H}_{n+1}$, and one has $\Delta = \lambda^{\frac{1}{\mathfrak{c}}}\left(\frac{x^0}{\mathfrak{x}}\right)^{\frac{1}{\mathfrak{c}}}$, hence, $\mathfrak{r}' = \mathfrak{r}\,\Delta^{\mathfrak{c}} = \lambda\, x^0 = x^{0'}$.

1. $S_{\lambda\mu}^{\cdot\cdot\nu} = 0$,
2. $P_{\lambda}^{\cdot\nu} = P\,A_{\lambda}^{\nu}$, $\quad P = c - 1 = \dfrac{-1}{(n+1)\mathfrak{c}} = \mathit{constant},$
3. $N_{\omega\mu\rho}^{\cdot\cdot\cdot\rho} = 0$,
4. $N_{\rho\mu\lambda}^{\cdot\cdot\cdot\rho} = 0$,
5. $U_{\omega\mu} = 0 \qquad \left(Q_{\mu} = \dfrac{1}{\mathfrak{x}}\nabla_{\mu}\mathfrak{x}\right).$

§ 15.

Concluding remarks.

51. We have repeatedly emphasized that the theory that is presented here is not purely projective geometric. To conclude, we would now like to sketch how it is also possible to construct a purely projective geometric differential geometry.

For this, it is necessary that one free oneself from degree conditions, at least from the ones that reduce the transformation group. Instead of the group $\mathfrak{H}_{n+1}$ one must therefore establish an enveloping subgroup $\mathfrak{K}_{n+1}$ of the group $\mathfrak{G}_{n+1}$, which is defined such that the ratios of the $x^{\nu'}$ are functions of the *ratios* of the x^{ν}, or else one would have to say something about the proportionality factor. We can then always give the transformation in the form:

$$x^{\nu'} = \chi\xi^{\nu'}, \tag{230}$$

in which $\xi^{\nu'}$ is *homogeneous of null degree* in the x^{ν}, whereas χ is a *completely arbitrary function* of the x^{ν} with $x^{\mu}\,\partial_{\mu}\,\chi \neq 0$. For the functional matrix, one then finds:

$$A_{\mu}^{\nu'} = \chi\,\partial_{\mu}\,\xi^{\nu'} + x^{\nu'}\,\partial_{\mu}\log\chi\,. \tag{231}$$

The differentials dx^{ν} transform as usual according to (17a). We now also establish a definition of general quantities. By a contravariant (covariant, resp.) point, we understand a system of $n + 1$ numbers that transform, *up to a common arbitrary factor*, like the differentials dx^{ν} (contragrediently, resp.). This factor may be a completely arbitrary function of the coordinates, hence:

$$v^{\nu'} = \varphi\,A_{\mu}^{\nu'}\,v^{\mu}\,, \qquad v^{\nu} = \frac{1}{\varphi}A_{\mu'}^{\nu}v^{\mu'}, \tag{232}$$

$$w_{\mu'} = \psi\,A_{\mu'}^{\nu}\,w_{\nu}\,, \qquad w_{\mu} = \frac{1}{\varphi}A_{\mu}^{\nu'}w_{\nu'}\,. \tag{233}$$

Moreover, we replace the homogeneity condition with a weaker condition: *The ratios of the values of a projector are pure functions of position*; hence they are homogeneous of null degree. An arbitrary contravariant point v^ν can therefore always be given the form:

(234) $$v^\nu = v\, u^\nu ,$$

in which u^ν is a contravariant point of null degree and v is an *arbitrary* scalar factor ≥ 0. The criterion that v^ν must satisfy the homogeneity condition (234) reads like:

(235) $$x^\omega \partial_\omega v^\nu = \varphi\, v^\nu ,$$

or also:

(236) $$x^\omega (\partial_\omega v^{[\nu}) v^{\mu]} = 0 ;$$

i.e., the differential operator $x^\omega \partial_\omega$ must be a multiplier. Since the $\xi^{\nu'}$ are homogeneous of null degree and this makes $x^\rho \partial_\rho\, \xi^{\nu'} = 0$, we find, by means of (202), that:

(237) $$A_\mu^{\nu'} x^\mu = \psi\, x^{\nu'} , \qquad \psi = x^\rho \partial_\rho \log \chi ,$$

such that the contact position exists as a projective quantity here, as well.

