INTRODUCTION TO ALGEBRAIC GEOMETRY

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FOREWORD

From the lectures in the worthwhile Ergebnisse textbook “Algebraic Surfaces” by O. ZARISKI, the thought of writing an introduction to algebraic geometry took on a definite form for me. Such an introduction should contain the “elements” of algebraic geometry in the classical sense of the word; i.e., it should provide the necessary foundations for going further into the deeper theory. Also, Herr GEPPERT, who intended to write a book on algebraic surfaces in this collection, emphasized the necessity of such an introduction, to which he could then refer, and encouraged me to write this book.

What I learned in the course of my own very substantial lectures on algebraic curves and surfaces proved useful to me in the process of writing; I can therefore employ a lecture preparation that was prepared by Dr. M. DEURING and Dr. V. GARTEN. Thus, much material was borrowed from my series of articles in the Mathematischen Annalen “Zur algebraischen Geometrie.”

In the choice of material, it was not the aesthetic viewpoint, but ultimately the distinction between being necessary and being dispensable that was definitive. Everything that is derived from the “elements” without qualification will have to be, I hope, assumed. The theory of ideals, which led me to my earlier investigations, seems to be dispensable for laying the groundwork; the far-reaching methods of the Italian school will take its place. For the explanation of the methods and extension of the problem statement substantial individual geometric problems would have to be addressed; for that reason, I have also sought to restrict that extension to a certain degree here, since otherwise the scope would easily grow without bound.

I was assisted in the correction process by Herren Prof. H. GEPPERT, Dr. O.-H. KELLER, Dr. H. REICHARDT, and Prof. G. SCHAAKE, who also pointed out many improvements, and for this they have my deepest gratitude. Herr Dr. REICHARDT made the sketches of the figures. The publisher has given the book their well-known impeccable attention, which is the culmination of my most special wish.

Leipzig, February 1939.

B. L. VAN DER WAERDEN

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Introduction

Algebraic geometry came about through the organic blending of the highly developed theory of algebraic curves and surfaces in Germany with the higher-dimensional geometry of the Italian school; function theory and algebra both share its cradle. The creator of algebraic geometry in the strict sense of the term was MAX NOETHER; its final unfolding into a mature flower was the work of the Italian geometers SEGRE, SEVERI, ENRIQUES, and CASTELNUOVO. A second blossom was the algebraic geometry of our own time, since topology has placed itself at our service, while algebra was simultaneously developing from the examination of its foundations.

This little book will not go any further into these foundations. The algebraic basis for algebraic geometry is now flourishing to such an extent that it would not be possible to present the theory "from the top down." Starting from an arbitrary ground field, one can develop the theory of algebraic manifolds in n-dimensional space just like the theory of fields of algebraic functions in one variable. By specialization, one would then obtain plane algebraic curves, space curves, and surfaces. The connection with function theory and topology will subsequently present itself when one chooses the ground field to be the field of complex numbers.

This type of presentation will not be chosen here. Rather, the historical development will be followed consistently, although one also considers the examples in a somewhat abbreviated and distorted form. We will therefore always strive to first make the necessary intuitive material available before we develop the general notions. First, we treat the elementary structures in projective spaces (linear subspaces, quadrics, rational normal curves, collineations, and correlations), then the plane algebraic curves (with occasional glimpses of surfaces and hypersurfaces), and then manifolds in n-dimensional space. At first, the ground field will be the field of complex numbers; later, the use of more general fields will be introduced, but always ones that include the field of all algebraic numbers. We will seek to do this in each case, while drawing upon the most elementary lemmas possible, even when the theorems in question themselves prove to be special cases of more general theorems. As an example, I cite the elementary theory of point groups for curves of third order, in which use is made of neither elliptic functions nor the fundamental theorem of NOETHER.

This manner of presentation has the advantage that the beautiful methods and results of the classical geometers such as PLÜCKER, HESSE, CAYLEY, and CREMONA, up until the school of CLEBSCH, once again take their rightful place. Moreover, the connection with the function-theoretic way of looking at things is likewise present from the onset in the theory of curves, in which the notion of the branch of a plane algebraic curve will be clarified with the help of PUISEUX's series development. The oft-heard reproach that this method is not purely algebraic is easily refuted. I know full well that the theory of valuations makes possible a beautiful and general algebraic foundation, but it seems to me that for a correct understanding it is important that the reader be first familiar with PUISEUX's series and have an intuition for the singularities of algebraic curves.

In chapter 4, we first encounter the general theory of algebraic manifolds. At the center of this, we have their decomposition into irreducible manifolds, along with the notions of general points and dimension.

An important special case of an algebraic manifold is given by the algebraic correspondences between two manifolds, to which chapter 5 is dedicated. The simplest theorem concerning irreducible correspondences – in particular, the principle of constant count – generates numerous applications. Chapter 6 introduces the essential feature of the Italian way of treating things: viz., the linear families that lie at the basis of the theory of birational invariants of algebraic manifolds. In chapter 7, the fundamental theorem of NOETHER will be presented, along with its *n*-dimensional generalizations and various corollaries, among which is the BRILL-NOETHER remainder theorem. Finally, chapter 8 will give a brief outline of theory of points that are "infinitely close" to plane curves.

Whoever is somewhat familiar with *n*-dimensional projective geometry (chap. 1) and the basic notions of algebra (chap. 2) can just as well begin the lectures in this book with either chapter 3 or chapter 4; both of them are independent of each other. Chapters 5 and 6 refer to only chapter 4 in an essential way. The first time that we will use everything that preceded our discussion will be in chapter 7.

CHAPTER ONE.

Projective geometry of n-dimensional spaces.

Only the first seven sections and §10 in this chapter will be required throughout this book. The remaining sections have only the goal of introducing intuitive material and simple examples that can be treated without the aid of higher algebraic concepts, and can therefore prepare one for the general theory of algebraic manifolds later on.

§ 1. The projective space S_n and its linear subspaces.

For quite some time, it has been found to be convenient to extend the domain of real points to that of complex points in the projective geometry of planes and spaces. Whereas a real point of the projective plane will be given by three *real* homogeneous coordinates (y_0, y_1, y_2) that are not all zero and can be multiplied by a factor $\lambda \neq 0$, a "complex point" will be given by three *complex* numbers (y_0, y_1, y_2) that are also not all zero and can be multiplied by a factor $\lambda \neq 0$.

One can define the notion of a complex point in a purely geometric way, as in VON STAUDT ([1]). It is, however, much simpler to define the notion algebraically and to understand a complex point of the plane to be simply the totality of all triples of numbers $(y_0 \lambda, y_1\lambda, y_2 \lambda)$ that can be obtained from a fixed triple of complex numbers (y_0, y_1, y_2) by multiplying with an arbitrary factor λ. Analogously, a complex point of space will be defined to be the totality of all proportional quadruples of numbers. These algebraic definitions will be established in the sequel.

Once one has been so far removed from geometric intuition in this way, by regarding points as purely algebraic structures, nothing else stands in the way of making an n-dimensional generalization. One understands a *complex point of n-dimensional space* to mean the totality of all $(n+1)$-tuples of numbers$(y_0 \lambda, y_1\lambda, \ldots, y_n \lambda)$ that can be obtained from a fixed $(n+1)$-tuple of complex numbers $(y_0, y_1, \ldots, y_n)$ that are not all zero by multiplying with an arbitrary factor λ. The totality of all points that are defined in this way is called the *n-dimensional complex projective space S_n*.

One can pursue this generalization even further. Namely, one can consider an arbitrary *commutative field* $\mathbb{K}$, in the sense of algebra, in place of the field of complex numbers, a field that we shall only assume is, like the field of complex numbers, algebraically closed; i.e., that any non-constant polynomial $f(x)$ over the field $\mathbb{K}$ can be completely decomposed into linear factors. Examples of algebraically closed fields are: the field of algebraic numbers, the field of complex numbers, and the field of algebraic functions of k indeterminates. All of these fields lead to projective spaces that agree in their properties so closely that we can treat all of them the same way.

([1]) Cf., the thorough presentation of G. JUEL, *Vorlesungen über projective Geometrie*, Berlin 1934, which appears in this collection.

It is now convenient to relate the notion of projective space to that of vector space. An n-tuple $(y_1, \dots, y_n)$ of elements of $\mathbb{K}$ is called a *vector*. The totality of all vectors is called the *n-dimensional vector space* E_n. Vectors can be added, subtracted, and multiplied by field elements in a well-known way. Any m vectors $\overset{1}{v}, \dots, \overset{m}{v}$ are called *linearly independent* when $\overset{1}{v}\gamma_1 + \dots + \overset{m}{v}\gamma_m = 0$ always implies that $\gamma_1 = \dots = \gamma_m = 0$. Any n linearly independent vectors $\overset{1}{v}, \dots, \overset{m}{v}$ span the entire vector space; i.e., any vector v may be written as a linear combination $\overset{1}{v}\gamma_1 + \dots + \overset{n}{v}\gamma_n = v$. The totality of all linear combinations of m linearly independent vectors $\overset{1}{v}, \dots, \overset{m}{v}$ $(m \leq n)$ is called an *m-dimensional linear subspace* E_m of the vector space E_n. The *dimension m* is the number of linearly independent basis vectors $\overset{1}{v}, \dots, \overset{m}{v}$ ([2]).

In particular, a one-dimensional subspace consists of all vectors $\overset{1}{v}\lambda$, where $\overset{1}{v} = (y_0, y_1, \dots, y_n)$ is fixed vector that is different from zero. A point of the projective space S_n, with the definition above, is nothing but a one-dimensional subspace, or *ray*, in E_{n+1}. *S_n is then the totality of all rays in the vector space E_{n+1}.*

A subspace S_m of S_n can now be defined as the totality of all rays in a subspace E_{m+1} of E_{n+1}. Therefore, S_m is comprised of all points y whose coordinates depend linearly upon the coordinates $(y_0, y_1, \dots, y_m)$ of $m + 1$ linearly independent points $\overset{0}{y}, \dots, \overset{m}{y}$:

$$y_k = \overset{0}{y}_k\,\gamma_0 + \overset{1}{y}_k\,\gamma_1 + \dots + \overset{m}{y}_k\,\gamma_m \qquad (k = 0, 1, \dots, n)\,. \tag{1}$$

The field elements $\gamma_0, \dots, \gamma_m$ can be referred to as *homogeneous coordinates* (or *parameters*) in the subspace S_m. The points $\overset{0}{y}, \dots, \overset{m}{y}$ are the basis points of this coordinate system. Thus, since every point of the subspace is determined by $m + 1$ homogeneous coordinates $\gamma_0, \dots, \gamma_m$ the notation S_m for the subspace will be naturally justified. The one-dimensional subspaces are called *lines*, the two-dimensional subspaces, *planes*, and the $(n - 1)$-dimensional subspaces are called *hyperplanes* in S_n. An S_0 is a point.

Formula (1) will thus still be valid when $m = n$, when S_m agrees with all of the space S_n. The parameters $\gamma_0, \dots, \gamma_m$ will then be new coordinates for the point y that are connected with the old coordinates by the linear transformation (1). We now write it as follows ([3]):

$$y_k = \sum \overset{i}{y}_k\,\gamma_i\,.$$

([2]) For the proof of this, see perhaps B. L. VAN DER WAERDEN: *Moderne Algebra I*, § 28 or II, § 105.

([3]) Here and in the sequel, a Σ-sign with no other givens means that one sums over any two identical indices (preferably one above and one below).

Since the points $\overset{0}{y}, \ldots, \overset{m}{y}$ were assumed to be linearly independent, one can solve these equations for the γ_i:

$$\gamma_i = \sum \vartheta_i^k y_k \ .$$

The γ_i are called *general projective coordinates* (in the plane: triad coordinates, in space: tetrad coordinates). In particular, if $\left(\overset{i}{y}_k\right)$ is the identity matrix then the γ_k will be the original y_k .

d independent homogeneous linear equations in the coordinates $y_0, y_1, \ldots, y_n$ define a S_{n-d} in S_n; as is well known, their solutions can then be linearly obtained from $n - d + 1$ linearly independent solutions. In particular, a single linear equation:

$$u^0 y_0 + u^1 y_1 + \ldots + u^n y_n = 0 \tag{2}$$

defines a hyperplane. The coefficients $u^0, u^1, \ldots, u^n$ are called the *coordinates* of the hyperplane u. They are determined only up to a common factor $\lambda \neq 0$, since equation (2) may be multiplied by precisely such a factor.

We denote the left-hand side of equation (2), once and for all, by u_y, or $(u\ y)$. We then set:

$$(u\ y) = u_y = \sum u^i y_i = = u^0 y_0 + u^1 y_1 + \ldots + u^n y_n \, .$$

Any linear space S_d in S_n be defined by $n - d$ linearly independent linear equations. If S_d is determined by the points $\overset{0}{y}, \ldots, \overset{d}{y}$ then the $d + 1$ linear equations:

$$(u\ \overset{0}{y}) = 0 \ , \ \ (u\ \overset{1}{y}) = 0 \ , \ \ldots, (u\ \overset{d}{y}) = 0$$

in the unknowns $u^0, u^1, \ldots, u^d$ will have precisely $n - d$ linearly independent solutions. Any of these solutions defines a hyperplane, and the intersection of these $n - d$ hyperplanes is an S_d that includes the points $\overset{0}{y}, \overset{1}{y}, \ldots, \overset{d}{y}$, and therefore must be identical with the given S_d .

Problems. 1. n linearly independent points $\overset{1}{y}, \ldots, \overset{n}{y}$ determine a hyperplane u. Show that the coordinates u^ν of this hyperplane are proportional to the n-rowed sub-determinants of the matrix $\left(\overset{i}{y}_k\right)$.

2. n linearly independent hyperplanes $u_1, \ldots, u_n$ determine a point y. Show that the coordinates y_ν of this point are proportional to the r-rowed sub-determinants of the matrix (u_i^k).

3. Being given the basis points $\overset{0}{y}, \ldots, \overset{m}{y}$ in a space S_m does not uniquely determine the coordinates $\gamma_0, \ldots, \gamma_m$ of a point y, since one can multiply the coordinates of the basis points by arbitrary non-zero factors $\lambda_0, \ldots, \lambda_m$. Show that the coordinates for each point y are uniquely determined up to a common factor $\lambda \neq 0$ as long as one is given the "unity point" e, which has the coordinates $\gamma_0 = 1, \ldots, \gamma_m = 1$. Can the unity point in S_m be chosen arbitrarily?

4. Show that an S_{m-1} in S_m is given by a linear equation in the coordinates $\gamma_0, \ldots, \gamma_m$.

5. Show that the transition from one system of parameters $\gamma_0, \ldots, \gamma_m$ in an S_m to another parameter system (defined by other basis points) for the points of the same S_m can be mediated by a *linear* parameter transformation:

$$\gamma_i' = \sum \alpha_i^k \gamma_k \, .$$

§ 2. The projective combination theorems.

From the definitions in §1, there immediately follow the mutually dual combination theorems:

I. *$m + 1$ points in S_n that do not line in an S_q with $q < m$ determine an S_m.*

II. *d hyperplanes in S_n that have no S_q with $q > n - d$ in common determine an S_{n-d}.*

We now prove, in addition that:

III. *When $p + q \geq n$, an S_p and an S_q in S_n will have a linear space S_d of dimension $d \geq p + q - n$ for their intersection.*

Proof: S_p is defined by $n - p$ independent linear equations, and S_q is defined by $n - q$ linear equations. Collectively, that is $2n - p - q$ linear equations. If they are independent then they will define a space of dimension $n - (2n - p - q) = p + q - n$. If they are dependent then one can omit some of them, and the dimension of the intersection space will go up.

IV. *An S_p and an S_q that have an S_d in common lie in an S_m with $m \leq p + q - d$.*

Proof. The intersection space S_d is determined by $d + 1$ linearly independent points. In order to determine S_p, one must add $p - d$ more points to these $d + 1$ points in order to obtain $p + 1$ linearly independent points. In order to determine S_q one must likewise add $q - d$ more points. All of these:

$$(d + 1) + (p - d) + (q - d) = p + q - d + 1$$

points will determine an S_{p+q-d}, in the event that they are linearly independent. The S_m that is thus determined with $m \leq p + q - d$ will contain all of the points that determine S_p, as well as the ones that determine S_q; hence, it will contain S_p and S_q themselves.

If there is no intersection space S_d then the same argument will teach us that:

V. *An S_p and an S_q always lie in an S_m with $m \leq p + q + 1$.*

With the help of III, one can sharpen IV and V to:

VI. *An S_p and an S_q whose intersection is an S_p (is empty, resp.) lie in a uniquely determined S_{p+q-d} (S_{p+q+1}, resp.).*

Proof. First, let the intersection be S_d. From IV, S_p and S_q lie in an S_m with $m \leq p + q - d$. On the other hand, from III:

$$d \geq p + q - m, \qquad \text{hence,} \qquad m \geq p + q - d .$$

From this, it follows that $m = p + q - d$. If S_p and S_q were contained in yet another S_m then the intersection of these two S_m would have a smaller dimension, which, from what we just proved, would not be possible.

Now, let the intersection be empty. From V, S_p and S_q lie in an S_m with $m \leq p + q + 1$. If $m \leq p + q$ then, from III, S_p and S_q would have a non-empty intersection. Hence, one must have $m = p + q + 1$. Just as in the first case, one further gets that S_m is unique.

The space S_{p+q-d} (S_{p+q+1}, resp.) that is defined by VI is called the *join* of S_p and S_q.

Problems. 1. Derive the combination axioms for the plane S_2, the space S_3, and the space S_4 by specializing I, II, III, VI.

2. If one projects all of the points of a space S_m in S_n onto another space S'_m in S_n, in which both are linked by a given S_{n-m-1} and the coupling space S_{n-m} always intersects S'_m, then there will be a one-to-one map of the points of S_m onto the points of S'_m, assuming that S_{n-m-1} has points in common with either S_m or S'_m.

§ 3. The duality principle. Further concepts. Double ratio.

A space S_p is called *incident* with an S_q when either S_p is contained in S_q or S_q is contained in S_p. In particular, a point y will be incident with a hyperplane u when the relation $(u\, y) = 0$ is valid.

Since a hyperplane, like a point of S_n, is given by $n + 1$ homogeneous coordinates u^0, …, u^n (y_0, …, y_n, resp.), which can be multiplied by a factor $\lambda \neq 0$, and since the incidence relation $(u\, y) = 0$ involves both u and y in the same way, one will then have the *n-dimensional duality principle, which says that in any correct statement about the incidence of points and hyperplanes, these two notions may be interchanged without influencing the validity of the statement.* For example, in the plane, the notions of points and lines, and in space, the notions of points and planes can be interchanged in any theorem that treats only the incidence of points and lines (planes, resp.).

One can also formulate the duality principle as: Any figure that consists of points and hyperplanes may be associated with a figure that consists of hyperplanes and points and exhibits the same incidence relations as the original one. Namely, one can associate any point y with a hyperplane u with the same coordinates y_0, …, y_n, and any hyperplane u may be associated with a point with the same coordinates u^0, …, u^n. The relation $(u\, y) = 0$ will thus remain true. The association itself is a particular *correlation* or *duality*. The space of points (u^0, …, u^n) is also called the *dual space* to the original S_n.

We would now like to investigate what a linear space S_m corresponds to under duality. S_m will be given by $n - m$ independent linear equations in the point coordinates y. If one now regards the y as the coordinates of a hyperplane then one has $n - m$ independent linear equations that express that the hyperplane y shall go through $n - m$ linearly independent points. These $n - m$ points determine an S_{n-m-1}, and the linear equations state that the hyperplane y shall be contained in the space S_{n-m-1} .Thus, any S_m corresponds to an S_{n-m-1} under duality, and the points of S_m correspond to the hyperplanes through S_{n-m-1} .

Now let S_p be contained in an S_q, i.e., let all of the points of S_p be likewise points of S_q. Dually, S_p corresponds to an S_{n-p-1} and S_q to an S_{n-q-1}, such that all hyperplanes through S_{n-p-1} likewise go through S_{n-q-1} . However, that means that S_{n-q-1} is obviously contained in an S_{n-p-1} . *The relation of inclusion of linear spaces is thus inverted under duality.*

On the basis of this consideration, one can apply the principle of duality to not only figures that consist of points and hyperplanes, but also figures that consists of arbitrary linear spaces S_p, S_q, …, as well as theorems concerned with such figures. Duality associates any S_p with an S_{n-p-1} , and all incidence relations for S_p remain true: When S_q is contained in S_p, the S_{n-p-1} that corresponds to S_p will be contained in the S_{n-q-1} that corresponds to S_q.

A series of derived notions arise from the basic notions of projective geometry that were defined in § 1, and we shall summarize the most important ones here.

The totality of points on a line is called a (linear) family of points. The line is called the *carrier* of the family of points. Dual to this is the totality of all hyperplanes in S_n that contain an S_{n-2} . One calls this totality a *pencil of hyperplanes* (n = 2: pencil of rays, n = 3: pencil of planes) and the S_{n-2} , the *carrier* of the pencil. For the pencil, just as for the family of points, one will have a parametric representation:

$$u^k = \lambda_0\, s^k + \lambda_1\, t^k . \tag{1}$$

The totality of points in a plane S_2 is called a planar *point field* with the *carrier* S_2. Dual to this is the notion of a *net* or a *bundle* of hyperplanes in S_n that contain an S_{n-3} as a *carrier* of the bundle. The parametric representation of a net is:

$$u^k = \lambda_0\, r^k + \lambda_1\, s^k + \lambda_2\, t^k .$$

The totality of all linear spaces through a point y in S_n is called a *star* with *carrier* y.

If u, v, x, y are four different points on a line, and one sets:

$$\begin{cases} x_k = u_k \lambda_0 + v_k \lambda_1, \\ y_k = u_k \mu_0 + v_k \mu_1 \end{cases} \tag{2}$$

then one calls the quantities:

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \frac{\lambda_1 \mu_0}{\lambda_0 \mu_1} \tag{3}$$

the *double ratios* of the four points u, v, x, y. The double ratio obviously does not change when the coordinates of u or v, or x or y are multiplied by a factor $\lambda \neq 0$; thus, it depends only upon the four points, not their coordinates.

One also defines the double ratio of four hyperplanes in a pencil (or perhaps four lines in a planar pencil of rays) by precisely the same formulas (2), (3).

Problems. 1. Under duality, the intersection of two linear spaces corresponds to the join, and conversely.

2. Prove the following transformation principle by projecting onto an S_n in S_{n+1} from a point of S_{n+1}: Any valid theorem concerning the incidence of points, lines, …, hyperplanes in an S_n corresponds to an equally valid theorem concerning the incidence of lines, planes, …, hyperplanes of a star in S_{n+1}.

3. Prove the formulas:

$$\begin{bmatrix} u & v \\ x & y \end{bmatrix} = \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} v & u \\ y & x \end{bmatrix} = \begin{bmatrix} y & x \\ v & u \end{bmatrix},$$

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix}\begin{bmatrix} y & x \\ u & v \end{bmatrix} = 1\,,$$

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} + \begin{bmatrix} x & u \\ y & v \end{bmatrix} = 1\,.$$

4. If *a*, *b*, *c*, *d* are four points in a plane, no three of which lie in a line, then their coordinates can be normalized so that one has:

$$a_k + b_k + c_k + d_k = 0\,.$$

The “diagonal point” *p*, *q*, *r* of the “complete rectangle” *abcd*, i.e., the intersection point of *ab* with *cd*, of *ac* with *bd*, and *ad* with *bc*, can then be represented by:

$$\begin{aligned} p_k &= a_k + b_k = -c_k - d_k \\ q_k &= a_k + c_k = -b_k - d_k \\ r_k &= a_k + d_k = -b_k - c_k\,. \end{aligned}$$

5. With the help of the formulas, and with the notation of Problem 4, prove the *Complete Rectangle Theorem*, which says that the diagonal points *p* and *q* lie harmonically with the intersection points *s* and *t* of *pq* with *ab* and *bc*, i.e., the double ratio is:

$$\begin{bmatrix} p & q \\ s & t \end{bmatrix} = 1\,.$$

6. How does the theorem in the projective geometry of the plane that is dual to the Complete Rectangle Theorem read?

§ 4. Multiply-projective spaces. Affine space.

The totality of pairs of points (x, y), where x is a point of an S_m and y is a point of an S_n, is the *doubly-projective space* $S_{m,n}$. A *point* of $S_{m,n}$ is thus a pair of points (x, y). Analogously, one defines triply and multiply-projective spaces. One considers the number $m + n$ to be the dimension of the space $S_{m,n}$.

The goal of the introduction of multiply-projective spaces is to treat all problems in manifolds of point pairs, point triples, etc., or equations in which more homogeneous families of variables appear in a manner that is analogous to the corresponding problems in manifolds of points and homogeneous equations in one family of variables.

One understands an *algebraic manifold* in a multiply-projective space $S_{m,n}$ to mean the totality of points $(x, y, \ldots.)$ in the space that satisfy a system of equations $F(x, y, \ldots) = 0$ that is homogeneous in each family of variables. The solution of a *single* equation $F(x, y, \ldots) = 0$ of degree numbers $g, h, \ldots$ will define an *algebraic hypersurface* in $S_{m,n}$ of degree numbers $g, h, \ldots$

A hypersurface in an ordinary projective space S_n has only one degree number: viz., the *degree* or *order* of the hypersurface. A hypersurface of degree 2, 3, or 4 is also called a quadratic, cubic, or bi-quadratic hypersurface. A hypersurface in S_2 or $S_{1,1}$ is called a *curve*, and a hypersurface in S_3, a *surface*. A curve of degree 2 in S_2 is called a *cone*, and a hypersurface of degree 2 is generally called a *quadric*.

One can map the points of a doubly-homogeneous space $S_{m,n}$ to the points of an algebraic manifold $S_{m,n}$ in an ordinary projective space S_{mn+m+n} in a one-to-one manner. To this end, one sets:

$$z_{ik} = x_i\, y_k \qquad (i = 0, 1, \ldots, m;\ k = 0, 1, \ldots, n) \tag{1}$$

and regards the $(m + 1)(n + 1)$ elements z_{ik}, which are not all zero, as the coordinates of a point in S_{mn+m+n} . Conversely, one can determine the x and y uniquely from the z_{ik} , up to a common factor λ . Hence, when perhaps $y_0 \neq 0$, from (1), the $x_0, \ldots, x_m$ will be proportional to $z_{00}, z_{10}, \ldots, z_{m0}$. The z_{ik} will be coupled by the $\binom{m+1}{2}\binom{n+1}{2}$ equations:

$$z_{ik}\, z_{jl} = z_{il}\, z_{jk} \qquad (i \neq j,\ k \neq l). \tag{2}$$

The manifold $S_{m,n}$ will thus be defined by a system of $\binom{m+1}{2}\binom{n+1}{2}$ quadratic equations. They are called *rational*, since their points admit the rational parametric representation (1).

The simplest case of the map (1) is the case $m = 1$, $n = 1$. Equations (2) will then define a quadratic surface in three-dimensional space:

$$z_{00}\, z_{11} = z_{01}\, z_{10}\ , \tag{3}$$

and any non-singular quadratic equation (viz., an equation of a quadric with no double points) can be brought into the form (3) by a projective transformation. We thus have a map of point pairs of second degree to the points of an arbitrary double-point-free quadric before us. This map will be used in the sequel, in order to study the properties of points, lines, and curves on the quadric.

Problems. 1. Two systems of linear spaces S_m (S_n, resp.) lie on the manifold $S_{m,n}$, which can be obtained when one holds the x or the y constant [special case: two families of lines on the surface (3)]. Any two spaces in different families have a point in common, and any two spaces in the same family have no point in common.

2. An equation $f(x, y) = 0$ that is homogeneous of degree l in x_0, x_1 and homogeneous of degree m in y_0, y_1 defines a *curve* $C_{l,m}$ of degree (l, m) on the quadratic surface (3). Show that a line on the surface has degree (1, 0) or (0, 1), a planar section of the surface has degree (1,1), and an intersection with a quadratic surface has degree (2, 2).

3. A curve of degree (k, l) on the quadratic surface (3) will intersect a plane at $k + l$ points, in general. Prove this assertion, and then make the expression "in general" more precise by enumerating all of the possible cases. (Write down the equation of the curve and that of a planar section, and then eliminate x or y from these equations.)

If one omits all points of the hyperplane $y_0 = 0$ from the projective space S_n then what remains will be the *affine space* A_n. One has $y_0 \neq 0$ for the points of affine space, but one can multiply the coordinates by a factor such that $y_0 = 1$. The remaining coordinates $y_1, \dots, y_n$ – viz., the *inhomogeneous coordinates* of the point y – will then be determined uniquely. Any point of the affine space A_n is thus in one-to-one correspondence with a system of n coordinates $y_1, \dots, y_n$.

If one distinguishes a point $(0, \dots, 0)$ in affine space then it will become a vector space. There is then a one-to-one correspondence between points $(y_1, \dots, y_n)$ and vectors $(y_1, \dots, y_n)$. (Conversely, one can likewise regard any vector space as an affine space.)

Vector spaces and affine spaces are simpler from an algebraic standpoint than projective spaces, since one can recognize their points to be in one-to-one correspondence with n elements $y_1, \dots, y_n$ of the field $\mathbb{K}$. Geometrically, however, the projective space S_n is simpler and more interesting.

For the algebraic treatment of the projective space S_n it is frequently convenient to refer it back to an affine space or a vector space. From the above, there exist two possibilities: Either one regards the points of S_n as rays in a vector space E_{n+1}, or one omits the hyperplane $y_0 = 0$ from S_n, and thus obtains an affine space of dimension n. One also calls the hyperplane $y_0 = 0$ the *imaginary hyperplane*, and the points with $y_0 \neq 0$, the *real points* of S_n. By an appropriate renumbering of the coordinates $y_0, y_1, \dots, y_n$, one can make any point y into a real point, as long as one y_i is $\neq 0$.

One may also go from the points of a multiply projective space to the points of a space whose points can be represented by a one-to-one correspondence with inhomogeneous coordinates $x_1, \dots, x_m, y_1, \dots, y_n$ by omitting the points with $x_0 = 0$, the points with $y_0 = 0$, etc., and therefore we will again recognize this space to be an affine space. A doubly projective space $S_{m,n}$ yields an affine space A_{m+n} in this way. This is the basis by which we can consider $S_{m,n}$ to be an $(m+n)$-dimensional space.

Under the substitution $x_0 = 1$, $y_0 = 1$, a homogeneous equation in the homogeneous coordinates x, y goes to a not-necessarily-homogeneous equation in the remaining x and y. One then defines an algebraic manifold (hypersurface, resp.) in an affine space to be the totality of all solutions of an arbitrary system of algebraic equations (one such equation, resp.) in the inhomogeneous coordinates.

Conversely, one can change any inhomogeneous equation in $x_1, \dots, x_m, y_1, \dots, y_n$ into a homogeneous one by the introduction of $x_0, y_0, \dots$. Any algebraic manifold in an affine space A_n ($A_{m+n+\dots}$, resp.) thus belongs to at least one algebraic manifold in the projective space S_n (in the multiply projective space $S_{m+n+\dots}$, resp.).

§ 5. Projective transformations.

A non-singular linear transformation of the vector space E_{n+1}:

(1) $$y'_i = \sum_0^n \alpha_i^k y_k$$

takes any linear subspace E_m to another linear subspace E'_m; in particular, any ray E_1 goes to a ray E'_1. It thus induces a one-to-one transformation of the points of the projective spaces S_n that can be given by the formulas:

(2) $$\rho\; y'_i = \sum_0^n \alpha_i^k y_k \qquad (\rho \neq 0).$$

Such a transformation (2) is called a *projective transformation*, or also a *linear correlation.*

A projective transformation takes lines to lines, planes to planes, and S_m to S'_m, and leaves the incidence relations (e.g., S_m lies in S_q or S_q contains S_m) unchanged. The converse of this theorem is *not* true: Not every one-to-one point transformation that takes lines to lines (and therefore also planes to planes, etc) is a projective transformation. A counterexample is the anti-linear transformation $y'_k = \bar{y}_k$ that takes any point to its complex conjugate point. The most general one-to-one point transformation that takes lines to lines is given by the formula:

$$\rho\; y'_i = \sum_0^n \alpha_i^k S y_k ,$$

in which S is an automorphism of the ground field $\mathbb{K}$.

From (2), a projective transformation is given by a non-singular quadratic matrix $A = (\alpha_i^k)$. Proportional matrices A and ρA ($\rho \neq 0$) define the same projective transformation. The product of two projective transformations is again a projective transformation, and its matrix is the product matrix. The inverse of a projective transformation is again a projective transformation, and its matrix is the inverse matrix A^{-1}. The projective transformations of S_n thus define a group, namely, the *projective group* $PGL(n, \mathbb{K})$ ([1]).

Projective geometry in S_n is the study of the properties of constructions in S_n that remain invariant under projective transformations.

If one introduces general projective coordinates z and z' for the points y and y', as in § 1, by a coordinate transformation:

([1]) PGL = projective general linear. For the properties of this group, see, B. L. VAN DER WAERDEN, *Gruppen von linearen Transformationen*. Berlin 1935.

$$(3) \qquad \begin{cases} y_k = \sum \beta_k^l \, z_l , \\ y'_i = \sum \gamma_i^j \, z'_j \end{cases}$$

then, on the basis of (2) and (3), the z'_i will be again linear functions of the z_l:

$$(4) \qquad \rho z'_j = \sum_0^n d_j^l \, z_l ,$$

with the matrix:

$$D = \left(d_i^l\right) = C^{-1}AB .$$

In particular, if the same coordinate system is chosen for both y and y' then $C = D$ and:

$$D = B^{-1}AB .$$

We now prove the following:

Main theorem about projective transformations: *A projective transformation T of the space S_n is uniquely determined when one is given n + 2 points $\overset{0}{y}, \overset{1}{y}, \ldots, \overset{n}{y}, \overset{*}{y}$, and their image points $T\overset{0}{y}, T\overset{1}{y}, \ldots, T\overset{n}{y}, T\overset{*}{y}$, assuming that no n + 1 of the points y or their image points lie in a hyperplane.*

Proof. We choose the points $\overset{0}{y}, \overset{1}{y}, \ldots, \overset{n}{y}, \overset{*}{y}$ to be the basis points of a new coordinate system for the point y of S_n, and likewise choose the points $T\overset{0}{y}, T\overset{1}{y}, \ldots, T\overset{n}{y}, T\overset{*}{y}$ to be the basis points for a coordinate system for the image point Ty. The matrix D of the transformation T will then be necessarily a diagonal matrix:

$$D = \begin{pmatrix} \delta_0 & & & \\ & \delta_1 & & \\ & & \ddots & \\ & & & \delta_n \end{pmatrix}.$$

The condition that the transformation T shall take the given point $\overset{*}{y}$ with the coordinates z_k to the given point $T\overset{*}{y}$ with the coordinates z' now says that, from (4):

$$(5) \qquad \rho z'_j = \delta_j \, z_j \qquad (j = 0, 1, \ldots, n).$$

Since the z, like the z', are different from zero, from (5), the δ_j are established uniquely, up to a common factor r. However, since a factor ρ does not enter into (4), the transformation T will be determined uniquely.

One will derive the following corollary from this proof: *Two projective transformations are identical only when their matrices* (α_i^k) *and* $('\alpha_i^k)$ *differ from each other by only a numerical factor* λ: $(\alpha_i^k) = \lambda('\alpha_i^k)$.

The definition of a projective transformation and resulting proof will remain the same when one considers, not a projective transformation of S_n into itself, but a projective transformation from a space S_n onto another S'_n. In particular, we would like to consider projective transformations from S_m onto S'_m when both spaces are contained in the same larger space. Whereas we used the term "the coordinates y_k" in our definition, we must now replace these coordinates with parameters $\gamma_0, \ldots, \gamma_m$. In that case, the formula for a projective transformation will then read:

$$\rho\gamma'_i = \sum \alpha_i^k \gamma_k .$$

We now have the:

Projection theorem. *Let S_m and S'_m be two subspaces of the same dimension in S_n. A third subspace S_{n-m-1} has points in common with either S_m or S'_m. If the point y of S_m is projected onto S'_m in such a way that it is linked with an S_{n-m-1}, as well as an S_{n-m}, which always intersect S'_m, then that projection will be a projective transformation.*

Proof. S_{n-m-1} has the equations:

$$(6) \qquad (\overset{0}{u}\, z) = 0\,, \; (\overset{1}{u}\, z) = 0, \; \ldots, \; (\overset{m}{u}\, z) = 0\,.$$

All points of the join S_{n-m} are linear combinations of y and $n - m$ points $\overset{1}{z}, \overset{2}{z}, \ldots, \overset{n-m}{z}$ of S_{n-m-1} for which (6) is true. This is true, in particular, for the intersection point y' of S_{n-m} with S'_m. One then has:

$$(7) \qquad y'_k = \lambda\, y_k + \lambda_1 \overset{1}{z}_k + \lambda_2 \overset{2}{z}_k + \ldots + \lambda_{n-m} \overset{n-m}{z}_k\,.$$

Since $\lambda \neq 0$, one can choose $\lambda = 1$. It now follows from (6) and (7) that:

$$(8)\qquad \begin{cases} (\overset{0}{u}\,y') = (\overset{0}{u}\,y) = \beta_0 \\ (\overset{1}{u}\,y') = (\overset{1}{u}\,y) = \beta_1 \\ \dots\dots\dots\dots\dots \\ (\overset{m}{u}\,y') = (\overset{m}{u}\,y) = \beta_m. \end{cases}$$

By means of the parametric representation of S_m , the y_k, and therefore also the β_i, will be linear combinations of the parameters $\gamma_0, \dots, \gamma_m$ of the point y:

$$(9)\qquad \beta_i = \sum \delta_i^k \gamma_k \,.$$

Likewise, the y'_k, and therefore also the β_i, will be linear combinations of the parameters $\gamma'_0, \dots, \gamma'_m$ of the point y':

$$(10)\qquad \beta_i = \sum \varepsilon_i^k \gamma'_k \,.$$

Since S_m and S'_m have no point in common with S_{n-m-1}, the linear transformations (9) and (10) will be invertible if the linear forms on the right-hand side never assume the value zero at the same time. Hence, the γ'_k will be linear functions of the β_i and the β_i are linear functions of the γ_l, so they will be linear functions of the γ_l (and conversely), and the projection theorem is proved.

A projective transformation of S_m to S'_m that one constructs by way of the projection theorem is called a *perspectivity*.

The most important theorems of projective geometry follow from the projection theorem and the main theorem above, viz., DESARGUE's theorem and the theorem of PAPPUS (cf., the problems below).

Problems. 1. A projective transformation of a line to itself that leaves three distinct points fixed is the identity.

2. A projective relation between two intersecting lines that takes the point of intersection to itself is a perspectivity.

3. DESARGUE's Theorem. If the six distinct points $A_1, A_2, A_3, B_1, B_2, B_3$ in space or in the plane lie in such a way that the lines A_1B_1, A_2B_2, A_3B_3 are distinct and go through a point P then A_2A_3 and B_2B_3, A_3A_1 and B_3B_1, A_1A_2 and B_1B_2 will intersect in three points C_1, C_2, C_3 that lie on a line.

(Project the sequence of points PA_2A_2 onto PA_3A_3 from C_1, then onto PA_1A_1 from C_2, and finally back to PA_2A_2 from C_3, and then apply Problem 1.)

4. Theorem of PAPPUS. If one has six distinct points of a plane, $A_1, A_2, A_3, A_4, A_5, A_6$, such that the points with the odd and even indices lie on distinct lines then the three intersection points, P, of A_1A_2 and A_4A_5, Q, of A_2A_3 and A_5A_6 , R, of A_3A_4 and A_6A_1 , will all lie in a line.

(Project the sequence of points A_4A_5 onto A_4A_6 from A_1, then onto A_4A_6 from A_3, and finally back to A_4A_5 from R, then apply Problem 1.)

5. A projective relationship between two skew lines g, h in a space S_3 is always a perspectivity. (Connect the three points A_1, A_2, A_3 of g with their image points B_1, B_2, B_3 on h and construct a line s through a third point of A_1B_1 that intersects A_2B_2 and A_3B_3. Project g onto h from s.)

6. Give a construction for a projective transformation that takes three given points of line to three given points of another line on the basis of the projection theorem.

7. Construct the projective transformation that is uniquely determined by the main theorem that would take five given points A, B, C, D, E in the space S_3 to the same five points geometrically. (Project the space onto CD from AC and apply Problem 6 to the resulting sequence of points. Likewise, project onto BD from AC, etc.)

§ 6. Degenerate projectivities. Classification of projective transformations.

In addition to one-to-one projective transformations, it is occasionally useful to consider degenerate projective transformations. These will be defined by the same formula (2) (§5), in which, however, the matrix $A = (\alpha_i^k)$ has rank $r \leq n$. The point y may therefore belong to a space S_n and the image point y', to a space S_m. For certain points y all of the coordinates y'_k will be zero; these points y, which define an S_{n-m}, will therefore have no well-defined image point y'. From (2) (§5), all of the image points y' will be linear combinations of n points α^k with coordinates α_i^k, of which, r of them will be linearly independent. The image points y' thus define a space S_{r-1} in S_m. Hence:

A degenerate projective transformation of rank $r \leq n$ maps the space S_n to an image space S_{r-1}, except for a subspace S_{n-r}, for whose points, the transformation will be undefined.