52. However, some difficulties arise in the definition of a projective derivative that will only be briefly discussed here, and whose resolution we will however defer to a later occasion. Naturally, we would like to avoid the introduction of a covariant point, since otherwise a hyperplane ${}^nE$ would be distinguished in every ${}^{n+1}E$, which would be regarded as "imaginary," such that geometry, at least infinitesimally, would again take on an affine character. If one then defines:

(238) $$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Pi_{\lambda\mu}^\nu v^\lambda ,$$

then $\nabla_\mu\, v^\nu$ satisfies the homogeneity condition when this is the case with v^ν and the $\Pi_{\lambda\mu}^\nu$ are *homogeneous of degree* -1. Namely, a brief calculation shows that from $x^\rho \partial_\rho v^\nu = \varphi\, v^\nu$ it follows that:

(239) $$x^\rho \partial_\rho \nabla_\mu v^\nu = (\varphi - 1) \nabla_\mu v^\nu .$$

The difficulty is, however, that $\nabla_\mu v^\nu$ does not have a covariant character. Namely, if one transforms the $\Pi_{\lambda\mu}^\nu$ according to (53) then one has:

(240) $$\nabla_{\mu'} v^{\mu'} = \varphi (A_{\sigma\mu'}^{\nu'\rho} \nabla_\rho v^\sigma + v^{\nu'} \partial_{\mu'} \log \varphi) .$$

A term thus appears that is proportional to the differentiated quantities, and whose cofactor is a gradient. One can also not include this term in the transformation laws for the $\Pi^{\nu}_{\lambda\mu}$ since φ is not, as in the theory of densities or pseudo-quantities of Schouten and Hlavatý, a power of a *fixed* factor here, but it can depend on the v^{ν}.

53. The question is now raised of whether of whether one can once more extend the notion of quantity. In order to do this, we thus introduce an equivalence relation, and indeed we call the *contact point* and any *gradient null equivalent*, along with multiplicities of the two with arbitrary (not necessarily scalar) cofactors. We now call two projectors *equivalent* when a linear combination of them with non-vanishing coefficients is null equivalent, and we extend the notion of quantity by calling a system of numbers a *projector in the broader sense* when it transforms like a projector in the narrower sense, *up to terms that are null equivalent.*

Another possibility resolution is consequently to restrict oneself to alternating products and only to operate with derivative quantities like:

$$(\nabla_{\mu}\, v^{[\nu}) \, v^{\rho]}$$

(and not with $\nabla_{\mu}\, v^{\rho}$ itself), which, as one easily verifies, possess a completely covariant character.

However, we shall pursue the consequences of this far-reaching step, which promises to be rich in results, no further – in particular, when one introduces condition IIa – so it will remain just this fleeting hint.

Supplement [47]).

The later work on projective differential geometry has shown that the foregoing investigation can be simplified considerably in its formal respects, and indeed, in the following ways:

If $X^{\cdot}_{\cdot}$ is a projector with *t* contravariant and *s* covariant indices then we call *t* the *contravariant, s*, the *covariant, t* – *s* the *algebraic,* and *t* + *s* the *total valence* (or simply, the *valence*) [48]) of $X^{\cdot}_{\cdot}$. If $X^{\cdot}_{\cdot}$ is homogeneous of $\mathfrak{r}^{\text{th}}$ degree then we call the surplus $\varepsilon = \mathfrak{r} - (t - s)$ of the degree over the algebraic valence the *excess* of $X^{\cdot}_{\cdot}$. [49]) This number is invariant under addition, contraction with x^{ν}, and partial or covariant differentiation. The excess of a product is the sum of the excesses of the factor. The contact point x^{ν} has null excess. Even though $\Pi^{\nu}_{\lambda\mu}$ is not a projector, we also define its excess as its degree plus one.

Condition I then states that $\Pi^{\nu}_{\lambda\mu}$ and Q_{μ} have null excess. If we further introduce the functions:

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[47]) [Added on 6.1. 1932.]

[48]) From a suggestion of J.A. Schouten, we will apply the word "valence" to the difference of the degrees of homogeneity to describe the notion that we have called "degree" up till now (e.g., in R.K. pp. 23).