One obtains an example of a degenerate projective transformation of rank r when one projects all of the points of S_n from an S_{n-r} in S_n onto an S_{r-1} that does not meet S_{n-r}; the projection is undefined for the points of S_{n-r}. For the remaining points y and their projections y', one has formulas of the form (8) and (10), as in §5, with $m = r - 1$, which one can once more solve for γ'_k. The parameters γ'_k of y thus depend linearly upon the β_1, …, β_m, and these, in turn, depend linearly upon $y_0, \ldots, y_n$. Thus, one has, in fact:

$$\gamma'_i = \sum \alpha_i^k y_k , \tag{1}$$

in which the matrix (α_i^k) has rank $r = m + 1$.

One can simplify the formulas somewhat more, by considering the β_i to be coordinates in S_m, instead of the γ'_k. This is allowed, because, from (10) § 5, the β_i are coupled to the γ'_k by an invertible linear transformation. The formula for the projection then reads simply:

$$\beta_i = \left(\overset{i}{u}\, y\right) = \sum \overset{i}{u}{}^k\, y_k .$$

In this expression, the $\overset{i}{u}$ are completely arbitrary hyperplanes that are subject to only the condition that they determine an S_{n-m-1}; i.e., $\left(\overset{i}{u}{}^k\right)$ is an *arbitrary* matrix (with $m + 1$ rows and $n + 1$ columns) of rank $m + 1$. From this, it follows that: *Any degenerate projective transformation of rank $r = m + 1$ implies a projection of the space S_n, except for a subspace S_{n-m-1}, onto a subspace S_m of S_n that is distinct from that subspace.*

A projective transformation T of S_n into itself that has the matrix A has, as we saw, the matrix $D = B^{-1}AB$ relative to another coordinate system. By a suitable choice of B, one can, as is well known ([1]), now bring this matrix into "Jordan normal form," whose diagonal sequence of box matrices has the form:

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}, \tag{2}$$

in which there is a "characteristic root" λ in the main diagonal, whereas in the slanted row above the main diagonal there is an arbitrary non-zero number, which can be chosen to be 1. If the box matrices in (2) have degree (= number of rows) 1 then there are no ones above the diagonal, and the boxes will contain nothing but the element λ. From SEGRE, the JORDAN normal form can be characterized by a schema involving whole numbers that give the degrees (= number of rows) in the boxes. If more boxes appear with the same root λ then their degrees will be enclosed in a round bracket. The total SEGRE symbol will ultimately be enclosed in a square bracket. Thus, there are – e.g., in the case of the plane ($n = 2$) – the following possible normal forms:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \left(\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_2 \end{array}\right), \left(\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_1 \end{array}\right), \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Their SEGRE symbols are: [111], [(11)1], [(111)], [21], [(21)], [3].

If one allows the root λ to have the value 0, as well, then classification above will also include the degenerate projective transformations. We thus restrict ourselves to one-to-one transformations in the following discussion.

([1]) See, perhaps, B. L. VAN DER WAERDEN: *Moderne Algebra II*, § 109. For a purely geometric derivation, see ST. COHN-VOSSEN: Math. Ann. Bd. **115** (1937), 80-86.

The JORDAN normal form is very closely connected with the question of points, lines, etc., that are invariant under T. Namely, each box (2) with e lines is associated with the following basis vectors in the vector space:

An "eigenvector" v_1 with $Av_1 = \lambda v_1$

a vector v_2 with $Av_2 = \lambda v_2 + v_1$

……………………………………….

etc., up to v_e with $Av_e = \lambda v_e + v_{e-1}$.

The ray (v_1) is therefore invariant under the transformation T, just as the spaces (v_1, v_2), (v_1, v_2, v_3), etc., are. In projective space, this therefore yields an invariant point, an invariant line through this point, an invariant plane through this line, etc., up to an invariant space S_{e-1}. Linear combinations of eigenvectors with the same eigenvalue are again eigenvectors. If we thus assume that for an eigenvalue λ there are, perhaps, g boxes A_r then there will also be g linearly independent eigenvectors of eigenvalue λ, which will span a subspace E_g. The rays E_1 of E_g are each invariant under the transformation T, and together they define a pointwise invariant linear subspace S_{g-1} in S_n. The same thing will again be true for every characteristic root λ. This transformation does not possess any other invariant points, since the matrix A has no other eigenvectors.

There are some special cases of interest:

1. The "general case" [111…1], in which D is a diagonal matrix with roots $\lambda_1, \ldots, \lambda_n$ in the diagonal that are all different. The invariant points are the vertices of the fundamental simplex of the new coordinate system and the invariant linear spaces are the edges of this simplex.

2. The "central collineations," which are characterized by the property that all of the points of a hyperplane transform to themselves. Their SEGRE symbols are [(111…1)1] or [(211…1)]. Besides the points of the invariant hyperplane, there is also an invariant point – the "center" – with the property that all of the linear spaces through the center are invariant. The center does not exist in the case [(11…1)1], and in the other cases it will always be in the invariant hyperplane.

3. The projective transformations with period 2 – or "involutions" – whose squares are the identity. Since the characteristic roots of the matrix A^2 are the squares of the characteristic roots of A, and since, on the other hand, $A^2 = \mu E$, A can only have two characteristic roots $\lambda = \pm\sqrt{\mu}$. Since one can multiply A by a factor, one can assume that $\lambda = 1$. If one now squares the boxes (2) then that will yield that only one-rowed boxes appear. D will then be a diagonal matrix with the elements + 1 and – 1. There will be two spaces S_r and S_{n-r-1}, whose points will each remain invariant. The line that connects a non-invariant point y to its image point y' will meet S_r and S_{n-r-1} at two points that lie harmonically with y and y'. Thus, we will assume that the characteristic of the basic field is not equal to 2.

Problems. 1. Give the invariant points and invariant lines for each type of projective transformations of the plane.

2. A central collineation is given completely by the data of an invariant hyperplane S_{n-1} and the image points x' and y' of two points x and y, such that xy and $x'y'$ must intersect S_{n-1}. Give a projective-geometric construction of a collineation from these data.

3. Under a central collineation, the connecting line from a non-invariant point y with its image point y' will always go through the center.

4. An involution in the line always possesses two different fixed points, and there exist point pairs (y, y') that lie harmonically with these fixed points.

§ 7. PLÜCKERian S_m-coordinates.

Let an S_m in S_n be given by $m + 1$ points. As an example, we take $m = 2$ and $m + 1$ points x, y, z. We now define:

$$\pi_{ikl} = \sum \pm x_i\, y_k\, z_l = \begin{vmatrix} x_i & x_k & x_l \\ y_i & y_k & y_l \\ z_i & z_k & z_l \end{vmatrix}.$$

The quantities π_{ikl} are not all = 0, since the points x, y, z are linearly dependent. Switching any two indices will change the sign of π_{ikl}. If two indices are equal then π_{ikl} = 0. Thus, there are just as many essentially different, not necessarily vanishing, π_{ikl} as there are combinations of $n + 1$ indices taken 3 at a time. For an arbitrary m the number of π_{ikl} will be equal to $\binom{n+1}{m+1}$.

We now show that the π_{ikl} depend upon only the plane S_2, not upon the choice of points x, y, z in it, up to a proportionality factor. Namely, if x', y', z' are three other points that determine the plane then, since x', y', z' belong to the linear space that is determined by x, y, z, one will have:

$$\begin{aligned} x'_k &= x_k\,\alpha_{11} + y_k\,\alpha_{12} + z_k\,\alpha_{13}, \\ y'_k &= x_k\,\alpha_{21} + y_k\,\alpha_{22} + z_k\,\alpha_{23}, \\ z'_k &= x_k\,\alpha_{31} + y_k\,\alpha_{32} + z_k\,\alpha_{33}, \end{aligned}$$

and therefore, from the multiplication theorem for determinants:

$$\begin{vmatrix} x'_i & x'_k & x'_l \\ y'_i & y'_k & y'_l \\ z'_i & z'_k & z'_l \end{vmatrix} = \begin{vmatrix} x_i & x_k & x_l \\ y_i & y_k & y_l \\ z_i & z_k & z_l \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix},$$

or:

$$\pi'_{ikl} = \pi_{ikl}\,\alpha.$$

Secondly, we show that the plane S_2 is determined by the quantities π_{ikl}. To that end, we state necessary and sufficient conditions for a point ω of the plane S_2. They are based in the fact that all four-rowed sub-determinants of the matrix:

$$\begin{pmatrix} \omega_0 & \omega_1 & \cdots & \omega_n \\ x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \\ z_0 & z_1 & \cdots & z_n \end{pmatrix}.$$

vanish. If one develops such a sub-determinant along the first row then one will obtain the condition:

$$\omega_i\, \pi_{jkl} - \omega_j\, \pi_{ikl} + \omega_k\, \pi_{ijl} - \omega_l\, \pi_{ijk} = 0\,. \tag{1}$$

We can regard condition (1) as the *equations for the plane* S_2 in the point coordinates ω_i. However, a linear space is determined uniquely by its equations.

Precisely the same equations are valid for arbitrary m ($0 \leq m < n$). Since the $\pi_{ik\ldots l}$ of the space S_m are determined uniquely, we can regard them as the coordinates of S_m. They are called PLÜCKER*ian* S_m*-coordinates*. They are homogeneous coordinates, since they are determined only up to a factor λ and not all of them can be equal to zero.

If we hold all of the indices up to the last one fixed, but let λ range through all values, then we can regard these π_{ghl} as the coordinates of a point π_{gh}. This point will belong to the space S_2, since it is:

$$\pi_{ghl} = \begin{vmatrix} y_g & y_h \\ z_g & z_h \end{vmatrix} x_l + \begin{vmatrix} z_g & z_h \\ x_g & x_h \end{vmatrix} y_l + \begin{vmatrix} x_g & x_h \\ y_g & y_h \end{vmatrix} z_l\,.$$

The vector π_{gh} is then a linear combination of the vectors x, y, and z. Furthermore, one has $\pi_{ghg} = 0$ and $\pi_{ghh} = 0$. The point π_{gh} thus belongs to the space S_{n-2} with the equations $\omega_g = \omega_h = 0$. S_{n-2} is one side of the basic coordinate simplex. *The point π_{gh} is therefore the intersection point of the space S_2 with the side S_{n-2} of the coordinate simplex.*

Naturally, all of this will be valid only when not all of the π_{ghl} (g and h fixed, $l = 0, 1, \ldots, n$) are equal to zero. If this is the case then one can show that S_2 and S_{n-2} will have at least one S_1 in common, and conversely. We shall not go into this any further.

Relations exist between the π_{ikl}. We obtain them when we express the idea that the point belongs to the space S_2 in any case; hence, the equations (1) must be satisfied. This yields:

$$\pi_{ghi}\, \pi_{jkl} - \pi_{ghj}\, \pi_{ikl} + \pi_{ghk}\, \pi_{ijl} - \pi_{ghl}\, \pi_{ijk} = 0\,. \tag{2}$$

Now, if we let π_{ikl} be any quantities whatsoever that are not all zero then the sign will change when we switch two indices and relations (2) are satisfied. We would like to prove that the π_{ikl} will then be the PLÜCKERian coordinates of a plane.

In order to prove this, we assume, perhaps, that $\pi_{012} \neq 0$. Three points are defined by way of:

$$x_i = \pi_{12i}\,,$$

$$y_i = - \pi_{02i} ,$$
$$z_i = \pi_{01i} ,$$

and they span a plane with the PLÜCKERian coordinates:

$$p_{ikl} = \pi_{012}^{-2} \cdot \begin{vmatrix} x_i & x_k & x_l \\ y_i & y_k & y_l \\ z_i & z_k & z_l \end{vmatrix} .$$

(We will likewise see that $p_{012} \neq 0$, which means the three points are linearly independent.) For this plane, one likewise has relations (2) in the form:

(3) $$p_{ghi}\, p_{jkl} - p_{ghj}\, p_{ikl} + p_{ghk}\, p_{ijl} - p_{ghl}\, p_{ijk} = 0 .$$

We now compute the p_{01i} :

$$p_{01i} = \pi_{012}^{-2} \cdot \begin{vmatrix} x_0 & x_1 & x_i \\ y_0 & y_1 & y_i \\ z_0 & z_1 & z_i \end{vmatrix} = \pi_{012}^{-2} \cdot \begin{vmatrix} \pi_{120} & 0 & \pi_{12i} \\ 0 & -\pi_{021} & -\pi_{02i} \\ 0 & 0 & \pi_{01i} \end{vmatrix}$$

$$= \frac{\pi_{012}\pi_{012}\pi_{01i}}{\pi_{012}^2} = \pi_{01i} .$$

One likewise finds that:

$$p_{02i} = \pi_{02i} \quad \text{and} \quad p_{12i} = \pi_{12i} .$$

We thus see that all of the p_{ghi} for which the two indices g and h have the values 0, 1, or 2 agree with the corresponding π_{ghi} . In particular, $p_{012} = \pi_{012} \neq 0$. We would now like to prove that, in general, one has:

(4) $$p_{ghi} = \pi_{ghi} .$$

It follows from (2) and (3) that:

(5) $$\pi_{ghi} = \pi_{012}^{-1} (\pi_{gh0}\, \pi_{i12} - \pi_{gh1}\, \pi_{i02} + \pi_{gh2}\, \pi_{i01}),$$

(6) $$p_{ghi} = p_{012}^{-1} (p_{gh0}\, p_{i12} - p_{gh1}\, p_{i02} + p_{gh2}\, p_{i01}).$$

Now, when one of the indices g or h equals 0, 1, or 2 then the right-hand side of (5) will agree with that of (6). Hence, $\pi_{ghi} = p_{ghi}$, as long as one of the indices g, h has the value 0, 1, or 2. Moreover, it then follows that when none of the indices g, h has the value 0, 1, or 2 the right-hand sides of (5) and (6) will also agree. Hence, (4) will be true in general.

We summarize: *Necessary and sufficient conditions for the quantities π_{ikl} to represent* PLÜCKER*ian coordinates of a plane in S_n are that they do not collectively vanish, they change sign under the exchange of any two indices, and they satisfy relations*

(2). *If – say –* $\pi_{012} \neq 0$ *then all of the* π_{ikl} *will be rationally expressible in terms of* π_{12i} , π_{02i} , π_{01i} .

All of the considerations up to now will be valid with no essential changes for the PLÜCKERian coordinates of S_m in S_n . In the general case, relations (3) read as follows:

(7)
$$\pi_{g_0 g_1 \cdots g_d} \pi_{a_0 a_1 \cdots a_d} - \sum_0^m \pi_{g_0 \cdots g_{\lambda-1} a_0 g_{\lambda+1} \cdots g_m} \pi_{g_\lambda a_1 \cdots a_m} = 0,$$

and in the case of a line ($m = 1$):

(8)
$$\pi_{gi}\, \pi_{kl} - \pi_{gk}\, \pi_{il} + \pi_{gl}\, \pi_{ik} = 0.$$

For further details on S_m-coordinates, in particular, for the introduction of dual S_m-coordinates $\pi^{ij\ldots l}$ with $n - m$ indices and their reduction to the π_{ikl} , I refer the reader to the textbook of R. WEITZENBÖCK ([1])

If one regards the $\binom{n+1}{m+1}$ quantities $\pi_{ij\ldots l}$ as the coordinates of a point in a space S_N :

$$N = \binom{n+1}{m+1} - 1$$

then the quadratic relations (7) will define an algebraic manifold M in this space. Conversely, any point of this manifold M will correspond to a unique subspace S_m in S_n .

The simplest interesting case of this map is the case of the lines S_1 in the space S_3 . In this case, there is only one relation (7), namely:

(9)
$$\pi_{01}\, \pi_{23} + \pi_{02}\, \pi_{31} + \pi_{03}\, \pi_{12} = 0 .$$

It defines a hypersurface M of degree 2 in S_5 . *The lines of the space* S_3 *may thus be mapped in a one-to-one manner to the points of a quadratic hypersurface in* S_5 .

A pencil of lines will correspond to a line in M under this map. If x is the center of the pencil and $y = \lambda_1 y' + \lambda_2 y''$ is the parametric representation of one of the lines in the plane of the pencil that does not go through x then one will obtain the PLÜCKERian coordinates of all lines of the pencil in the form:

$$\begin{aligned}\pi_{kl} \quad &= x_k(\lambda_1 y'_l + \lambda_2 y''_l) - x_l(\lambda_1 y'_k + \lambda_2 y''_k) \\ &= \lambda_1(x_k y'_l - x_l y'_k) + \lambda_2(x_k y''_l - x_l y''_k) \\ &= \lambda_1 \pi'_{kl} + \lambda_2 \pi''_{kl} .\end{aligned}$$

Conversely: If a line $\pi_{kl} = \lambda_1 \pi'_{kl} + \lambda_2 \pi''_{kl}$ lies completely in M, hence, the π_{kl} satisfy the condition (9) identically in λ_1, λ_2, then it will follow with no further assumptions that:

([1]) WEITZENBÖCK, R.: *Invariantentheorie,* pp. 117-120. Groningen, 1923.

$$\pi'_{01}\pi''_{23} + \pi'_{02}\pi''_{31} + \pi'_{03}\pi''_{12} + \pi'_{23}\pi''_{01} + \pi'_{31}\pi''_{02} + \pi'_{12}\pi''_{03} = 0 ,$$

or, in determinant form, when one sets:

$$\pi'_{kl} = x'_k y'_l - x'_l y'_k , \qquad \text{and} \qquad \pi''_{kl} = x''_k y''_l - x''_l y''_k ,$$

one will have:

$$\begin{vmatrix} x'_0 & x'_1 & x'_2 & x'_3 \\ y'_0 & y'_1 & y'_2 & y'_3 \\ x''_0 & x''_1 & x''_2 & x''_3 \\ y''_0 & y''_1 & y''_2 & y''_3 \end{vmatrix} = 0 .$$

The points x', y', x'', y'' will thus lie in a plane, so the two lines π' and π'' will intersect and will thus determine a pencil. One of the lines that lie in M will thus always correspond to a pencil of lines.

A plane in the space S_3 will be obtained when one couples a fixed point P with all of the points of a line RS by means of lines. If this plane is to lie in M completely then, at the very least, the lines PR, PS, and RS must lie in M. The points P, R, S must therefore correspond to three mutually intersecting lines π, ρ, σ in S_3 that do not belong to a pencil. However, three such lines will either lie in a plane or go through a point. If one now connects the line π with all of the lines of the pencil $\rho\,\sigma$ by means of some pencil then the totality of lines so obtained will either be a line field or a star of lines. Conversely, any line field or star of lines can be obtained in that manner. *Hence, there are precisely two types of planes that lie in M: One type corresponds to a field of lines and the other, to a star of lines in* S_3. Furthermore, FELIX KLEIN has proved the theorem: Any projective transformation of the space S_3 into itself corresponds to a projective transformation of the space S_5 that leaves the hypersurface M invariant, and in this way, one also obtains all projective transformations of M into itself that do not exchange the two families of planes ([1]).

Problems. 1. The connecting space of an S_m with a point ω that lies outside of S_m has the PLÜCKERian coordinates:

$$\rho_{ijk\ldots l} = \omega_i\, \pi_{jk\ldots l} - \omega_j\, \pi_{ik\ldots l} + \omega_k\, \pi_{ij\ldots l} + \ldots + (-1)^{m+1}\, \omega_l\, \pi_{ijk\ldots l} .$$

2. The intersection of an S_m with a hyperplane u that is not contained in it has the PLÜCKERian coordinates:

$$\sigma_{k\ldots l} = \text{o}\; u^i\, \pi_{ik\ldots l} .$$

3. The condition for two lines π, ρ in the space S_n to intersect or coincide reads:

$$\pi_{gi}\, \rho_{kl} - \pi_{gk}\, \rho_{il} + \pi_{gl}\, \rho_{ik} + \pi_{kl}\, \rho_{gi} - \pi_{il}\, \rho_{gk} + \pi_{ik}\, \rho_{gl} = 0 .$$

4. A ruled surface in S_3 (consisting of all lines that intersect three skew lines) corresponds to a conic section on M, namely, the section of M that contains a plane S_2 in the space S_5 .

([1]) For the proof, cf., B. L. VAN DER WAERDEN: *Gruppen von linearen Transformationen*. Ergebn. Math., Bd. IV, 2 (1935), § 7.

§ 8. Correlations, null systems, and linear complexes.

A (projective) *correlation* is an association of each point y in S_n with a corresponding hyperspace v in S_n, and its coordinates are given by:

$$\rho\, v^i = \sum_k \alpha^{ik} y_k\,, \tag{1}$$

in which the α^{jk} shall define a non-singular matrix. The association is therefore one-to-one; its inverse is given by:

$$\sigma y_k = \sum \beta_{kl}\, v^l\,, \tag{2}$$

in which (β_{kl}) is the inverse of the matrix (α^{jk}). If the point y lies on a hyperplane u then one will have $\sum u^k y_k = 0$, and it will follow from (2) that:

$$\sum\sum u^k \beta_{kl}\, v^l = 0,$$

i.e., the hyperplane v will contain the star with the midpoint:

$$x_l = \sum \beta_{kl}\, u^k\,. \tag{3}$$

Conversely, If the hyperplane v contains the star with the midpoint x then o $v^i x_i = 0$, and it will follow from (1) that:

$$\sum\sum \alpha^{jk} x_i y_k = 0, \tag{4}$$

and therefore the point y will lie in a hyperplane u with the coordinates:

$$u^k = \sum \alpha^{jk} x_i\,. \tag{5}$$

The product of two correlations is obviously a projective collineation. The product of a collineation with a correlation is a correlation. The projective collineations and correlations together thus define a group.

Formulas (3), (5) define a second one-to-one transformation that takes hyperplanes u to points x, and which is connected with the original transformation (1), (2) by the following properties: *If y lies in u then v will go through x, and conversely.*

We regard the associated transformations $y \leftrightarrow v$ and $u \leftrightarrow x$ as an association that we also call a *complete correlation*, or a *duality*. *A complete correlation thus associates each point y in S_n with a hyperplane v and each hyperplane u with a point x in a one-to-one way such that the incidence relations between points and hyperplanes thus remain valid.*

As in § 3, in which we considered a special correlation $v^i = y_i$, one proves that a correlation associates each subspace S_m of S_n with a subspace S_{n-m-1} and that the relation of inclusion is thus inverted.

A correlation, just like a projective transformation, is determined uniquely as long as the images of $n + 2$ given points, no $n + 1$ of which lie in a hyperplane, are known. The proof is the same as the main theorem in § 5. The construction of a correlation from this data can happen in the way that was suggested for projective transformations in Problem 7 (§ 5).

Two correlations, just like two projective transformations, are identical when and only when their matrices differ from each other by only a numerical factor λ:

$$\alpha'_{ik} = \lambda\, \alpha_{ik}\,.$$

We now seek to determine the *involutory correlations,* in particular, i.e., the ones that are identical with their inverse correlations. Since the inverse correlation to (1) will be given by formula (5), for an involutory correlation it is necessary and sufficient that:

$$\alpha^{ki} = \lambda\, \alpha^{jk}\,, \qquad (\lambda \neq 0). \tag{6}$$

It will immediately follow from (6) that:

$$\alpha^{jk} = \lambda\, \alpha^{ki} = \lambda^2\, \alpha^{jk},$$

and since at least one $\alpha^{jk} \neq 0$, one will have:

$$\lambda^2 = 1.$$

There are therefore two cases: the case $\lambda = 1$, for which the matrix (α^{jk}) is *symmetric:*

$$\alpha^{ki} = \alpha^{jk}\,,$$

and the case $\lambda = -1$, for which the matrix is *anti-symmetric:*

$$\alpha^{ki} = -\,\alpha^{jk}\,. \tag{7}$$

In the first (symmetric) case one calls the correlation a *polar system*, or a *polarity*. In this case, the symmetric matrix (α^{jk}) defines a quadratic form:

$$\sum\sum \alpha^{jk}\, x_i\, x_k\,,$$

and the hyperplane that is given by (1) is the *polar* of the point y relative to this form.

By contrast, in the anti-symmetric case the correlation is called a *null system*, or a *null correlation*. As is well known, a non-singular anti-symmetric matrix is possible only when the number of rows $n + 1$ in the matrix is even, hence, when the dimension n is odd. It follows from (7), in particular, that $\alpha^{ji} = 0$, and furthermore, that:

$$\rho \sum v^i\, y_i = \sum\sum \alpha^{jk}\, y_i\, y_k = 0\;;$$

hence, the hyperplane v – viz., the *null hyperplane* of y – goes through the point y – viz., the *null point* of v.

This latter property is also characteristic of the null correlation. If a correlation associates each point y with a hyperplane that goes through y then that will be true for the point (1, 0. …, 0), from which it will follow that $\alpha^{00} = 0$. Likewise, one shows that $\alpha^{ii} = 0$ for each i. If one now makes the same argument starting with the point (1, 1, 0, …, 0) then it will follow that:

$$\alpha^{01} + \alpha^{10} = 0 , \qquad \text{hence,} \qquad \alpha^{01} = -\alpha^{10};$$

similarly, one will again have $\alpha^{jk} = -\alpha^{ki}$.

Along with the previously considered non-singular null correlations, we now also discuss the degenerate ones, for which the anti-symmetric matrix (α^{jk}) is singular, and correspondingly, the null hyperplanes of a point can be undetermined. Two points x, y are called *conjugate* for a null system or polar system when one of them lies in the null (polar, resp.) hyperplane of the other one. Equation (4) is definitive of this, and its meaning does not change when one exchanges x and y. The conjugacy relation is therefore symmetric in the points x and y: When x lies in the null hyperplane of y then y will lie in the null hyperplane of x.

We now consider the totality of all lines g that go through a point y and that lie in the (one, resp.) null hyperplane of this point. If x is second point on such a line then (4) will be valid, from which (7) will also allow one to write:

$$\sum_{i<k} \alpha^{ik} (x_i\, y_k - x_k\, y_i) = 0 . \tag{8}$$

The bracketed quantities are the PLÜCKERian coordinates π_{ik} of the line g; (8) is then equivalent to:

$$\sum_{i<k} \alpha^{ik}\, \pi_{ik} = 0 . \tag{9}$$

In this form, one sees that the character of the line g is completely independent of the choice of point y on the line. One calls the totality of all lines g whose PLÜCKERian coordinates satisfy a linear equation (9) a *linear line complex.*

Conversely, if one starts with a linear line complex (9) then all of the line complexes through a point y will lie in a hyperplane whose equation (8) will be given as long as (8) is not satisfied identically in x. If one writes (8) in the form (4) with $\alpha^{jk} = -\alpha^{ki}$ then one will again obtain equations (1) for the coordinates v_k of the plane. Hence:

To each linear complex of lines (9) *there belongs one (possibly degenerate) null system* (1), *and conversely, in such a way that the line complex through a point y will satisfy the null hyperplane of y precisely. If the null hyperplane of y is indeterminate then all of the lines through y will be ray complexes, and conversely.*

The *projective classification* of the null system – and thus the linear complex, as well – is a very simple concern. If P_0 is a point whose null hyperplane is not indeterminate and P_1 is a point that is not conjugate to P_0 – i.e., does not lie in the null hyperplane of P_0

– then the null hyperplane of P_1 will likewise not be indeterminate, and since it does not go through P_0, it will be different from P_0. Both null hyperplanes will therefore intersect in a space S_{n-2}. The connecting line of P_0P_1 will touch the null hyperplane of P_0 only at P_0, and that of P_1, only at P_1, so it will have no point in common with S_{n-2} whatsoever.

We now choose P_0 and P_1 to be the basic points of a new coordinate system, while the remaining points will be chosen from S_{n-2}. If any two points in S_{n-2} are conjugate then we will choose $P_2, \ldots, P_n$ arbitrarily: these points will therefore all be conjugate to each other, as well as to P_0 and P_1. If this is not the case then we will choose P_2 and P_3 in S_{n-2} in such a way that they are conjugate to each other. The null hyperplanes of P_2 and P_3 do not include S_{n-2}; hence, they will each intersect S_{n-2} in a S_{n-3}. These two S_{n-3} in S_{n-2} will be different, and they will thus intersect in an S_{n-4}, which (as before) will have no point in common with the connecting line P_2P_3.

We then proceed. The basic points $P_4, \ldots, P_n$ are chosen in S_{n-4}. If all of the points of S_{n-4} are mutually conjugate then we will choose $P_4, \ldots, P_n$ arbitrarily in S_{n-4}, otherwise, we will choose P_4 and P_5 in such a way that they are not conjugate, and we construct the intersection of their polar hyperplane with S_{n-4}, etc.

We finally obtain a system of linearly independent basic points $P_0, P_1, \ldots, P_{2r-1}, \ldots, P_n$, in such a way that:

$$\begin{gathered} P_0 \text{ and } P_1, \\ P_2 \text{ and } P_3, \\ \ldots\ldots\ldots\ldots \\ P_{2r-2} \text{ and } P_{2r-1}, \end{gathered}$$

are not conjugate, and, by contrast, all of the remaining pairs of basic points are conjugate. Therefore, $\alpha^{01}, \alpha^{23}, \ldots, \alpha^{2r-2,\,2r-1}$ are non- zero, and all of the other are zero. For a suitable choice of unit points, one will have $\alpha^{01} = \alpha^{23} = \ldots = \alpha^{2r-2,\,2r-1} = 1$. Moreover, equation (2) for the line complex that is associated with the null system will read like:

$$\pi_{01} + \pi_{23} + \ldots + \pi_{2r-2,\,2r-1} = 0 .$$

The matrix (α^{jk}) has rank $2r$ $(0 < 2r \leq n + 1)$; hence, the number r is a projective invariant of the null system. We thus conclude the projective classification of linear complexes with:

The rank of the anti-symmetric matrix (α^{jk}) *of a null system is always an even number* $2r$. *When one is given the rank, the null system, and therefore also the associated linear complex, will be determined uniquely, up to a projective transformation.*

In the case $n = 1$, there is only one null system: viz., the identity, which associates any point of the line with itself. In the case $n = 2$, there are only singular null systems of rank 2 that associate any point with its connecting line with a fixed point O. The associated linear complex is a pencil of lines with center O.

In the case of ordinary space ($n = 3$), there are *singular* (or *special*) linear complexes of rank 2 and *regular* (or non-special) linear complexes of rank 4. A singular linear complex has the equation $\pi_{01} = 0$, and thus consists of all lines that intersect a fixed line, namely, the *axis* of the singular complex. A regular linear complex has the equation $\pi_{01} + \pi_{23} = 0$ and belongs to a non-singular null system.

One obtains a non-singular null system in S_3 by the following projective construction: Any vertex of a spatial pentahedron will be associated with the plane through it and two neighboring vertices. These five planes might all be different from each other. A correlation K is then determined by these five points and five associated planes. It is a null correlation for which all pairs of consecutive vertices represent conjugate point pairs. Proof: There is at least one linear complex $\sum \alpha^{jk}\,\pi_{ik} = 0$ that includes the 5 sides of the pentagon; these 5 sides then give only 5 conditions for the six quantities α^{jk}. If Γ is such a complex then Γ will be non-singular when there is no axis that meets all 5 sides. Thus Γ defines a null correlation. The null plane of a vertex must include both of the faces that go through this vertex, since it is a line complex. Hence, the null correlation in the 5 points and 5 associated planes agrees with the correlation K, and is therefore identical with it.

One obtains an intuitive picture of a null system when one subjects a chosen point to a uniform screwing motion (a translation along an axis a, coupled with a rotation about a, both with constant speed), and then associates each point y with the plane that is perpendicular to the velocity vector at this point. When the axis a is assumed to be the z-axis, and when ρ is the ratio of the translational to rotational velocity, one will find the equation for this plane to be:

$$(x_1\,y_2 - x_2\,y_1) - \rho\,(x_3\,y_0 - x_0\,y_3) = 0\,.$$

In fact, this equation has the form of (8).

Problems. 1. Show that the equation for a non-singular null system can always be brought into the form (8) through an orthogonal coordinate transformation, and that any such null system is thus associated with a screw motion.

2. Extend Problem 1 to dimension $2n + 1$.

3. A null system $\sum \alpha^{jk}\,\pi_{ik} = 0$ in S_3 is special when and only when one has:

$$\alpha^{01}\,\alpha^{23} + \alpha^{02}\,\alpha^{31} + \alpha^{03}\,\alpha^{12} = 0\,;$$

(9) will represent the condition for the line π to intersect a given line in precisely this case.

4. A linear complex of rank 2 in S_n always consists of the lines that intersect a given S_{n-2}.

5. A null correlation determines not only a line complex of lines, but also (dual to that) a linear complex of spaces S_{n-2} that are the intersections of any two conjugate hyperplanes.

§ 9. Quadrics in S_r and the linear spaces that lie in them.

In the sequel, we shall understand a *quadric* $\mathfrak{Q}_{r-1}$ to be a quadratic hypersurface in a space S_r. Thus, a quadric $\mathfrak{Q}_0$ is a point pair, a quadric $\mathfrak{Q}_1$ is a conic section, and a quadric $\mathfrak{Q}_2$ is a quadratic surface. We assume that the equation of a quadric takes the form:

$$(1) \qquad \sum_{j,k=0}^{r} a^{jk}\,x_j\,x_k = 0 \qquad (a^{jk} = a^{kj}).$$

If we intersect the quadric (1) with a line:

(2) $$x_k = \lambda_1 y_k + \lambda_2 z_k ,$$

in which we substitute (2) in (1), then we will obtain a quadratic equation for λ_1, λ_2 :

(3) $$\lambda_1^2 \sum_{j,k} a^{jk} y_j y_k + 2\lambda_1\lambda_2 \sum_{j,k} a^{jk} y_j z_k + \lambda_2^2 \sum_{j,k} a^{jk} z_l z_k = 0 .$$

Thus, when the line does not lie completely within the quadric, there will be two (different or coincident) intersection points.

If the middle coefficient in (3) is equal to zero:

(4) $$\sum_{j,k} a^{jk} y_j z_k = 0,$$

then both of the roots $\lambda_1 : \lambda_2$ in equation (3) will be equal and opposite, i.e., both intersection points will either lie harmonically with the two points y, z or they will agree with the point y or the point z . When y is held fixed and z is varied, equation (4) will define a hyperplane with the coordinates:

(5) $$u^k = \sum_j a^{kj} y_j ,$$

which is the *polar* to y in the polar system that is defined by the quadric. When the point y is uniquely determined by u, it will be called the *pole* of u . The point z, which satisfies equation (4), and therefore lies in the polar to y, will be called *conjugate* to y relative to the quadric. If z is conjugate to y then y will also be conjugate to z .

If the polar of y is indeterminate:

(6) $$\sum_j a^{kj} y_j = 0 \qquad (k = 0, 1, \ldots, r)$$

then the first two terms in (3) will vanish identically; hence, each line through y will either have two intersection points with the quadric that agree with y or it will lie completely within the quadric. In this case, the point y will be called a *double point* of the quadric. The quadric is then a *cone* with the *vertex* y ; i.e., it has what are called *generators* through the point y .

If the determinant $| a^{ik} |$ of the system of equations (6) is non-zero then the quadric will be free of double points. In that case, the polar system (5) will be a non-singular correlation. This not only associates each point y with a unique polar u, but also, conversely, each hyperplane u with a unique pole y , and generally, each space S_p with a polar space S_{r-p-1} . This association is involutory, i.e., the polar space to S_{r-p-1} is again S_p. Hence, if all of the points of S_{r-p-1} are conjugate to all points of S_p then all of the points of S_p will be conjugate to all of the points of S_{r-p-1} .

If y is not a double point, but a point of the quadric, then one will call the lines through y that intersect the quadric doubly or lie within it the *tangents* to the surface at y. The condition for this is that not only the first term in (3), but also the second one, must vanish, hence, that z will lie in the polar hyperplane to y. Thus, all of the tangents lie within the polar hyperplane of y, which is therefore also called the *tangent hyperplane,* or the *tangential hyperplane*, to the quadric at the point y. In particular, the tangential hyperplane includes all of the lines that lie in the quadric and go through y, hence, all of the linear subspaces that lie in the quadric and go through y.

If the point y lies outside the quadric then all of the points z that are harmonically separated from two points of the surface will lie in the polar of y, just like all of the contact points z of the tangents that go through y. The latter will generate a cone with vertex y whose equation is found by setting the discriminant of the quadratic equation (3) equal to zero:

$$(\sum a^{jk}\, y_j\, y_k)(\sum a^{jk}\, z_j\, z_k) - (\sum a^{jk}\, y_j\, z_k)^2 = 0\,.$$

If (a'_{jk}) is the inverse matrix to the non-singular matrix (a^{ik}) then one can solve equation (5) for y with its help:

$$y_j = \sum a'_{jk}\, u^k\,. \tag{7}$$

The hyperplane u will be a tangent hyperplane when and only when it goes through its pole y, hence, when:

$$\sum_{j,k} a'_{jk}\, u^j\, u^k = 0\,. \tag{8}$$

The tangential hyperplane of a double-point-free quadric thus defines a quadric in the dual space, or, as one says, a *hyperplane of the second class*.

As is well-known, equation (1) can always be brought into the form:

$$x_0^2 + x_1^2 + \cdots + x_{\rho-1}^2 = 0$$

by a coordinate transformation; thus, ρ is the rank of the matrix (a^{jk}). *Thus, two quadrics of equal rank are always projectively equivalent.* A quadric of rank 2 decomposes into two hyperplanes, whereas a quadric of rank 1 is a hyperplane that is counted twice.

One can find the intersection of the quadric $\mathfrak{Q}_{r-1}$ with a subspace S_p of S_n:

$$x_k = \lambda_0 \overset{0}{y}_k + \lambda_1 \overset{1}{y}_k + \cdots + \lambda_p \overset{p}{y}_k\,, \tag{9}$$

which will be found when one substitutes (9) in (1); this will yield a homogeneous quadratic equation in $\lambda_0, \ldots, \lambda_p$. Thus, the intersection will be a quadric $\mathfrak{Q}_{p-1}$ in the space S_p, as long as the space S_p is not completely included in the given quadric $\mathfrak{Q}_{r-1}$.

Problems. 1. An involutory projective transformation of the line S_1 into itself (i.e., an *involution*) consists of all point pairs that are harmonic with a given point. A degenerate involution consists of point pairs that include a fixed point.

2. The point pairs that are conjugate, relative to the quadric $\mathfrak{Q}_{r-1}$, to a given line of the space S_r and do not lie in Q_{r-1} define a point pair.

3. If one connects all of the points of a quadric $\mathfrak{Q}_{r-1}$ with a fixed point B that lies outside of the space S_r then one will obtain a quadric $\mathfrak{Q}_r$ with a double point at B.

4. Give an affine classification of the quadrics $\mathfrak{Q}_{r-1}$.

The foregoing was only, on the one hand, a multi-dimensional generalization of well-known facts from the analytic geometry of conic sections and quadratic surfaces. We now come to the discussion of linear spaces that lie in quadrics. We thus consider quadrics without double points exclusively.

As is well-known, there are two families of lines that lie in a quadratic surface $\mathfrak{Q}_2$ in S_3. As we will show in § 7 with the help of line geometry, there are two families of planes that lie in a quadric $\mathfrak{Q}_4$ in S_5. We would now like to show that, in general, two families of S_n lie in a quadric $\mathfrak{Q}_{2n}$, but, by contrast, only one family of S_n lies on a quadric $\mathfrak{Q}_{2n+1}$, and that the quadrics in both cases cannot include any linear spaces of higher dimension.

What are we to understand a family to be then? If we already have the concept of an irreducible algebraic manifold then we can clarify the notion of a family by saying that it is such an irreducible algebraic manifold. However, we would like our family to exhibit something more than just irreducibility and continuous connectedness: we will assume that any family in S_n has a rational parametric representation such that precisely one element S_n of the family belongs to each system of values, and that the entire family can be exhausted by the parametric representation. In this sense, we will prove the existence of *a rational family of S_n on $\mathfrak{Q}_{2n+1}$* (*two disjoint rational families of S_n on $\mathfrak{Q}_{2n}$*, resp.). In addition, we will show that two spaces S_n on $\mathfrak{Q}_{2n}$ that have an S_{n-1} as their intersection will always belong to *different* families.

In order to prove all of these statements, we use complete induction on n. For $n = 0$, a quadric consists of two separated points, whereas a quadric $\mathfrak{Q}_1$ – hence, a conic section – contains a single rational family of points. Namely, if one brings the equations of the conic section into the form:

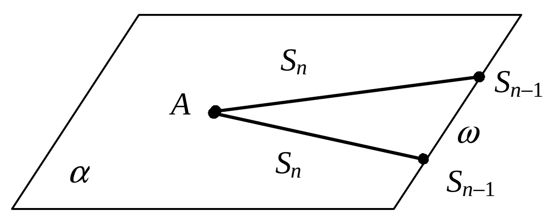
$$x_1^2 - x_0 x_2 = 0$$

then all of the points of the conic section will be given by the parametric representation:

$$\begin{cases} x_0 = t_1^2 \\ x_1 = t_1 t_2 \\ x_2 = t_2^2. \end{cases}$$

Now, we may assume that our statements are true for the quadrics Ω_{2n-2} and Ω_{2n-1}. We consider a quadric Ω_{2n}. (The case of Ω_{2n+1} can be handled in an analogous fashion, so the reader can omit it.)

We would next like to prove that the spaces S_n that lie in Ω_{2n} and go through a fixed point A of Ω_{2n} define two disjoint rational families. These spaces all lie in the tangent hyperplane α. Now, if ω is a fixed space S_{2n-1} that is contained in α and does not go through A (such a space exists, since α is an S_{2n}) then the intersection of Ω_{2n} with ω will be a *double-point-free* quadric Ω_{2n-1}. Namely, if Ω_{2n-2} had a double point D then it would be conjugate to all of the points of ω and to A, hence, the polar to D would coincide with α, which is not true, since α has only the pole A. The lines that connect A with the points of Ω_{2n-2} lie completely within Ω_{2n} since they contact this quadric at A and contain every other point of it elsewhere. Thus, if one connects a space S_{n-1} that lies in Ω_{2n-2} with A then the connecting space S_n will lie completely in Ω_{2n}. Conversely: If an S_n lies in Ω_{2n} and goes through A then it will also lie in the tangent hyperplane α, and will therefore have an S_{n-1} in common with α that lies in Ω_{2n-2}. – From the induction hypothesis, two rational families of S_{n-1} lie in Ω_{2n-2} and no space of higher dimension; hence, two rational families of spaces S_n will also go through A in Ω_{2n}, namely, the connecting space of A with each S_{n-1}, and no space of dimension higher than n. Furthermore, from the induction hypothesis, two spaces S_{n-1} in Ω_{2n-2} that have an S_{n-2} in common always belong to two different families. From this, it follows that two spaces S_n that go through A and have an S_{n-1} in common will also belong to two different families. We call these families $\Sigma_1(A)$ and $\Sigma_2(A)$.



It should be remarked that each space S_n that is in A and lies in Ω_{2n} also goes through A and thus belongs to $\Sigma_1(A)$ or to $\Sigma_2(A)$. Namely, if S_n does not go through A then the connecting space S_{n+1} of S_n with A would lie completely in Ω_{2n}, which is impossible.

Now, in order to free the two families of S_n from the point A and to make them span the entire quadric, we proceed as follows: We choose one of the spaces S_n that goes through A and lies in the quadric and consider all possible spaces S_{n+1} that go through it. They do not lie in Ω_{2n}, and they thus intersect Ω_{2n}, as well as a quadric Ω_{2n} that contains an S_n as a component, and thus decomposes into two S_n. It is impossible that they coincide, since every point of S_n would then be a double point of Ω_{2n} and would thus be conjugate to all points of S_{n+1}, which is not true, since this polar space is only an S_{n-1}. We denote the two spaces S_n that the quadric decomposes into by S_n and S'_n. If S_n and S_{n+1} are given then S'_n may be computed rationally when one draws $(n + 1)$ arbitrary lines through a point B of S_n that do not lie in S'_n and together span S_{n+1}, makes them intersect the quadric $\mathfrak{Q}_{2n}$, and determines the linear space S'_n by the intersection points $B_1, \ldots, B_{n+1}$ that differ from B. All of these steps are rational. Now, if we let S_n range through the entire family $\Sigma_2(A)$, and also let S_{n+1} range through all spaces through S_n then we will obtain a rational family of spaces S'_n; we denote it by Σ'_1. Likewise, if we let S_n range through the entire family $\Sigma_1(A)$ then we will obtain a second rational family of spaces S_n, which we denote by Σ'_2.

It was not an abuse of notation, but our explicit intent, that made us derive Σ'_1 from $\Sigma_2(A)$ and Σ'_2 from $\Sigma_1(A)$. Namely, if we choose the space S_{n+1} to be in α, in particular, then S_n and S'_n will both lie in α and will thus go through A. (Namely, if S'_n did not go through A then A would not be a double point of the quadric $\mathfrak{Q}_{2n}$ that is determined by S_n and S'_n, and an arbitrary line g through A in S_{n+1} would meet $\mathfrak{Q}_n$, and thus also $\mathfrak{Q}_{2n}$, at two different points, which cannot happen, since g lies in α, and is therefore tangent to $\mathfrak{Q}_{2n}$ at A.) S_n and S'_n will have an intersection S_{n-1} since they lie in S_{n+1}, and will thus belong to different families; thus, if S_n belongs to $\Sigma_2(A)$ then S'_n will belong to $\Sigma_1(A)$, and conversely. The spaces of the family Σ'_1 that go through A thus belong to the family $\Sigma_1(A)$, and the spaces of the family that go through Σ'_2 belong to $\Sigma_2(A)$.

We now show that each of the spaces S'_n that lie in $\mathfrak{Q}_{2n}$ belong to one and only one of the families Σ'_1, Σ'_2. For the spaces that go through A, this is already clear from the preceding: They belong to $\Sigma_1(A)$ when they belong to Σ'_1, and they belong to Σ'_2 when they belong to $\Sigma_2(A)$. Now, if a space S'_n in $\mathfrak{Q}_{2n}$ does not go through A then the connecting space of A with S'_n will be an S'_{n-1} whose intersection with $\mathfrak{Q}_{2n}$ will be a quadric $\mathfrak{Q}_n$ that decomposes into S'_n and another S_n through A. S'_n will belong to Σ'_2 or to Σ'_1, depending upon whether S_n belongs to $\Sigma_1(A)$ or to $\Sigma_2(A)$, respectively.

The families Σ'_1 and Σ'_2, which we denote by Σ_1 and Σ_2 from now on, are thus disjoint and exhaust the totality of all S_n in the quadric $\mathfrak{Q}_{2n}$. A continuous transition from one family to the other is impossible since the families would have to have an element in common. Were we to start with another point A', instead of A, we would obtain the same families as a result, but with a different parametric representation.

If two spaces S'_n, S''_n that lie in $\mathfrak{Q}_{2n}$ have an intersection S_{n-1} then one can always choose the point A to be in this intersection and then conclude from the foregoing considerations that S'_n and S''_n belong to different families $\Sigma_1(A)$ and $\Sigma_2(A)$, hence, to different families Σ_1, Σ_2, as well. Thus, all of our assertions about Q_{2n} are proved – assuming that they are valid for Q_{2n-1}. The induction is thus complete.

Last of all, we prove that *two spaces* S'_n, S''_n *of the same family always have an intersection of dimension* $n - 2k$*, and, by contrast, two spaces of different families always* *have an intersection of dimension* $n - 2k - 1$, *in which k is a whole number.* Thus, an empty intersection will be considered to be one of dimension – 1.

Once more, we employ complete induction on n. The assertion is trivial for $n = 0$ since each family will then consist of a single S_0, and the intersection of an S_0 with itself will be of dimension 0, although its intersection with another S_0 will have dimension – 1. We thus assume that the assertion is true for the S_{n-1} in $\mathfrak{Q}_{2n-2}$.

If one projects both of the families of S_{n-1} in $\mathfrak{Q}_{2n-2}$ from A, as above, then one will obtain both families $\Sigma_1(A)$ and $\Sigma_2(A)$ of spaces S_n through A. Under projection, one raises the dimension of the intersection space, like the space S_{n-1} itself, by one; an intersection of dimension $(n - 1) - 2k$ will become one of dimension $n - 2k$. Hence, our assertion is valid for the spaces of the families $\Sigma_1(A)$ and $\Sigma_2(A)$, and since the point A can be chosen arbitrarily the assertion will be valid for any two spaces S_n that have a point in common.

Now, let S_n' and S_n'' be two spaces that have no point in common. We choose A to be in S_n''. The connecting space S_{n+1} of A with S_n' has only the point A in common with S_n''. It intersects $\mathfrak{Q}_{2n}$ in a quadric $\mathfrak{Q}_n$ that decomposes into S_n' and another S_n, which goes through A. Our assertion is already proved for both S_n and S_n'', since both of them go through A; i.e., only outside of the intersection that consists of A does it have dimension n $2k$ when S_n and S_n'' belong to the same families and dimension $n - 2k - 1$ when S_n and S_n'' belong to different families. In the first case, however, S_n and S_n'' will belong to different families, and their intersection will, in fact, have dimension $- 1 = 0 - 1 = (n - 2k - 1) - 1 = n - 2(k + 1)$. In either case, the assertion is therefore proved to be true.

§ 10. Maps of hypersurfaces to points. Linear families.

The higher-dimensional spaces are not only interesting in themselves, but they also define an indispensable aid in the study of systems of algebraic curves in planes and surfaces in ordinary space. This rests upon the following:

One can map the planar algebraic curves and the algebraic surfaces in S_3, and generally the hypersurfaces of degree g of a given space S_n to the points of a projective space S_N in a one-to-one manner, in which we have set:

$$N = \binom{g+n}{n} - 1 .$$

Such a hypersurface will, in fact, be given by an equation:

$$a_0 x_0^g + a_1 x_0^{g-1} x_1 + \cdots + a_N x_N^2 = 0 ,$$

whose left-hand side may be multiplied by a non-zero factor λ, and whose coefficients may all be zero. The number of coefficients is well-known ([1]) to be equal to:

$$\binom{g+n}{n} = \binom{g+n}{g} = N + 1.$$

One can thus regard the coefficients $a_0, \ldots, a_N$ as the coordinates of a point a in a space S_N, from which the stated map is obtained. If one is dealing with curves of degree g in the plane then:

$$N = \binom{g+2}{2} - 1 = \tfrac{1}{2}g(g + 3).$$

The curves of degree g in S_0 may thus be mapped to points in a space of dimension $\frac{1}{2}g(g + 3)$ in a one-to-one manner.

Under the map, a linear subspace S_r of S_N corresponds to a family of hypersurfaces that one calls a *linear family* of *dimension r*. Special cases are: one-dimensional linear families, or *pencils*, whose elements are given by:

$$a_k = \lambda_1\, b_k + \lambda_2\, c_k\,,$$

and two-dimensional linear families, or *nets*, whose elements are given by:

$$a_k = \lambda_1\, b_k + \lambda_2\, c_k + \lambda_3\, d_k\,.$$

One can write this equation in another way: If $B = 0$ and $C = 0$ are two hypersurfaces that determine a pencil then the equations of the hypersurfaces in the pencil will obviously be given by:

$$\lambda_1\, B_k + \lambda_2\, C_k = 0\,.$$

Analogously, the formula:

(1) $$\lambda_1\, B_k + \lambda_2\, C_k + \ldots + \lambda_r\, D_r = 0$$

defines an r-dimensional linear family when the forms $B_0, \ldots, B_r$ are linearly independent.

By means of the map of the points of S_N to hypersurfaces and linear subspaces to linear families of dimension r, one can carry over all theorems that pertain to linear spaces in S_N to linear families of hypersurfaces with no further assumptions. In this way, one will obtain, among other things, the theorem: *$N - r$ linearly independent linear equations in the coordinates $a_0, \ldots, a_N$ define a linear family of hypersurfaces of dimension r.*

As an example, the hypersurfaces that go through $N - r$ given points define a linear family of dimension r, assuming that these points impose *independent* linear constraints

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([1]) The proof is obtained quite easily by complete induction on $n + g$, when one converts the form $f_g(x_0, \ldots, x_n)$ of degree g into the form $f_g(x_0, \ldots, x_{n-1}) + x_n\, f_{g-1}(x_0, \ldots, x_n)$.

on the hypersurfaces. In order to apply this to any particular case, if that is the case, one arranges the given points into a particular sequence: $P_1, \ldots, P_{N-r}$, and establishes whether there is a hypersurface of degree g that goes through $P_1, \ldots, P_{k-1}$, but not through P_k. If that is the case for any value of k with $1 < k \leq N - r$ then the linear conditions that the points impose on the hypersurfaces will be independent. By skillfully choosing the sequence of points, one can very frequently choose the hypersurfaces through $P_1, \ldots, P_{k-1}$ to be decomposable.

By this method, one effortlessly proves, e.g., that there are at most five points in the plane, no four of which lie in a line, that always impose independent conditions on the conic sections in the plane. If one can always draw a pair of lines through $k - 1$ (≤ 4) points, and does not go through a given k^{th} point, then one must have that this k^{th} point lies on a line with three others. From this, it follows that:

Three given points always determine a conic section. Four points that do not lie in a line always determine a pencil of conic sections. Five given points, no four of which lie in a line, determine a single conic section.

Problems. 1. One proves by the same method that eight points in a plane, no five of which lie in a plane and no eight of which lie on a conic section, always determine a pencil of curves of third order. [Hint: For curves through $k - 1$ given points, use those third-order curves that decompose into a conic section and a line or into three lines.]

2. If a, b, c, d are four given non-collinear points in the plane and if (xyz) always denotes the determinant of the coordinates of three points x, y, z then the pencil of conic sections that go through a, b, c, d will be given by the equation:

$$\lambda_1\,(abx)\,(cdx) + \lambda_2\,(acx)\,(bdx) = 0\,.$$

3. In the notation of problem 2, the conic section through five given points will be given by the equation:

$$(abx)\,(cdx)\,(ace)\,(bde) - (acx)\,(bdx)\,(abe)\,(cde) = 0\,.$$

4. Seven points in the space S_3, no four of which lie in a line, no six of which lie in a conic section, and no seven of which lie in a plane, always determine a net of second order surfaces.

[Hint: For the surfaces through $k - 1$ points, again use decomposable surfaces or, if that does not work, cones.]

A single curve of order g goes through $\frac{1}{2}g(g+3)$ points of the plane "in general," i.e., when these points represent independent constraints for the curves of order g. The exceptional case is the one for which the point P_k belongs to all of the curves of order g through $P_1, \ldots, P_{k-1}$, which obviously can come about only for particular positions of the point P_k relative to $P_1, \ldots, P_{k-1}$.

If the hypersurfaces $B_0, \ldots, B_r$ that determine a linear family (1) have one or more points – or even an entire manifold – in common then these points, or that manifold, will obviously belong to all hypersurfaces of the family. These points will be then be called the *base points* of the family, while the manifold will be called the *base manifold* of the family. In particular, it can happen that all of the forms $B_0, \ldots, B_m$ have a common factor A; in this case, all of the hypersurfaces (1) of the family will include $A = 0$ as a *fixed component.* For example, the conic sections in the plane that have a given triangle for polar triangle define a net with no base point, while the conic sections through three given points define a net with three base points, or also (when the three points lie in a line) a net with one fixed component.

A pencil of quadrics is given by:

$$a^{jk} = \lambda_1\, b^{jk} + \lambda_2\, c^{jk}\,, \tag{2}$$

in which the equation of a quadric is assumed to take the form:

$$\sum_j \sum_k a^{jk} x_j x_k = 0\ .$$

If D is the determinant of the matrix (a^{jk}) then the condition for a double point will read:

$$D = 0\ . \tag{3}$$

By means of (2), D is a form of degree $n + 1$ in λ_1 and λ_2. Equation (3) is either satisfied identically by λ_1, λ_2, or it has $n + 1$ (not necessarily distinct) roots. There is therefore at least one and at most $n + 1$ cones in the pencil (2), or else all of the hypersurfaces of the pencil are cones.

In the case of a pencil of conics, it will thus follow that a pencil of conics includes at least one decomposable conic. If one looks at all possible positions that a pair of lines can have relative to another conic then one will effortlessly obtain a complete classification of all pencils of conics (and their base points). Since a pair of lines has four (not necessarily distinct) points of intersection with another conic or a component in common with it, a pencil of conics will have either a fixed component or four (not necessarily distinct) base points. If the four base points are actually distinct then, from the above, the pencil will be determined by these four points: It will consist of all conics through these four points. The three decomposable conics of the pencil will then be the three pairs of opposing sides of a complete rectangle.

For more on the theory of pencils of conics, one may confer the classical work of CORRADO SEGRE ([1]).

We will later see that a pencil of n^{th}-order curves in the plane has n^2 (not necessarily distinct) base points or a fixed component. Likewise, a net of n^{th}-order surfaces in the space S_3 possesses either n^3 base points or a base curve or a fixed component.

For example, a pencil of 3rd-order plane curves has nine base points in general, of which, from Problem 1, for certain assumptions, any eight of them will already determine the pencil. Likewise, a net of quadratic surfaces in space has eight base points, in general, of which, under certain assumptions, any seven of them will determine the net, and therefore also the eight points.

Problems. 5. There are the following types of pencils of conics:

I. Pencils with four distinct base points and one common tangent whose double point defines a common polar triangle for all curves of the pencil.

II. Pencils with three distinct base points and one common tangent at one of these points. Two decomposable examples.

([1]) SEGRE, C.: "Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimension. " Torino Mem., 2nd Series, **36**.

III. Pencils with two distinct points and fixed tangents at these points. Two decomposable examples, and a double line.

IV. Pencils with two distinct base points with given tangents and given curvature at each of these points. One decomposable curve.

V. Pencils with one four-fold base point and a decomposable curve (namely, a double line).

VI. Pencils of decomposable conics with a fixed component.

VIII. Pencils of decomposable conics with fixed double points (Involutions of line pairs).

§ 11. Cubic space curves.

1. The rational normal curve. If one applies the map that was discussed in § 10 to hypersurfaces in S_1, in particular – i.e., to groups of n points in a line – then one will obtain a map of this group of points to the points of a space S_n. In order to have something definite in mind, we consider the case $n = 3$, although most of the following remarks will be valid for an arbitrary n .

Let the triple of points to be examined be given by equations of the form:

$$f(x) = a_0 x_1^3 - 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 - a_3 x_2^3 = 0\ ; \tag{1}$$

it will therefore be mapped to points (a_0, a_1, a_2, a_3) in S_3. One must pay particular attention to the triple that consists of three coincident points; for them, one will have:

$$f(x) = (x_1\,t_2 - x_2\,t_1)^2 = x_1^3 t_2^3 - 3x_1^2 x_2 t_1 t_2 + 3x_1 x_2^2 t_1^2 t_2 - x_2^3 t_1^3\ ;$$

hence, the image point will have the coordinates:

$$\begin{cases} y_0 = t_2^3 \\ y_1 = t_1 t_2^2 \\ y_2 = t_1^2 t_2 \\ y_3 = t_1^3 . \end{cases} \tag{2}$$

By means of (2), the t-line S_1 is mapped to a curve in the space S_3 (S_n, resp.) that one generally (for arbitrary n) calls a *rational normal curve*, and in the special case of $n = 3$, one calls it a *cubic space curve* ([1]). The projective transformations of such a curve will again be called cubic space curves.

The word "cubic" thus means that an arbitrary plane u intersects the curve in three (not necessarily distinct) points. Namely, if one substitutes (2) in the equation for the plane u then one will obtain a third degree equation:

$$u_0 t_2^3 + u_1 t_1 t_2^2 + u_2 t_1^2 t_2 + u_3 t_1^3 = 0 \tag{3}$$

that defines three points on the t-axis.

([1]) In the case $n = 2$, the normal curve will be a conic.

If these three points are q, r, s then for a certain choice of the arbitrary factors one will have:

$$u_0 t_2^3 + u_1 t_1 t_2^2 + u_2 t_1^2 t_2 + u_3 t_1^3 = (q_1\, t_2 - q_2\, t_2)\,(r_1\, t_2 - r_2\, t_2)\,(s_1\, t_2 - s_2\, t_2)$$

identically in t_1, t_2; hence, by equating coefficients, one will have:

$$(4)\qquad \begin{cases} u_0 = q_1 r_1 s_1 \\ u_1 = q_1 r_1 s_2 + q_1 r_2 s_1 + q_2 r_1 s_1 \\ u_2 = q_1 r_2 s_2 + q_2 r_1 s_2 + q_2 r_2 s_1 \\ u_3 = q_2 r_2 s_2 . \end{cases}$$

By means of (4), each triple of points p, q, r on the t-axis will correspond to a uniquely-determined plane u that intersects the curve at the points P, Q, R with the parameter values p, q, r, resp. It is therefore not only the points of S_3, but also, at the same time, the planes of S_3 that are mapped to triples of points on the parameter line S_1 in a one-to-one manner.

In particular, if the points P and Q coincide then u will be called a tangential plane at the point Q. Since the u depend upon the parameters s for a fixed $Q = R$, the tangential plane will define a pencil whose carrier goes through Q and will be called the *tangent* at the point Q. If all three P, Q, R coincide then u will be called the *osculating plane* to the point Q.

Theorem. *Any curve that permits a rational parametric representation by way of functions of degree three:*

$$(5)\qquad y_k = a_k t_2^3 + b_k t_1^2 t_2 + c_k t_1 t_2^2 + d_k t_2^3$$

is projectively equivalent to a cubic space curve or a projection of a cubic space curve onto the space S_2 or S_1.

Proof: The projective transformation:

$$y'_k = a_k\, y_0 + b_k\, y_1 + c_k\, y_2 + d_k\, y_3$$

obviously takes the curve (2) to the curve (5). If this transformation is degenerate then, from § 6, it will amount to a projection onto a subspace S_{r-1}.

If one projects the cubic space curves from a point of the curve onto a plane S_2, then, as we shall see, one will obtain a conic section. If one projects from a point that lies outside of the curve then one will obtain a plane curve that will obviously intersect any line in three points, hence, (cf. below, § 17), a plane curve of degree three. Finally, if one projects onto a line then one will obtain this line itself, covering itself several times. Other projections will not be considered.

2. The null system linked with the curve. Since each point of S_3 corresponds to a point-triple in S_1 in a one-to-one fashion, and since each such point-triple again corresponds to a plane there will also be a one-to-one map of the points of S_3 onto the plane u. One obtains its equations when one writes the same form $f(x)$, once in the form (1) and once in the form (3) (with x_1, x_2 instead of t_1, t_2), and equates the coefficients. If one writes z_0, z_1, z_2, z_3, instead of a_0, a_1, a_2, a_3, then one will obtain the equations:

$$\begin{cases} u_0 = & & & -z_3 \\ u_1 = & & 3z_2 & \\ u_2 = & -3z_1 & & \\ u_3 = z_0. & & & \end{cases} \tag{6}$$

Since the matrix of this linear transformation is skew-symmetric it will represent a *null system* ([1]).

If one takes z in (6) to be, in particular, a point y of the curve in the parametric representation (2) then one will immediately see that the plane u is the osculating plane at this point. Hence: *The null system associates each point of the curve with its osculating plane.* From this, one obtains a simple construction of the null points of a plane that is not a tangential plane: One attaches the osculating plane to the three intersection points of the plane with the curve. They intersect it at null points of the plane. Since each point can be regarded as the null point of its null plane, it follows that: *Three (not necessarily distinct) osculating planes go through each point of a cubic space curve. The connecting plane of their osculating points is the null plane of the points.*

3. The chords of the curve. In addition to the chords that connect two points of a curve, in the sequel we will also tacitly compute the tangents to the curve. We now prove:

Precisely one chord goes through each point outside of the curve.

Proof: We intersect the point A with all possible planes u. One then has:

$$a_0\, u_0 + a_1\, u_1 + a_2\, u_2 + a_3\, u_3 = 0;$$

hence, from (4), if P, Q, R are the intersection points of the plane with the curve then:

$$\begin{cases} a_0 q_1 r_1 s_1 + a_1 (q_1 r_1 s_2 + q_1 r_2 s_1 + q_2 r_1 s_1) \\ \qquad + a_1 (q_1 r_1 s_2 + q_1 r_2 s_1 + q_2 r_1 s_1) + a_3 q_2 r_2 s_2 = 0. \end{cases} \tag{7}$$

For a given Q and R, the linear equation (7) will generally determine a unique ratio $s_1 : s_2$, and therefore, the plane u. However, if AQR is a chord then each arbitrary point S of the curve will come about as the third intersection point with a plane through the APQ in question; hence, (7) will then be satisfied identically in s_1 and s_2. That will yield:

([1]) If, as we did at the start of this section, one chooses an arbitrary number n to be the dimension of the space then for even n one will obtain a polar system and for odd n, a null system.

$$(8) \qquad \begin{cases} a_0 q_1 r_1 + a_1(q_1 r_2 + q_2 r_1) + a_2 q_2 r_2 = 0 \\ a_0 q_1 r_1 + a_2(q_1 r_2 + q_2 r_1) + a_3 q_2 r_2 = 0. \end{cases}$$

From these two equations, one can uniquely determine the ratios:

$$q_1\, r_1 : \ (q_1\, r_2 + q_2\, r_1) : q_2\, r_2\,,$$

hence, the ratios of the coefficients of the quadratic equation whose values $q_1 : q_2$ and $r_1 : r_2$ are determined uniquely. The assertion follows from this.

On the other hand, in each plane there are obviously three (not necessarily distinct) chords. For this reason, one says that the chords of a cubic space curve define a "congruence of field degree 3 and bundle degree 1" (cf. below, § 34).

4. Projective generation of cubic space curves. Let QR and $Q'R'$ be two chords of the curve. The pencil of planes that one obtains when one projects all of the points S of the curve onto QR is represented by (4). As one sees, s_1 and s_2 are the projective parameters of the pencil. However, the same is true for every other chord $Q'R'$. Hence: *If one connects any two chords (or tangents) with all points of the curve by planes then one will obtain two projective pencils of planes that each cover each other.*

As is well-known, two projective pencils of planes generate a quadratic surface that goes through the lines that carry them. *Hence, one can pass a second-order surface of that includes a cubic space curve through any two chords of the curve.* If we next take both chords to be skew then the surface will include two distinct families of lines, and since they are skew, both chords will belong to the same family. In the exceptional case, each plane through one of the chords will intersect the curve at the endpoints only once; hence, each line in the other family will have only one point of intersection with the curve. Exceptionally, any plane through such a secant with the curve will intersect the space curve at the points of intersection of the secant with the curve only twice; hence, any line of the first family will again be a chord.

Secondly, if we let both chords pass through the same points of the curve then the quadric that they contain will be a cone. *Hence, the cubic space curve will be projected through a quadratic cone from each of its points.*

If we now consider three chords, the third of which not shall belong to the ruled family that is determined by the first two, then we will obtain three projective pencils of planes that cover each other, in such a way that each three corresponding planes will always intersect in a point of the curve. *Hence, a cubic space curve will be generated by the intersection of three mutually projective pencils of planes.*

Conversely: *Three projective pencils of planes generally generate a cubic space curve. Exceptions are:*

1. When corresponding planes of the three pencils always have a line in common.

2. When the intersection points of corresponding triples of planes lie in a fixed plane.

Proof: Let:

(9) $$\begin{cases} \lambda_1 l_1 + \lambda_2 l_2 = 0, \\ \lambda_1 m_1 + \lambda_2 m_2 = 0, \\ \lambda_1 n_1 + \lambda_2 n_2 = 0 \end{cases}$$

be the equations of three projective pencils of planes. If one uses these equations to compute the coordinates of the intersection point of the three associated planes then it will be represented by three-rowed determinants, hence, by forms of degree 3 in λ_1 and λ_2:

(10) $$y_0 : y_1 : y_2 : y_3 = \varphi_0(l) : \varphi_1(l) : \varphi_2(l) : \varphi_3(l) .$$

If the four forms φ_0, φ_1, φ_2, φ_3 are linearly independent then, from the theorem in no. 1 of this section, the curve that is represented by (10) will be a cubic space curve. However, if there exists a linear dependency:

$$c_0\varphi_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 = 0,$$

then this says that all of the points y will lie in a fixed plane. This plane will intersect the three projective pencils of planes in three projective pencils of lines that generate a conic or a line in it. However, if the three-rowed sub-determinants of which we spoke are identically zero then each of the three associated planes will go through a line; in general, this line will define a quadratic surface.

Suppose we assume that equations (9) actually define a cubic space curve. The equations of these curves will then be found by setting the two-rowed sub-determinants of the matrix:

$$\begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{pmatrix}$$

equal to zero. The cubic space curve will therefore be the complete intersection of three quadratic surfaces. Any two of these three surfaces, e.g.:

$$l_1 m_2 - l_2 m_1 = 0 , \qquad l_1 n_2 - l_2 n_1 = 0 ,$$

will have, besides the space curve, also the line $l_1 = l_2 = 0$, in common with each other.

Problems. 1. The residual intersection (*Restschnitt*) of two indecomposable quadratic surfaces with different vertices that have a common generator, but do not lie along it, is always a cubic space curve.

2. The quadratic surfaces that contain a given cubic space curve define a net.

3. A cubic space curve is uniquely determined by six of its points.

4. A cubic space curve always goes through six points, no four of which lie in a plane. (From Problems 3 and 4, one uses two cones, each of which have one of the six points for vertex and go through the remaining five.)

CHAPTER TWO

Algebraic functions

As its name implies, algebraic geometry deals with both geometric and algebraic concepts and methods. Whereas in the previous chapter the basic concepts of projective geometry were summarized, in this chapter the essential algebraic concepts and theorems will be discussed. The reader can find the proofs of the theorems presented, e.g., in my book that appears in this collection and is entitled "Moderne Algebra" ([1]).

§ 12. Concept and simplest properties of algebraic functions.

Let $\mathbb{K}$ be an arbitrary commutative field, say, the field of complex numbers. The elements of $\mathbb{K}$ are called *constants*. Let $u_1, \ldots, u_n$ be indeterminates, or, more generally, arbitrary quantities in an extension field of $\mathbb{K}$, between which there are no algebraic relations with constant coefficients. The field of rational functions of $u_1, \ldots, u_n$ is denoted by $\mathbb{K}(u)$ or $\mathbb{P}$.

We denote any element of an extension field of $\mathbb{K}(n)$ by ω and regard it as an *algebraic function* of $u_1, \ldots, u_n$ when it satisfies an algebraic equation $f(\omega) = 0$ with (not identically vanishing) coefficients in $\mathbb{K}(n)$. Among the polynomials $f(z)$ with the property $f(\omega) = 0$ there is a polynomial of least degree $\varphi(z)$, and one can prove algebraically that it has the following properties (cf., *Moderne Algebra I*, chap. 4):

1. $\varphi(z)$ is uniquely determined up to a factor in $\mathbb{K}(n)$.
2. $\varphi(z)$ is irreducible.
3. Any polynomial $f(z)$ in $\mathbb{P}[z]$ with the property $f(\omega) = 0$ is divisible by $\varphi(z)$.
4. For a given non-constant irreducible polynomial $\varphi(z)$ there is an extension field $\mathbb{P}(\omega)$ in which $\varphi(z)$ possesses a zero ω.
5. The field $\mathbb{P}(\omega)$ is uniquely determined by $\varphi(z)$ up to isomorphism, i.e., if ω_1 and ω_2 are two zeroes of the same polynomial $\varphi(z)$, which is irreducible over $\mathbb{P}(\omega)$, then one will have $\mathbb{P}(\omega_1) \cong \mathbb{P}(\omega_2)$, and this isomorphism will leave all elements of $\mathbb{P}$ fixed and take ω_1 to ω_2.

([1]) WAERDEN, B.L. VAN DER: *Moderne Algebra I*, 2nd ed., 1931; II, 1st ed., 1931; in particular, chaps. 4, 5, and 11.

Two such zeroes of a polynomial that is irreducible over $\mathbb{P}$ are called *conjugate* relative to $\mathbb{P}$. In general, two systems of algebraic quantities $\omega_1, \dots, \omega_n$ and $\omega_1', \dots, \omega_n'$ are called conjugate to each other when there is an isomorphism $\mathbb{P}(\omega_1, \dots, \omega_n) \cong \mathbb{P}(\omega_1', \dots, \omega_n')$ that leaves all elements of $\mathbb{P}$ fixed and takes each ω_r to ω_r'.

The divisibility property 3 may be further sharpened in our case of $\mathbb{P} = \mathbb{K}(n)$. $f(z)$ and $\varphi(z)$ only need to depend rationally upon $u_1, \dots, u_n$ alone. However, if one makes them *completely rational* in $u_1, \dots, u_n$ by multiplying them by a rational function of $u_1, \dots, u_n$ alone and then assumes that $\varphi(z)$ is *primitive* in $u_1, \dots, u_n$ – i.e., it includes no polynomial that depends upon $u_1, \dots, u_n$ alone as a factor (which one can obviously always arrange) – then $\varphi(z)$ will be an *irreducible polynomial* in $u_1, \dots, u_n, z$, and $f(z)$ will be divisible by $\varphi(z)$ *in the polynomial ring* $\mathbb{K}[u_1, \dots, u_n, z]$. All of this follows from a well-known lemma by GAUSS (cf., *Moderne Algebra I,* § 23.)

If $\omega_1, \dots, \omega_h$ are algebraic functions then all rational functions of $\omega_1, \dots, \omega_h$ and $u_1, \dots, u_n$ will define a field $\mathbb{P}(\omega_1, \dots, \omega_h) = \mathbb{K}(u_1, \dots, u_n, \omega_1, \dots, \omega_h)$ whose elements will be all of the algebraic functions of $u_1, \dots, u_n$; it will be an *algebraic function field.* Furthermore, one has the *transitivity theorem:* An algebraic extension of an algebraic function field is again an algebraic function field. If the extension is produced by the adjunction of *finitely many* algebraic functions then one will call it a *finite algebraic extension*.

Any polynomial $f(z)$ with coefficients in $\mathbb{P}$ possesses a *splitting field*, i.e., an algebraic extension field of $\mathbb{P}$ in which $f(z)$ is completely decomposed into linear factors. This splitting field is again uniquely determined up to isomorphism. If one decomposes $f(z)$ into nothing but linear factors then one will call $f(z)$, along with the zeroes of $f(z)$, *separable.* An algebraic extension field of $\mathbb{P}$ whose elements are all separable over $\mathbb{P}$ is called *separable over* $\mathbb{P}$. In this book, we will be concerned only with separable extension fields. When the field $\mathbb{P}$ includes the field of rational numbers (a field of characteristic zero), all algebraic extension fields of $\mathbb{P}$ will be separable.

For separable extension fields, one has the *primitive element theorem:* The adjunction of finitely many algebraic quantities $\omega_1, \dots, \omega_r$ may be replaced by the adjunction of a single quantity:

$$\theta = \omega_1 + \alpha_2 \omega_2 + \dots, \alpha_r \omega_r \qquad (\alpha_2, \dots, \alpha_r \text{ in } \mathbb{P});$$

i.e., $\omega_1, \dots, \omega_r$ may be expressed rationally in terms of θ.

When a separable algebraic function ω is identical with all of its conjugates relative to $\mathbb{P}$, it will be rational; i.e., it will belong to $\mathbb{P}$. The irreducible equation whose root is ω then has only one simple root, and can therefore only be linear.

The degree of transcendence. If $\omega_1, \ldots, \omega_m$ are elements of an algebraic function field, or at least an extension field of $\mathbb{K}$, then one will call them *algebraically independent* when any polynomial f with coefficients in $\mathbb{K}$ that has the property $f(\omega_1, \ldots, \omega_m) = 0$ will necessarily vanish identically. One can treat algebraically independent elements as indeterminates since their algebraic properties are the same. If $\omega_1, \ldots, \omega_m$ are not algebraically independent, but also not all algebraic over $\mathbb{K}$, then one can always find an algebraically independent subsystem $\omega_{i_1}, \ldots, \omega_{i_d}$ such that all of the ω_k will be algebraic functions of $\omega_{i_1}, \ldots, \omega_{i_d}$. The number d of these algebraically independent elements is called the *degree of transcendence* (or the *dimension*) of the system $\{\omega_1, \ldots, \omega_m\}$ relative to $\mathbb{K}$. If the $\omega_1, \ldots, \omega_m$ are all algebraic relative to $\mathbb{K}$ then the system $\{\omega_1, \ldots, \omega_m\}$ will have degree of transcendence zero.

In algebra, it is proved that the degree of transcendence d is independent of the choice of algebraically independent elements $\omega_{i_1}, \ldots, \omega_{i_d}$ (cf., *Moderne Algebra I*, § 64). If $\omega_1, \ldots, \omega_m$ are algebraic functions of $\theta_1, \ldots, \theta_n$, while, conversely, $\theta_1, \ldots, \theta_n$ also depend algebraically on $\omega_1, \ldots, \omega_m$, then the systems $\{\omega_1, \ldots, \omega_m\}$ and $\{\theta_1, \ldots, \theta_n\}$ will have the same degree of transcendence.

A polynomial $f(z_1, \ldots, z_n)$ in $\mathbb{P}[z_1, \ldots, z_n]$ is called *absolutely irreducible* when it remains irreducible under any extension of the ground field $\mathbb{P}$. The following theorem is true: *A finite algebraic extension of $\mathbb{P}$ is sufficient to decompose a given polynomial f into absolutely irreducible factors.*

Proof. Let the degree of f be less than c. We replace each z_r in $f(z_1, \ldots, z_n)$ with $t^{c^{\nu-1}}$, and thus define:

$$F(t) = f(t, t^c, t^{c^2}, \ldots, t^{c^{n-1}}).$$

Each term $z_1^{b_1} z_2^{b_2} \cdots z_n^{b_n}$ of $f(z_1, \ldots, z_n)$ then corresponds to a term:

$$t^{b_1 + b_2 c + \cdots + b_n c^{n-1}}$$

in $F(t)$. Different terms of f yield different terms of F, since a whole number can be described in only one way as $b_1 + b_2 c + \ldots + b_n c^{n-1}$ with $b_r < c$. The coefficients of the terms of f are thus also coefficients in $F(t)$. If f decomposes in some extension field of $\mathbb{P}$ then $F(t)$ will also decompose in the same field if the fact that:

$$f(z) = g(z)h(z)$$

implies that:

$$f(t, t^c, t^{c^2}, \ldots) = g(t, t^c, t^{c^2}, \ldots)\, h(t, t^c, t^{c^2}, \ldots)$$

or that:

$$F(t) = G(t)H(t),$$

and the coefficients of $g(z)$ and $h(z)$ will also be coefficients of $G(t)$ and $H(t)$, resp.. Now, a finite extension of $\mathbb{P}$ will suffice to completely decompose $F(t)$ into linear factors. The coefficients of the factors $G(t)$ and $H(t)$ will also belong to this extension field, hence, those of $f(z)$ and $g(z)$, as well. The assertion is thus proved.

§ 13. The values of algebraic functions. Continuity and differentiability.

Let ω be an algebraic function of $u_1, \ldots, u_n$ that is defined by an irreducible rational equation:

(1) $$\varphi(u, \omega) = a_0(u)\, \omega^g + a_1(u)\, \omega^{g-1} + \ldots + a_g(u) = 0 .$$

Thus, the $a_0, \ldots, a_g$ are assumed to be polynomials in the u with no common terms.

We understand a value ω' of the function ω for particular values u' of the indeterminates u to mean any solution ω of the equation $\varphi(u, \omega) = 0$. When $a_0(u') \neq 0$, there will be g values ω associated with any system of values u' for u, which we denote by $\omega^{(1)}, \ldots, \omega^{(g)}$, and which will be defined by:

(2) $$\varphi(u', \omega) = a_0(u')\prod_1^g (z - \omega^{(r)}) .$$

Some of the roots $\omega^{(r)}$ will be equal to each other if and only if $D(u') = 0$, where $D(u)$ is the discriminant of equation (1). $D(u)$ is not identically zero in the u. The values u' for which $a_0(u')\, D(u') = 0$ are called *critical values* of the function ω. In general, they are associated with less than g different values ω', and in some situations, even none at all.

Theorem. *Any correct algebraic relation $f(u, \omega) = 0$ will remain correct under the replacement of u with any value u' and ω with one of the associated values ω'.*

Proof. It follows from § 12 that if $f(u, \omega) = 0$ then there will be a factorization:

$$f(u, \omega) = \varphi(u, \omega)\, g(u, \omega)$$

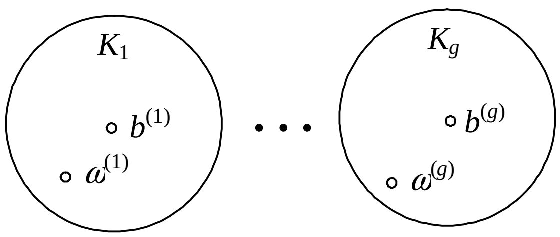
and therefore upon replacing u' and ω', one will be able to assert that:

$$f(u', \omega') = 0 .$$

We now assume that the ground field $\mathbb{K}$ from which the values u' and ω' will be chosen is the field of complex numbers. We investigate the continuous dependency of the values of the function ω' on the values of the argument u'. We thus restrict ourselves to the values u' in a neighborhood $U(a)$ of a non-critical locus a (here, a locus will mean simply a system of values $a_1, \dots, a_n$ of the independent values $u_1, \dots, u_n$); thus we assume that $|u_i' - a_i| < \delta$, in which δ is a positive number that is yet to be determined.

Since a is not critical, the locus a will be associated with g different values $b^{(1)}, \dots, b^{(g)}$ of the function ω, which can be understood to be points of the complex number plane. We draw circles $K_1, \dots, K_g$ around these points that have an arbitrarily small radius ε and have no interior points in common.

Any locus u' in a sufficiently small neighborhood $U(a)$ is associated with g values $\omega'^{(1)}, \dots, \omega'^{(g)}$ of the function ω. One can now express the *theorem of the continuity of algebraic functions* in the following way:



For a suitable choice of the neighborhood $U(a)$ (hence, of the number δ), in any circle K_r there lies precisely one value $\omega'^{(\mu)}$ such that one can enumerate the $\omega'^{(r)}$ in such a fashion that $\omega'^{(r)}$ lies in K_r. With this enumeration, any $\omega'^{(r)}$ is a single-valued function of u' that is continuous in the entire neighborhood $U(a)$.