[49]) The excess is identical with the number that Veblen introduced (in a completely different way), and in the beginning [**5**, pp. 147] he called it "weight," but later [**6**, pp. 61] called it "index." Since both words already occur with other meanings, we prefer the word "excess."

$$\overset{*}{\Pi}{}^{\nu}_{\lambda\mu} = \overset{1}{\Pi}{}^{\nu}_{\lambda\mu} = \Pi^{\nu}_{\lambda\mu} + A^{\nu}_{\lambda} Q_{\mu} \; ,$$

instead of the $\Pi^{\nu}_{\lambda\mu}$ (cf., (61), which are associated with covariant derivatives of contravariant points of first degree, hence, of null excess, and ε is the excess of the differentiated quantities, then (59), (62), (63), turn into:

(59*) $$\nabla_{\mu}\, q = \partial_{\mu}\, q + \varepsilon\, Q_{\mu}\, q \, ,$$

(62*) $$\nabla_{\mu}\, v^{\nu} = \partial_{\mu}\, v^{\nu} + \overset{*}{\Pi}{}^{\nu}_{\lambda\mu}\, v^{\lambda} + \varepsilon\, Q_{\mu}\, v^{\nu} ,$$

(63*) $$\nabla_{\mu}\, w_{\lambda} = \partial_{\mu}\, w_{\lambda} - \overset{*}{\Pi}{}^{\nu}_{\lambda\mu}\, w_{\nu} + \varepsilon\, Q_{\mu}\, w_{\lambda} \, .$$

Furthermore, one has:

(68*) $$T_{\lambda\mu}^{\cdot\cdot\nu} = \overset{*}{S}{}_{\lambda\mu}^{\cdot\cdot\nu} ,$$

such that quasi-symmetry simply turns into the symmetry of the $\overset{*}{\Pi}{}^{\nu}_{\lambda\mu}$.

(69*) $$U_{\omega\mu} = \nabla_{[\omega}\, Q_{\mu]} + \overset{*}{S}{}_{\omega\mu}^{\cdot\cdot\rho}\, Q_{\rho} = \partial_{[\omega}\, Q_{\mu]} \, ,$$

(67*) $$\nabla_{[\omega\mu]}\, q = \overset{*}{S}{}_{\omega\mu}^{\cdot\cdot\rho}\, \nabla_{\rho}\, q + \varepsilon\, U_{\omega\mu}\, q \, ,$$

(70*) $$\nabla_{[\omega\mu]}\, v^{\nu} = -\frac{1}{2} \overset{*}{N}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\, v^{\lambda} + \overset{*}{S}{}_{\omega\mu}^{\cdot\cdot\rho}\, \nabla_{\rho}\, v^{\nu} + \varepsilon\, U_{\omega\mu}\, v^{\nu} ,$$

(71*) $$\nabla_{[\omega\mu]}\, w_{\lambda} = +\frac{1}{2} \overset{*}{N}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\, w_{\nu} + \overset{*}{S}{}_{\omega\mu}^{\cdot\cdot\rho}\, \nabla_{\rho}\, w_{\lambda} + \varepsilon\, U_{\omega\mu}\, w_{\lambda} \; ,$$

in which:

(55*) $$\overset{*}{N}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = -2\,\partial_{[\omega}\, \overset{*}{\Pi}{}^{\nu}_{|\lambda|\mu]} - 2\, \overset{*}{\Pi}{}^{\nu}_{\rho[\omega}\, \overset{*}{\Pi}{}^{\rho}_{|\lambda|\mu]} = N_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} + A^{\nu}_{\lambda} U_{\omega\mu} \, .$$

For the second identity, one finds:

(73*) $$\overset{*}{N}{}_{[\omega\mu\lambda]}^{\cdot\cdot\cdot\nu} = 2\,\nabla_{[\omega}\, \overset{*}{S}{}_{\mu\lambda]}^{\cdot\cdot\nu} + 4\, \overset{*}{S}{}_{[\omega\mu}^{\cdot\cdot\rho}\, \overset{*}{S}{}_{\lambda]\rho}^{\cdot\cdot\nu} ,$$

and for the Bianchi identity:

(74*) $$\nabla_{[\kappa}\, \overset{*}{N}{}_{\omega\mu]\lambda}^{\cdot\cdot\cdot\nu} = -2\, \overset{*}{S}{}_{[\kappa\omega}^{\cdot\cdot\rho}\, \overset{*}{N}{}_{\mu]\rho\lambda}^{\cdot\cdot\cdot\nu} ,$$

whereby we likewise note the identity:

($74\frac{1}{2}$*) $$\nabla_{[\kappa}\, U_{\omega\mu]} = -2\, \overset{*}{S}{}_{[\kappa\omega}^{\cdot\cdot\rho}\, U_{\mu]\rho} \, .$$

If we further introduce, instead of the quantities:

(56*) $\overset{1}{P}{}^{\cdot\nu}_{\lambda} = \overset{*}{\Pi}{}^{\nu}_{\lambda\mu} x^{\mu}$, $\overset{1}{Q}{}^{\nu}_{\cdot\mu} = \overset{*}{\Pi}{}^{\nu}_{\lambda\mu} x^{\lambda}$,

the quantities:

(56*) $\overset{*}{P}{}^{\cdot\nu}_{\lambda} = \overset{*}{\Pi}{}^{\nu}_{\lambda\mu} x^{\mu} + A^{\nu}_{\lambda} = P^{\cdot\nu}_{\lambda} + Q\,A^{\nu}_{\lambda}$,

(57*) $\overset{*}{Q}{}^{\nu}_{\cdot\mu} = \overset{*}{\Pi}{}^{\nu}_{\lambda\mu} x^{\lambda} + A^{\nu}_{\mu} = Q^{\nu}_{\cdot\mu} + A^{\nu}_{\mu} + x^{\nu} Q_{\mu}$,

then one has:

(75*) $\overset{*}{N}{}^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} x^{\lambda} = -2\nabla_{[\omega} \overset{*}{Q}{}^{\nu}_{\cdot\mu]} + 2\overset{*}{S}{}^{\cdot\cdot\rho}_{\omega\mu} \overset{*}{Q}{}^{\nu}_{\cdot\rho}$

(76*) $x^{\omega} \overset{*}{N}{}^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} = \nabla_{\mu} \overset{*}{P}{}^{\cdot\nu}_{\lambda}$,

(76½*) $x^{\omega} U_{\omega\mu} = -\tfrac{1}{2} \nabla_{\mu} Q$,

(77*) $\nabla_{\mu} x^{\nu} = \overset{*}{Q}{}^{\nu}_{\cdot\mu} = \overset{*}{P}{}^{\cdot\nu}_{\lambda} - 2\overset{*}{S}{}^{\cdot\cdot\nu}_{\lambda\mu} x^{\mu}$,

(78*) $x^{\lambda} \nabla_{\lambda} v^{\nu} = \overset{*}{P}{}^{\cdot\nu}_{\lambda} v^{\lambda} + \varepsilon Q\, v^{\nu}$,

(79*) $x^{\lambda} \nabla_{\lambda} w_{\mu} = -\overset{*}{P}{}^{\cdot\nu}_{\lambda} w_{\nu} + \varepsilon Q\, w_{\mu}$.

The existence condition for the covariant differential of an arbitrary projector of excess ε is:

(IIa*) $\overset{*}{P}{}^{\cdot\nu}_{\lambda} = P\,A^{\nu}_{\lambda}$,

along with:

(85*) $\dfrac{\overset{*}{P}}{Q} = \text{const.} = \varepsilon$.

Theorem 3 preserves the same form (up to a replacement of the quantities without stars with ones with stars). The form of the existence conditions II, III for the geodetic lines also remains unchanged. Thus, one has:

$$\overset{*}{p}_{\lambda} = p_{\lambda}, \qquad \overset{*}{q}_{\mu} = p_{\mu} + Q_{\mu},$$

$$\overset{*}{P} = P + Q, \qquad \overset{*}{p} = p, \qquad \overset{*}{q} = q + Q - 1.$$

If one replaces $N'{}^{\cdot\cdot\cdot d}_{abc}$ in the assumptions of theorem 6a with $\overset{*}{N}'{}^{\cdot\cdot\cdot d}_{abc}$ then the form of (144), (147) stays the same, whereas (145), (146) simplify to:

(145*) $\nabla_{(\lambda} \overset{*}{p}_{\mu)} + \overset{*}{p}_{(\lambda} \overset{*}{p}_{\mu)} = 0$,

(146*) $\overset{*}{q}_{\mu} = \overset{*}{P}$.

The corresponding formal changes in the other theorems and formulas may be easily carried out. On the whole, they therefore change nothing.

In order to make the comparison with the older theories more comfortable, we then decide to also write (218), (219) in the form:

$$A_\nu^{\underline{0}} = \frac{1}{x^0} A_\nu^0, \quad A_\nu^{\underline{k}} = \frac{2}{x^0} A_\nu^{[k} x^{0]}, \tag{218a}$$

$$A_{\underline{0}}^\nu = x^\nu, \qquad A_{\underline{i}}^\nu = x^0 A_i^\nu . \tag{219a}$$

The *covariant* Veblen index 0 (which is our $\underline{0}$) thus means contraction by x^ν, from which, the scalar nature of the index is clarified. On the other hand, the *contravariant* Veblen index 0 (as Veblen himself remarked) has no invariant meaning (one must give the field x^0 or $\mathfrak{r}$). In general, one has:

$$v^{\underline{0}} = \frac{1}{x^0} v^0, \quad v^{\underline{k}} = \frac{1}{x^0}\left(v^k - \frac{x^k}{x^0} v^0\right), \text{ [50]} \tag{A}$$

$$w_{\underline{0}} = w_\lambda x^\lambda, \qquad w_{\underline{i}} = x^0 w_i . \tag{B}$$