Proof. If one sets $z = b^{(1)}$ and takes the absolute value then it will follow from (2) that:

$$\left|\frac{\varphi(u', b^{(1)})}{a_0(u')}\right| = \prod_1^g \left|b^{(1)} - \omega^{(r)}\right|.$$

Now, $\dfrac{\varphi(u', b^{(1)})}{a_0(u')}$ is a continuous function of u' that takes the value zero for $u' = a$. In a sufficiently small neighborhood $U(a)$, one therefore has:

$$\left|\frac{\varphi(u', b^{(1)})}{a_0(u')}\right| < \varepsilon^n;$$

hence, also:

$$\prod_1^g \left|b^{(1)} - \omega^{(r)}\right| < \varepsilon^n.$$

If all of the factors on the left were $\geq \varepsilon$ then this inequality would be false. Hence, at least one factor must be $< \varepsilon$, i.e., at least one $\omega'^{(r)}$ lies in the circle K_1 of radius ε around $b^{(1)}$. The same is also true for the circles $K_2, \dots, K_g$. Since there are just as many points $\omega'^{(\nu)}$ as circles K_ν and the circles are distinct from each other, any circle $K^{(\nu)}$ must contain precisely one point $\omega^{(\nu)}$, and we can choose the normalization of $\omega^{(\nu)}$ in such a way that $\omega^{(\nu)}$ lies in K_ν. Any $\omega^{(\nu)}$ is then determined uniquely. Furthermore, on the basis of the proof that we just carried out $|\omega^{(\nu)} - b^{(\nu)}| < \varepsilon$ for an arbitrarily small ε, as long as $|u_i' - a_i| < \delta$. Thus, the function $\omega^{(\nu)}$ is continuous at the locus $u' = a$. Since one can ultimately replace the locus a with any other non-critical locus, hence, in particular, with any locus inside of $U(a)$, the function $\omega^{(\nu)}$ will be continuous everywhere in $U(a)$. Since the loci inside of $U(a)$ are not critical, it will follow that the associated values $\omega^{(1)}, \dots, \omega^{(n)}$ will all be different.

The differentiability of the algebraic functions follows very easily from their continuity. Thus, since we are only concerned with the partial differentiability with respect to one of the variables $u_1, \dots, u_n$, we can restrict ourselves to the case of a single indeterminate u. One may associate the functional value b to a value a of u, and the value $\omega' = b + k$ to $u' = a + h$. One will then have:

$$\varphi(a, b) = 0, \qquad \varphi(a + h, b + k) = 0. \tag{3}$$

We now have to prove that lim k/h exists when $h \to 0$. The partial derivatives of the polynomial $\varphi(u, z)$ may be denoted by φ_u and φ_z. They refer to the coefficients of the first powers in h in the development of $\varphi(a + h, z)$ [$\varphi(u, b + k)$, resp.]. If one now develops $\varphi(a + h, b + k)$ in powers of h and then in powers of k then one will get:

$$\begin{cases} \varphi(u+h, z+k) = \varphi(u, z+k) + h\varphi_1(u, h, z+k) \\ \qquad\qquad\qquad\quad = \varphi(u, z) + h\varphi_1(u, h, z+k) + k\varphi_2(u, z, k), \end{cases} \tag{4}$$

with:

$$\varphi_1(u, 0, z) = \varphi_u\,,$$
$$\varphi_2(u, 0, z) = \varphi_z\,.$$

If one replaces $u = a$, $z = b$ in (4) then, due to (3), it will follow that:

$$0 = h\varphi_1(a, h, b + k) + h\varphi_2(a, b, k).$$

Since a was not a critical value, one will have $\varphi_z(a, b) \neq 0$, and therefore one will also have $\varphi_2(a, b, k) \neq 0$ for sufficiently small k. One can therefore divide the functions and get:

$$\frac{k}{h} = -\frac{\varphi_1(a, h, b+k)}{\varphi_1(a, b, k)}.$$

If one now lets h tend to zero then, due to the continuity of the function ω', k will also tend to zero. Thus, $\varphi_2(a, b, k)$ will tend to $\varphi_2(a, b)$ and $\varphi_2(a, h, b + k)$ will tend to $\varphi_2(a, b)$. It will then follow that:

$$\frac{d\omega'}{du'} = \lim\ \frac{k}{h} = -\frac{\varphi_u(a, b)}{\varphi_z(a, b)}.$$

Therefore, the differentiability is proved at any non-critical locus. One likewise shows that the differential quotient at any such locus will have the value:

$$\frac{d\omega'}{du'} = -\frac{\varphi_u(u',\omega')}{\varphi_z(u',\omega')}. \tag{5}$$

In my book *Moderne Algebra*, § 65, I showed that one can also define the differential quotients of a separable algebraic function ω, independently of any continuity properties and for an arbitrary ground field, by means of:

$$\frac{d\omega}{du} = -\frac{\varphi_u(u,\omega)}{\varphi_z(u,\omega)},$$

and all of the rules of differentiation can be derived from this definition immediately.

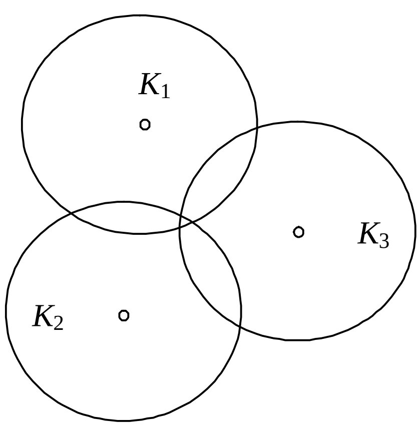
The analyticity of a complex-valued function of complex variables follows from its differentiability. *Therefore, the values $\omega^{(1)}, \ldots, \omega^{(g)}$ of an algebraic function in the neighborhood of a non-critical locus a are regular analytic functions of the complex variables u'.* Incidentally, the same thing is true for the values of an algebraic function of more than one variable at a non-critical locus.

§ 14. Series development for algebraic functions of one variable.

A regular analytic function of one variable u' may always be developed into a power series. In particular, the regular function elements $\omega^{(1)}, \ldots, \omega^{(g)}$ that were examined in § 2 have series developments at any non-critical locus a:

$$\omega^{(r)} = c_0^{(r)} + c_1^{(r)}\tau + c_2^{(r)}\tau^2 + \ldots \quad (\tau = u' - a).$$

They converge in any circle around a that contains no critical loci.



For a critical locus, the situation is somewhat more complex. Let a be such a critical locus in the u'-plane. We next assume that the initial coefficient $a_0(u)$ in equation (1), § 13, does not vanish at the locus $u = a$. One now draws a sequence of circles K_1, K_2, K_3 in the u'-plane that go through a, such that any two of them have a common neighborhood, which, however, contains no other critical values, so there are regular function elements $\omega^{(1)}, \ldots, \omega^{(g)}$ in any of these circles g. In the common neighborhood of two circles, the $\omega^{(1)}, \ldots, \omega^{(g)}$ of one circle must agree with the $\omega^{(1)}, \ldots, \omega^{(g)}$ of the other circle for any series. If one now starts, say, from $\omega^{(1)}$ in the first circle K_1, and then seeks the element in K_2 that agrees with $\omega^{(1)}$ in the common neighborhood, and then goes to K_3, until one again returns to K_1, then it can happen that one will either obtain the same functional element $\omega^{(1)}$ or that one will obtain the "analytic continuation" of another element – say $\omega^{(2)}$ – by the process just described. However, in either case, one must again return to $\omega^{(1)}$ after a finite number of orbits. If this is true after k orbits then one will have k functional elements $\omega^{(1)}, \ldots, \omega^{(k)}$ that are analytically connected in a neighborhood of the point a, and which will define a *cycle*. The n functional elements $\omega^{(1)}, \ldots, \omega^{(g)}$ will break up into a certain number of cycles $[\omega^{(1)}, \ldots, \omega^{(k)}]$, $[\omega^{(k+1)}, \ldots, \omega^{(k+l)}]$, …, $[\omega^{(m+1)}, \ldots, \omega^{(g)}]$ in this manner.

In order for us to exhaustively describe the sort of multi-valuedness in our analytic function ω at the locus $u = a$, we would like to convert the multi-valued function ω into a single-valued one in a neighborhood of $u = a$ by the introduction of the "position uniformization" $\tau = \sqrt[k]{u' - a}$. This comes about as a consequence of the following argument:

$u' = a + \tau^k$ is an analytic function of τ, and each $\omega^{(r)}$ is an analytic function of u' in one of the circles that we just described. One will get $\omega^{(r)}$ as an analytic function of τ by combining these analytic functions. If one now rotates the point τ around the zero point once then $u' - a$ will rotate around the zero point k times; u' will therefore encircle the point a k times. Since $\omega^{(1)}, \ldots, \omega^{(k)}$ can be cyclically permuted by a single orbit around the locus a, they will go back to themselves precisely after k such orbits. In a neighborhood of the locus $\tau = 0$ (that does not include this locus itself) the $\omega^{(1)}, \ldots, \omega^{(k)}$ are therefore single-valued analytic functions of τ. However, these functions will be restricted under a reduction of the neighborhood about the locus $\tau = 0$, since it is a well-known elementary fact that one can estimate the behavior of the roots of an algebraic function by the behavior of its coefficients. There, the locus $\tau = 0$ will either be an essential singularity or a pole, i.e., the locus will not be a singularity, at all. One can thus specify the values of $\omega^{(1)}, \ldots, \omega^{(k)}$ at the locus $\tau = 0$ in such a way that these functions will be analytic in the entire neighborhood of $\tau = 0$, and can therefore be developed into a power series in τ:

$$\begin{cases} \omega^{(1)} = \alpha_0^{(1)} + \alpha_1^{(1)}\tau + \alpha_2^{(1)}\tau^2 + \cdots \\ \omega^{(2)} = \alpha_0^{(2)} + \alpha_1^{(2)}\tau + \alpha_2^{(2)}\tau^2 + \cdots \\ \qquad \cdots \\ \omega^{(k)} = \alpha_0^{(k)} + \alpha_1^{(k)}\tau + \alpha_2^{(k)}\tau^2 + \cdots \end{cases} \tag{1}$$

But there is more! If one describes a complete circle around the point 0, or only the k^{th} part of it:

$$\tau = r\, e^{i\theta}, \qquad 0 \le \theta \le \frac{2\pi}{k},$$

then u' will describe a complete circle, and $\omega^{(1)}$ will therefore go to $\omega^{(2)}$, $\omega^{(2)}$ will go to $\omega^{(3)}$, …, and $\omega^{(k)}$ will go to $\omega^{(1)}$. Hence, any power series $\omega^{(1)}, \ldots, \omega^{(k)}$ will originate in the previous one when one makes the replacement:

$$\tau \to \tau\,\zeta, \qquad\qquad \zeta = e^{\frac{2\pi}{k}}.$$

This becomes immediately remarkable in the form of the power series (1), since they define a cycle.

In all of the previous considerations, nothing essential would change if $a_0(a) = 0$. One could then introduce – e.g. – $a_0(a)\,\omega$ instead of ω as a new function; the values of this function would remain restricted for $u = a$. One would also obtain a cycle of power series in this particular case for which one would now have that only a finitely many negative powers of τ could appear.

If one substitutes $\tau = (u' - a)^{1/k}$ in the development (1) then these fractional powers of $u - a$ will become power series that one calls PUISEAUX *series*; we will denote these power series by P_ν. If one substitutes them everywhere in equation (2), § 13 then it will follow that:

$$\varphi(u', z) = a_0(u')\prod_1^g (z - P_\nu).$$

Since this equation is valid for all of the u' in a neighborhood of the locus a, one can replace $u' - a$ in it with an indeterminate x and obtain the factor decomposition:

$$\frac{\varphi(x,z)}{a_0(x)} = \prod_1^g (z - P_\nu), \tag{2}$$

in which the P_ν are power series in the fractional powers of x with finitely many negative exponents.

The derivation of the power series developments P_ν that was used here, which originated with PUISEAUX, is indeed the simplest and most natural, but it leaves everything unknown, since, in reality, it treats a *purely algebraic* situation in the series development, and it also gives no means of effectively computing the power series. We thus present a second purely algebraic derivation of the power series development of algebraic functions that originates in the simpler form of OSTROWSKI, and is valid for arbitrary ground fields of zero characteristic. The convergence of the series will thus generally remain outside of consideration; it is uninteresting from an algebraic standpoint and is, moreover, already proved by the foregoing function-theoretic considerations. From now on, we will simply deal with *formal* power series $P_1, \ldots, P_n$, which involve fractional powers of $u - a$ and satisfy equation (2) in a purely formal way.

The denominator $a_0(x)$ in (2) can be decomposed into linear factors, and its inverse can be developed into a geometric series in x:

$$(\alpha x - \beta)^{-1} = \beta^{-1}\left(1 - \frac{\alpha x}{\beta} + \frac{\alpha^2 x^2}{\beta^2} - \cdots\right),$$

as long as it has the form $(\alpha x - \beta)^{-1}$ with $\beta \neq 0$. The left-hand side of (2) will then take on the form of a polynomial in x whose coefficients are power series in x divided by a power of x, hence, power series with finitely many negative exponents.

HENSEL's lemma. *If $F(x, z)$ is a polynomial in x of the form:*

$$F(x, z) = z^n + A_1 z^{n-1} + \dots + A_n$$

whose coefficients are complete power series in x:

$$A_\nu = a_{\nu 0} + a_{\nu 1}\, x + a_{\nu 2}\, x^2 + \dots$$

and $F(0, z)$ decomposes into two relatively prime factors of degree p and q with $p + q = n$:

$$F(0, z) = g_0(z) \cdot h_0(z): \qquad (g_0(z), h_0(z)) = 1$$

then $F(x, z)$ will decompose into two factors of the same degrees in z:

$$F(x, z) = G(x, z) \cdot H(x, z)$$

whose coefficients are, in any case, complete power series in x. One thus has:

$$G(x, z) = g_0(z), \qquad H(x, z) = h_0(z).$$

Proof. We order $F(x, z)$ in powers of x:

$$F(x, z) = F(0, z) + x\, f_1(z) + x^2 f_2(z) + \dots,$$
$$f_1(z) = a_{1k}\, z^{n-1} + \dots + a_{nk}\,,$$

and make the Ansatz for $G(x, z)$ and $H(x, z)$ that:

$$G(x, z) = g_0(z) + x\, g_1(z) + \ + x^2\, g_2(z)\dots,$$
$$H(x, z) = h_0(z) + x\, h_1(z) + \ + x^2\, h_2(z)\dots$$

With this Ansatz, the polynomials $g_1(z)$, $g_2(z)$, … will have degree at most $p - 1$, and the polynomials $h_1(z)$, $h_2(z)$, … will have degree at most $q - 1$. If we now form the product $G(x, z) \cdot H(x, z)$ and equate it with $F(x, z)$ then we will obtain a sequence of equations of the form:

(1) $$g_0(z)\, h_0(z) + g_1(z)\, h_{k-1}(z) + \dots + g_k(z)\, h_0(z) = f_k(z) \qquad (k = 1, 2, \dots).$$

If we now assume that have already determined $g_1, \dots, g_{k-1}$ and $h_1, \dots, h_{k-1}$ from the first $k - 1$ equations in (1) then, from (1), we will have an equation for the determination of g_k and h_k:

(2) $$g_0(z)\, h_k(z) + h_0(z)\, g_k(z) = B_k(z),$$

in which $B_k(z)$ will be a given polynomial of degree at most $n - 1$.

This equation is, however, well known to always be soluble, and indeed, in such a way that g_k and h_k have degrees at most $p - 1$ and $q - 1$, resp. (cf., *Moderne Algebra I*, § 29). Therefore, one can determine all of the g_k and h_k as series from (1). The power series that they define for $G(x, z)$ [$H(x, z)$, resp.] will be polynomials in z of degree p (q, resp.) that go to $g_0(z)$ [$h_0(z)$, resp.] for $x = 0$. The lemma is thus proved.

Theorem. *Any polynomial:*

$$F(x, z) = z^n + A_1 z^{n-1} + \ldots + A_n$$

whose coefficients are power series in x with only finitely many negative exponents will decompose completely into linear factors:

$$F(x, z) = (z - P_1)(z - P_2) \ldots (z - P_n),$$

in which $P_1, \ldots, P_n$ are power series, each of which lead to powers of a suitable fractional power of x.

Proof. We may assume that $A_1 = 0$, since otherwise we would need only to introduce $z - \frac{1}{n} A_1$ in place of z as a new variable. If A_ν is not identically zero then the development of A_ν will begin with $a_\nu x^{\sigma_\nu}$, where $a_\nu \neq 0$. If all of the $A_\nu = 0$ then there will be nothing to prove. Otherwise, let σ be the smallest of the numbers σ_ν / ν such that $A_\nu \neq 0$. It is then obvious that:

$$\sigma_\nu - \nu\sigma \geq 0 \qquad (\nu = 1, 2, \ldots, n),$$

in which the equality is valid for at least one ν. If we now introduce a new variable ζ by:

$$z = \zeta x^\sigma$$

then our polynomial will be converted into:

$$F(x, z) = F_1(x, \zeta)\, x^{n\sigma} (\zeta^n + A_2 x^{-2\sigma} \zeta^{n-2} + \ldots + A_n x^{-n\sigma}). \tag{4}$$

If one now has $\sigma = p / q$ with $q > 0$, and one sets:

$$\xi = x^{1/q}, \qquad x = \xi^q,$$

then the brackets on the right-hand side of (4) can also be written:

$$\Phi(\xi, \zeta) = \zeta^n + B_2(\xi)\zeta^{n-2} + \ldots + B_n(\xi),$$

with:

$$B_\nu(\xi) = A_\nu(\xi)\, x^{-\nu p}.$$

As long as it is not identically zero, the power series $B_\nu(\xi)$ will begin with:

$$a_\nu \xi^{\sigma_\nu - \nu p} = a_\nu \xi^{q(\sigma_\nu - \nu\sigma)},$$

which will then be a complete power series x whose constant term $B_\nu(0)$ will be non-zero for at least one ν. The polynomial:

$$\varphi(\zeta) = \Phi(0, \zeta) = \zeta^n + \ldots + a_\nu\, \zeta^{n-\nu} + \ldots$$

will therefore not be equal to ζ^n. On the other hand, since the coefficient of ζ^{n-1} is null, $\varphi(\zeta)$ cannot be the n^{th} power of a linear factor $\zeta - \alpha$. $\varphi(\zeta)$ thus possesses at least two different roots and may thus be decomposed into two relatively prime factors:

$$\varphi(\zeta) = g_0(\zeta) \cdot h_0(\zeta) .$$

From HENSEL's lemma, $\Phi(x, z)$, and therefore, also $F(x, z)$, will now decompose into two factors of the same degree as $g_0(\zeta)$ and $h_0(\zeta)$ whose coefficients will be power series in ξ.

If we apply the same reasoning to the two factors of $F(x, z)$ then we can also further decompose these factors and proceed in this way until the decomposition of $F(x, z)$ into linear factors is complete.

It is self-explanatory that one can examine the behavior of the function ω in the neighborhood of $u = \infty$ in precisely the same way as in the neighborhood of $u = a$, if one now sets $u^{-1} = x$ instead of $u - a = x$. The roots $\omega^{(1)}, \ldots, \omega^{(n)}$ will then be power series in increasing powers of $x = u^{-1}$.

Problem. 1. Determine the initial terms in the power series development of the roots of the polynomial:

$$F(u, z) = z^3 - uz + u^3$$

for the neighborhood of the locus $u = 0$.

§ 15. Elimination.

In what follows, we will need some theorems from elimination theory that we now summarize briefly.

The resultant. Two polynomials with undetermined coefficients:

$$f(z) = a_0\, x^n + a_1\, x^{n-1} + \ldots + a_n ,$$
$$g(z) = b_0\, x^m + b_1\, x^{m-1} + \ldots + b_m$$

possess a resultant:

$$R = \begin{vmatrix} a_0 & a_1 & \cdots & \cdots & \cdots & a_0 & & & \\ & a_0 & a_1 & \cdots & \cdots & \cdots & a_n & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & & a_0 & a_1 & \cdots & a_n & \\ b_0 & b_1 & \cdots & \cdots & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & \cdots & \cdots & b_m & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & & b_0 & b_1 & \cdots & b_m & \end{vmatrix}$$

with the following properties:

1. For special values of the a_j and b_k one will have $R = 0$ if and only if either $a_0 = b_0 = 0$ or $f(x)$ and $g(x)$ have a common factor $\varphi(x)$.
2. Any term of R has degree m in the coefficients a_j, degree n in the b_k, and weight (viz., the sum of the indices of the factors a_j and b_k together) $m \cdot n$.
3. One has an identity of the form:

$$R = A\, f(x) + B\, g(x),$$

in which A and B are a_j, b_k, and x, and A has degree at most $m - 1$ and B has degree at most $n - 1$.

4. If $\xi_1, \ldots, \xi_n$ is the zero locus of $f(x)$ and $\eta_1, \ldots, \eta_m$, that of $g(x)$ then one will also have the following expression for the resultant R:

$$R = a_0^m \prod_1^n g(\xi_\nu) = (-1)^{mn} b_0^n \prod_1^m f(\eta_\mu)$$
$$= a_0^m b_0^n \prod_1^n \prod_1^m (\xi_\nu - \eta_\nu)\,.$$

One understands the resultant of two homogeneous forms in x_1 and x_2:

$$F(x) = a_0 x_1^n + a_0 x_1^{n-1} x_2 + \cdots + a_n x_2^n\,,$$
$$G(x) = b_0 x_1^m + b_0 x_1^{m-1} x_2 + \cdots + b_m x_2^m$$

to mean the above determinant R, in any case. It will be zero if and only if $F(x)$ and $G(x)$ have a common factor. By switching the roles of x_1 and x_2, the resultant will be multiplied by $\varepsilon = (-1)^{mn}$, which follows easily from the form of the determinant.

The resultant system of a sequence of polynomials. Let $f_1(x), \ldots, f_r(x)$ be polynomials of degree $\leq n$ with undetermined coefficients $a_1, \ldots, e_\omega$. There is then one system of forms $R_1, \ldots, R_s$ in the coefficients $a_1, \ldots, e_\omega$ with the following properties:

1. The expressions $R_1, \ldots, R_s$ vanish for special values of the $a_1, \ldots, e_\omega$ if and only if $f_1, \ldots, f_r$ have a common factor, or the initial coefficient vanishes in all of these polynomials.

2. All of the terms of $R_1, \ldots, R_s$ have the same degree in the coefficients of any individual polynomial and the same weight as all of the coefficients together.

3. One has identities of the form:

$$R_j = \sum A_{jk} f_k(x),$$

in which the A_{jk} are polynomials in the $a_1, \ldots, e_\omega$, and x.

The resultant system of a sequence of homogeneous forms. Let $f_1, \ldots, f_r$ be forms in $x_0, x_1, \ldots, x_n$ with undetermined coefficients $a_1, \ldots, e_\omega$. There is then a system of forms $R_1, \ldots, R_s$ with the following properties:

1. The forms $R_1, \ldots, R_s$ vanish for special values of $a_1, \ldots, e_\omega$ if and only if $f_1, \ldots, f_r$ possess a non-trivial common zero locus – i.e., one that is different from (0, 0, …, 0) – in a suitable extension field.

2. $R_1, \ldots, R_s$ are homogenous in the coefficients of any individual form $f_1, \ldots, f_r$.

3. One has identities of the form:

$$x_\nu^\sigma R_j = \sum A_{\nu jk} f_k ,$$

in which the $A_{\nu jk}$ are forms in the $a_1, \ldots, e_\omega, x_0, \ldots, x_n$.

One also calls the construction and the setting to zero of the resultant system of the polynomials (resp., forms) $f_1, \ldots, f_r$ the *elimination* of x (resp., of $x_0, \ldots, x_n$) from the equations $f_1 = 0, \ldots, f_r = 0$.

If the equations $f_1, \ldots, f_r$ are homogeneous in further sequences of variables then the elimination of such a sequence will yield a system of resultants that is again homogeneous in the other sequences, such that one can continue the elimination. There is therefore also a resultant system for forms that are homogeneous in further sequences of variables, a resultant system with properties that are completely analogous to properties 1, 2, 3.

(For the proof, see *Moderne Algebra II*, chap. 11.)

CHAPTER THREE

Plane algebraic curves

In this chapter, x, y, z, u will mean indeterminates, while η, ζ, … will mean complex numbers. The ξ and ω that will be introduced later on will be algebraic functions of one indeterminate u.

§ 16. Algebraic manifolds in the plane

Let there be given a system of homogeneous equations:

$$f_\nu(\eta_0, \eta_1, \eta_2) = 0 \qquad (\nu = 1, 2, \ldots, r). \tag{1}$$

One calls the totality of points η in the plane that satisfy equations (1) an *algebraic manifold.* However, the totality of points that satisfy a single homogeneous equation is called an *algebraic curve*.

We would like to show that every algebraic manifold in the plane is composed of an algebraic curve and finitely many isolated points. To that end, we define the greatest common factor $g(y)$ of the polynomial $f_\nu(\eta_0, \eta_1, \eta_2)$ and set:

$$f_\nu(y) = g(y)\, h_\nu(y).$$

The solutions of (1) will then reside on the points of the curve:

$$g(\eta) = 0 \tag{2}$$

and the solutions of the system:

$$h_\nu(\eta) = 0 \qquad (v = 1, 2, \ldots, r). \tag{3}$$

Therefore, the polynomial $h_\nu(y)$ has the greatest common factor unity. If one regards it as a polynomial in y_2 with coefficients that are rational in y_0 and y_1 then it is known that the greatest common factor can itself be represented as a linear combination of polynomials:

$$1 = a_1(y_2)\, h_1(y) + \ldots + a_r(y_2)\, h_r(y).$$

The $a_\nu(y_2)$ are completely rational in y_2 and rational in y_0 and y_1 . If one makes them completely rational in y_0 and y_1 by multiplying them by the least common denominator $b(y_0, y_1)$ then one will obtain:

$$b(y_0, y_1) = b_1(y)\, h_1(y) + \ldots + b_r(y)\, h_r(y). \tag{4}$$

Should $b(y_0, y_1)$ not be homogeneous, one would look for the terms of a fixed degree in $b(y_0, y_1)$ that will define a homogeneous, non-vanishing polynomial $c(y_0, y_1)$, and likewise look for the terms of the same degree in the right-hand side of (4); one would thus obtain:

(5) $$c(y_0, y_1) = c_1(y)\, h_1(y) + \dots + c_r(y)\, h_r(y).$$

It follows from (5) that all solutions of the system of equations (3) will simultaneously be solutions of:

$$c(y_0, y_1) = 0.$$

However, this homogeneous equation determines only finitely many values of the ratios $\eta_0 : \eta_1$; one likewise finds finitely many values for $\eta_1 : \eta_2$ and $\eta_2 : \eta_0$. Therefore, the system of equations (3) has only finitely many solutions $\eta_0 : \eta_1 : \eta_2$. These, together with the points of the curve (2) will constitute all solutions of the original system of equations (1).

If one decomposes the polynomial $g(y)$ into irreducible factors:

$$g(y) = g_1(y) \dots g_s(y)$$

then the curve (2) will obviously decompose into *irreducible curves:*

$$g_1(y) = 0, \dots, g_s(y) = 0;$$

i.e., ones that are defined by irreducible forms. *Thus, any algebraic manifold* (1) *decomposes into finitely many irreducible curves and finitely many isolated points.* Naturally, one can also treat only curves or only isolated points; one can also encounter the case in which the system of equations (1) has no solutions at all. Finally, if the system of equations (1) is empty, or if all of the f_ν are identically zero then the manifold that it defines will be the entire plane.

A curve $g(\eta)$ contains infinitely many points. If – say – η_2 actually enters into the polynomial $g(\eta)$ then the equation:

$$g(\eta) = a_0(\eta_0, \eta_1)\eta_2^m + a_0(\eta_0, \eta_1)\eta_2^{m-1} + \dots + a_m(\eta_0, \eta_1) = 0$$

will define at least one value (and at most m values) η_2 for each value ratio $\eta_0 : \eta_1$ for which $a_0(\eta_0, \eta_1) \neq 0$.

If an equation $f(\eta) = 0$ is true for all or almost all (i.e., all except for finitely many) points of an irreducible curve $g(\eta) = 0$ then the form $f(y)$ will be divisible by $g(y)$. Otherwise, $f(y)$ and $g(y)$ would be relatively prime, and from that it would follow, as before, that the equations $f(\eta) = 0$ and $g(\eta) = 0$ would have only finitely many common solutions.

The last theorem is also true for hypersurfaces in the space S_n (as well as in affine and multiply-projective spaces):

The STUDY Lemma ([1]). *Let f and g be polynomials in $y_1, \dots, y_n$. If all (or almost all) solutions of the irreducible equation $g(\eta) = 0$ also satisfy the equation $f(h) = 0$ then the polynomial $f(y)$ will be divisible by $g(y)$.*

([1]) The STUDY Lemma is a special case of the HILBERT Nullstellensatz (*Moderne Algebra II*, Chap. 11).

Proof. If $f(y)$ and $g(y)$ were relatively prime then, assuming that y_n actually enters into $g(y)$, the resultant $R(y_1, \ldots, y_{n-1})$ of $f(y)$ and $g(y)$ would not vanish identically, and it would be:

$$R(y) = a(y)\, f(y) + b(y)\, g(y). \tag{6}$$

If one now chooses $\eta_1, \ldots, \eta_{n-1}$ such that $R(\eta_1, \ldots, \eta_{n-1}) \neq 0$ and such that the coefficient of the highest power of y_n in $g(y)$ is likewise non-vanishing for $y_1 = \eta_1, \ldots, y_{n-1} = \eta_{n-1}$ then one can determine η_n from the equation $g(\eta_1, \ldots, \eta_{n-1}) = 0$. For all (or almost all) such $\eta_1, \ldots, \eta_{n-1}$ one will then also have $f(\eta) = 0$; thus, the right-hand side of (6) will vanish, but not the left-hand side, which is a contradiction.

Corollary. *If the equations* $f(\eta) = 0$ *and* $g(\eta) = 0$ *represent the same hypersurface then the forms* $f(y)$ *and* $g(y)$ *will be composed of the same factors, possibly with differing exponents.*

Therefore, from the STUDY Lemma, every irreducible factor of $f(y)$ must appear in $g(y)$, and conversely.

§ 17. The degree of a curve. BEZOUT's theorem.

If $g_1, \ldots, g_s$ are various irreducible forms in y_0, y_1, y_2 then the equation:

$$g_1(\eta)^{q_1} g_2(\eta)^{q_2} \cdots g_s(\eta)^{q_s} = 0$$

will define the same curve as the equation:

$$g_1(\eta) \ldots, g_s(\eta) = 0.$$

On the basis of this, we can always assume that the equation of a plane curve is free of multiple factors. If this is the case then one will call the degree n of the form $g_1(\eta) \ldots, g_s(\eta)$ the *degree* or the order of the curve $g = 0$ (1).

The degree also has a geometric meaning. Namely, if we intersect a line with the curve, after we have introduced the parametric representation:

$$\eta = \lambda_1 p + \lambda_2 q$$

into the equation of the curve $g(\eta) = 0$, then we will obviously obtain an n^{th}-degree equation for the determination of the ratio $\lambda_1 : \lambda_2$. Therefore, there will be at most n points of intersection, in the event that the equation does not vanish identically, in which case, all of the points of the line will lie on the curve. From the STUDY Lemma, it will follow that in the latter case the equation of the line will be contained in the equation of the curve as a factor.

In the next paragraph, we will see that there are always lines that actually have n different points of intersection with the curve. *The degree n of the curve is then the maximum number of its intersection points with a line that is not contained in it.*

An extremely important question is that of the number of intersection points of two plane curves $f(\eta) = 0$ and $g(\eta) = 0$. Let the forms $f(y)$ and $g(y)$ be relatively prime; then, from § 1, only finitely many intersection points $\eta^{(0)}, \ldots, \eta^{(h)}$ will be present in any case. Now, *Bezout's Theorem* states that one can provide these intersection points with such (positive whole number) multiplicities that the sum of these multiplicities will be equal to the product $m \cdot n$ of the degree of the forms f and g.

In order to grasp the intersection points algebraically and define their multiplicities we first consider two undetermined points p and q and their connecting line in the parametric representation:

(1) $$\eta = \lambda_0 p + \lambda_1 q.$$

If we substitute (1) into the curve equation then we will obtain two forms of degrees m and n in λ_0 and λ_1 whose resultant $R(p, q)$ will depends upon only p and q. $R(p, q)$ will vanish if and only if the connecting line $\overline{pq}$ includes an intersection point of the two curves, thus, when one of the determinants:

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(1) Occasionally, the degree of a polynomial f with multiple factors is also called the degree or order of the curve $f = 0$. The irreducible components of the curve will then be multiply counted.

$$(p\ q\ \eta^{(\nu)}) = \begin{vmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \\ \eta_0^{(\nu)} & \eta_1^{(\nu)} & \eta_2^{(\nu)} \end{vmatrix}$$

vanishes. From the corollary to the STUDY Lemma (§ 16), it will then follow that $R(p, q)$ is composed of the same irreducible factors as the product:

$$\prod_{\nu=1}^{h} (p\,q\,\eta^{(\nu)}) .$$

One will then have:

(2) $$R(p, q) = c\prod_{\nu=1}^{h} (p\,q\,\eta^{(\nu)})^{\sigma_\nu} ,$$

in which c does not depend upon p and q and is $\neq 0$. We now define σ_ν to be the *multiplicity* of the intersection point $\eta^{(\nu)}$ of $f = 0$ and $g = 0$.

BEZOUT's Theorem now states that *the sum of the multiplicities of all intersection points is equal to $m \cdot n$:*

(3) $$\sum \sigma_\nu = m \cdot n.$$

In order to prove it, we now need to determine the degree of $R(p, q)$ in the p. If we set:

$$f(\eta) = f(\lambda_0\, p + \lambda_1\, q) = a_0\lambda_1^m + a_1\lambda_1^{m-1}\lambda_0 + \cdots + a_m\lambda_0^m ,$$
$$g(\eta) = g(\lambda_0\, p + \lambda_1\, q) = b_0\lambda_1^n + b_1\lambda_1^{n-1}\lambda_0 + \cdots + b_n\lambda_0^n$$

then each a_k and each b_k will be homogeneous of degree k in the p. Since, from § 15, the resultant $R(p, q)$ has the weight $m \cdot n$ it will be homogeneous of degree $m \cdot n$ in the p. The assertion (3) will follow immediately from this, on account of (2).

The multiplicities σ_ν are invariant under projective transformations. Namely, under a projective transformation that acts on the points η, p, q, $\eta^{(1)}$, …, $\eta^{(k)}$ in the same way the determinants $(p\ q\ \eta^{(\nu)})$ will remain invariant, up to a constant factor, while the resultant $R(p, q)$ was already defined in an invariant way.

There exists a series of methods for the effective evaluation of the multiplicities σ_ν that can be derived from formula (2) by specialization. If we first set $\lambda_0 = 1$, $\lambda_1 = \lambda$, $p = (1, u, 0)$, $q = (0, v, 1)$, so, from (1):

$$\eta_0 = 1,$$
$$\eta_1 = u + \lambda\, v,$$
$$\eta_2 = \lambda,$$

then $R(p, q) = N(u, v)$ will be the resultant of $f(1, u + v\lambda, \lambda)$ and $g(1, u + v\lambda, \lambda)$ in λ, and, from (2), one will have:

$$N(u, v) = c\prod_{\nu=1}^{h}(u\eta_0^{(\nu)} - \eta_1^{(\nu)} - v\eta_2^{(\nu)})^{\sigma_\nu} . \tag{4}$$

One calls $N(u, v)$ the *NETTO resolvent.* Its factorization allows a direct computation of the multiplicities σ_ν. If one carries out the specialization even further, when one sets v = 0, then one will obtain the resultants of $f(1, u, z)$ and $g(1, u, z)$ in z:

$$R(u) = c\prod_{\nu=1}^{h}(u\eta_0^{(\nu)} - \eta_1^{(\nu)})^{\sigma_\nu} . \tag{5}$$

It allows the determination of the σ_ν only under the assumption that no two intersection points $\eta^{(\mu)}$, $\eta^{(\nu)}$ have the same ratio $\eta_0 : \eta_1$.

Formulas (4) and (5) indeed appear to be truly simple, although the practical computation of the multiplicities on the basis of these formulas seems truly tedious, firstly because the resultants are large determinants, and above all, because the entire extent of the curves $f = 0$ and $g = 0$ enters into it, whereas, in truth, the intersection multiplicity depends upon only the behavior of the curves in the immediate neighborhood of an intersection point. Expressing this is, however, possible only when one carries out the PUISEAUX series development of the algebraic functions. We will come back to this in § 20.

Problems. 1. The multiplicities of the intersection of a line with a curve are the same as the multiplicities of the roots of the equation that one obtains when one solves the equation of the line for one coordinate and substitutes it into the equation of the curve.

2. If the equations $f = 0$ and $g = 0$ are ordered into increasing powers of η_0 that begin with:

$$a_1\eta_0^{m-1}\eta_1 + a_2\eta_0^{m-1}\eta_2 + \cdots = 0,$$
$$b_1\eta_0^{n-1}\eta_1 + b_2\eta_0^{n-1}\eta_2 + \cdots = 0$$

then the multiplicity of the intersection point (1, 0, 0) will be equal to 1 or greater than it, depending upon whether $a_1 b_2 - a_2 b_1$ is $\neq 0$ or $= 0$, resp.

§ 18. Intersection points of lines and hypersurfaces. Polars.

The intersection point of a line with an m^{th}-degree plane curve – or, more generally, with a hypersurface in the space S_n – is calculated most conveniently by substituting a parametric representation:

$$\eta = \lambda_1 r + \lambda_2 s$$

for the line into the equation of the hypersurface $f(\eta) = 0$. One gets:

$$f(\lambda_1 r + \lambda_2 s) = \lambda_1^m f_0 + \lambda_1^{m-1}\lambda_2 f_1 + \cdots + \lambda_2^m f_m = 0. \tag{1}$$

Therefore, $f_0 = f(r)$ is of degree m in r, and likewise $f_m = f(s)$ is of degree m in s, while $f_k (0 \leq k \leq m)$ is homogeneous of degree $m - k$ in r and of degree k in s. The expressions $f_0, f_1, \ldots, f_m$ are called *polars* of the form f. Their defining rule comes to light when one develops the left-hand side of (1) in a TAYLOR series in powers of λ_2 ; one finds, when ∂_k means the partial derivative of $f(x)$ with respect to x_k , that:

$$f_0 = f(r),$$
$$f_1 = \sum_k s_k \partial_k f(r),$$
$$f_2 = \frac{1}{2!} \sum_k \sum_l s_k s_l \partial_k \partial_l f(r),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

One also calls the hypersurfaces polars whose equations are given by $f_1 = 0$, $f_2 = 0$ for fixed s and variable r, and indeed one calls $f_1 = 0$ the *first polar* of the point s, $f_2 = 0$ the *second* one, etc. By contrast, for fixed r and variable s, $f_1 = 0$ is the $(m - 1)^{\text{th}}$ polar of r, $f_2 = 0$ is the $(m - 2)^{\text{th}}$ polar, etc.

In the case of a plane curve, the multiplicities of the roots of (1) *will agree with the multiplicities of the intersection points of the curve with the line, as defined in* § 17.

Proof: The resultant $R(p, q)$ of § 17 is, in this case, the resultant of a linear form in λ_0 , λ_1 and a form of degree m; it can be calculated when one substitutes a root of the linear form into the form of degree m. The root of the linear form belongs to the intersection point of the line $\overline{p\,q}$ with the line $\overline{r\,s}$; this intersection point is, from the computations of § 10, problem 2:

$$t = (p\;q\;r)\;s - (p\;q\;s)\;r.$$

If we substitute this into $f(t)$ then we will obtain the desired resultant:

$$R(p, q) = f((p\;q\;r)\;s - (p\;q\;s)\;r).$$

It will therefore be equal to the form $f(\lambda_1 r + \lambda_2 s)$ for $\lambda_1 = -(p\;q\;s)$ and $\lambda_2 = (p\;q\;r)$. If the form $f(\lambda_1 r + \lambda_2 s)$ then decomposes into linear factors with multiplicities σ_k then $R(p, q)$will correspondingly decompose into linear factors with the same multiplicities, which was to be proved.

We now come to the practical determination of these multiplicities. The root $\lambda_2 = 0$ of equation (1) is k-fold when the left-hand side of the equation is divisible by λ_2^k; thus, when one has:

$$f_0 = 0, \quad f_1 = 0, \; \ldots, \; f_{k-1} = 0. \tag{2}$$

It follows from this that: *The point r is a k-fold intersection point of the line g with the hypersurface* $f = 0$ *when equations* (2) *are valid for any second point s of this line.* The first of these equations says only that r lies on the hypersurface $f = 0$. The others are the terms of the series that are linear, quadratic, …, up to degree $k - 1$ in s.