From (219a), it then follows that:

$$\partial_{\underline{0}} A_{\underline{0}}^\nu = x^\nu, \qquad \partial_{\underline{0}} A_{\underline{i}}^\nu = \partial_{\underline{i}} A_{\underline{0}}^\nu = x^0 A_i^\nu, \qquad \partial_{\underline{k}} A_{\underline{i}}^\nu = 0, \tag{219b}$$

hence:

$$\begin{aligned}
&\overset{*}{\Pi}{}^{\underline{0}}_{\underline{00}} = \frac{1}{x^0}\overset{*}{b}{}^0, && \overset{*}{\Pi}{}^{\underline{k}}_{\underline{00}} = \frac{1}{(x^0)^2}\left(\overset{*}{b}{}^k x^0 - \overset{*}{b}{}^0 x^k\right), \\
&\overset{*}{\Pi}{}^{\underline{0}}_{\underline{i0}} = \overset{*}{P}{}_i^{\cdot 0}, && \overset{*}{\Pi}{}^{\underline{k}}_{\underline{i0}} = \frac{1}{x^0}\left(\overset{*}{P}{}_i^{\cdot k} x^0 - \overset{*}{P}{}_i^{\cdot 0} x^k\right), \\
&\overset{*}{\Pi}{}^{\underline{0}}_{\underline{0j}} = \overset{*}{Q}{}^{0}_{\cdot j}, && \overset{*}{\Pi}{}^{\underline{k}}_{\underline{0j}} = \frac{1}{x^0}\left(\overset{*}{Q}{}^{k}_{\cdot i} x^0 - \overset{*}{Q}{}^{0}_{\cdot i} x^k\right), \\
&\overset{*}{\Pi}{}^{\underline{0}}_{\underline{ij}} = x^0 \overset{*}{\Pi}{}^{0}_{ij}, && \overset{*}{\Pi}{}^{\underline{k}}_{\underline{ij}} = \overset{*}{\Pi}{}^{k}_{ij} x^0 - \overset{*}{\Pi}{}^{0}_{ij} x^k,
\end{aligned} \tag{C}$$

in which we have set:

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[50]) In order to avoid ambiguity, we recommend that the Veblen coordinates not be denoted (as we did in § 14) by x^ν, but by, e.g., ξ^ν:

$$\xi^{\underline{0}} = \log x^0, \qquad \xi^{\underline{k}} = \frac{x^k}{x^0}.$$

From (A):

$$x^{\underline{0}} = 1, \qquad x^{\underline{k}} = 0,$$

are the *values* of the contact point x^ν, whereas the $\xi^{\underline{\nu}}$ are its Veblen *coordinates*.

(D) $$\overset{*}{b}{}^{\nu} = \overset{*}{P}{}_{\lambda}^{\cdot\nu} x^{\lambda} = \overset{*}{Q}{}^{\nu}_{\cdot\mu} x^{\mu} = \overset{*}{\Pi}{}^{\nu}_{\lambda\mu} x^{\lambda} x^{\mu} + x^{\nu} .$$

Ultimately, since they are both valid, one must decide whether to identify the Veblen $\Pi^{\nu}_{\lambda\mu}$ with our $\Pi^{\nu}_{\underline{\lambda\mu}}$ or our $\overset{*}{\Pi}{}^{\nu}_{\underline{\lambda\mu}}$, since for Veblen, one has $Q_{\mu} = 0$, whereas the theory of Sch. and G. can be identified with ours in two kinds of ways. Namely, if one demands that the values of the Sch. and G. points relative to the *homogeneous* coordinates be pure functions of position then the point quantities correspond to our projectors of null degree. However, if one then demands that the values relative to the *Veblen* coordinates be pure functions of position then the point quantities correspond to our quantities with null *excess*. In the latter case, equations (220), (221), (223), (224) are replaced with the corresponding equations with a *, whereas, from the anomalous definition of $\overset{*}{P}{}_{\lambda}^{\cdot\nu}$, (222) turns into:

(222*) $$\overset{*}{P}{}_{\lambda}^{\cdot\nu} = c\, A^{\nu}_{\lambda} .$$

Corresponding statements are valid for theorem 12. We also remark that for Sch. and G. the weight of the quantities relates to the determinant Δ_0, hence, from (215), upon multiplying by $\frac{c-1}{c}$, it emerges from the projective weight (relative to Δ).

(Received on 11. 5. 1931.)

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