If equations (2) are satisfied identically in s – so *each* line through r intersects the curve at the point r *at least* k times (hence, not necessarily precisely k times) – then one will call r a *k-fold point* of the hypersurface. For example, in this nomenclature any multiple point will also be called a double point.

A line through the k-fold point r that intersects the hypersurface at r more than k times is called a *tangent* to the hypersurface at r. If g is such a tangent then every point s of g will satisfy the equation:

(3) $$f_k = 0,$$

in addition to equations (2). *The tangents to r will thus define a conic hypersurface* whose equation is given by (3). The equation is of degree k, so the cone will be of degree at most k. In the case of a plane curve, the cone will decompose into at most k lines through r. *There are thus at most k tangents to a k-fold point of a plane curve.*

In the case of a simple point, (3) will represent a plane with the equation:

$$\sum s_k \, \partial_k f(r) = 0.$$

All tangents to a simple point r of a hypersurface will thus lie in a hyperplane whose coefficients are given by:

(4) $$u_k = \partial_k f(r);$$

it is called the *tangent hyperplane*. In the case of a given curve, there is a single tangent u to a simple point that is given by (4).

We now ask which tangents one can draw from a point s outside of the hypersurface to the hypersurface $f = 0$. If r is the contact point of such a tangent then the equations:

(5) $$f_0 = 0, \qquad f_1 = 0$$

must be true. They are of degree m ($m - 1$, resp.) in r. They are, however, satisfied not only when r is the contact point of a tangent, but also when r is a multiple point of the hypersurface $f = 0$. In order to study them more closely, we think of a given point s as being located at the point $(0, 0, …, 1)$. Equations (5) will then read:

(6) $$f(r) = 0, \qquad \partial_n f(r) = 0.$$

If the form $f(x)$ is free of multiple factors then, as is well-known, $f(x)$ and its derivative with respect to x_n will have no common factor. In the case of a plane curve, the two curves (6) will have finitely many – namely, at most $m(m - 1)$ – intersection points. *One can therefore draw at most m* (m – 1) *tangents from a point s to a plane curve of degree m. Its contact points, as well as the double points of the curve, will be the intersection points of the curve with the first polar of the point s.* In particular, it follows that a plane algebraic curve can have only finitely-many double points.

One will find the equations of the tangents at the point $(0, 0, …, 1)$ to the hypersurface $f = 0$ when one constructs the resultant for r_n from the two equations (6).

One will obtain a conic hypersurface $R(r_0, \ldots, r_{n-1})$ of degree m $(m - 1)$ with its vertex at $s = (0, 0, \ldots, 1)$. The generating lines of the cone will be the tangent or go through the multiple points of the hypersurface. *All remaining lines through the point s will intersect the hypersurface to m different points.*

Problems. 1. The k^{th} polar of a point r relative to the l^{th} polar of the same point is the $(k + l)^{\text{th}}$ polar of r.

2. The k^{th} polar of r relative to the l^{th} polar of q is equal to the l^{th} polar of q relative to the k^{th} polar of r.

3. If $f(s) = \sum\sum \ldots \sum a_{ij\ldots l}\ s_i \ldots s_l$ then the successive polars of a point r will be given by:

$$f_1 = \sum\sum \ldots \sum a_{ij\ldots l}\ r_i\, s_j \ldots s_l,$$

$$f_2 = m\,(m-1) \sum \ldots \sum a_{ijk\ldots l}\ r_i\, r_j\, s_k \ldots s_l,$$

etc. On this, cf., the theory of quadric polars!

4. The coordinate origin (1, 0, 0) is a k-fold point of the curve $f = 0$ if and only if the terms in the polynomial f whose degree in y_1 and y_2 is less k are absent

§ 19. Rational transformations of curves. The dual curve.

We speak of a *rational transformation* of an irreducible curve $f = 0$ when each point η of the curve (possibly with finitely many exceptions) is uniquely associated with a point ζ of the plane whose coordinate ratios are rational functions of the coordinate ratios of the point η:

$$(1) \qquad \begin{cases} \dfrac{\zeta_1}{\zeta_0} = \varphi\left(\dfrac{\eta_1}{\eta_0}, \dfrac{\eta_2}{\eta_0}\right), \\[2ex] \dfrac{\zeta_2}{\zeta_0} = \psi\left(\dfrac{\eta_1}{\eta_0}, \dfrac{\eta_2}{\eta_0}\right). \end{cases}$$

If one writes the functions φ and ψ as quotients of complete rational functions, puts them over the same common denominator, and then multiplies the numerator and denominator by a suitable power of η_0 then, from (1), one will have:

$$\frac{\zeta_1}{\zeta_0} = \frac{g_1(\eta_0, \eta_1, \eta_2)}{g_0(\eta_0, \eta_1, \eta_2)},$$
$$\frac{\zeta_2}{\zeta_0} = \frac{g_2(\eta_0, \eta_1, \eta_2)}{g_0(\eta_0, \eta_1, \eta_2)},$$

or also:

$$(2) \qquad \zeta_0 : \zeta_1 : \zeta_2 = g_0(h) : g_1(h) : g_2(h).$$

The g_i will be forms of the same degree that are not all three divisible by the form f, since otherwise the ratios (2) would be undetermined. However, there can be finitely many points η on $f = 0$ for which it happens that $g_0(\eta) = g_1(\eta) = g_2(\eta) = 0$; the image point ζ of these points η will then be undetermined.

Theorem 1. *Under the rational transformations* (2) *of an irreducible curve* $f = 0$, *the image points* ζ *will all lie on an irreducible curve* $h = 0$. *It will be determined uniquely unless the point* ζ *is a constant point.*

In order to prove this, we first introduce the notion of a *general point* of an irreducible curve $f = 0$. Let u be an indeterminate and let ω be an algebraic function of u that is defined by the equation $f(1, u, \omega) = 0$. We then call $(\xi_0, \xi_1, \xi_2) = (1, u, \omega)$ a *general point* of the curve. ξ is indeed not a point in the sense of Chap. 1, since the coordinates of ξ are not complex numbers, but algebraic functions, although we can still treat ξ as a point insofar as its coordinates are elements of a field, so the algebraic rules of calculation will still apply.

A general point has the following property: *If a homogeneous equation* $g(\xi_0, \xi_1, \xi_2) = 0$ *with constant coefficients is true for a general point x then the form* $g(x_0, x_1, x_2)$ *will be divisible by* $f(x_0, x_1, x_2)$, *and therefore the equation* $g(\eta_0, \eta_1, \eta_2) = 0$ *will be true for all points* η *of the curve.* From $g(1, u, \omega) = 0$, it will then follow, from § 12, that $g(1, u, z)$ is divisible by $f(1, u, z)$:

$$g(1, u, z) = f(1, u, z)\, q(1, u, z).$$

If one makes this equation homogeneous then the asserted divisibility of $g(x_0, x_1, x_2)$ by $f(x_0, x_1, x_2)$ will follow.

The rational transformation (2) associates the general point ξ with a point ζ^* whose coordinates are:

$$\zeta_0^* = 1,$$
$$\zeta_1^* = \frac{g_1(\xi)}{g_0(\xi)} = \frac{g_1(1, u, \omega)}{g_0(1, u, \omega)},$$
$$\zeta_2^* = \frac{g_2(\xi)}{g_0(\xi)} = \frac{g_2(1, u, \omega)}{g_0(1, u, \omega)}.$$

ζ_1^* and ζ_2^* are algebraic functions of u, so the system (ζ_1^*, ζ_2^*) has a degree of transcendence of at most 1. There are therefore two possibilities: Either ζ_1^*, ζ_2^* are both algebraic over the constant field K – hence, since it is algebraically closed, they will be constants in K – or one of the two quantities – say, ζ_1^* – is transcendental and the other one ζ_2^* is an algebraic function of ζ_1^*. In the latter case, there will exist a single irreducible equation $h(\zeta_1^*, \zeta_2^*) = 0$, or, when it is made homogeneous:

$$h(\zeta_1^*, \zeta_2^*, \zeta_3^*) = 0.$$

From the meaning of ζ^*, this says that:

$$h(g_0(\xi), g_1(\xi), g_2(\xi)) = 0. \tag{3}$$

Equation (3) is valid for the general point ξ, and thus, for every point of the curve $f = 0$. Hence, when ζ is determined using (2) one will always have the equation:

$$h(\zeta_0, \zeta_1, \zeta_2) = 0.$$

With this, Theorem 1 is proved.

Theorem 1 is also true – with a small alteration – for rational maps of hypersurfaces in S_n. Here, there is also a general point $(1, u_1, \ldots, u_{n-1}, \omega)$ whose image point $(1, \zeta_1^*, \cdots, \zeta_n^*)$ has a degree of transcendence of at least $n - 1$. There is thus *at least* one irreducible equation $h(\zeta_1^*, \cdots, \zeta_n^*) = 0$, and therefore at least one irreducible hypersurface $h(\zeta_0, \ldots, \zeta_n) = 0$ on which the image points lie. In the case of degree of transcendence $n - 1$, there will indeed be *precisely* one irreducible hypersurface, but all values of the degree of transcendence from 0 to $n - 1$ will be possible.

One will obtain an important example of a rational map of a curve when one associates each point η of the curve with the tangent v to the curve and regards v_0, v_1, v_2 as point coordinates in a second plane: viz., the *dual plane*. From § 17, the equations of the map will read:

$$v_0 : v_1 : v_2 = \partial_0 f(\eta) : \partial_1 f(\eta) : \partial_2 f(\eta) .$$

The map will be undetermined at only finitely many double points. The ratios of the v will be constant only when the constant line v contains all curve points η; hence, when the curve is a line. In all other cases, the image point v will lie in the dual plane, so from Theorem 1, on a single irreducible curve: viz., the *dual curve* $h(v) = 0$.

The tangents to the simple points of the original curve will correspond to points of the dual curve. However, we will see that, conversely, the tangents to the dual curve will also correspond to points of the original curve. Namely, one will have:

Theorem 2. *The dual curve of the dual curve is the original one. If the tangent at η corresponds to the point v of the dual curve then the tangent at v will correspond to the point η.*

Proof. Let $\xi = (1, u, \omega)$ again be a general point of the curve $f = 0$. One then has:

$$f(\xi_0, \xi_1, \xi_2) = 0,$$

and from this, after differentiating by *u:*

$$\partial_0 f(\eta) \cdot d\xi_0 + \partial_1 f(\eta) \cdot d\xi_1 + \partial_2 f(\eta) \cdot d\xi_2 = 0,$$

or, if v^* is the tangent at a general point:

(3) $$v_0^* d\xi_0 + v_1^* d\xi_1 + v_2^* d\xi_2 = 0.$$

Furthermore, since the tangent contains the point itself, one will have:

(4) $$v_0^* \xi_0 + v_1^* \xi_1 + v_2^* \xi_2 = 0.$$

If one differentiates (4) by u and subtracts it from (3) then it will follow that:

(5) $$\xi_0^* dv_0 + \xi_1^* dv_1 + \xi_2^* dv_2 = 0.$$

(5) is dual to (3), while (4) is dual to itself. For v^*, one has the equation:

$$h(v_0, v_1, v_2) = 0.$$

If one now denotes the tangent to this curve at the point v^* by ξ^* then one will get equations that are analogous to (3), (4):

(6) $$v_0^* \xi_0^* + v_1^* \xi_1^* + v_2^* \xi_2^* = 0,$$

(7) $$\xi_0^* dv_0^* + \xi_1^* dv_1^* + \xi_2^* dv_2^* = 0.$$

These determine the point ξ^* uniquely; otherwise, all two-columned sub-determinants of the matrix:

$$\begin{pmatrix} v_0^* & v_1^* & v_2^* \\ dv_0^* & dv_1^* & dv_2^* \end{pmatrix}$$

would vanish, and that would mean that:

$$\frac{d}{du}\frac{v_1^*}{v_0^*} = 0 \qquad \text{and} \qquad \frac{d}{du}\frac{v_2^*}{v_0^*} = 0;$$

hence, the ratios $v_0^* : v_1^* : v_2^*$ would be constant. However, we saw before that this is the case only for curves of degree 1. Hence, the point ξ^* that is determined from (6) and (7) will coincide with the point ξ that is determined from (5) and (4), which one can express through the equations:

$$\xi_j^* \xi_k - \xi_k^* \xi_j = 0.$$

However, since these equations are valid for the general curve point they will also be valid for every particular curve point η. Thus, if the tangent at η corresponds to the point v of the dual curve then the tangent to this curve at the point v will correspond to the point η. With this, Theorem 2 is proved.

We will give a second proof of Theorem 2 later on that is based upon PUISEAUX's series development, and which is valid for the tangents at multiple points. The proof above is, however, more elementary, and can be easily carried over to hypersurfaces, as long as they possess a uniquely determined dual hypersurface, which is not always the case. E.g., if $f = 0$ represents a developable ruled surface or a cone in the space S_3 then the image points v of the tangent plane will not define a surface in the dual space, but only a curve. The developable ruled surfaces are then defined by saying that all of their points possess a generator of the same tangent plane, such that the tangent plane at a general point ξ will depend algebraically upon not two parameters, but only one.

The *degree* of the dual curve is equal to the maximum number of its intersection points with a line, or, what amounts to the same thing, the maximum number of tangents that one can draw from a point r of the plane to the original curve. This number is called the *class* of the curve $f = 0$. From § 18, the class of a curve of degree m amounts to at most $m(m - 1)$, and it will then be smaller when the curve possesses multiple points. In order to compute the class more precisely, one must know how many intersection points of the curve will be absorbed into the polar of an arbitrary point of the multiple points. The means to do this is given by the power series expansion of the curve branches, which we will discuss more thoroughly in the next paragraph.

Problems. 1. Every double point is an at least two-fold intersection point of the curve and its polar, and will thus reduce the class by at least 2 (cf., § 17, prob. 2).

2. An irreducible curve of order 2 (conic section) has class 2. An irreducible curve of order 3 can have only one of the classes 6, 4, or 3

§ 20. The branches of a curve.

Let $f(\eta) = 0$ be an irreducible curve, and let $\xi = (1, u, \omega)$ be a general point of this curve; ω will then be any solution of the equation $f(1, u, \omega) = 0$. However, from § 14, these solutions will be power series in fractional powers of $u - a$ or u^{-1}. In the former case, one has:

$$u - a = \tau^k \qquad \text{or} \qquad u = a + \tau^k \qquad (k > 0),$$
$$\omega = c_h \tau^h + c_{h+1} \tau^{h+1} + \dots \qquad (h > 0,\ h = 0, \text{ or } h < 0);$$

hence:

$$\text{(1)} \qquad \begin{cases} \xi_0 = 1, \\ \xi_1 = a + \tau^k, \\ \xi_2 = c_h \tau^h + c_{h+1} \tau^{h+1} + \cdots \end{cases}$$

In the latter case, one has:

$$u^{-1} = \tau^k \qquad \text{or} \qquad u = \tau^{-k}$$
$$\omega = c_h \tau^h + c_{h+1} \tau^{h+1} + \dots;$$

hence:

$$\text{(2)} \qquad \begin{cases} \xi_0 = 1, \\ \xi_1 = \tau^{-k}, \\ \xi_2 = c_h \tau^h + c_{h+1} \tau^{h+1} + \cdots \end{cases}$$

In both cases, ξ_0, ξ_1, ξ_2 are therefore power series in the position uniformization τ. Any k power series that go into each other under the substitution $\tau \to \zeta\tau$, $\zeta^k = 1$ will define a cycle. Such a cycle is called a *branch* of the curve $f = 0$.

We now consider, more generally, any non-constant point of the curve whose coordinates are each a power series in one variable σ:

$$(3)\qquad \begin{cases} \rho\xi_0 = a_p\sigma^p + a_{p+1}\sigma^{k+1} + \cdots, \\ \rho\xi_1 = b_p\sigma^p + b_{p+1}\sigma^{k+1} + \cdots, \\ \rho\xi_2 = c_p\sigma^p + c_{p+1}\sigma^{k+1} + \cdots \end{cases}$$

Since the quotient of two power series will again be a power series, we can divide all three $\rho\xi_\nu$ by $\rho\xi_0$, and obtain the normalized coordinates:

$$(4)\qquad \begin{cases} \xi_0 = 1, \\ \xi_1 = d_g\sigma^g + d_{g+1}\sigma^{g+1} + \cdots, \\ \xi_2 = e_h\sigma^h + e_{h+1}\sigma^{h+1} + \cdots \end{cases}$$

The power series for ξ_1 cannot exist as just a constant term, since if ξ_0 and ξ_1 were constant then, on the grounds of the equation $f(\xi) = 0$, ξ_2 would also be constant; hence, ξ would be a constant point, contrary to the assumption.

We will now show that any power series triple (4) can be brought into one of the forms (1) or (2) by the introduction of a new variable τ, instead of σ.

In order to prove this, we distinguish between the cases $g \geq 0$ and $g < 0$. In the case of $g \geq 0$, we write the power series for ξ_1 as follows:

$$\xi_1 = a + d_k\sigma^k + d_{k+1}\sigma^{k+1} + \ldots \qquad (d_k \neq 0).$$

Using the development theorem of § 14, we now solve the equation:

$$\tau^k = d_k\sigma^k + d_{k+1}\sigma^{k+1} + \ldots \qquad (d_k \neq 0)$$

by a power series:

$$\tau = b_1\sigma + b_2\sigma^2 + \ldots \qquad (b_1 \neq 0)$$

It will follow that:

$$\xi_1 = a + \tau^k.$$

It is not difficult to transform the powers series:

$$(5)\qquad \xi_1 = e_h\sigma^h + e_{h+1}\sigma^{h+1} + \ldots$$

into a power series in τ. If the powers τ^h, τ^{h+1}, … are power series in s that begin with terms in σ^h, σ^{h+1}, … then one can obtain the power series (5) by a suitable linear combination of these power series. We will thus obtain:

$$(6)\qquad \begin{cases} \xi_0 = 1, \\ \xi_1 = a + \tau^k, \\ \xi_2 = c_h\tau^h + c_{h+1}\tau^{h+1} + \cdots \end{cases}$$

In the event that the exponents that enter into the power series ξ_1, ξ_2 have a common factor d, one can introduce τ^d as a new variable, and thus force the exponents to be relatively prime. The power series representation thus obtained will have the form (1) and must also agree with one of the developments (1). Namely, if one introduces:

$$\tau = (u-a)^{\frac{1}{k}}$$

into (6), where u is an indeterminate, then this will make $\xi_0 = 1$, $\xi_1 = u$, and ξ_2 will become a power series in fractional powers of $u - a$ that satisfies the equation $f(1, u, \xi_2) = 0$. On the basis of the factor decomposition:

$$f(1, u, z) = a_0 \prod_1^m \left(z - \omega^{(\nu)}\right)$$

that is valid in the domain of this power series, ξ_2 must then agree with one of the power series $\omega^{(\nu)}$, which was to be proved.

The second case $g < 0$ is then treated in a completely analogous way. We then set $g = -k$ and have, from (4):

$$\xi_1 = d_{-k}\, \sigma^{-k} + d_{-k+1}\, \sigma^{-k+1} + \dots \qquad (d_{-k} \neq 0)$$

We now solve the equation:

$$\tau^k (d_{-k}\, \sigma^{-k} + d_{-k+1}\, \sigma^{-k+1} + \dots) = 1$$

by a power series:

$$\tau = b_1 \sigma + b_2 \sigma^2 + \dots \qquad (b_1 \neq 0)$$

and then have $\tau^k \xi_1 = 1$; hence:

$$\xi_1 = \tau^{-k}.$$

The power series:

$$\xi_2 = e_h\, \sigma^h + e_{h+1}\, \sigma^{h+1} + \dots$$

can be further transformed into a power series in τ:

$$\xi_2 = c_h\, \tau^h + c_{h+1}\, \tau^{h+1} + \dots$$

We thus come to a power series development of the form (2), which, on the basis of the argument followed above (possibly with the introduction of τ^d in place of τ), must agree with the development (2).

We thus see: *Any power series development* (3) *belongs to a certain branch of the curve, and may be reduced, to one of the power series developments* (1) *or* (2) *of that branch by the introduction of new variables.*

From this theorem, it will now follow easily that the concept of branch *is invariant* under projective transformations, and in fact, more generally, *under arbitrary rational transformations.* Namely, if such a rational map is given by:

(7) $$\zeta_1 : \zeta_2 : \zeta_3 = g_0(\xi) : g_1(\xi) : g_2(\xi)$$

and one substitutes the power series (3) for $\xi_0 : \xi_1 : \xi_2$, then one will again obtain power series in τ for $\zeta_1 : \zeta_2 : \zeta_3$, which, from the theorem above, will belong to a certain branch of the image curve. *Thus, any branch of the curve $f = 0$ will correspond to a certain branch of the image curve under the rational map* (7).

The proportionality factor ρ in (3) is arbitrary. If one chooses ρ to be a power of σ whose exponent is equal to the smallest of the numbers p, q, r then will one find developments for $\xi_0 : \xi_1 : \xi_2$ in which no negative powers appear, while the constant terms are not all three equal to zero. In the sequel, we shall *always* assume this normalization of the proportionality factor ρ. If one now sets $\sigma = 0$ then one will retain only the constant terms of the power series, and one will get a certain point of the plane: viz., the *starting point* of the branch in question. In (1), e.g., for $h \geq 0$, the starting point will be the point $(1, a, c_0)$, although, in the case $h < 0$ the point will be $(0, 0, c_h)$. In (2), for $h > -k$ it will be the point $(0, 1, 0)$, for $h = -k$ it will be the point $(0, 0, 1)$, and for $h < -k$ it will be the point $(0, 0, c_h)$. If the point $(0, 0, 1)$ does not lie on the curve, which one can always insure for some choice of coordinate system, then one must always have $h \geq 0$ in (1) and $h \geq -k$ in (2).

The starting point of a branch is always a point of the curve, since the equation $f(\xi_0, \xi_1, \xi_2) = 0$ is valid identically in σ; hence, also for $\sigma = 0$. However, one also has conversely: *Any curve point h is the starting point of at least one branch.* In order to prove this, we further assume that the point $(0, 0, 1)$ does not lie on the curve. In the equation:

$$f(1, u, z) = a_0 z^m + a_1(u) z^{m-1} + \ldots + a_m(u)$$

one will then have $a_0 \neq 0$. Now, if at first $\eta_0 \neq 0$ – say, $\eta_0 = 1$, $\eta_1 = a$, $\eta_2 = b$ – then we will assume the factor decomposition:

(8) $$f(1, u, z) = a_0 \prod_1^m (z - \omega_\nu)$$

at the locus $u = a$. For $u = a$, $z = b$ the left-hand side will become zero, hence, a factor on the right-hand side will be zero, as well, which will make one of the power series ω_ν assume the value b for $u = a$, $\tau = 0$.

Secondly, if $\eta_0 = 0$, $\eta_1 \neq 0$ – say, $\eta_1 = 1$, $\eta_2 = b$ – then we will form the factor decomposition (8) at the locus $u = \infty$, and thus assume that the power series ω_ν is in u^{-1}. We multiply both sides by u^{-m} and obtain:

$$f(u^{-1}, 1, z\,u^{-1}) = a_0 \prod_1^m (zu^{-1} - u^{-1}\omega_\nu)\,.$$

If we set $u^{-1} = x$ and $z\,u^{-1} = y$ then it will follow that:

$$(9) \qquad f(x, 1, y) = a_0 \prod_1^m (y - x\omega_\nu).$$

Therefore, $x\omega_\nu$ will be a power series in non-negative fractional powers of $x = u^{-1} = \tau^k$, namely:

$$(10) \qquad x\omega_\nu = \tau^k (c_h \tau^h + c_{h+1} \tau^{h+1} + \ldots) = c_h \tau^{h+h} + c_{h+1} \tau^{h+h+1} + \ldots$$

If we now substitute $x = 0$, $y = 0$ in (9) then the left-hand side will become zero, hence, one of the factors on the right-hand side, as well. With that, one of the power series (10) will assume the value b for $t = 0$, and everything will also be proved in this case. What remains is the second case, which can also revert to the first one by a projective transformation.

One understands the *order* of a non-zero power series in τ to mean the exponent of the lowest power of τ that appears in it. The order will remain unchanged when a new variable σ is introduced in place of τ by way of $\tau = b_1\sigma + b_2\sigma^2 + \ldots$, with $b_1 \neq 0$; it can be positive, zero, or negative. When one substitutes the power series ξ_0, ξ_1, ξ_2 for a branch $\mathfrak{z}$ into a form $g(\xi_0, \xi_1, \xi_2)$, that will yield a power series that likewise possesses a certain order that is *positive* or *zero*, depending upon whether the curve $g = 0$ does or does not include the starting point η of the branch $\mathfrak{z}$, resp. We will call this order the *order of the form g on the branch* $\mathfrak{z}$, or also the *intersection multiplicity of the curve* $g = 0$ *with the branch* $\mathfrak{z}$. It is obviously invariant under projective transformations.

We now prove the extremely important theorem:

The multiplicity of an intersection point η *of the curves* $f = 0$ *and* $g = 0$ *is equal to the sum of the orders of the form g on the branches of the curve* $f = 0$ *that have their starting point at* η.

Proof. We choose the coordinate system such that $\eta_0 \neq 0$, the point (0, 0, 1) does not lie on the curve $f = 0$, and no two intersection points of the curves $f = 0$ and $g = 0$ have the same ratio $\eta_0 : \eta_1$. For an intersection point, let $\eta_0 = 1$, $\eta_1 = a$, $\eta_2 = b$. From §17, the multiplicity of η as the intersection point of $f = 0$ and $g = 0$ will then be equal to the multiplicity of $u - a$ in the factor decomposition of the resultant $R(u)$ of $f(1, u, z)$ and $g(1, u, z)$. One now has the formulas:

$$f(1, u, z) = a_0 \prod_1^m (z - \omega_\mu)$$

$$(8) \qquad R(u) = a_0^n \prod_1^m g(1, u, \omega_\mu).$$

In them, a_0 is the coefficient of z^m in $f(1, u, z)$, and $\omega_1, \ldots, \omega_m$ are power series in fractional powers of $u - a$.

The factor $g(1, u, \omega^{(1)})$ has the order s_1 as a power series in the position uniformization $\tau = (u - a)^{1/h}$. The order s_1 is the same for all power series $g(1, u, \omega^{(1)})$,

…, $g(1, u, \omega^{(h)})$ that belong to the same cycle. The product of the power series of this cycle:

$$\prod_1^m g(1, u, \omega_\mu) \tag{9}$$

has the order $k\, s_1$ as a power series in τ, and thus the order s_1 as a power series in $u - a = \tau^k$. Correspondingly, the remaining branches of the point $(1, a, b)$ yield products like (9) of order s_2 , …, s_r . However, the branches that belong to the other points give rise to only factors $g(1, u, \omega_\mu)$ of order zero, since one has $g(1, a, b') \neq 0$ for all points $(1, a, b')$ with $b' \neq b$ that lie on $f = 0$. The total order of the product (8) as a power series in $u - a$ is therefore equal to $s_1 + s_2 + \ldots + s_r$. With that, the theorem is proved.

A quotient of two forms of equal degree:

$$\varphi(\xi) = \frac{g(\xi_0, \xi_1, \xi_2)}{h(\xi_0, \xi_1, \xi_2)}$$

is a function that depends upon the ratios $u = \xi_1 : \xi_0$ and $\omega = \xi_2 : \xi_0$. One calls $\varphi(\xi) = \varphi(u, \omega)$ a *rational function of the general curve point* ξ – or, briefly – *a rational function of the curve.* Such a function has a certain *order* at each branch $\mathfrak{z}$ of the curve, namely, the difference between the orders of the numerator and the denominator. If the order is positive then one will speak of a *zero* of the function $\varphi(\xi)$; if it is negative then $\varphi(\xi)$ will have a *pole*. The sum of the orders of the function $\varphi(\xi)$ on all branches is equal to the sum of the orders of the numerator minus that of the denominator. Hence, from BEZOUT's theorem, it will be zero, since the numerator and denominator have the same degree. It the follows that:

The sum of the orders of the zero loci and poles of a rational function on an irreducible curve is zero.

ZEUTHEN's rule. If one assumes that $g = 0$ also includes the point (0, 0, 1) then one can decompose $f(1, u, z)$, as well as $g(1, u, z)$, in the domain of the power series into linear factors:

$$g(1, u, z) = c_0 \prod_1^n (z - \zeta_\nu)\,.$$

One will then have the expression:

$$R(u) = a_0^n c_0^m \prod_{\mu=1}^m \prod_{\nu=1}^n (\omega_\mu - \zeta_\nu) \tag{10}$$

for the resultant $R(u)$. The differences $\omega_\mu - \zeta_\nu$ are power series in fractional powers of $u - a$. Each of them has a certain order χ, i.e., it begins with a certain power $(u - a)^\chi$. From (10), the order of $R(u)$ is equal to the sum of the orders of the differences $\omega_\mu - \zeta_\nu$.

If ω_μ, ζ_ν, or both of them belong to branches that do not include the point (1, a, b), then the difference $\omega_\mu - \zeta_\nu$ will have order zero. One will then get **ZEUTHEN's rule:**

The multiplicity of an intersection point (1, a, b) *of the curves* $f = 0$ *and* $g = 0$ *is equal to the sum of the orders of the power series* $\omega_\mu - \zeta_\nu$ *as functions of* $u - a$, *where* (1, u, ω_μ) *and* (1, u, ζ_ν) *are the power series developments of those branches of the curves* $f = 0$ *and* $g = 0$ *that have the point* (1, a, b) *as starting point.*

ZEUTHEN's rule shows that the intersection multiplicity is composed of contributions that originate in the individual branch pairs of f and g. The computation of this contribution takes on an especially simple form when the branches are *linear*; i.e., when they exist as power series in integer powers of $u - a$. If the power series ω_μ and ζ_ν then agree in the terms $c_0 + c_1(u - a) + \ldots + c_{s-1}(u - a)^{s-1}$, but differ in the terms with $(u - a)^s$, then s will be the contribution of the branch pair to the total multiplicity of the intersection point (1, a, b).

Problem. 1. Compute the multiplicities of the three intersection points of the circle $\eta_1^2 + \eta_2^2 - \eta_0\eta_1 = 0$ with the cardioid $(\eta_1^2 + \eta_2^2)^2 - 2\eta_0\eta_1(\eta_1^2 + \eta_2^2) - \eta_0^2\eta_2^2 = 0$.

§ 21. The classification of singularities.

For a closer examination of the branch of a curve $f = 0$, we assume that the point O = (1, 0, 0) is the starting point of a branch. We thus have the development:

$$(1) \qquad \begin{cases} \xi_0 = 1, \\ \xi_1 = \tau^k, \\ \xi_2 = c_h\tau^h + c_{h+1}\tau^{h+1} + \cdots \end{cases}$$

The ratio $\xi_2 : \xi_1$ is a power series that begins with τ^{h-k}. For $\tau = 0$, this ratio will assume a definite value when $h \geq k$; however, if $h < k$ then we will say that the ratio $\xi_2 : \xi_1$ "becomes infinite" for $\tau = 0$. In each case, however, the value of $\xi_2 : \xi_1$ for $\tau = 0$ will define a certain direction at the starting point whose direction constant is exactly this value. The line that is defined by this direction is called the *tangent* to the curve branch. The tangent is, by definition, the limit point of a chord, one end of which is the starting point O. We will likewise see that the notion of tangent defined here agrees with the previously defined (§18) notion of curve tangent.

If we define the coordinate system such that the tangent falls on the axis $\eta_2 = 0$ then $h > k$; say, $h = k + l$. One calls (k, l) the *characteristic numbers* of the branch $\mathfrak{z}$. One can characterize them geometrically as follows: Any tangent that differs from the lines through the point O will intersect the branch $\mathfrak{z}$ at O with the multiplicity k, but the tangent will intersect it with multiplicity $k + l$. Namely, if one substitutes the power series (11) for $\eta_1 : \eta_2$ in the equation $g(\eta) = a_1\,\eta_1 + a_2\,\eta_2 = 0$ of such a line then, in the case of $a_1 \neq 0$, $g(\xi)$ will be divisible by τ^k, but in the case $a_1 = 0$ it will be divisible by τ^{h+k}, and that means that the intersection multiplicity of $\mathfrak{z}$ and $g = 0$ will equals k in the former case and $k + l$ in the latter. The number k is sensibly called the *multiplicity* of the point O for the branch $\mathfrak{z}$. For $k = 1$, one will have a *linear branch*.

If r branches with the multiplicities $k_1, \ldots, k_r$ come together at a point O, then the multiplicity of the point O on the curve will be $k_1 + \ldots + k_r$; each line through O that contacts no branch will then intersect the individual branches at O with multliplicities $k_1, \ldots, k_r$, and the total curve thus intersects it with multiplicity $k_1 + \ldots + k_r$. However, if the line is tangent to a branch then the multiplicity will be increased. *The tangents to the curve at the point O are thus precisely the tangents to the individual curve branch at O.*

Theorem. *If the curve $f = 0$ has a p-fold point and $g = 0$ has a q-fold point then the intersection multiplicity of O will always be $\geq pq$. The equality sign will be valid if and only if the tangents to one curve at O are different from those of the other curves.*

Proof. We apply ZEUTHEN's rule and assume that no tangent goes through the point (0, 0, 1). There are p power series ω_μ and q power series ζ_ν. The difference $\omega_\mu - \zeta_\nu$ will have order one at u when the branch tangents are different; otherwise, one order would be > 1. The assertion follows from this.

The dual curve. We would like to compute the branch of the dual curve that corresponds to the branch (1). For the computation of the tangents v^* at the general point x, we use formulas (3) and (4) of § 4. In our case (1), this yields:

$$\begin{cases} v_1^* d\xi_1 + v_2^* d\xi_2 = 0, \\ v_0^* + v_1^*\xi_1 + v_2^*\xi_2 = 0, \end{cases}$$

or:

$$\begin{cases} v_1^* k\tau^{k-1} d\tau + v_2^*\{(k+l)c_{k+l}\tau^{k+l-1} + \cdots\} d\tau = 0, \\ v_0^* + v_1^*\tau^k + v_2^*\{c_{k+l}\tau^{k+l-1} + \cdots\} = 0, \end{cases}$$

or finally, if one chooses $v_2^* = 1$ then:

$$\begin{cases} v_2^* = 1, \\ v_1^* = -\dfrac{k+l}{k} c_{k+l}\tau^k + \cdots, \\ v_0^* = \left(\dfrac{k+l}{k} c_{k+l}\tau^l + \cdots\right)\tau^k - (c_{k+l}\tau^{k+l} + \cdots), \\ \quad = \dfrac{l}{k} c_{k+l}\tau^{k+l} + \cdots \end{cases}$$

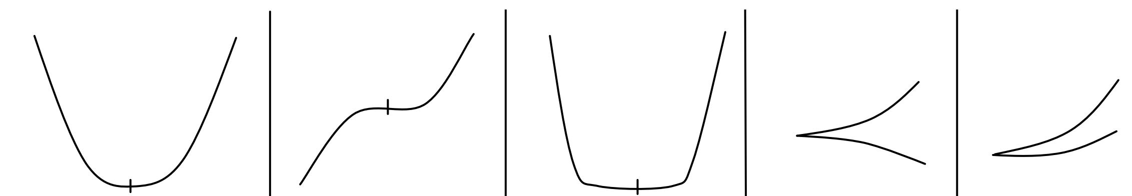
The starting point of this branch $\mathfrak{z}^*$ is the point $v = (0, 0, 1)$, which is the image point of the tangent to the branch $\mathfrak{z}$. The tangent to the branch $\mathfrak{z}^*$ is the line $v_0^* = 0$ with the coordinates (1, 0, 0), which is the image line of the point O = (1, 0, 0) of the original plane. The characteristic numbers of the branch $\mathfrak{z}^*$ are (l, k), which are equal and opposite by comparison to those of the branch $\mathfrak{z}$. Hence:

There exists a one-to-one correspondence between the branches $\mathfrak{z}$ of a curve and the branches $\mathfrak{z}^*$ of the dual curve. Thus, the starting point of $\mathfrak{z}$ corresponds to the tangent of $\mathfrak{z}^*$ and the tangent to $\mathfrak{z}$, to the starting point of $\mathfrak{z}^*$. The characteristic numbers of $\mathfrak{z}^*$ are those of $\mathfrak{z}$ in the opposite sequence.

Classification of the branch. Almost all points of a curve are simple points (i.e., there are only finitely many multiple points). Only one linear branch can have its starting point at a simple point. For almost all branches, one will thus have k = 1. Since the same is true for the dual curve, one will also almost always have l = 1. Almost all branches will thus have the characteristic (1, 1). One calls such a branch an *ordinary branch*, and its starting point, in the event that it carries only one branch, an *ordinary point* of the curve.

If a linear branch has the characteristic (1, 2) then the tangent will intersect the branch at the point O three times. Such a point is called an *inflection point* and its tangent, an *inflection tangent.* A point that carries a branch with the characteristic (1, l) with l > 2, is called a *higher inflection point*; for l = 3 in particular, it is called a *flat point*. The tangent intersects the branch at a flat point four times.

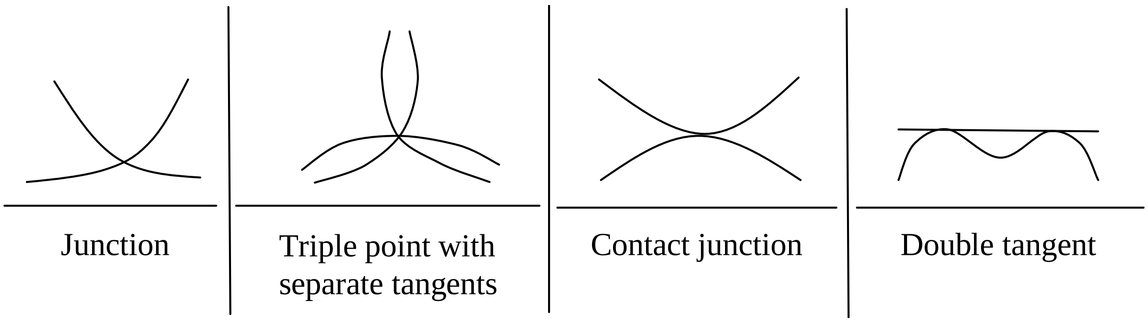
The inflection point corresponds, dually, to the *cusp*, whose characteristic is (2, 1). If the point O is a double point of the branch then the tangent will intersect it precisely three times. For the characteristic (2, 2), the tangents will intersect a branch four times, and one will speak of a *beak*. Therefore, the most frequently-occurring singularities are described by the individual branches. The figures show how the curves appear over the reals in the neighborhood of the point O.



Ordinary point	Inflection point	Flat point	Cusp	Beak
(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)

One obtains another type of singularity when several branches come together at a point. If two linear branches with different tangents have precisely the same starting point then one calls this a *junction;* if there are r linear branches then one will speak of an *r-fold point with separate tangents.* However, when two linear branches contact each other at the point O then one will call this a *contact junction*.

One will obtain singularities of the dual curve when several branches have the same tangent. The corresponding duals to the junction point and the r-fold point with separate tangents are the *double tangent* and the *r-fold tangent with r different contact points.* The contact junction is clearly dual to itself.



The class. We would now like to examine the influence that the various types of singularities have on the class of a curve. The class is the number of intersection points of the dual curve with a line q, or, what amounts to the same thing, the number of tangents to the original curve at a point Q, where the multiplicities with which these tangents are to be counted are to be computed on the dual curve according to a rule that is well-known to us. Therefore, Q is completely arbitrary; we can thus choose Q to be external to the curve and external to the tangents to the multiple points O'.

We will thus obtain the tangents to Q when we eliminate the multiple point O', with its respective intersection multiplicities, from the $m(m-1)$ intersection points of the curve $f = 0$ with the first polar $f_1 = 0$ of the point Q, and connect the remaining intersection point O with Q. If one can still establish that the multiplicities of the remaining intersection point O (when calculated in the plane of the curve $f = 0$) agree with the multiplicities of the tangents that correspond to them (when calculated in the dual plane) then it will follow that the desired number of tangents is equal to $m(m-1)$, minus the sum of the multiplicities of the O' as intersection points of $f = 0$ and $f_1 = 0$.

Let $Q = (0, 0, 1)$ and $O' = (1, 0, 0)$. The decomposition of $f(1, u, z)$ into linear factors reads:

$$f(1, u, z) = (z-\omega_1)(z-\omega_2)\dots(z-\omega_m). \tag{2}$$

By differentiating with respect to z, it will follow that:

$$f_1(1, u, z) = \sum_{i=1}^{m}(z-\omega_1)\dots(z-\omega_{i-1})(z-\omega_{i+1})\dots(z-\omega_m). \tag{3}$$

The multiplicity of the intersection of the polar $f_1 = 0$ with the branch $\mathfrak{z}$, which belongs to the power series ω_1, will be found when one substitutes $z = \omega_1$ in (3) and then examines the order of the corresponding product:

$$(\omega_1-\omega_2)(\omega_1-\omega_3)\dots(\omega_1-\omega_m) \tag{4}$$

as a power series in τ. Summing over all branches at the point O' will yield those multiplicities as intersection points of f and f_1.

First, if O' is an h-fold point with separate tangents then all of the differences ($\omega_j - \omega_k$) will have the order 1; the product (4) will thus have order $n - 1$, and the point O' will have the multiplicity $h(h - 1)$. In particular, one will obtain the value 2 for an ordinary junction.

If O' is a cusp then $\tau = u^{1/2}$ will be the position uniformization, and:

$$\omega_1 = c_3\,\tau^3 + \ldots,$$
$$\omega_2 = -\,c_3\,\tau^3 + \ldots,$$

from which, $(\omega_2 - \omega_1)$ will have the order 3. A cusp will thus have multiplicity 3 as an intersection point of f and f_1. All types of singular points will be treated analogously.

We now still have to calculate the multiplicities of the simple point O whose tangents go through Q as an intersection point of f and f_1. The point O has the characteristic $(1, l)$; the power series development of the branch of the curve $f = 0$ is then given by:

$$\begin{aligned} u &= \tau^{l+1}, \\ \omega_1 &= c_1\tau + c_2\tau^2 + \ldots && (c_1 \neq 0), \\ \omega_2 &= c_1\zeta\tau + c_2\,\zeta^2\tau^2 + \ldots && (\zeta^{l+1} = 1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_{l+1} &= c_1\zeta^{l+1} + \ldots \end{aligned}$$

The differences $(\omega_1 - \omega_k)$ all have order 1 in τ, so the product (4) will have order l. The multiplicity of the tangent OQ to the corresponding point in the dual plane as an intersection point of the dual curve with the line q that does not contact it will be, however, likewise equal to l if we assume that only one branch of the dual curve has this point as its starting point. The two multiplicities will thus agree, in fact.

It follows that: *The class m' of a curve of m^{th} order that has no other singularities than only d junctions and s cusps will be given by the "PLÜCKER formula*:"

$$m' = m(m-1) - 2d - 3s. \tag{5}$$

If other singularities are present then one must subtract further terms that can be calculated as intersection multiplicities of f and f_1, as above.

Problems. 1. Examine the singular points of the "CARTESIAN leaf:"

$$x^3 + y^3 = 3xy,$$

the "heart line" (cardioid):

$$(x^2 + y^2)\,(x - 1)^2 = x^2,$$

the four-leafed rosette:

$$(x^2 + y^2)^2 = 4\,x^2\,y^2.$$

2. A contact junction has multiplicity 4 as an intersection point of f and f_1 (or higher, if the two branches exhibit a higher contact, but in any case, an even number).

3. A beak has multiplicity 5 as an intersection point of f and f_1 (or higher, when the power series of the branch lacks the τ^5 term).

4. If the branch tangent does not go through Q, then the part of the product (4) that relates to the k power series of a single cycle will have order at least $(k + 1)(k - 1)$; hence, in the case of a nonlinear branch, its order will be at least $3(k - 1)$.

§ 22. Inflection points. The HESSian curve.

If η is an inflection point (the higher inflection points are excluded) of the curve $f = 0$ then one will have the followingequations for all points ζ of the tangent g:

$$(1) \qquad \begin{cases} f_0(\eta) = 0, \\ f_1(\eta, \zeta) = 0, \\ f_2(\eta, \zeta) = 0. \end{cases}$$

For variable ζ, the third equation will represent a conic section K: viz., the *quadratic polar* of the point η. In our case, the tangent g will be included as a component in K; K will then decompose into two lines.

Conversely, if η is a simple point of the curve whose quadratic polar divides K then η will be an inflection point. One proves this as follows: The polar of η relative to K is the linear polar $f_1(\eta, \zeta) = 0$; hence, the tangent g. Since η lies on K, it will follow that g is included in K as a component. Now, when K decomposes, as well, g will be included as a component in K. Equations (1) will then be valid for all points ζ of g, so the line g will cut the curve at least three times at η. One then has:

Theorem 1. *The simple points of the curve* $f = 0$ *whose quadratic polar decomposes are its inflection points (and higher inflection points).*

One must further remark that the quadratic polar of a double point likewise decomposes – namely, into two double point tangents. Moreover, one must remark that in the case of an inflection point, the second component h of K cannot go through the point h, since otherwise the polar h relative to K – hence, the linear polar $f_1(\eta, \zeta) = 0$ – would vanish identically, whereas, by contrast, it represents the tangent.

The necessary and sufficient condition for the decomposability of the quadratic polar:

$$\sum \sum \zeta_i \zeta_k \partial_i \partial_k f(\eta) = 0$$

is the vanishing of the *HESSian determinant:*

$$H(\eta) = \begin{vmatrix} \partial_0\partial_0 f(\eta) & \partial_0\partial_1 f(\eta) & \partial_0\partial_2 f(\eta) \\ \partial_1\partial_0 f(\eta) & \partial_1\partial_1 f(\eta) & \partial_1\partial_1 f(\eta) \\ \partial_2\partial_0 f(\eta) & \partial_2\partial_1 f(\eta) & \partial_2\partial_2 f(\eta) \end{vmatrix}.$$

The equation $H = 0$ defines a curve of degree $3(m - 2)$ – viz., the *HESSian curve.* From Theorem 1, it now follows:

Theorem 2. *The intersection points of the curve* $f = 0$ *with its HESSian curve are its inflection points and its multiple points.*

The following theorem is important for the calculation of the number of inflection points:

Theorem 3. *The ordinary (not higher) inflection points have multiplicity one as intersection points of the curves* $f = 0$ *and* $H = 0$.

Proof. Let $\eta^2 = 0$ be the tangent to the inflection point (1, 0, 0). The development of the form $f(x)$ in increasing powers of x_0 reads, since the terms x_0^m, $x_0^{m-1}x_1$, $x_0^{m-2}x_1^2$ are absent:

$$f(x) = x_0^{m-1}ax_2 + x_0^{m-1}(bx_1x_2 + cx_2^2) + x_0^{m-3}(dx_1^3 + \cdots) + \cdots$$

We now develop the determinant $H(x)$, but only look at the terms that are divisible by either x_2 or x_1^2. It becomes:

$$H(x) = \begin{vmatrix} 0+\cdots & 0+\cdots & (m-1)a\,x_0^{m-2}+\cdots \\ 0+\cdots & 6d\,x_0^{m-3}x_1+\cdots & b\,x_0^{m-2}+\cdots \\ (m-1)a\,x_0^{m-2}+\cdots & b\,x_0^{m-2}+\cdots & 2c\,x_0^{m-2}+\cdots \end{vmatrix}$$

$$= -\,6(m-1)^2\, d\, a^2\, x_0^{3m-7}x_1 + \ldots$$

If $r = (1, 0, 0)$ is a simple point of $f = 0$ then $a \neq 0$. If r is an ordinary inflection point then $d \neq 0$. With these assumptions, the curve $H = 0$ will also have just one simple point at (1, 0, 0) and its tangent will be different from the tangent to the curve $f = 0$. It will follow from this that the point r is a simple intersection point of the two curves.

From BEZOUT's theorem, the curves $f = 0$ and $H = 0$ will have $3m(m - 2)$ intersection points. These will then decompose into the inflection points and multiple points of the curve. It thus follows that:

Theorem 4. *A double-point-free curve of order m has* $3m(m - 2)$ *inflection points. Therefore, ordinary inflection points are to be counted simply, while higher inflection points are to be counted multiply (corresponding to their multiplicity as intersection points of the curves* $f = 0$ *and* $H = 0$*). The presence of double points or multiple points will decrease the number of inflection points.*

In particular, a double-point-free curve of order 3 has nine inflection points. There will be no higher inflection points in this case, since the inflection tangent cannot intersect more than three times.

In conclusion, we would like to derive a remarkable property of the HESSian curve of a curve of order 3. The points q of the HESSian curve are defined in such a way that their polar conic section:

(2) $$\sum q_k\, \partial_k\, f(\zeta) = 0$$

possesses a double point p; i.e., one has:

$$\sum_j p_j\, \partial_j \left(\sum q_k\, \partial_k\, f(z)\right) = 0 \qquad \text{identically in } z$$

or:

(3) $$\sum \sum p_j\, q_k\, \partial_j\, \partial_k\, f(z) = 0 \qquad \text{identically in } z.$$

Equation (3) is symmetric in p and q. The point p thus also belongs to the HESSian curve, and its polar conic section has a double point at q. It follows that:

Theorem 5. *The HESSian curve of a plane cubic curve is also the locus of the double points of all decomposable polar conic sections* (2). *Its points define pairs* (p, q) *such that the polars of p will always have their double points at q, and conversely.*

Problems. 1. Show that a flat point – in the sense of theorem 4 – must be counted as two inflection points, and, in general, a point with the characteristic (1, l), as l – 1 inflection points.

2. One can characterize the pair (p, q) of theorem 5 by the fact that it is conjugate to all of the conic sections of the net (2).

§ 23. Third-order curves.

Projective generation. A pencil of conic sections:

$$\lambda_1\, Q_1(\eta) + \lambda_2\, Q_2\,(\eta) = 0$$

and a pencil of lines:

$$\lambda_1\, l_1(\eta) + \lambda_2\, l_2\,(\eta) = 0$$

that is projective to it will generate a curve of order three:

$$Q_1(\eta)\, l_1(\eta) + Q_2\,(\eta)\, l_2\,(\eta) = 0$$

when corresponding elements of the two pencils intersect each other.

Any curve of order three can be obtained in this way. Therefore, if an arbitrary point of a curve can be chosen to be the vertex (1, 0, 0) of a coordinate triangle then only terms that are divisible by η_1 or η_2 can appear in the equation of the curve; the equation of the curve will then read:

$$Q_1(\eta)\eta_1 + Q_2\,(\eta)\eta_2 = 0.$$

Subdivision. We would like to ascertain the possible forms that an irreducible curve of order three can have. Such a curve cannot have two double points, since the connecting line between two double points would cut the curve at each double point twice, and thus, in total, four times, which is impossible. On the same basis, no triple points can be present if the connecting line of the triple point with a simple point also cuts the curve four times. If a double point with two different (linear) branches is present, then they cannot contact each other, since otherwise the common tangent to both branches would double each branch, and thus cut the curve four times. Finally, if a double point with a branch is present then it will have the characteristic (2, 1), and will therefore be an ordinary cusp, since otherwise the tangent to the branch would cut it more than three times. There are thus three types:

I. Cubic curve with no double point.
II. Cubic curve with a junction.
III. Cubic curve with a cusp.

Normal forms. In case I, we choose the coordinate system in such a way that the point (0, 0, 1) is an inflection point and $\eta_0 = 0$ is the inflection tangent. (If the coefficients of the curve equation are real then, because the number of inflection points is odd, there will be a real inflection point.) The equation is then:

$$a\eta_1^3 + b\eta_0\eta_1^2 + c\eta_0\eta_1\eta_2 + d\eta_0\eta_2^2 + e\eta_0^2\eta_1 + f\eta_0^2\eta_2 + g\eta_0^3 = 0 \qquad (a \neq 0).$$

One must have $d \neq 0$, since otherwise the point (0, 0, 1) would be a double point. By the substitution:

$$\eta_2' = \eta_2 + \frac{c}{2d}\eta_1 + \frac{f}{2d}\eta_0,$$

one can arrive at the result that $c = f = 0$. By the substitution:

$$\eta_1' = \eta_1 + \frac{b}{3a}\eta_0,$$

one can further arrive at the fact that $b = 0$. The equation will then assume the form:

$$a\eta_1^3 + d\eta_0\eta_2^2 + e\eta_0^2\eta_1 + g\eta_0^3 = 0,$$

or, when written inhomogeneously ($\eta_0 = 1$):

$$a\eta_1^3 + d\eta_2^2 + e\eta_1 + g = 0.$$

By a suitable choice of unit point, one can ultimately demand that $d = -1$ and $a = 4$ ([1]). What will then remain is the equation:

(1) $$\eta_2^2 = 4\eta_1^3 - g_2\eta_1 - g_3.$$

([1]) The factor 4 was chosen in order to arrive at the connection with the well-known equation from the theory of elliptic function:

$$\wp'(u)^2 = 4\,\wp(u)^2 - g^2\,\wp(u) - g^3.$$

The first polar of the inflection point (0, 0, 1) exists, moreover, on the sides $\eta_0 = 0$ and $\eta_2 = 0$ of the coordinate triangle. The second polar of their intersection point (0, 1, 0) is the third side $\eta_1 = 0$. Thus, if one of the nine inflection points is chosen to be the vertex (0, 0, 1) then the coordinate triangle will be determined invariantly, and the individual coordinate transformations that do not disturb the form (1) will have the form:

$$\begin{cases} \eta_0' = \lambda^3 \mu \eta_0, \\ \eta_1' = \lambda \mu \eta_1, \\ \eta_2' = \mu \eta_2. \end{cases}$$

The quantity:

$$I = \frac{g_2^3}{g_3^2}$$

will remain invariant under this transformation. It is therefore a projective invariant of the curve that depends upon at most the choice of inflection point used.

In order for the curve (1) to have, in fact, no double point, the discriminant of the polynomial $4x^3 - g_2\, x - g_3$ would have to be non-zero.

In case II, we choose the two tangents of the double point to be the sides $\eta_1 = 0$ and $\eta_2 = 0$ of the coordinate triangle. The equation of the curve will then read:

$$a\eta_0\eta_1\eta_2 + b\eta_1^3 + c\eta_1^2\eta_2 + d\eta_1\eta_2^2 + e\eta_2^3 = 0 \qquad (a \neq 0).$$

By the substitution:

$$\begin{aligned} \eta_0' &= a\,\eta_0 + c\,\eta_1 + d\,\eta_2\,, \\ \eta_1' &= -\beta\,\eta_1 && (\beta^3 = b), \\ \eta_2' &= -\gamma\,\eta_2 && (\gamma^3 = b), \end{aligned}$$

one immediately brings the equation into to the form:

$$\eta_0\ \eta_1\ \eta_2 = \eta_1^3 + \eta_2^3. \tag{2}$$

All third order curves with a double point are thus projectively equivalent.

In case III, we choose the cusp to be the vertex (1, 0, 0) and the cusp tangent to be the side $\eta_2 = 0$ of the coordinate system. The equation of the curve will take on the form:

$$a\eta_0\eta_2^2 + b\eta_1^3 + c\eta_1^2\eta_2 + d\eta_1\eta_2^2 + e\eta_2^3 = 0 \qquad (a \neq 0,\ b \neq 0).$$

By the substitution:

$$\eta_1' = \eta_1 + \frac{c}{3b}\eta_3\,,$$

one will make $c = 0$. Thus, by the substitution:

$$-\eta_0' = a\eta_0 + d\eta_1 + e\eta_2,$$

one will arrive at the ultimate form:

(3) $$\eta_0\eta_2^2 = \eta_1^3 .$$

It then follows: *All curves of order three with a cusp are projectively equivalent to each other.*

The curves (2) and (3) possess rational parameter representations, namely:

(4) $$\begin{cases} \xi_0 = t_1^3 + t_2^3, \\ \xi_1 = t_1^2 t_2, \\ \xi_2 = t_1 t_2^2, \end{cases}$$ (5) $$\begin{cases} \xi_0 = t_1^3, \\ \xi_1 = t_1 t_2^2, , \\ \xi_2 = t_2^3, \end{cases}$$ resp.

On the basis of things that we will explain later, the curve (1) possesses no rational parameter representation, expect for a multi-valued one by means of algebraic functions, or a single-valued one by means of elliptic functions:

(5) $$\xi_0 = 1, \qquad \xi_1 = \wp(u), \qquad \xi_2 = \wp'(u).$$

Remark. The form of equation (1) can also be employed for third order curves with double points (cusps, resp.). Namely, for $I = 27$ the equation (1) represents a curve with a double point, and for $g_2 = g_3 = 0$ it represents a curve with a cusp.

Tangents. From formula (5), § 21, the curve (1) will have class 6, the curve (2) will have class 4, and the curve (3) will have class 3. At a point Q outside of the curve (1) one can therefore draw six tangents to the curve. Their contact points will lie on a conic section, namely, on the polar of the point Q. Of the six tangents, two of them will coincide if the tangent in question is an inflection tangent. If Q lies on the curve then two of the six tangents will coincide with the tangent to the point Q; if Q is an inflection point then three of the six will coincide with the inflection point. In all other cases, the six tangents will be different from each other, as one will immediately recognize upon considering the dual curve. For the curves (2) and (3), the number of tangents will reduce to two and three, resp. One can then draw four, two, and one tangent, resp., at a point Q of the curve (1), (2), or (3), resp., to the curve (besides the tangent at Q). These numbers will be reduced by one when Q is an inflection point.

Transformations of the curve into itself. The curve (3) possesses ∞^1 projective transformations into itself:

$$\begin{cases} \eta_0' = \lambda^3 \eta_0, \\ \eta_1' = \lambda \eta_1, \\ \eta_2' = \eta_2. \end{cases}$$

The curve (2) has six projective transformations into itself:

$$\begin{cases} \eta_0' = \eta_0, \\ \eta_1' = \rho\eta_1, \\ \eta_2' = \rho^2\eta_2, \end{cases} \qquad \begin{cases} \eta_0' = \eta_0, \\ \eta_1' = \rho\eta_2, \\ \eta_2' = \rho^2\eta_1, \end{cases} \qquad (\rho^3 = 1).$$

As we will see the curve (1) admits, a group of at least 18 projective transformations that permute the nine inflection points transitively. Namely, one has:

Theorem 1. *Each inflection point w is associated with one projective reflection of the curve into itself that permutes the remaining inflection points pair-wise.*

One can read off the theorem from (1) directly: The reflection is given by $\eta_2' = -\eta_2$. One can also prove the theorem without coordinate transformations, if one starts with the fact that the polar of the inflection point w will decompose into two lines, namely, into the inflection tangent and a line g that goes through w. Now, if s is a point of g then the intersection point of the line $\overline{ws}$ with the curve will be found from the equation:

$$f_0(w)\lambda_1^3 + f_1(w,s)\lambda_1^2\lambda_2 + f_2(w,s)\lambda_1\lambda_2^2 + f_3(s)\lambda_2^3 = 0.$$

Now, one has $f_0(w) = 0$ and $f_2(w, s) = 0$ in this, because w lies on the curve and s on the first polar of w. Thus, $-\lambda_1 : \lambda_2$ is also a solution of the equation, along with $\lambda_1 : \lambda_2$. The projective reflection that takes the point $\lambda_1 w + \lambda_2 s$ to $-\lambda_1 w + \lambda_2 s$ will then take the curve into itself.

Any two points that are permuted by the reflection will lie on a line through w. The connecting line of w with another inflection point will then always include a third inflection point, and since w was an arbitrary inflection point, it will follow that:

Theorem 2. *The connecting line between two inflection points always includes a third inflection point.*

This theorem is also valid, as its proof shows, for curves with a double point. In fact, as one immediately recognizes by constructing the HESSian curve, the curve (2) has precisely three inflection points, which lie on the line $\eta = 0$. The theorem will find no application to the curve (3), since it possesses only one inflection point (0, 0, 1).

Theorem 3. *Any two inflection points will be permuted by one of the reflections that were mentioned in Theorem 1.*

From theorem 2, their connecting line will then include yet a third inflection point w, which belongs to a reflection that, from theorem 1, will permute any two inflection points that lie in a line with w.

It will first follow from theorem 3 that the projective invariant I of the curve (1) does not depend upon the choice of the inflection point that was chosen to be the vertex (0, 0, 1). It will follow further that the group $\mathfrak{G}$ of projective transformations of the curve into itself will permute the inflection points *transitively*. From theorem 2, the subgroup of $\mathfrak{G}$

that leaves the point w fixed will have order at least two. Its cosets (*Nebenklassen*) will take w to all nine inflection points of the curve. Therefore, the group $\mathfrak{G}$ will be of order at least 18.

The inflection point configuration. We would now like to examine a configuration that is defined by the nine inflection points of a double-point-free cubic curve. The examination proceeds in a purely combinatorial manner.

Four lines emanate from an inflection point w that will include any other two inflection points. If one chooses w to be the sequence of all nine inflection points then one will obtain $9 \cdot 4/3 = 12$ connecting lines that, along with the nine inflection points, will define a "configuration $9_4\ 12_3$" (nine points, through each point of which there are four lines, and 12 lines, on each of which there are three points).

If a_1, a_2, a_3 are three inflection points on a line g then three more lines will go through a_1, a_2, a_3 that are all different. We will then get (together with g) $1 + 9 = 10$ lines through a_1, a_2, or a_3. Two lines will remain, which will go through either a_1, a_2, or a_3. If h is one of these and b_1, b_2, b_3 are the inflection points that lie on h then, along with g, h, and the nine lines $a_i\, b_k$ will exhaust the set of all lines that go through a_1, a_2, a_3, b_1, b_2, or b_3. One line will remain from the 12 lines that will go through either a_1, a_2, a_3 or b_1, b_2, b_3. It is called l and goes through c_1, c_2, c_3.

Any line like g will thus belong to one single triple of lines (g, h, l) that includes precisely all nine of the inflection points. Since each of the 12 lines belongs to one and only one such triple, there will be four such triples. We thus have:

Theorem 4. *The nine inflection points can be decomposed into three triples in four ways, such that each triple lies on a line.*

If we denote a decomposition into triples by:

$$a_1, a_2, a_3 \mid b_1, b_2, b_3 \mid c_1, c_2, c_3$$

then we can choose the numbering of b_k and c_k in such a way that a second decomposition will be given by:

$$a_1, b_1, c_1 \mid a_2, b_2, c_2 \mid a_3, b_3, c_3\ .$$

The third and fourth decomposition can then only read as follows:

$$a_1, b_2, c_3 \mid a_2, b_3, c_1 \mid a_3, b_1, c_2\ ,$$
$$a_1, b_3, c_2 \mid a_2, b_1, c_3 \mid a_3, b_2, c_1\ .$$

If one chooses the coordinate system in such a way that the four points a_1, a_3, c_1, c_3 take on the following inhomogeneous coordinates:

$$a_1(1, 1); \qquad a_3(1, -1); \qquad c_1(-1, 1); \qquad a_3(-1, -1)$$

then, on the basis of the position relations between the nine points, which are given by our 12 lines, the remaining points will inevitably have the following coordinates:

$$a_2(1, w); \qquad b_1(-w, 1); \qquad b_3(w, -1); \qquad c_2(-1, -w); \quad b_3(0, 0) \qquad [w^2 = -3].$$

The positions of the nine inflection points will then be determined uniquely, up to a projective transformation, and independently of the invariant I of the curve. The inflection point configuration cannot be realized by real points, since the equation $w^2 = -3$ is not soluble by real numbers.

If one regards the previous four triples of lines as decomposable curves of order three then two of them will determine a pencil whose basis points are our nine inflection points. This pencil will also belong to the other two triples of lines, as well as the original curve C, since they all go through the nine basis points of the pencil. On the same grounds, the HESSian curve will also belong to the pencil. The pencil is therefore also given by:

$$\lambda_1 C + \lambda_2 H = 0.$$

One calls it the *syzygetic pencil* ($\sigma\acute{\upsilon}\zeta\upsilon\gamma o\varsigma$ = yoked together) of the curve C.

Theorem 5. *If nine different points of the plane have the position that was described in theorem 4, and one determines a pencil of third-order curves by two of the four triples of lines, which naturally also belongs to the two other triples of lines, then all curves of this pencil will have their inflection points at the nine given points.*

Proof. If w is one of the nine points then the first polar of w relative to the curves of the pencil will define a pencil of conic sections. The point w will be the inflection point of a curve C that goes through w if and only if the first polar of w relative to C decomposes. Now, there are four exemplars of the pencil that have an inflection point at w, namely, the four triples of lines that were mentioned in theorem 4. There are then four decomposable conic sections in the pencil of conic sections. However, when not all elements of the pencil decompose, a pencil of conic sections will contain at most three decomposable conic sections. Hence, all of the conic sections of the pencil will decompose; i.e., w will be an inflection point for all curves of the pencil.

It follows from theorem 5 that all curves $\lambda_1 C + \lambda_2 H$ of the syzygetic pencil have the same inflection points as the curve C.

Problems. 1. There is a group of 216 projective transformations that transforms the inflection point configuration and the syzygetic pencil into themselves. It includes, as normal a subgroup, the group of 18 collineations that, from theorem 1, is generated by the reflections that transform each curve of the pencil into itself.

2. The parameter values s, t, u of the three points on the curve (4) ((5), resp.) that is cut out of a line satisfy the equation:

$$s_1\, t_1\, u_1 + s_2\, t_2\, u_2 = 0,$$

resp.:

$$s_1\, t_2\, u_2 + s_2\, t_1\, u_2 + s_2\, t_2\, u_1 = 0,$$

or, after introducing the inhomogeneous parameters $s = s_1 : s_1$, etc.:

$$s\, t\, u = -1,$$

resp.:

$$s + t + u = 0.$$

3. The well-known addition theorem for elliptic functions may be expressed as follows: The parameter values u, v, w of three intersection points of a line with the third-order curve that is represented by the parametric representation (6) satisfy the relation:

$$u + v + w = 0 \qquad \text{(mod periods)}.$$

§ 24. Point groups on a third-order curve.

We would like to examine the point groups ([1]) on a third-order curve K_3 that will be intersected by other curves K_m . Thus, the multiple intersection points will be counted with the correct multiplicity. It will be generally assumed that multiple points of K_3 do not appear in the point group considered; the curves K_m that intersect the curve K_3 shall therefore avoid the possible multiple points of K_3 .

Theorem 1. *If, among the* $3m$ *intersection points of a curve* K_m *of order m with a curve* K_3 *of order three, three of them intersect a line G outside of* K_3 *then the remaining* $3(m - 1)$ *of them will intersect a curve* K_{m-1} *of order* $m - 1$ *that is outside of* K_3 .

Proof. The line G has the equation $\eta_0 = 0$, the curve K_3 is $f = 0$, and the curve K_m is likewise $F = 0$. Thus, we first have to show that a μ-fold intersection point of K_3 and G is also at least a μ-fold intersection point of K_m and G. We show this as follows: The branch development of the linear branch $\mathfrak{z}$ of K_3 at the point S agrees with the branch development of the line G in its terms in $1, \tau, \ldots, \tau^{\mu-1}$. Thus, if the form F has order $\geq \mu$ at the branch $\mathfrak{z}$ then it will also have order at least μ at the corresponding branch of the line G. The point S will therefore be at least a μ-fold intersection point of K_m with G.

If one now sets $x_0 = 0$ in both $F(x_0, x_1, x_2)$ and $f(x_0, x_1, x_2)$ then the three zero loci of the form $f(0, x_1, x_2)$ will appear in the zero locus of the form $F(0, x_1, x_2)$ with the correct multiplicity, and therefore $F(0, x_1, x_2)$ will be divisible by $f(0, x_1, x_2)$:

$$F(0, x_1, x_2) = f(0, x_1, x_2) \cdot g(x_1, x_2).$$

If one now adds the terms containing the factor x_0 back into F and f then it will follow that:

$$F(x_0, x_1, x_2) = f(x_0, x_1, x_2) \cdot g(x_1, x_2) + x_0 \cdot h(x_0, x_1, x_2). \tag{1}$$

It follows from (1) that the order of the form $F(x)$ on each branch of the curve $f = 0$ will equal to the order of the form $x_0 \cdot h$. The $3m$ intersection points of $F = 0$ and $f = 0$ thus divide the three intersection points of $x_0 = 0$ with $f = 0$ and the $3(m - 1)$ intersection points of $h = 0$ and $f = 0$.

Theorem 2. *If one connects the six intersection points of a conic section* K_2 *and a curve* K_3 *pairwise with three lines* g_1 , g_2 , g_3 *that cut the curve* K_3 *three times at* P_1, P_2 , P_3 *then* P_1, P_2 , P_3 *will be the intersection points of* K_3 *with a line.* (Arbitrarily many of

([1]) The phrase "point group" has nothing to do with the concept of "group." Rather, it denotes a finite number of points in which the same point might appear many times.

the 6 + 3 points can coincide, but the conic section can contain no double point of the curve K_3.)

Proof. K_2 and $\overline{P_1P_2}$ may, together with the curve K_m and g_1, define the line G of theorem 1. $\overline{P_1P_2}$ cuts K_3 for the third time at Q and g_i cuts K_2 at A_i, B_i. It then follows that $A_2A_3B_2B_3P_2Q$ will lie on a conic section K_2'.

3 of these 6 intersection points will lie on a line, namely, A_2, B_2, P_2. Thus (again, from theorem 1), A_3, B_3, Q will be the intersection points of K_3 with a line K_1. This is, however, g_3; hence, $Q = P_3$.

When K_3 decomposes into a conic section and a line and K_2 decomposes into two lines, theorem 2 will include the special case of PASCAL's theorem, with all of its asymptotic cases. (One needs to draw a figure!)

One can also prove theorem 2 directly when one chooses an exemplar from the pencil of curves that is determined by the curves K_3 and $g_1g_2g_3$, such that it includes any seven points Q of the conic section K_2. This exemplar, since it has seven points in common with the conic section, must then include the conic section as one component. The other component will be a line that includes the points P_1, P_2, P_3.

If one lets the conic section of theorem 2 degenerate into two coincident lines then one will get:

Theorem 3. *The three tangents to the three intersection points of a line g with a third-order curve K_3 cut the curve again at three points P_1, P_2, P_3 that lie on a line.*

If one chooses g to be the connecting line of two inflection points then one will obtain theorem 2 of the previous paragraph all over again: On the connecting line between two inflection points, there will always lie a third inflection point.

From now on, the curve K_3 will be assumed to be irreducible. We choose a fixed point P_0 of the curve (naturally, not a double point) and now define a *sum* of two arbitrary points P, Q in the following way: The connecting line PQ cuts the curve again at R', and the connecting line P_0R' further cuts the curve at R. We then write $P + Q = R$. ([1])

The addition thus described is obviously commutative and uniquely invertible. The point P_0 is the zero element of addition:

$$P + P_0 = P.$$

We will prove that the addition is also associative:

([1]) One is led to this definition – which seems remarkably strange, at first – when one starts with either the theory of divisor classes in algebraic function fields or the theory of elliptic functions. Namely, if one represents the coordinates of the point of the curve as elliptic functions of u, such that P_0 belongs to the parameter value 1, P to the value u_P, Q to the value u_Q, and R to the value u_R then one will have $u_P + u_Q \equiv u_R$ (mod periods). Proof: If $l_1 = 0$ and $l_2 = 0$ are the equations of the lines PQR' and P_0RR' then the quotient $l_1 : l_2$ will be a rational function of the coordinates of a variable curve point, hence, an elliptic function of u with the zeroes u_P and u_Q and the poles u_R and 0. Now, the sum of the zeroes of an elliptic function, minus the sum of the poles, is always a period. It follows that $u_P + u_Q - u_R \equiv 0$.

$$(P + Q) + R = P + (Q + R).$$

We set $P + Q = S$, $S + R = T$, $Q + R = U$, and have to prove that $P + U = T$. By the definition of the addition:

$P\ Q\ S'$	may be cut out of a line			g_1
$P_0\ S\ S'$	"	"	"	h_1
$S\ R\ T'$	"	"	"	g_2
$P_0\ T\ T'$	"	"	"	l
$Q\ R\ U'$	"	"	"	h_2
$P_0\ U\ U'$	"	"	"	g_3 .

We would like to prove that $P\ U\ T'$ can also be cut out of a line h_3 ; we apply theorem 1. The points $P\ Q\ S'\ S\ R\ T'\ P_0\ U\ U'$ will be cut out of a third-order curve $g_1g_2g_3$, but $P_0\ S\ S'$ will be cut out of h_1; hence, the remaining points $P\ Q\ R\ T'\ U\ U'$ will be cut out of a conic section. However, $Q\ R\ U'$ will be cut out of h_2 ; thus (again, from theorem 1), $P\ T\ U'$ will be cut out of a line h_3 . From this, it immediately follows that $P + U = T$ if $P_0\ T\ T'$ will be cut out from a line h_3 . Thus, all of the rules of ordinary addition are valid.

Now, we prove the decisive:

Theorem 4. *The* $3m$ *intersection points* $S_1, \ldots, S_{3m}$ *of* K_3 *with a curve* K_m *of order* m *satisfy the equation:*

(2) $$S_1 + S_2 + \ldots + S_{3m} = m\, P_1 .$$

Therefore, P_1 *is a fixed point, namely, the third intersection point of the tangent at* P_0 *with* K_3 *.*

Proof, by complete induction on m. For $m = 1$, the assertion will follow immediately from the definition of the sum $S_1 + S_2 + S_3 = (S_1 + S_2) + S_3$. Namely, if R is the third intersection point of $S_3\, P_0$ with the curve then, since the $S_1\ S_2\ S_3$ will lie on a line, $S_1 + S_2 = R$ and $R + S_3 = P_1$. We now assume that the assertion is true for curves of degree $(m - 1)$. S_1S_2 will cut the curve for a third time at P, and likewise, $S_3\ S_4$ at Q, and PQ at R. The points $S_1, \ldots, S_{3m}$, P, Q, R of a curve K_3 of degree $(m + 1)$ that exists on K_m and the line $P\ Q\ R$ will then be cut out of K_3 . Of these points, S_1, S_2, P will be cut out of a line; hence, from theorem 1, the group $S_3, \ldots, S_{3m}\ Q\ R$ will come from a curve of order m, but then again, $S_3\ S_4\ Q$ will come from a line, hence, $S_5, \ldots\ S_{3m}$, R comes from a curve K_{m-1} of order $(m - 1)$. From the induction assumption, one will then have:

$$S_5 + \ldots + S_{3m} + R = (m - 1)\, P_1 .$$

One adds to this:

$$S_1 + S_2 + P = P_1 ,$$
$$S_3 + S_4 + Q = P_1 ,$$

and obtains:

$$S_1 + S_2 + \ldots + S_{3m} + P + R = (m + 1)\, P_1 .$$

If one subtracts $P + Q + R = P_1$ from this then the assertion (2) will follows.

It follows from theorem 4 that: *Each of the* $3m$ *intersection points of a fixed curve* K_3 *with a curve* K_m *that does not go through a double point of* K_3 *is determined uniquely by the* $3m - 1$ *remaining ones.*

We now show that one can choose the $3m - 1$ intersection points $S_1, \ldots, S_{3m}$ arbitrarily on K_3 , except for the double point; in other words, at least one curve of order m that does not contain K_3 will go through any $3m - 1$ points of K_3 . The assertion will be clear for $m = 1$ and $m = 2$; we thus assume that $m \geq 3$. The linear family of all K_m that go through the $3m - 1$ given points has a dimension of at least:

$$\frac{m(m+3)}{2} - (3m-1) = \frac{m(m-3)}{2} + 1 .$$

The linear family of all K_m that include K_3 as a component $K_m = K_3\, K_{m-3}$, however, has the dimension:

$$\frac{(m-3)m}{2} .$$

The former dimension is greater; hence, there are actual curves K_m through $S_1, \ldots, S_{3m}$ that do not contain K_3 as a component. The $3m^{\text{th}}$ intersection point of K_m with K_3 is the point S_{3m} that is determined by (2). We thus have:

Theorem 5. *The necessary and sufficient condition for* $3m$ *points of* K_3 *to be cut out of a* second *curve* K_m *is condition* (2).

A generalization of theorems 1 and 2 follows immediately from theorem 5:

Theorem 6. *If, of the* $3(m + n)$ *intersection points of a* K_3 *with a* K_{m+n} *, any* $3m$ *of them on* K_3 *are cut out of a* K_m *then the remaining* $3n$ *will be cut out of a* K_n .

Then, from:

$$S_1 + S_2 + \ldots + S_{3m+3n} = (m + n)\, P_1$$

and:

$$S_1 + S_2 + \ldots + S_{3m} = m\, P_1$$

it will follow by subtraction that:

$$S_{3m+1} + \ldots + S_{3m+3n} = n\, P_1 .$$

Finally, we prove:

Theorem 7. *If* K_m *and* K'_m *cut out the same group of* $3m$ *points from* K_3 *then a decomposable curve* $K_3\, K_{m-3}$ *will be present in the pencil of curves* K_m *and* K'_m*, and the rest of the* $m^2 - 3m = m(m - 3)$ *intersection points of* K_m *and* K'_m *will lie on* K_{m-3} .

Proof. Let Q be any point of K_3 that does appear in the group of $3m$ points. There is a curve through the point Q in the pencil that is spanned by K_m and K'_m. It has $3m + 1$ points in common with K_3 ; hence, it will contain K_3 as a component, and we can denote it by $K_3\,K_{m-3}$. The intersection points of K_m and K'_m are the basis points for the pencil. Hence, so are the intersection points of K_m and $K_3\,K_{m-3}$; i.e., they are the intersection points of K_m and K_3, augmented by those of K_m and K_{m-3} .

Theorems 5, 6, 7 admit very many applications, only a few of which will be selected. First, we once more come back to the inflection point configuration. There is always an inflection point, so we can assume that P_0 is the inflection point. One will then have $P_1 = P_0$; we will denote this point by O (i.e., the origin). The determination of the inflection point W comes from the solution of the equation:

$$3W = O.$$

If there is yet another solution U, in addition to the solution $W = O$, then $2U = U + U$ will also be a solution, and one will have:

$$O + U + 2U = O;$$

i.e., the three inflection points O, U, $2U$ will lie in a line. If there is another inflection point V, along with O, U, $2U$, then there will be nine different inflection points:

$$\text{(3)} \qquad \left\{ \begin{array}{lll} O, & U, & 2U, \\ V, & U+V, & 2U+V, \\ 2V, & U+2V, & 2U+2V. \end{array} \right.$$

That is also the maximum number. In fact, we saw that the three curve types III, II, I of the series possess one, three, and nine inflection points, resp.. The configuration of nine inflection points should be removed immediately from the schema (3); it is only when three of the nine points yield the sum O that they will lie on a line. That is the case for the points of the rows and columns in schema (3), as well as for the triple that (like determinant terms) includes precisely one point from each row and column.

It now follows that: *A real curve of order three has one or three real inflection points.*

One gets the fact that there is one real inflection point from the fact that the imaginary inflection points can occur only in complex conjugate pairs. Thus, we can choose a real inflection point for P_0 . If U is then a second real inflection point then $2U$ will also be real, and there will be three real inflection points O, U, $2U$. There cannot be a fourth real inflection point, since then the entire inflection point configuration (3) would be real, which, from § 23, is impossible.

We understand the *tangential point* of a point P on the curve K_3 to mean the third intersection point of the tangent at P with the curve. The tangential point Q will be defined by:

$$2P + Q = P_1 .$$

For a given tangential point Q, there will be four points P on a curve of type I, two points P on a curve of type II, and one point P on a curve of type III, resp. Therefore, the equation:

$$2X = P_1 - Q \tag{4}$$

will always have four solutions (two solutions, one solution, resp.).

We now consider a curve of type I; hence, a K_3 without double points. If X and Y are two solutions of (4) then the difference $X - Y$ will be a solution of:

$$2(X - Y) = P_1 - P_1;$$

hence, it will be one of the four points whose tangential point is P_1 . Let P_0 , D_1 , D_2 , D_3 be these four points. Thus, all solutions X of the equation (4) will arise from a solution Y by adding P_0, D_1, D_2, or D_3. For each i, the association:

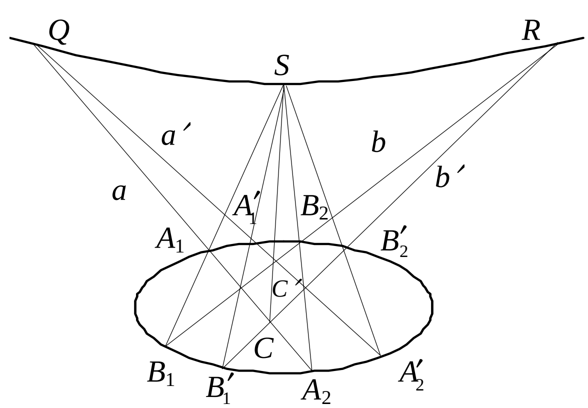
$$X = Y + D_i \qquad (i = 1, 2, 3)$$

will be a one-to-one correspondence of period 2: If $X = Y + D_i$ then one will also have $Y = X + D_i$. *There are thus three involutions of the point pairs* (X, Y) *on the curve such that one always has* $X \neq Y$, *while* X *and* Y *always have the same tangential point. Any point* X *will be in one-to-one correspondence with a point* Y *under any involution, and the point* Y, *on the other hand, will be associated with* X *in the same way.*

The tangents to a varying curve point A have the remarkable property that their double ratio is constant. This follows from:

Theorem 8. *If one draws all possible lines* a *through a fixed curve point* Q *that may cut the curve at two further points* A_1 *and* A_2, *and then one further links both* A_1 *and* A_2 *with a fixed curve point* S, *and seeks the third curve points* B_1, B_2 *of these lines with the curve, then the connecting lines* $b = B_1B_2$ *will all go through a fixed curve point* R. *If the line* a *runs through the pencil* Q *then* b *will run through the pencil* R, *and this association* $a \to b$ *will be a projectivity.*

Proof. We have:



$$Q + A_1 + A_2 = P_1 ,$$
$$A_1 + S + A_2 = P_1 ,$$
$$A_2 + S + A_2 = P_1 ,$$
$$B_1 + B_2 + R = P_1 .$$

From this, by addition and subtraction, one will get:

$$Q + R - 2S = 0, \tag{5}$$

from which, R will, in fact, be constant (independent of the line a). The association $a \to b$ is obviously one-to-one. In order to show that it is a projectivity, we choose a fixed position a' of the line a, properly construct the points A_1', A_2', B_1', B_2', and the line b' from it, denote the intersection point of a and b' by C, that of a' and b by C', and then prove that S, C, C' will all lie on one line. To that end, we apply theorem 7 with $m = 4$. K_m exists on the four lines a, b, $A_1' S B_1'$, $A_2' S B_2'$, just as K_m' lies on the four lines a', b', A_1SB_1, A_2SB_2. K_m and K_m' cut out the same point groups $QA_1A_2A_1'A_2'SSB_1B_2B_1'B_2'R$ from the curve K_3. Hence, from theorem 7, the rest of the four intersection points S, S, C, C' will lie on a line. The association $a \to b$ may be arranged in the following way: One cuts a with b', projects from S onto a', and connects with R. The association will thus be a projectivity.

For a given Q and R one can always find a suitable S on the basis of equation (5).

If one chooses a tangent for a, in particular, then $A_1 = A_2$, $B_1 = B_2$, and therefore also b, will be a tangent. Hence, the four tangents at Q to the four tangents at R will be projective and will have the same double ratio. Since Q and R are arbitrary curve points, it follows that:

Theorem 9. *The double ratio of the four tangents that one can draw from a point Q of the curve K_3 to the curve is independent of the choice of point Q.*

If one chooses an inflection point for Q then one of the four tangents will be the inflection point. If we put Q at (1, 0, 0) and the tangent to the line at $x_0 = 0$ then it will follow that the double ratio that was mentioned in theorem 9 is equal to the partial ratio $(e_1 - e_2)/(e_1 - e_3)$ of the three roots e_1, e_2, e_3, of the polynomial $4x^3 - g_2\, x - g_3$ that appeared in the normal form (1), § 23.

Problems. 1. A cubic curve with no double point possesses three systems of triply-contacting conic sections. In each system, one can choose two of the three contact points arbitrarily; the third one will then be determined uniquely.

2. There are 27 non-decomposable conic sections that contact a double-point-free curve of order three at each point with the multiplicity 6. Its contact points will found when one draws the three tangents to the curve from each of the nine inflection points.

§ 25. The resolution of singularities.

Let $f(\eta_0, \eta_1, \eta_2) = 0$ be a non-decomposable plane algebraic curve of degree $n > 1$. We would like to transform this curve into another one that possesses no other singularities besides r-fold points with r distinct tangents. The Lemma for this defines a very simple transformation of the plane (viz., *Cremona transformation*) that is rational in both directions, and is given by the formulas:

$$\zeta_0 : \zeta_1 : \zeta_2 = \eta_1\eta_2 : \eta_2\eta_0 : \eta_0\eta_1 , \tag{1}$$

$$\eta_0 : \eta_1 : \eta_2 = \zeta_1\zeta_2 : \zeta_2\zeta_0 : \zeta_0\zeta_1 . \tag{2}$$

It is clear that (2) is the solution of (1) in the case $\eta_0\eta_1\eta_2 \neq 0$. The transformation (1) is therefore its own inverse. It is one-to-one, except for the sides of the fundamental triangle, but all points of the side $\eta_0 = 0$ will go to the opposite corner $\zeta_1 = \zeta_2 = 0$; corresponding statements are true for the remaining sides. The transformation (1) will be undetermined for the vertices of the fundamental triangle.

If one substitutes the value (2) of the ratio in the equation $f(\eta_0, \eta_1, \eta_2)$ of the present curve then one will obtain a transformed equation:

$$f(\zeta_1\zeta_2,\ \zeta_2\zeta_0,\ \zeta_0\zeta_1) = 0. \tag{3}$$

If the original curve $f = 0$ does not go through a corner of the fundamental triangle then each point of this curve will originate uniquely from a point of the curve (3), and (from § 19) the latter will be irreducible. However, if $f = 0$ goes through a vertex – say, through (1, 0, 0) – then all of the terms in $f(y_0, y_1, y_2) = 0$ will be divisible by y_1 or y_2 , and therefore the factor z_0 will split off from $f(z_1 z_2, z_2 z_0, z_0 z_1)$. If $f = 0$ has an r-fold point at (1, 0, 0) then it will be the factor z_0^r precisely that indeed splits off from $f(z_1 z_2, z_2 z_0, z_0 z_1)$. We thus set:

$$f(z_1 z_2, z_2 z_0, z_0 z_1) = z_0^r z_1^s z_2^t\, g(z_0, z_1, z_2)\,, \tag{4}$$

and call $g(z) = 0$ the *transformed curve* of $f = 0$.

By the substitution:

$$z_0 = y_1 y_2\,, \qquad z_1 = y_2 y_0\,, \qquad z_2 = y_0 y_1\,,$$

one will obtain, from (4):

$$\begin{aligned}(y_0, y_1, y_2)^n f(y) &= y_0^{s+t} y_1^{t+r} y_2^{r+s} g(y_1y_2, y_2y_0, y_0y_1)\\ g(y_1 y_2, y_2 y_0, y_0 y_1) &= y_0^{n-s-t} y_1^{n-t-r} y_2^{n-r-s} g(y_0, y_1, y_2)\,.\end{aligned} \tag{5}$$

Hence, one will also have, conversely, that $f = 0$ is the transformed curve of $g = 0$. If $g(y_0, y_1, y_2)$ were decomposable then, from (5), $f(y_0, y_1, y_2)$ would also be decomposable, contrary to the assumption. Hence, $g(z) = 0$ is a *non-decomposable* curve.

By differentiation with respect to z_2 , it will follow from (4) that if f_0', f_1', f_2' are the derivatives of f and g_0', g_1', g_2' are those of g then:

$$\begin{aligned}&z_1 f_0'(z_1z_2, z_2z_0, z_0z_1) + z_0 f_1'(z_1z_2, z_2z_0, z_0z_1)\\ &\qquad = t\, z_0^r z_1^s z_2^{t-1} g(z_0, z_1, z_2) + z_0^r z_1^s z_2^t g_2'(z_0, z_1, z_2)\,.\end{aligned}$$

If one multiplies this equation on both sides by z_2 and applies the EULER identity:

$$y_0 f_0'(y) + y_1 f_1'(y) + y_2 f_2'(y) = n\, f(y)$$

then it will follow that:

$$n\, f(z_1 z_2, z_2 z_0, z_0 z_1) - z_0 z_1 f_2'(z_1z_2, z_2z_0, z_0z_1) = t\, z_0^r z_1^s z_2^t g(z) + z_0^r z_1^s z_2^{t+1} g_2'(z)\,. \tag{6}$$

Analogous equations are naturally valid for the other two derivatives f_0', f_1'.

As for any rational map, each branch of the curve $f = 0$ corresponds to a unique branch of the curve $g = 0$, and conversely. If $\eta_0(\tau)$, $\eta_1(\tau)$, $\eta_2(\tau)$ are the power series developments of a branch $\mathfrak{z}$ of the curve $f = 0$ then one will obtain the corresponding branch $\mathfrak{z}'$ of the curve $g = 0$, in which one first forms the product $\eta_1(\tau)\eta_2(\tau)$, $\eta_1(\tau)\eta_0(\tau)$, $\eta_0(\tau)\eta_1(\tau)$ and then removes a possible common factor τ^λ, from the three power series:

$$\begin{cases} \zeta_0(\tau)\tau^\lambda = \eta_1(\tau)\eta_2(\tau), \\ \zeta_1(\tau)\tau^\lambda = \eta_2(\tau)\eta_0(\tau), \\ \zeta_2(\tau)\tau^\lambda = \eta_0(\tau)\eta_1(\tau). \end{cases}$$

The factor τ^λ appears only when the starting point of the branch $\mathfrak{z}$ is a vertex of the coordinate triangle. If we assume, perhaps, that this is the vertex (1, 0, 0) and the tangent to the branch is not a side of the coordinate triangle then the power series development of the branch will read thusly:

$$\text{(7)} \qquad \begin{cases} \eta_0(\tau) = 1, & \\ \eta_1(\tau) = b_k\tau^k + b_{k+1}\tau^{k+1} + \cdots & (b_k \neq 0), \\ \eta_2(\tau) = c_k\tau^k + c_{k+1}\tau^{k+1} + \cdots & (c_k \neq 0). \end{cases}$$

One will then find that $\lambda = k$ and:

$$\text{(8)} \qquad \begin{cases} \zeta_0(\tau) = b_k c_k \tau^k + (b_k c_{k+1} + b_{k+1} c_k)\tau^{k+1} + \cdots, \\ \zeta_1(\tau) = c_k + c_{k+1}\tau + \cdots, \\ \zeta_2(\tau) = b_k + b_{k+1}\tau + \cdots \end{cases}$$

In this case, the starting point $\mathfrak{z}'$ will thus lie on the opposite side of the coordinate triangle. Conversely, if one forms:

$$\begin{aligned} \eta_0(\tau) &= \zeta_1(\tau)\zeta_2(\tau), \\ \eta_1(\tau) &= \zeta_2(\tau)\zeta_0(\tau), \\ \eta_2(\tau) &= \zeta_0(\tau)\zeta_1(\tau), \end{aligned}$$

while starting from the branch $\mathfrak{z}'$, then one will get back the original branch $\mathfrak{z}$, except for an inessential factor of $\zeta_1(\tau)\zeta_2(\tau)$.

We now go on to the "resolution of singularities." We select a certain singularity – i.e., a multiple point O – on the curve $f = 0$ that we would like to resolve – i.e., convert into simple singularities. We place the vertex (1, 0, 0) of the coordinate triangle at O, and choose the three other corners outside the curve such that the sides of the coordinate triangle are not curve tangents and include no more points of the curve beyond O. In equation (4), one will then have $s = t = 0$, while r will give the multiplicity of the point O. We now have three things to examine:

1. The effect of the transformation on the branch of the point O,
2. Its effect on the branches at the intersection points of the triangle with the curve,
3. Its effect on the remaining curve points and its branches.

We introduce a measure for the complexity of a singularity, namely, the multiplicity of the intersection point of O as an intersection point of the curve $f = 0$ with the polar of a point P, which is chosen such that this multiplicity will be as small as possible. If O is a simple point then this measure will have the value zero, while it will always be > 0 at multiple points.

The intersection multiplicity of the curve with the polar combines contributions that originate in the different branches of the point O. We will now show that for each such branch $\mathfrak{z}$ of the point O, the contribution will always be reduced under the Cremona transformation above in the event that O is actually a multiple point; hence, when $r > 1$.

We understand ζ_0, ζ_1, ζ_2 to mean the power series (8), and η_0, η_1, η_2 to mean the power series:

$$\eta_0 = \zeta_1\zeta_2, \qquad \eta_1 = \zeta_2\zeta_0, \qquad \eta_2 = \zeta_0\zeta_1$$

that are proportional to (7) and represent the branch $\mathfrak{z}$. The polar of a point $P(\pi_0, \pi_1, \pi_2)$ will have the equation:

$$\pi_0 f_0'(\eta) + \pi_1 f_1'(\eta) + \pi_2 f_2'(\eta) = 0$$

and will cut the branch $\mathfrak{z}$ with a multiplicity that will be ≥ the minimum of the orders of the power series $f_0'(\eta)$, $f_1'(\eta)$, $f_2'(\eta)$ and which will be equal to this minimum, in general (except for special locations of the point P). We can assume that the vertex (0, 0, 1) of the coordinate triangle has no such special location, and thus, that the order μ of the power series $f_2'(\eta)$ is already equal to the minimum in question.

If one now substitutes the power series ζ_0, ζ_1, ζ_2 for z_0, z_1, z_2 in (6) then, since $s = t = 0$, $f(\eta) = 0$, $g(\eta) = 0$, it will follow that:

$$-\,\zeta_0\zeta_1 f_2'(\eta_0, \eta_1, \eta_2) = \zeta_0^r\zeta_2 g_2'(\zeta_0, \zeta_1, \zeta_2)\,,$$

or, after canceling ζ_0:

(9) $$-\,\zeta_1 f_2'(\eta_0, \eta_1, \eta_2) = \zeta_0^{r-1}\zeta_2 g_2'(\zeta_0, \zeta_1, \zeta_2)\,.$$

The left-hand side will have order μ precisely, since, from (8), ζ_1 has order zero. The factor ζ_0^{r-1} on the right will have order $(r-1)\,k$ and ζ_2 will have order 0. Hence, the factor $g_2'(\zeta_0, \zeta_1, \zeta_2)$ will have the order:

$$\mu - (r-1)\,k < \mu.$$

The minimum of the orders of $g'_0(\zeta)$, $g'_1(\zeta)$, $g'_2(\zeta)$ will be $< \mu$, moreover. Thus, the minimal intersection multiplicity of the branch with the polar has, in fact, diminished under the Cremona transformation.

We now turn to the second issue of the intersection points of the triangle with the curve. If such a point lies – say – on the triangle side $\eta_2 = 0$ then, since the intersection point shall be a simple one, η_2 will have order 1, while η_2 and η_1 have order zero:

$$\eta_0 = a_0 + a_1\tau + \ldots \qquad (a_0 \neq 0),$$
$$\eta_1 = b_0 + b_1\tau + \ldots \qquad (b_0 \neq 0),$$
$$\eta_2 = c_1\tau + \ldots \qquad (c_1 \neq 0).$$

The transformed branch will read:

$$\zeta_0 = \eta_1\eta_2 = b_0\, c_1\ \tau + \ldots,$$
$$\zeta_1 = \eta_2\eta_0 = a_0\, c_1\ \tau + \ldots,$$
$$\zeta_2 = \eta_0\eta_1 = a_0\, b_0 + \ldots$$

One is thus dealing with a linear branch at the point (0, 0, 1) whose tangent direction is given by the ratio $b_0 : a_0$; thus, it will depend upon the location of the point $(a_0, b_0, 0)$ on the opposing side of the triangle, which is where we started from. Since the intersection point of the curve $f = 0$ with the triangle sides outside of O were all assumed to be different, we will obtain nothing but distinct tangents for the transformed linear branches at the vertices of the triangle. Thus, *new singularities* will appear under the Cremona transformation, namely, *multiple points with nothing but linear branches with separate tangents.*

In the third case, we must consider points that lie either at a vertex or on a side of the fundamental triangle. The Cremona transformation will be one-to-one for these points. It will transform linear branches into linear branches (as one easily verifies), and it will also transform the tangent directions of the branch at such a point in a one-to-one manner. Simple points will thus go to simple points, and q-fold points with q separate tangents will again go to others of that sort. If one is dealing with a singular point then it will follow from formula (9), which will also be true for this case, that the intersection multiplicity of the branches with the polars will remain unchanged in this case; thus, the measure of the singularity will not be raised.

If one now defines a whole number $\mu(f)$ for each curve $f = 0$ to be the sum of the singularity measures over all of the singular points that are not merely multiple points with separate tangents then it will follow from the foregoing that the number $\mu(f)$ can always be diminished by a suitable Cremona transformation when it is non-zero. After finitely many such transformations one will have $\mu(f) = 0$, and we will get the theorem:

Any irreducible curve $f = 0$ *may be converted into one that possesses only "normal" singularities (i.e., multiple points with separate tangents) by a birational transformation.*

Problem. Show that the theorem proved above is also true for decomposable curves.

§ 26. The invariance of the genus. The PLÜCKER formulas.

Let m be the degree of a plane irreducible curve K and let m' be its class. We compute the characteristics (k, l) for all non-ordinary points of K and form the sums:

$$s = \sum (k - 1)$$
$$s' = \sum (l - 1).$$

s is called the "number of cusps" and s' is the "number of inflection points." In fact, when there are no other extraordinary branches than the cusp (2, 1) and the inflection point (1, 2), s will actually mean the number of vertices, and s' will mean the number of inflection points.

We now set:

(1) $$m' + s - 2m = 2p - 2$$

and call the rational number p that is defined by (1) the *genus* of the curve. We will later see that p is a whole number ≥ 0, and that p will remain invariant under all birational transformations of the curve.

We first give the definition of the genus a somewhat different form. We again assume, for the sake of simplicity, that the point (0, 0, 1) does not lie on the curve. We consider a general point $(1, u, \omega)$ of the curve K, in which ω is thus an algebraic function of u, and consider the branching points of this function ω – i.e., the values $u = a$ or ∞ at which several power series developments $\omega_1, \ldots, \omega_h$ come together (*zusammentreten*) into a cycle. The number h – viz., the order of the branching – is the order of the function $u - a$ (u^{-1}, resp.) on the branch in question. If one now addresses the classification of the branch (§ 21) then one will see that:

$h = k$ when the branch tangent does not go through (0, 0, 1),
$h = k + l$ when the branch tangent goes through (0, 0, 1).

When one sums over all $h > 1$, it will then follow that:

$$\sum (h - 1) = \sum (k - 1) + \sum' l,$$

in which the last sum is taken over only the branches whose tangents go through the point (0, 0, 1). $\sum' l$ will then be the sum of the multiplicities of the tangents to (0, 0, 1), or the *class* m'. The sum $\sum (h - 1)$ is called the *branching number* w of ω as an algebraic function of u. Finally, one has $\sum (k - 1) = s$, so:

$$w = s + m'.$$

If one substitutes this into (1) then it will follow that:

(2) $$w - 2m = 2p - 2.$$

In words: *The branching number of an algebraic function* ω*, minus twice its degree, is equal to* $2p - 2$*, when* p *is the genus of the associated algebraic curve.*

It is not difficult to also prove this theorem for the case in which the point (0, 0, 1) – which, up till now, has been assumed to lie outside the curve – is a q-fold point of the curve with nothing but ordinary branches. In this case, the degree n of the function ω will not be equal to m, but $m - q$, and also $\sum' l$ will not be equal to m', but $m' - 2q$, and it will follow that:

$$w - 2n = 2p - 2.$$

The genus is closely connected with the *differentials* of the function field $K(u, \omega)$. By this, we mean the following: The differential of the independent variables du shall be merely a symbol, or, if one wishes, an indeterminate. Moreover, if η is any function of the field then we set:

$$d\eta = \frac{d\eta}{du}\,du\,.$$

We understand the *order* of the differential du on any branch of the curve to mean the order of the differential quotients $d\eta / d\tau$ with respect to the position uniformization τ. The order of $d\eta$ is, correspondingly, the order of:

$$\frac{d\eta}{d\tau} = \frac{d\eta}{du}\frac{du}{d\tau}\,.$$

If $u - a$ has order h on a branch:

$$u - a = c_h\,\tau^h + \ldots$$

then du will have the order $h - 1$, and then it will follow by differentiation that:

$$\frac{du}{d\tau} = h\,c_h\,\tau^{h-1} + \ldots$$

h is different from 1 only at the branching points; there are thus also only finitely many branches for the differential du on which its order is different from zero. If $u = \infty$ on a branch then one will have:

$$u^{-1} = c_h\,\tau^h + \ldots,$$

$$u = c_h^{-1}\tau^{-h} + \ldots,$$

$$\frac{du}{d\tau} = -\,h\,c_h^{-1}\tau^{-h-1} + \ldots;$$

hence, the order of du will become $-h - 1$ there. Now, one has:

$$-h - 1 = (h - 1) - 2h.$$

The sum of the orders of the differential *du* on all of the branches will equal:

$$\sum (h - 1) = 2h = \sum_\infty w - 2m = 2p - 2,$$

in which $\sum_\infty$ means the sum over all branches with $m = \infty$, hence, over all intersection points of the curve with the line $\eta_0 = 0$. Since this intersection point has, at the same time, the multiplicity h on any branch $\sum_\infty h$ will be equal to the degree of the curve m. Thus:

The sum of the orders of the differential du over all curve branches is equal to $2p - 2$.

Now, if $d\eta / d\tau$ is a function of the field then this will be true for not only *du*, but also for any differential:

$$d\eta = \frac{d\eta}{du} du ,$$

and the sum of the orders of such a function over all branches will be equal to zero (§ 20).

What now follows immediately from this remark is the **Theorem of the invariance of the genus:**

If two curves $f = 0$ *and* $g = 0$ *can be mapped to each other birationally then they will have the same genus.*

Thus, if (u, ω) is a general point of the one curve and (v, θ) is a point of the other then any function $h(u, \omega)$ will correspond to a function $h'(v, \theta)$ by means of birational map, and each branch will correspond to a branch. The position uniformization τ of the branch will again correspond to the position uniformization, the differential quotients will again correspond to the differential quotients, and it will follow that the orders of the differential $d\eta$ will be preserved, and therefore the sum $2p - 2$, as well.

As a first application of the theorem of the invariance of the genus, we prove that the genus is always a whole number ≥ 0. From § 25, we can convert any curve birationally into a curve K with nothing but "normal" singularities, namely, r-fold points with distinct tangents. If the curve has degree m then, from § 21, its class m' will equal:

$$m' = m(m - 1) - \sum r(r - 1),$$

in which the sum is taken over all multiple points. It will follow that:

$$2p - 2 = m' + s - 2m = m\,(m - 1) - \sum r\,(r - 1) - 2m$$
$$2p = (m - 1)\,(m - 2) - \sum r\,(r - 1).$$

The right-hand side is an even number, so p will be a whole number. We can set:

$$\sum r\,(r - 1) = 2d,$$

and then call d the "number of the double point," if we count an r-fold point as $\binom{r}{2}$ double points. One will then have:

$$p = \frac{(m-1)(m-2)}{2} - d. \tag{3}$$

A curve that has at least an $(r - 1)$-fold point at each r-fold point of the curve K (with normal singularities!) is called an *adjoint curve* to K. In order for a given point to be an r-fold point of a curve $h = 0$, its coefficients must fulfill $r\,(r - 1) / 2$ linear equations, so if – say – $(1, 0, 0)$ is the point then the development of $h(x_0, x_1, x_2)$ in increasing powers of x_1 and x_2 must lack the terms of order $0, 1, \ldots, r - 1$. An adjoint curve thus has $\sum r\,(r - 1) / 2 = d$ (dependent or independent) linear conditions to fulfill. It cuts the curve in the multiple points $r\,(r - 1)$-fold, hence, $2d$-fold, in all.

There exist adjoint curves of order $m - 1$; e.g., the first polar of an arbitrary point. Since the total number of intersections of a curve and a polar amounts to $m\,(m - 1)$, it will follow that:

$$2d \leq m\,(m-1). \tag{4}$$

There are indeed adjoint curves of the order $m - 1$ that include, other than the multiple points, also:

$$\frac{(m-1)(m+2)}{2} - d$$

arbitrarily given points. A curve of order $m - 1$ will then have $m\,(m + 1) / 2$ coefficients, so that one can impose:

$$d + \frac{(m-1)(m+2)}{2} - d = \frac{m(m+2)}{2} - 1$$

conditions, which, when false, would imply that they all vanish. Since the number of intersection points again amounts to $m\,(m - 1)$, it will follow that:

$$2d + \frac{(m-1)(m+2)}{2} - d \leq m\,(m-1)$$

or:

$$d \leq \frac{(m-1)(m+2)}{2}, \tag{5}$$

or, from (3):

$$p \geq 0.$$

As the proof shows, the inequality (4) is valid, not only for irreducible curves, but also for arbitrary curves with no multiple components, while the inequality (5) is valid only for irreducible curves, but with arbitrary singularities. Both inequalities are the sharpest of their kind.

The notion of genus can be carried over to reducible curves; the definition (1) will remain the same. Since the class, cusp count, and degree of a decomposable curve are equal to the sums of the classes, cusp counts, and degree of the components, one will then have:

$$2p - 2 = (2p_1 - 2) + \ldots + (2p_r - 2)$$

or:

(6) $$p = p_1 + \ldots + p_r - r + 1$$

for a curve that decomposes into r components of genera $p_1, \ldots, p_r$.

The PLÜCKERian formula. From the theorem on the invariance of genus, the dual curve of an (irreducible) curve will have the same genus as the original curve. One will then have:

(7) $$m + s - 2m' = 2p - 2$$

dual to (1).

To the formulas (1), (3), one now adds formula (5) of § 21, which expresses the class m' in terms of the degree m and the type and number of singular points. If these exist as d junctions and s cusps then, from § 21, one will have:

(8) $$m' = m\,(m - 1) - 2d - 3s.$$

By a suitable definition of the number d, this formula will also be valid when the curve possesses higher singularities. For example, one must count an r-fold point with separate tangents as $r\,(r - 1) / 2$ junctions, and likewise, a contact junction as two junctions, etc. In each individual case, the methods of § 21 will give the possibility of computing the quantities that one must add to $m\,(m - 1)$ in order to obtain the class m', and one can always put this quantity into the form $2d + 3s$; then, from § 21, problem 4, it will always be $\geq 3s$, and it will always differ from $3s$ by an even number, since, from (1), $m' + s$ is an even number.

Dual to (8), one has the formula:

(9) $$m = m'\,(m' - 1) - 2d' - 3s',$$

in which d' means a suitably-defined number of double tangents.

We then summarize the formulas that we have found:

(1), (7) $$m + s - 2m = m + s' - 2m' = 2p - 2,$$

(8) $$m' = m\,(m - 1) - 2d - 3s,$$

(9) $$m = m'\,(m' - 1) - 2d' - 3s'.$$

In these, m means the degree of the curve, m', the class, s and s', the number of cusps and inflection points, resp., d and d', the number of double points and double tangents, resp., and finally, p is the genus.

It follows by subtracting (1) and (7) that:

(10) $$s' - s = 3(m' - m),$$

or, if the value of m' from (8) is substituted in this:

$$s' = 3m\,(m-2) - 6d - 8s\,. \tag{11}$$

Dual to this, one will have:

$$s = 3m'\,(m'-2) - 6d' - 8s'. \tag{12}$$

(8), (9), (11), (12) are called the *PLÜCKERian formulas.* One can calculate m', s', d' from them when m, s, d are given.

If one substitutes m' from (8) into (1) then what will follow, after a conversion, is the convenient genus formula:

$$p = \frac{(m-1)(m-2)}{2} - d - s. \tag{13}$$

As examples of its application, we compute the number of inflection points and double tangents of a double-point-free curve of order m.

From (8), it will first follow that the class is:

$$m' = m\,(m-1).$$

Thus, it will follow from (10) or (11) that the number of inflection points is:

$$s' = 3m\,(m-2),$$

and finally, from (9), that the number of double tangents is:

$$\left\{\begin{aligned} 2d' &= m'(m'-1) - m - 3s' \\ &= m(m-1)(m^2-m-1) - m - 9m(m-2) \\ &= m(m-2)(m^2-9) \\ d' &= \tfrac{1}{2}m(m-2)(m^2-9). \end{aligned}\right. \tag{14}$$

In particular, a double-point-free curve of order 4 has 28 double tangents ([1]).

([1]) These curves have very interesting geometric properties. See STEINER (J. reine angew. Math. Bd. 49), HESSE (J. reine angew. Math. Bd. 49 and 55), ARONHOLD (Mber. Akad. Berlin 1864), and M. NOETHER (Math. Ann. Bd. 15 and Abh. Akad. München Bd. 17). One finds a good introduction to the subject in H. WEBER's *Lehrbuch der Algebra II.*

CHAPTER FOUR

Algebraic manifolds

§ 27. Points in the broader sense. Relation-preserving specializations

Up till now, we have only considered points with constant coordinates in a fixed field K. Now, we extend the notion of a point by also allowing points whose indeterminates, or algebraic functions of indeterminates, or still more general elements, are in any extension field of K. A "point in the broader sense" of the vector space E_n is thus a system of n elements $y_1, \ldots, y_n$ of an arbitrary extension field of K, and a point in the broader sense of the projective space S_n will be defined accordingly. Furthermore, the notion of a linear space, hypersurface, etc., will be extended by regarding the particular points of the linear space in the broader sense (regarding the coefficients of the equation for the hypersurface as arbitrary elements of an extension field of K, resp.).

The extension field from which the elements $y_1, \ldots, y_n$ are taken is not to be thought of as a fixed field, but rather as an enlarged field that can be extended as often as necessary in the course of a geometric consideration, e.g., by the addition of new indeterminates and algebraic functions of the indeterminates. At the moment of the introduction of a sequence of new indeterminates all of the previously introduced indeterminates will be regarded as constant, and the ground field will be thought of as having been augmented. That means: By the introduction of new indeterminates $u_1, \ldots, u_m$, the ground field will become the field K' that comes about by adjoining all previously considered indeterminates $x_1, \ldots, x_n, \ldots$ to the ground field K.

The algebraic extension of a given field will always be *tacitly* carried out, when required. If, e.g., a hypersurface with coefficients in an extension field K' of K is intersected by a line then the intersection point will be obtained by solving an algebraic equation. We then always think of the field K' as being extended by the adjunction of all of the roots of this algebraic equation. In this sense, we can regard any algebraic equation as soluble in this enlarged field K' ([1]).

We understand the term *a general point* of the projective space S_n to mean a point whose coordinate ratios $x_1/x_0, \ldots, x_n/x_0$ are algebraically independent relative to the ground field K. Thus, there shall exist no algebraic equation $f(x_1/x_0, \ldots, x_n/x_0) = 0$, or, what amounts to the same thing, no homogeneous algebraic relation $F(x_0, \ldots, x_n) = 0$ with coefficients in K, unless the polynomial f (the form F, resp.) vanishes identically. One obtains a general point, e.g., when one chooses all of the coordinates $x_0, \ldots, x_n$ to be indeterminates, or also when one sets $x_0 = 1$ and chooses $x_1, \ldots, x_n$ to be indeterminates.

A *general hyperplane* in S_n is hyperplane u whose coefficient ratios $u_1/u_0, \ldots, u_n/u_0$ are algebraically independent relative to K. It is most convenient for one to simply

([1]) Through this sort of consideration, we avoid "transfinite induction," which is necessary in order to extend the field K' to one that is actually algebraically closed (cf., E. STEINITZ: *Algebraische Theorie der Körper*, Leipzig 1930). Transfinite induction is then necessary when one wishes to solve an infinite set of equations. However, only finitely-many equations will appear in geometric problems, and they can be solved among themselves in the order of their appearance without having to introduce transfinite induction.

choose $u_0, u_1, \ldots, u_n$ to be indeterminates. Analogously, a *general hypersurface of degree m* will be one whose equation coefficients are nothing but independent indeterminates.

A *general subspace* S_m is one whose Plücker coordinates satisfy no homogeneous algebraic relation with coefficients in K, except for relations that are valid for *any* subspace S_m. One can obtain a general S_m, for example, as the intersection of $n - m$ general (mutually-independent) hyperplanes or as the join of $m + 1$ independent general points.

Relation-preserving specializations. A point (in the broader sense) η is called a *relation-preserving specialization* of the same point ξ when all homogeneous algebraic equations $F(\xi_0, \ldots, \xi_n) = 0$ with coefficients in the ground field K that are valid for the point ξ are also valid for the point η, and thus it always follows from $F(\xi) = 0$ that $F(\eta) = 0$ for any form F. For example, any point of a space is a relation-preserving specialization of the general point in the same space. Another example: Let $\xi_0, \ldots, \xi_n$ be rational functions of the indeterminate parameter t and let $\eta_0, \ldots, \eta_n$ be the values of these rational functions for a particular value of t.

One defines a relation-preserving specialization of a point-pair (ξ, η), a point-triple (ξ, η, ζ), etc., analogously. If $(\xi, \eta) \to (\xi', \eta')$ is to be a relation-preserving specialization then all equations $F(\xi, \eta) = 0$ that are individually homogeneous in the ξ and η must remain true under the replacement of ξ with ξ' and η with η'.

The most important theorem on relation-preserving specializations, which will come before everything else in chap. 6, reads like:

Any relation-preserving specialization $\xi \to \xi'$ may be continued to a relation-preserving specialization $(\xi, \eta) \to (\xi', \eta')$ when (ξ, η) is any point-pair in the broader sense.

Proof. From the totality of all homogeneous equations $F(\xi, \eta) = 0$ one can, from the HILBERT basis theorem ([1]), select a finite number of them from which all of the other ones follow. One eliminates the η from these finitely-many forms – i.e., one constructs the resultant system $G_1, \ldots, G_k$. One will then have $G_1(\xi) = 0, \ldots, G_k(\xi) = 0$. Due to the relation-preserving specialization, it will follow from this that $G_1(\xi') = 0, \ldots, G_k(\xi') = 0$. From the meaning of the resultant system, the system of equations $F_1(\xi', \eta') = 0, \ldots, F_k(\xi', \eta') = 0$ will then be soluble for η'. That is, there will be a point η' such that all equations $F(\xi, \eta) = 0$ will also be valid for ξ', η'.

Essential use was made in the proof of the fact that one is dealing with homogeneous equations, and therefore homogeneous coordinates – at least, relative to the η. The theorem will no longer be true in affine spaces, in which the specialization $\xi \to \xi'$ can take the point η to infinity. On the contrary, it is essential that one deals with only one point ξ and one point η, or else the theorem would be true for a whole sequence of further points $\overset{1}{\xi}, \ldots, \overset{r}{\xi}, \overset{1}{\eta}, \ldots, \overset{s}{\eta}$.

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[1]) Cf., *Moderne Algebra II*, § 80.

Problems. 1. A system of homogeneous linear equations always possesses a general solution, from which any solution will arise by relation-preserving specialization.

2. If η depends rationally upon ξ and some further parameter t, and these rational functions still make sense when ξ' is substituted for ξ and t', for t, and $\xi \to \xi'$ is a relation-preserving specialization then $(\xi, \eta) \to (\xi', \eta')$ will also be a relation-preserving specialization.

3. If η is the general solution to a system of linear equations whose coefficients are homogeneous rational functions of ξ, and if ξ is specialized to ξ' in a relation-preserving way, where the rank of the system is not reduced, and if η' is a solution of the specialized system of equations then $(\xi, \eta) \to (\xi', \eta')$ will be a relation-preserving specialization. (One represents the solution η' with the help of determinants, and likewise, the general solution η, and applies problem 2.)

§ 28. Algebraic manifolds. Decomposition into irreducible ones

An *algebraic manifold* in the projective space S_n is the totality of all points (in the broader sense) whose coordinates $\eta_0, \ldots, \eta_n$ satisfy a system of finitely or infinitely many algebraic equations:

$$f_1(\eta_0, \ldots, \eta_n) = 0 \tag{1}$$

with coefficients in the constant field $\mathbb{K}$. If there is no such point then one calls the manifold *empty*. We will always exclude this case from consideration.

Due to the HILBERT basis theorem, one can replace an infinite system of equations by an equivalent finite system.

Similarly, one defines an algebraic manifold in double projective space $S_{m,\,n}$ by a system of homogeneous equations in two sequences of homogeneous variables:

$$f_i(\xi_0, \ldots, \xi_m, \eta_0, \ldots, \eta_n) = 0\,. \tag{2}$$

If equation (1) ((2), resp.) is made inhomogeneous by the substitution $\eta_0 = 1$, ($\xi_0 = 1$, resp.) then one will obtain the equations of an algebraic manifold in an affine space A_n (A_{m+n}, resp.). From now on, we always write $f(x)$, $f(\eta)$, $f(\xi, \eta)$, etc., instead of $f(x_0, \ldots, x_n)$, $f(\eta_0, \ldots, \eta_n)$, $f(\xi_0, \ldots, \xi_m, \eta_0, \ldots, \eta_n)$, etc.

The notion of algebraic manifold can be generalized still further by considering, in place of the point η and the point-pair ξ, η, other geometric objects that are given in terms of homogeneous coordinates; e.g., hypersurfaces, linear subspaces S_m in S_n, etc. For example one can, speak of the manifold of all planes in S_n; its equations are given by (2), § 7.

The intersection $M_1 \cap M_2$ of two algebraic manifolds M_1 and M_2 is obviously again an algebraic manifold. However, the union [1] or *sum* of two algebraic manifolds is also one. Namely, if $f_i(\eta) = 0$ and $g_j(\eta) = 0$ are the equations of the two manifolds being united then the equations of the union will be:

$$f_i(\eta)\, g_j(\eta) = 0\,.$$

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[1] The word "union" (or sum) is used with its set-theoretic meaning. Components, such as M_1 and M_2 in the sum are to be counted only once, and no more. Multiply-counted manifolds will first be introduced much later (§ 36 and § 37).

An algebraic M in S_n is called *decomposable* or *reducible* when it is the sum of distinct – i.e., disjoint from each other – submanifolds. An indecomposable manifold is called *irreducible*.

Lemma. *If an irreducible manifold M is contained in the union of two algebraic manifolds M_1 and M_2 then M will be contained in either M_1 or M_2.*

Proof. Any point of M belongs to either M_1 or M_2, hence, to the intersection $M \cap M_1$ or the intersection $M \cap M_2$. Hence, M is the union of $M \cap M_1$ and $M \cap M_2$. However, since M is irreducible, one of these manifolds $M \cap M_1$ or $M \cap M_2$ must coincide with M itself; i.e., M will be contained in either M_1 or M_2.

This lemma may be immediately carried over to several manifolds $M_1, \ldots, M_r$ by complete induction.

A special case:

If a product $f_1 f_2$ of two forms is zero at all points of an irreducible manifold then f_1 or f_2 will have the property of being zero at all points of M.

On the other hand, if M is decomposable – perhaps into M_1 and M_2 – then there will be, firstly, a form f_1 among the defining equations of M_1 that is zero at all points of M_1, but not all points of M_2, and likewise there will be a form f_2 that is zero at all points of M_2, but not all points of M_1. The product $f_1 f_2$ will then be zero at all of the points of M, but neither of the factors f_1, f_2 has this property. Thus, we have:

First irreducibility criterion. *A necessary and sufficient condition for the decomposability of a manifold M is the existence of a product $f_1 f_2$ that is zero at all points of M, without either of the forms f_1, f_2 being zero on M.*

Moreover, for algebraic manifolds, we have the:

Chain theorem. *A sequence of manifolds:*

$$M_1 \supset M_2 \supset \ldots \tag{3}$$

in A_n or S_n in which M_{r+1} is a proper submanifold of M_r must terminate after finitely many terms.

Proof. The equations of the manifolds $M_1, M_2, \ldots$ allow us to write down the sequence as:

$$f_1 = 0, \quad f_2 = 0, \quad \ldots f_h = 0; \qquad f_{h+1} = 0, \ldots, f_{h+h} = 0; \ldots$$

From the HILBERT basis theorem, all of these equations follow from finitely many of them. However, that means that the equations of $M_1, \ldots, M_l$ collectively comprise the equations of all further manifolds, hence, after M_l , no further proper submanifolds can be given in the sequence.

We now come to the fundamental:

Decomposition theorem. *Any algebraic manifold is (either irreducible or) the sum of finitely many irreducible manifolds:*

$$M = M_1 + M_2 + \dots + M_r . \tag{4}$$

Proof. Assume that there is a manifold M that is not the sum ([1]) of irreducible manifolds. M is then decomposable, perhaps into M' and M''. If M' and M'' were the sums of irreducible manifolds then M would also be. Hence, M would possess a proper submanifold M' or M'' that would not be the sum of irreducible manifolds. The latter likewise would possess a proper submanifold, etc. One would thus obtain an infinite chain (3), which is impossible. Hence, any manifold is the sum of irreducible ones.

Uniqueness theorem. *The representation of a manifold M as an unshortenable sum (a sum is called "shortenable" when one summand in the sum contains the remaining ones; hence, one can omit them) of irreducible ones is unique, up to the order of the summands.*

Proof. Let $M = M_1 + \dots + M_r = M_1' + \cdots + M_s'$ be two unshortenable representations. It follows from the lemma that M_1 is contained in one of the manifolds M_r'. By altering the sequence of M_r', we can assume that M_1 is contained in M_1'. Likewise, M_1' is contained in some M_μ . If one had $\mu \neq 1$ then one would have $M_1 \subseteq M_1' \subseteq M_\mu$; hence, the sum $M_1 + \dots + M_\mu + \dots$ could be shortened, and therefore one could have $\mu = 1$ and $M_1 = M_1'$. Further summands M_{r+i}' can therefore no longer appear in the second sum, since it was unshortenable.

The irreducible manifolds that appear as an unshortenable sum in the representation of M are called the *irreducible components* of M.

The proof above still gives no means of effectively carrying out the decomposition of M into irreducible components when the equations of M are given. This means is first provided by the elimination theory that will be presented in § 31.

§29. The general point and dimension of an irreducible manifold

A point ξ is called a *general point* of a manifold M when ξ belongs to M and all homogeneous algebraic equations with coefficients in K that are valid for the point ξ are

([1]) We always understand "sum" to mean a finite sum, here. A sum can also consist of just one term.

valid for all points of M. In other words, ξ shall belong to M and all of the points of M shall emerge as relation-preserving specializations of the point.

Second irreducibility criterion. *If a manifold M possesses a general point then it will be irreducible.*

Proof. If M were decomposable then there would be a product fg of two forms that vanishes everywhere on M without one of the factors doing so. It would follow that:

$$f(\xi) \cdot g(\xi) = 0.$$

Hence, since $f(\xi)$ and $g(\xi)$ belong to a field:

$$f(\xi) = 0 \qquad \text{or} \qquad g(\xi) = 0,$$

and consequently one would have that either $f = 0$ at all points of M or $g = 0$ at all points of M, which would contradict the assumption.

Existence theorem. *Any non-empty irreducible manifold M possesses a general point ξ (for a suitable extension field of K).*

Proof. Any quotient of two forms of the same degree:

$$\frac{f(x_0, x_1, \cdots, x_n)}{g(x_0, x_1, \cdots, x_n)}$$

defines a rational function on the manifold M, as long as one assumes that the denominator is not zero at all points of M. Two such functions are said to be equal:

$$\frac{f}{g} = \frac{f'}{g'} \qquad \text{when} \qquad fg' = f'g \qquad \text{on } M.$$

Addition, subtraction, multiplication, and division of rational functions on M yields other rational functions on M. The rational functions on M then define a field that includes the constant field K.

We can assume that x_0 is not equal to zero at all points of M. We denote the rational functions:

$$\frac{x_1}{x_0}, \quad \frac{x_2}{x_0}, \quad \ldots, \quad \frac{x_n}{x_0}$$

by $\xi_1, \xi_2, \ldots, \xi_n$. Furthermore, we set $\xi_0 = 1$. $(\xi_0, \xi_1, \ldots, \xi_n)$ is then a general point of M. Then, from the fact that:

$$f(\xi_0, \xi_1, \ldots, \xi_n) = 0$$

or, what amounts to the same thing:

$$f\left(1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right) = 0 \text{ on } M,$$

it will follow, since f is homogeneous, that:

$$f(x_0, x_1, \ldots, x_n) = 0 \qquad \text{on } M,$$

and conversely. Hence, all homogeneous equations that are valid for the point ξ will be valid for all points of M, and conversely.

A point is called *normalized* when the first non-zero coordinate is equal to one; any point can be normalized in that way. By renumbering the coordinates, one can indeed assume that $\xi_0 \neq 0$, hence, that $\xi_0 = 1$. The $\xi_1, \ldots, \xi_n$ are then called the inhomogeneous coordinates of ξ.

Uniqueness theorem. *Any two normalized points ξ, η of a manifold M can be mapped to each other by a field isomorphism $K(\xi) \cong K(\eta)$ that fixes the elements of K. The algebraic properties of ξ and η thus agree precisely.*

Proof. From the definition of a general point, all homogeneous algebraic equations that are valid for ξ will also be valid for η, and conversely. Thus, if $\xi_0 = 0$ then one will also have $\eta_0 = 0$, and conversely. If ξ_i is the first non-zero coordinate of ξ then the same will be true for η_i. By a renumbering of the coordinates we can, by means of normalization, deduce that $\xi_0 = \eta_0 = 1$. Any polynomial in $\xi_1, \ldots, \xi_n$ can be made homogeneous by the introduction of ξ_0 factors to the individual terms. We now associate each such polynomial $f(\xi_1, \ldots, \xi_n)$ with the same polynomial in $\eta_1, \ldots, \eta_n$. If $f(\xi_1, \ldots, \xi_n) = g(\xi_1, \ldots, \xi_n)$ then one will have $f(\xi) - g(\xi) = 0$ and this relation, when made homogeneous, will also be true for η, from the original remarks:

$$f(\eta) - g(\eta) = 0, \qquad \text{hence} \quad f(\eta) = g(\eta) .$$

Our association $f(\xi) \rightarrow f(\eta)$ is therefore unique. On the same grounds, it is also valid in the opposite direction. It takes sums to sums and products to products, and is therefore an isomorphism. Moreover, it takes ξ_r to η_r. The isomorphism of the rings $K[\xi_1, \ldots, \xi_n]$ and $K[\eta_1, \ldots, \eta_n]$ thus obtained may be extended to an isomorphism of the quotient fields $K(\xi_1, \ldots, \xi_n)$ and $K(\eta_1, \ldots, \eta_n)$ with no further assumptions. With that, everything is proved.

Converse theorem. *Any point ξ (whose coordinates belong to any extension field of K, e.g., algebraic functions of undetermined parameters) is associated with an (irreducible) algebraic manifold M whose general point is ξ.*

Proof. One can choose a finite basis $(f_1, \ldots, f_r)$ with constant coefficients that has the property that $f(\xi) = 0$ for the totality of all forms $f(x_0, x_1, \ldots, x_n)$ by means of the

HILBERT basis theorem. The equations $f_1 = 0, \ldots, f_r = 0$ define an algebraic manifold M. The given point x is a general point of M. ξ will then belong to M, and all homogeneous equations that are valid for ξ will be consequences of the equations $f_1 = 0, \ldots, f_r = 0$, and will thus be true for all points of M.

On the grounds of the existence and uniqueness theorems, we can define the *dimension* of an irreducible manifold to be the number of algebraically independent coordinates of a normalized, general point ξ of M. One can also call this number the *dimension* of a general point ξ. The dimension of a decomposable manifold M is the highest dimension of the irreducible components, or – what amounts to the same thing – the highest dimension of a point of M. When all of the irreducible components of M have the dimension d, one calls M *purely d-dimensional*.

Dimension theorem. *If M and M' are irreducible and one has $M' \subset M$ then the dimension of M' is less than the dimension of M.*

Proof. We can assume that M', and therefore also M, do not lie in the ideal hyperplane $\eta_0 = 0$; thus, we can normalize a general point ξ of M and a general point ξ' of M' in such a way that $\xi_0 = \xi_0' = 1$. Any relation $f(\xi) = 0$ that is valid for the general point ξ of M can be made homogeneous by the introduction of ξ_0, and will thus be valid for ξ'.

Now, let, say, $\xi_1', \ldots, \xi_{d'}'$ be algebraically independent. Then $\xi_1, \ldots, \xi_{d'}$ will be, as well; hence, one will have $d \geq d'$. If one had $d = d'$ then all ξ_i would be algebraically dependent on $\xi_1, \ldots, \xi_d$. Since M' is a proper submanifold of M there will be a form g that is everywhere zero on M', but not on M. Thus, one has:

$$g(\xi) \neq 0, \qquad g(\xi') = 0.$$

$g(\xi)$ is algebraically dependent on $\xi_1, \ldots, \xi_d$; hence, it is a root of an algebraic equation:

$$a_0(\xi)\, g(\xi)^n + a_1(\xi)\, g(\xi)^{n-1} + \ldots + a_n(\xi) = 0,$$

where the a_ν are polynomials in $\xi_1, \ldots, \xi_d$ and $a_n(\xi) \neq 0$. If one replaces all of the ξ in this equation with ξ' then one will have $g(\xi') = 0$, hence, $a_n(\xi') = 0$, in contradiction to the assumption of the algebraic independence of $\xi_1, \ldots, \xi_{d'}$.

Corollary. *Any point ξ' of M (in the broader sense) has a dimension $d' \leq d$, where d is the dimension of the irreducible manifold M. If $d' = d$ then ξ' will be a general point of M.*

Proof. From the converse theorem, any point ξ' of M is a general point of a submanifold M' of M of dimension d'. From the dimension theorem, one has $d' < d$ for $M' \subset M$, and $d' = d$ for $M' = M$.

Therefore, any point of a zero-dimensional irreducible manifold M is algebraic over K and a general point of M. From the uniqueness theorem, all of these points are equivalent over K. With that, we have:

A zero-dimensional irreducible manifold in S_n is a system of conjugate points relative to the ground field K.

The only n-dimensional manifold in S_n is the entire space S_n. Thus, if ξ is a normalized n-dimensional point of space then one will have $\xi_0 = 1$ and $\xi_1, \dots, \xi_n$ will be algebraically independent over K. There is no relation $f(\xi_1, \dots, \xi_n) = 0$, and therefore also no homogeneous relation $f(\xi_0, \xi_1, \dots, \xi_n) = 0$ with coefficients in K that is identically valid in the ξ, hence, for any point of the entire space S_n.

A pure $(n - 1)$*-dimensional manifold M in S_n will be given through a single homogeneous equation* $h(\eta) = 0$*, and any form that has all points of M for its zero locus will be factorizable through* $h(x)$*.*

Proof. It suffices to carry out the proof for irreducible manifolds, since by multiplying the equations of the irreducible components, one will obtain the equations for a general manifold.

Let M be irreducible and let ξ be a general point. Let – say – $\xi_0 = 1$ and let $\xi_1, \dots, \xi_{n-1}$ be algebraically independent, and let ξ_n be linked to them by the irreducible equation $h(\xi_1, \dots, \xi_n) = 0$. Then, from field theory, any polynomial $f(\xi_1, \dots, \xi_{n-1}, z)$ with the zero locus ξ_n will be factorizable through $h(\xi_1, \dots, \xi_{n-1}, z)$, or – what amounts to the same thing – since one can also replace the algebraically-independent $\xi_1, \dots, \xi_{n-1}, z$ with other indeterminates $x_1, \dots, x_{n-1}, x_n$, any polynomial $f(x_1, \dots, x_n)$ with the zero locus ξ will be factorizable through $h(x_1, \dots, x_n)$. The factorizability remains true when one makes f and h homogeneous by the introduction of x_0. From the definition of a general point, this means that $h(x) = h(x_0, \dots, x_n)$ will have all points of M for its zero locus, and that any form $f(x)$ with this property is factorizable through $h(x)$. Thus, everything is proved.

One also easily proves that, conversely, any non-trivial homogeneous equation $f(\eta) = 0$ defines a purely $(n - 1)$-dimensional manifold. For the proof of this, we decompose the form f into irreducible factors $f_1 f_2 \dots f_r$. From § 19, any irreducible hypersurface $f_\nu = 0$ will possess a general point $(1, u_1, \dots, u_{n-1}, \omega)$ of dimension $n - 1$. Thus, the hypersurface $f = 0$ will decompose into nothing but irreducible components $f_\nu = 0$ of dimension $n - 1$. We thus obtain the theorem:

Any hypersurface $f(\eta) = 0$ *is a purely* $(n - 1)$*-dimensional manifold, and conversely.*

The manifolds of dimension less than $n - 1$ may not be defined by equations so simply. In the next section, we will therefore see that any irreducible d-dimensional manifold may be represented in a certain way as the partial intersection of $n - d$ hypersurfaces.

§ 30. Representation of manifolds as partial intersections of cones and monoids

If ξ is a general point of a d-dimensional irreducible manifold M in S_n then one can assume, with no loss of generality, that that $\xi_0 = 1$ and that $\xi_1, \dots, \xi_d$ are algebraically independent quantities upon which $\xi_{d+1}, \dots, \xi_n$ depend. We assume, moreover, that $\xi_{d+1}, \dots, \xi_n$ are separable algebraic quantities relative to $\mathbb{P} = K(\xi_1, \dots, \xi_d)$, which is always the case when the ground field has characteristic zero.

From the theorem on primitive elements, one can generate the field $\mathbb{P}(\xi_{d+1}, \dots, \xi_n)$ by the adjunction of the single quantity:

$$\xi'_{d+1} = \xi_{d+1} + \alpha_{d+2}\, \xi_{d+2} \;\dots + \alpha_n\, \xi_n \,.$$

We perform a coordinate transformation by which we introduce ξ'_{d+1} instead of ξ_{d+1} as new coordinates, and we omit the prime, from now on. One will thus have $\mathbb{P}(\xi_{d+1}, \dots, \xi_n) = \mathbb{P}(\xi_{d+1})$. The quantities ξ_{d+1}, which are algebraic over $\mathbb{P}$, satisfy an irreducible equation:

$$\varphi(\xi_1, \dots, \xi_d, \xi_{d+1}) = 0,$$

which one can make homogeneous by the introduction of ξ_0:

$$\varphi(\xi_0, \dots, \xi_d, \xi_{d+1}) = 0 \,. \tag{1}$$

The $\xi_{d+2}, \dots, \xi_n$ are rational functions of $\xi_1, \dots, \xi_{d+1}$:

$$\xi_i = \frac{\psi_i(\xi_1, \cdots, \xi_{d+1})}{\chi_i(\xi_1, \cdots, \xi_{d+1})}, \quad (i = d + 2, \dots, n). \tag{2}$$

If one multiplies the denominator χ_i and makes the equation homogeneous through the introduction of ξ_0 then it will follow that:

$$\xi_i\, \chi_i(\xi_0, \dots, \xi_{d+1}) - \psi_i(\xi_0, \dots, \xi_{d+1}) = 0 \,. \tag{3}$$

The $n - d$ equations (1), (3) are valid for the general point ξ of M, and thus for any particular point η of M:

$$\begin{cases} \varphi(\eta_0, \cdots, \eta_{d+1}) = 0, \\ \eta_i \chi_i(\eta_0, \cdots, \eta_{d+1}) - \psi_i(\eta_0, \cdots, \eta_{d+1}) = 0, \quad (i = d + 2, \cdots, n). \end{cases} \tag{4}$$

Henceforth, equations (4) will define an algebraic manifold D that, as we shall show, will contain M as an irreducible component.

Let χ be the least common multiple of the forms χ_i. We will show that all of the points of D for which $\chi \neq 0$ belong to M. Let η be a point with $\chi(\eta) \neq 0$ for which (4) is true. We have to show that η is a relation-preserving specialization of the general point ξ; hence, that $f(\eta_0, \ldots, \eta_n) = 0$ will always follow from $f(\xi_0, \ldots, \xi_n) = 0$ when f is a form.

If one substitutes the values that (3) yields for $\xi_{d+2}, \ldots, \xi_n$ into the equation $f(\xi_0, \ldots, \xi_n) = 0$ then one will obtain:

$$f\left(\xi_0,\cdots,\xi_{d+1},\frac{\psi_{d+2}(\xi)}{\chi_{d+2}(\xi)},\cdots,\frac{\psi_n(\xi)}{\chi_n(\xi)}\right) = 0. \tag{5}$$

It follows from this that the polynomial $g(1, x_1, \ldots, x_{d+1})$ is factorizable through the defining polynomial $\varphi(1, x_1, \ldots, x_{d+1})$ of the algebraic function ξ_{d+1}. The factorizability remains true when this polynomial is made homogeneous by the introduction of x_0:

$$g(x_0, \ldots, x_{d+1}) = \varphi(x_0, \ldots, x_{d+1}) \cdot g(x_0, \ldots, x_{d+1}) .$$

If one now replaces the indeterminates $\xi_0, \ldots, \xi_{d+1}$ with $\eta_0, \ldots, \eta_{d+1}$ then, due to (4), the right-hand side will become zero; hence:

$$g(\eta_0, \ldots, \eta_{d+1}) = 0 .$$

From the manner by which g was defined, this means that:

$$f\left(\eta_0,\cdots,\eta_{d+1},\frac{\psi_{d+2}(\eta)}{\chi_{d+2}(\eta)},\cdots,\frac{\psi_n(\eta)}{\chi_n(\eta)}\right) = 0;$$

however, due to (4):

$$f(\eta_0, \ldots, \eta_{d+1}, \eta_{d+2}, \ldots, \eta_n) = 0 ,$$

which we wished to prove.

The points η of D thus decompose into two classes: The ones with $\chi(\eta) \neq 0$, which belong to M, and the ones with $\chi(\eta) = 0$, which define a proper algebraic submanifold N of D. As a result, D will decompose into the two submanifolds M and N.

Since the $\eta_{d+2}, \ldots, \eta_n$ do not enter into the first equation in (4), it will represent a cone whose vertex can be taken to be an arbitrary point O of the space $\eta_1 = \ldots = \eta_{d+1} = 0$. We choose O in such a way that $\eta_{d+2} \neq 0, \ldots, \eta_n \neq 0$. Any further equation (4) will then represent a hypersurface that has a single intersection point besides O with a general line through O. Such a hypersurface is called a *monoid*.

In the case of a curve in S_3, equations (4) assume the form:

$$\varphi(\eta_0, \eta_1, \eta_2) = 0, \tag{6}$$

$$\eta_3\, \chi(\eta_0, \eta_1, \eta_2) = \psi(\eta_0, \eta_1, \eta_2) . \tag{7}$$

From the foregoing, the intersection of the cone (6) with the monoid (7) will consist of the curve M and a manifold N whose equations are given by (6), (7), and:

(8) $$\chi(\eta_0, \eta_1, \eta_2) = 0 .$$

Tt follows from (7) and (8) that:

(9) $$\psi(\eta_0, \eta_1, \eta_2) = 0 .$$

The pair-wise distinct equations (8), (9) define finitely many ratios $\eta_0 : \eta_1 : \eta_2$, thus, finitely many lines through the point $O(0, 0, 0, 1)$. If one eliminates from these lines the one that does not lie on the cone (6) then the remaining ones will define the manifold N.

The complete intersection of the cone (6) *with the monoid* (7) *thus consists of the curve M and finitely many lines through the point O.*

The representation by cones and monoids is most meaningful for the theory of space curves. HALPHEN ([1]) and NOETHER ([2]) have made it the foundation of their classification of algebraic space curves. SEVERI ([3]) has recently examined the monoidal representation of higher algebraic manifolds and utilized it for the theory of equivalence families on algebraic manifolds.

§ 31. The effective decomposition of a manifold into irreducible ones by means of elimination theory

Let a manifold M be given by a homogenous or inhomogeneous system of equations:

(1) $$f_i(\eta_1, \ldots, \eta_n) = 0 .$$

We are free to interpret the η as either inhomogeneous coordinates in affine space A_n or, in the case of homogeneous f_i , as homogeneous coordinates in a projective space S_{n-1} . Thus, we temporarily call any system of values $\eta_1, \ldots, \eta_n$ simply a "point," which is therefore based in the affine meaning. We can assume that the polynomial f_1 does not vanish identically.

In order to find all of the solutions of (1), one can – and this is the basic principle of elimination theory – successively eliminate $\eta_n, \ldots, \eta_1$ in (1) by constructing resultants. If the system of resultants $R_i(\eta_1, \ldots, \eta_{n-k})$ is identically zero after k steps then the $\eta_1, \ldots, \eta_{n-k}$ can be chosen arbitrarily, and (1) will have an $(n - k)$-dimensional solution manifold.

These simple basic principles will now be complicated by three kinds of circumstances: First, one will obtain not only the irreducible components of the highest dimension $n - k$ of the manifold M, but also all of the components of lower dimension. Thus, one may not go so far as to say that a system of resultants is identically zero, but rather, before any elimination step can be carried out, one must remove the greatest common divisor; the remaining polynomials will then remain relatively prime, and their

([1]) HALPHEN, G.: J. Ec. Poly., v. 52 (1882), pp. 1-200.
([2]) NOETHER, M.: J. reine angew. Math., v. 93 (1882), pp. 271-318.
([3]) SEVERI, F: Mem. Acad. Ital., v. 8 (1937), pp. 387-410.

system of resultants cannot be zero. Second, before any elimination step can be performed one must insure, by a linear coordinate transformation, that the highest power of the variable to be eliminated in one of the forms appears with a non-zero *constant* coefficient; only under these assumptions will the resultant theory be valid (cf., Chap. 2, § 15). Third, in order to put the equations of the manifolds obtained into a useful and formally beautiful form, following LIOUVILLE, one appropriately introduces one more unknown into the unknowns $\eta_1, \ldots, \eta_n$, namely:

$$\zeta = u_1 \eta_1 + \ldots + u_n \eta_n , \tag{2}$$

in which the $\eta_1, \ldots, \eta_n$ are indeterminates. One thus considers not only the equations (1), but also the system of equations (1), (2). When the η and ζ have been replaced with the indeterminates y and z, the left-hand sides of these equations will have no common divisors, since the linear polynomial $z - u_1 y_1 - \ldots - u_n y_n$ will enter into none of the polynomials $f(y_1, \ldots, y_n)$. This relative primeness thus guarantees that the following first step does not yield an identically vanishing resultant system.

The stepwise elimination of $\eta_1, \ldots, \eta_n$ will now be carried out in the following way:

Step 1. By an appropriate linear transformation:

$$\begin{aligned} \eta'_k &= \eta_k + v_k \, \eta_n \qquad (k = 1, \ldots, n-1), \\ \eta'_n &= v_n \, \eta_n , \end{aligned}$$

in which the v_k are suitably chosen constants, one can insure that the term $\eta_n'^{\rho}$, where ρ is the degree of f_1 in η, appears with one of the non-zero coefficients ([1]). The $u_1, \ldots, u_n$ in (2) will then be transformed accordingly, in such a way that $u_1\eta_1 + \ldots + u_n\eta_n$ will remain unchanged. After performing the transformation, the primes on η', u' may again be omitted.

Therefore, the resultant system of the system of equations (1), (2) for η_n will be defined by:

$$g_j(u_1, \ldots, u_n, \eta_1, \ldots, \eta_{n-1}, \zeta) = 0 . \tag{3}$$

Since equation (2) is homogeneous in $\zeta, u_1, \ldots, u_n$, the same will be true for the g_j.

One now replaces $u_1, \ldots, u_{n-1}, \zeta$ with the indeterminates $y_1, \ldots, y_{n-1}, z$, and the greatest common divisor $h(u, y, z)$ of the forms $g_j(u, y, z)$ defines the *first sub-resultant* of the system (1). As we already stipulated, the factor $h(u, y, z)$ must be removed from the g_j, and thus the second elimination step does not give a result that is identically zero. We therefore set:

$$g_j(u, y, z) = h(u, y, z) \cdot l_j(u, y, z), \tag{4}$$

([1]) The coefficient of $\eta_n'^{\rho}$ in the transformed polynomial f_1 will be equal to $f_1(v_1, \ldots, v_n)$, hence, for a suitable choice of $v \neq 0$ (in the ground field or an algebraic extension field).

in which the l_j are relatively prime polynomials. Any solution (η, ζ) of (1), (2) is simultaneously a solution of (3); hence, one will have either:

(5) $$h(u, y, z) = 0$$
or:
(6) $$l_j(u, y, z) = 0.$$

We will later see that the solutions of (5) yield precisely the purely $(n - 1)$-dimensional components of M, whereas those of (6) yield the components of lower dimension.

From (6), we would like to define a further system of equations that is independent of the ζ and u. To that end, we construct the resultant system $r_k(u, y)$ of the $l_j(u, y, z)$ for z, order the $r_k(u, y)$ in powers of u, and define the coefficients $e_j(y_1, \ldots, y_{n-1})$ of these products of powers. They do not all vanish identically, since the $l_j(u, y, z)$ are relatively prime. Any solution of (6) will then be simultaneously a solution of:

$$r_k(u, y) = 0,$$

and thus when $\eta_1, \ldots, \eta_n$ are independent of u ([1]) it will also be a solution of:

(7) $$e_j(\eta_1, \ldots, \eta_{n-1}) = 0.$$

Hence: Any solution (h, z) of (1), (2) in which the $\eta_1, \ldots, \eta_n$ are independent of u will also be a solution of either (5) or (6) and (7).

Conversely: Any solution $(\eta_1, \ldots, \eta_{n-1}, \zeta)$ of (5) or of (6) and (7) is also a solution of (3), and one can determine the η_n in it in such a way that one will obtain a solution of (1) and (2). Thus, if $\eta_1, \ldots, \eta_{n-1}$ are independent of u then so will η_n be, since η_n must satisfy an algebraic equation $f_1(\eta) = 0$ in which the term η_n^0 does not actually appear. Hence, for given $\eta_1, \ldots, \eta_{n-1}$, it can be only one of the finitely-many roots of this equation.

Step 2. One proceeds with equations (6), (7) exactly as one did for (1), (2). After an appropriate transformation of $\eta_1, \ldots, \eta_{n-1}$ [which is possible, because the $e_j(\eta_1, \ldots, \eta_{n-1})$ do not all vanish identically], one eliminates η_{n-1} from (6), (7), by which, one will obtain:

(8) $$g'_j(u, \eta_1, \ldots, \eta_{n-1}, \zeta) = 0,$$

splits off the greatest common divisor from the polynomials g'_j – the *second sub-resultant* $h'(u, y, z)$:

(9) $$g'_j(u, y, z) = h'(u, y, z) \cdot l'_j(u, y, z),$$

([1]) I.e., algebraic over the original constant field K, or also algebraic functions of other indeterminates, but not of $u_1, \ldots, u_n$.

further defines the resultant system of the l'_j for z, and obtains a system of equations:

$$l'_j(u_1, \ldots, u_n, \eta_1, \ldots, \eta_{n-2}, \zeta) = 0, \tag{10}$$

$$e'_j(\eta_1, \ldots, \eta_{n-2}) = 0 \tag{11}$$

by setting the $u_1, \ldots, u_n$ to zero identically.

Then again, this means that h' is a homogeneous form in $z, u_1, \ldots, u_n$, that the e'_j do not vanish identically, that any solution of (6) and (7) for which the $\eta_1, \ldots, \eta_{n-1}$ are independent of u will either be a solution of (10), (11) or a solution of:

$$h'(u, h, z) = 0, \tag{12}$$

and that conversely any such solution of (10), (11) or of (12) will give rise to a solution of (6) and (7), in which the η_{n-1} will also be independent of u.

One proceeds in this way until all of the η are eliminated. Since the process is arranged in such a way that the e_j, e'_j, … do not vanish identically, the final $e_j^{(n-1)}$ will be non-zero constants, and the final system of equations $e_j^{(n-1)} = 0$ will therefore be *contradictory*. The final sub-resultant $h^{(n-1)}(u, z)$ will contain only the u_i and z. If the original equations (1) are homogeneous in $\eta_1, \ldots, \eta_n$ then the resultants $h, h', \ldots, h^{(n-1)}$ will also be homogeneous in $y_1, \ldots, y_n, z$.

Any solution of (1), (2) for which the η do not depend on the u will be a solution of (5) or of (6) and (7); any such solution of (6), (7), will, however, again be a solution of (12) or of (10) and (11), etc., up until the repetition of the latter alternative ultimately leads to a contradictory system of equations. One must therefore choose the former alternative; i.e., there must be a vanishing sub-resultant. We thus have:

Theorem 1. *Any solution* (η, ζ) *of the system* (1), (2) *in which the* η *do not depend upon the u will simultaneously be a solution of one of the equations:*

$$h(u, \eta, \zeta) = 0,\ h'(u, \eta, \zeta) = 0, \ldots, h^{(n-1)}(u, \zeta) = 0\,. \tag{13}$$

Conversely, any solution $(\eta_1, \ldots, \eta_{n-r}, \zeta)$ of the r^{th} equation (13) can be completed to a solution of equations (1), (2), and if $\eta_1, \ldots, \eta_{n-r}$ are either constants or new indeterminates that are independent of the u then the remaining $\eta_{n-r+1}, \ldots, \eta_n$ will likewise not depend on the u. Hence, one has:

Theorem 2. *Any solution* ζ *of the* r^{th} *equation* (13) *for given (constant or indeterminate)* $\eta_1, \ldots, \eta_{n-r}$ *has the form:*

$$\zeta = u_1\,\eta_1 + \ldots + u_n\,\eta_n\,, \tag{14}$$

in which the η_k *are independent of the* u_i *and define a solution of* (1).

Any individual sub-resultant $h(u, y, z)$ or $h'(u, y, z)$ … can be further decomposed into irreducible factors. In order to have something specific in mind, we consider, e.g., the second sub-resultant:

$$h(u, y_1, .., y_{n-2}, z) = \Theta(y, u) \prod_{\mu} h'_{\mu}(u, y, z)^{\sigma_{\mu}} .$$

Should factors of $\Theta(y, u)$ that do not depend upon z appear during the decomposition, these factors can remain out of consideration, since they can never be zero for constant $\eta_1, \ldots, \eta_{n-2}$. If they were zero then they would be zero for arbitrary ζ, not just for ones of the form (14), which contradicts theorem 2.

On formal grounds, we replace the $y_k, \ldots, y_{n-2}$ in each factor h'_{μ} with new indeterminates $\xi_1, \ldots, \xi_{n-2}$. In a suitable algebraic extension field of $K(u, \xi)$, $h'_{\mu}(u, \xi, z)$ will decompose completely into linear factors $z - \zeta$, by which, from theorem 2, the zero loci ζ will all have the form (14) with $\eta_1 = \xi_1, \ldots, \eta_{n-2} = \xi_{n-2}$:

(15) $$h'_{\mu}(u, \xi, z) = \gamma_n \prod_{\nu} (z - u_1 \xi_1 - \cdots - u_{n-2} \xi_{n-2} - u_{n-1} \xi_{n-1}^{(\nu)} - u_n \xi_n^{(\nu)}) .$$

Thus, the various $\xi^{(r)}$ will be conjugate to each other relative to $\mathbb{P}(u, \xi)$, i.e., all systems of values $\xi^{(r)}$ will go over to one of them $\xi = \xi^{(1)}$ by field isomorphisms (i.e., they will be equivalent to ξ). If $\xi_1, \ldots, \xi_{n-2}$ are indeterminates and ξ_{n-1}, ξ_n are algebraic functions of them then this ξ will be an $(n-2)$-fold indeterminate point of the manifold M. The factor γ_{μ} will depend upon only $\xi_1, \ldots, \xi_{n-2}$, and we will no longer need to concern ourselves with it.

If we substitute the value (14) for ζ in $h'_{\mu}(u, \xi, z)$, develop in powers of products of the u_i, and set all of the individual coefficients to zero then we will obtain a system of equations:

(16) $$h'_{\mu 1}(\eta) = 0, \ldots, h'_{\mu m}(\eta) = 0$$

that define an algebraic manifold M'_{μ}. Theorems 1 and 2 now imply that the manifold M that is defined by (1) will be the union of all manifolds $M_{\mu}, M'_{\mu}, \ldots$ that are defined by the irreducible factors of the successive sub-resultants $h, h', \ldots$, by (16). We will see that all of these manifolds $M_{\mu}, M'_{\mu}, \ldots$ are irreducible, and that the point ξ that was defined above represents a general point of M'_{μ}, in the sharpened sense that *all* (not just homogeneous) equations that are valid for the point ξ will be valid for all points of M'_{μ}.

It is next clear that ξ is a point of M'_{μ}. Furthermore, it follows from the derivation of theorem 2 that the point η of M'_{μ}, or – what amounts to the same thing – the solutions (η, ζ) to the equation $h'_{\mu}(u, \eta, \zeta) = 0$ with $\zeta = u_1 \eta_1 + \ldots + u_n \eta_n$, will simultaneously be solutions of (7) and (1), that therefore η_{n-1} and η_n will be coupled with $\eta_1, \ldots, \eta_{n-2}$ by algebraic equations in which terms in η_{n-1} (η_n, resp.) actually appear. Thus, M'_{μ} will indeed contain the $(n - 2)$-dimensional point ξ, but nothing more than $(n - 2)$-dimensional points (1).

Lemma. *When a manifold M^* contains the point ξ of transcendence degree* $(n - 2)$, *but no points of higher transcendence degree, then when the aforementioned elimination process is applied to M^* it will yield a constant for the first sub-resultant, while the second sub-resultant will contain the factor h'_μ. M^* will thus include the manifold M'_μ that is defined by* (16).

Proof. If it would give a non-constant first sub0resultant then M^* would also contain a point ξ of transcendence degree $n - 1$, contradicting the assumption. The point ξ, however, lies on M^* and thus $\zeta = u_1\,\eta_1 + \ldots + u_n\,\eta_n$ will either be a zero locus of the second one or a higher sub-resultant. Since the higher sub-resultants only yield points of transcendence degree $< n - 2$, $\zeta = u_1\,\eta_1 + \ldots + u_n\,\eta_n$ must be a zero locus of the second sub-resultant $h'^*(u, \xi_1, \ldots, \xi_{n-2}, z)$. However, $h'^*(u, \xi_1, \ldots, \xi_{n-2}, z)$ must include the entire irreducible factor $h'_\mu(u, \xi_1, \ldots, \xi_{n-2}, z)$, whose zero locus is ζ.

Now, we can finally prove:

Theorem 3. *The submanifold M'_μ that is defined by* (16) *is irreducible and has the point ξ for its general point.*

Proof. The point ξ obviously belongs to M'_μ. We thus have only to prove that any equation $f(\xi) = 0$ with coefficients in K that is valid for the point ξ will also be valid for all of the points η of M'_μ.

The equations of M'_μ, together with the equation $f(\eta) = 0$, define a manifold M^* that is contained in M'_μ and contains ξ; hence, the assumptions of the lemma are satisfied. It follows that M^* contains the manifold M'_μ, and therefore that all of the points of M'_μ will, in fact, satisfy the equation $f(\eta) = 0$.

In the formulation and proof of theorem 3 we considered the case of a manifold M'_μ that arises from the second sub-resultant h' as only an example. It is self-explanatory that the consequences will persist precisely for any other sub-resultant, as long as the dimension of M'_μ is not $n - 2$, but any other number $n - 1, n - 3, \ldots, 1, 0$.

The elimination process that was just described thus provides – in the form of (16) – the equations of the irreducible manifolds M_μ, M'_μ, … of dimensions $n - 1, n - 2, \ldots, 0$, from which M is comprised; however, it will likewise provide a general point ξ for each of these manifolds. In order to obtain the *minimal decomposition of M into irreducible*

(1) Since we are looking at things from the standpoint of affine spaces A_n, we understand the dimension of a point to mean the number of algebraically independent coordinates (not coordinate ratios) for the point.

manifolds, one needs only to discard the manifolds $M'_\mu, M''_\mu, \ldots$ that are already included in another manifold of higher dimension M_λ or M'_λ. A criterion for – e.g. – a M''_μ to be contained in M'_λ is that the general point of M''_μ must satisfy the equations of M'_λ. Another criterion is that the elimination process, when applied to the equations of M'_λ and M''_μ together, must yield a power of h''_μ for the second resultant, and not a constant.

Likewise, our investigation teaches us how one obtains the equations of the irreducible manifold M_ξ for a given general point $(\xi_1, \ldots, \xi_n)$. We formulate this result as:

Theorem 4. *If $\xi_{d+1}, \ldots, \xi_n$ are complete algebraic functions* ([1]) *of the algebraically independent $\xi_1, \ldots, \xi_d$, and furthermore $u_1, \ldots, u_n$ are indeterminates, and $\zeta = u_1 \xi_1 + \ldots + u_n \xi_n$, as an algebraic function of $\xi_1, \ldots, \xi_n$, $u_1, \ldots, u_n$, is the zero locus of a polynomial $h(u, \xi, z) = h(u_1, \ldots, u_n, \xi_1, \ldots, \xi_n, z)$, then one will obtain the equations of an irreducible manifold M_ξ when one develops:*

$$h(u, h, u_1 \xi_1 + \ldots + u_n \xi_n)$$

in powers of u_i and sets all of the coefficients of these products of powers to zero. One will obtain the finitely many values $\eta_{d+1}, \ldots, \eta_n$ that belong to the given $\eta_1, \ldots, \eta_d$ from the zero locus $\zeta = u_1 \xi_1 + \ldots + u_n \xi_n$ of the polynomial $h(u, \eta, z)$.

This $h(u, \eta, z)$ is, in fact, then h'_μ of the results above.

If the equations (1) are homogeneous then they will represent a cone manifold that includes, not only any point $(\eta_1, \ldots, \eta_n)$ that is different from the origin O, but also all of the points $(\lambda\eta_1, \ldots, \lambda\eta_d)$ of a line through O. The irreducible components of the manifold (1) will also be cone manifolds. If one now interprets the lines through the origin as points of the projective space S_{n-1} then each d-dimensional cone manifold with $d > 0$ will yield a $(d - 1)$-dimensional manifold in S_{n-1}. Nothing will change in the formulas of this section, only their interpretation, and the dimension numbers will be lowered by 1.

The developments of this section obviously yield new proofs for the possibility of decomposing manifolds into irreducible ones, the existence of general points of irreducible manifolds, and the unique determination of manifolds by one of their general points. We ultimately prove:

Theorem 5. *An irreducible d-dimensional manifold M will remain purely d-dimensional under an arbitrary extension of the ground field K. A finite algebraic extension of K will suffice for the decomposition of M into absolutely irreducible manifolds, i.e., into ones that remain irreducible under further extensions of the ground field.*

([1]) This means that any of the quantities $\xi_{d+1}, \ldots, \xi_n$ satisfy an equation with constant coefficients in $K[\xi_1, \ldots, \xi_d]$ whose highest coefficient is equal to one.

Proof. From theorem 4, the equations of the manifold read:

(18) $$h(u, h, u_1 \xi_1 + \ldots + u_n \xi_n) = 0 \qquad \text{identically in } u.$$

The polynomial $h(u, \xi, z)$ is irreducible over K. Under an extension of K, $h(u, \xi, z)$ decomposes into conjugate factors:

$$h(u, \xi, z) = \prod_\nu h_\nu(u, \xi, z).$$

The manifold (18) will thus decompose into manifolds M_ν with the equations:

(19) $$h_\nu(u, \eta, u_1 \xi_1 + \ldots + u_n \xi_n) = 0 \qquad \text{identically in } u.$$

Any M_ν will belong to a polynomial $h_\nu(u, \xi_1, \ldots, \xi_d, z)$ in the same way that the original M belonged to h. Any M_ν will thus be an irreducible d-dimensional manifold.

From § 12, a finite algebraic extension of K will suffice in order to decompose the polynomial $h(u, \xi, z)$ completely into absolutely irreducible factors that do not decompose any more under further field extensions. The associated manifolds M_ν will then be also absolutely irreducible, from what was said above.

The absolutely irreducible factors $h_\nu(u, \xi, z)$ of $h(u, \xi, z)$ will be conjugate relative to K. Hence, the associated absolutely irreducible manifolds M_ν will be conjugate over K.

Appendix to Chapter Four

Algebraic manifolds as topological structures

From the standpoint of topology, the complex projective space S_n is not an n-dimensional manifold, but a $2n$-dimensional one, since its points in the neighborhood of a fixed point will depend upon n complex, hence, $2n$ real parameters. Likewise, as we will see, any d-dimensional algebraic manifold will be $2d$-dimensional in the eyes of topology.

The topology of algebraic manifolds is being thoroughly examined in the present era, especially by LEFSCHETZ. In this introduction, we can treat only the most general principles ([1]). We confine ourselves to the essentials of the proof that d-dimensional algebraic manifolds are $2d$-dimensional complexes, in the sense of topology, i.e., that they can be decomposed into finitely many curvilinear $2d$-dimensional simplexes.

Before we go on to the multi-dimensional case, we would like to treat the case of an algebraic curve in the complex projective plane. We would like to show that such a curve can be decomposed into finitely-many curvilinear triangles (viz., topological images of real rectilinear triangles), any two of which will have either one side or one

([1]) For far-reaching inverstigations, see S. LEFSCHETZ: *l'Analysis situs et la géométrie algébrique*, and B. L. VAN DER WAERDEN: "Topologische Begründung der abzählenden Geometrie," Math. Ann. **102** (1929), pp. 337, and O. ZARISKI: *Algebraic Surfaces*. Ergebn. Math. v. 3, 1935, issue 5.

vertex in common. Thus, we must assume that the elements of function theory are known.

Let the equation for the curve be regular in η_2 :

$$f(\eta_0, \eta_1, \eta_2) = \eta_2^n + a(\eta_0,\eta_1)\eta_2^{n-1} + \cdots + a_n(\eta_0,\eta_1)\,. \tag{1}$$

Any ratio $\eta_0 : \eta_1$ is associated with infinitely many values of η_2. The ratios $\eta_0 : \eta_1$ can be interpreted as points on a GAUSSIAN number sphere. As we know, there are then finitely many *critical points* on the number sphere, which are found by setting the discriminant of equation (1). We now subdivide the number sphere into curvilinear triangles, and indeed in such a way that the critical points thus appear as vertices and the points $\eta_0 = 0$ and $\eta_1 = 0$ do not lie in the same triangle. If, in such a triangle, one has, say, $\eta_0 \neq 0$ – i.e., the point $\eta_0 = 0$ lies outside the triangle – then one will normalize the coordinates so that $\eta_0 = 1$. The n roots $\eta_2^{(\nu)}$ of equation (1) are regular analytical functions of η_1 in the neighborhood of each point of the triangle. Since the triangle is simply connected, one can assume that these n functional elements will be single-valued over the entire triangle: there will thus be n single-valued analytic functions $\eta_1^{(1)},\ldots,\eta_2^{(n)}$ in the entire triangle. $\eta_1^{(1)},\ldots,\eta_2^{(n)}$ will be then regular and analytic on the sides of the triangle. The regular character can break down only at the critical vertices: thus, the functions will remain continuous there.

If one now selects any of these analytic functions $\eta_2^{(\nu)}$ on a triangle Δ then one can map the points (η_0, η_1, $\eta_2^{(\nu)}$) of the complex curve to the points (η_0, η_1) of the triangle Δ in a one-to-one and continuous way. They will thus define a curvilinear triangle $\Delta^{(\nu)}$ out of complex curves. Any triangle Δ will be associated with n such triangles $\Delta^{(\nu)}$, and all of these triangles will collectively cover the entire curve, since equation (1) can have no other solutions than the $\eta_2^{(\nu)}$. If two triangles Δ and Δ' abut on the sphere then any function $\eta_2^{(\nu)}$ on one of the triangles will agree with one of the functions $\eta_2^{(\nu)}$ on the other triangle on the common side; i.e., the triangles $\Delta^{(\nu)}$ and $\Delta'^{(\nu)}$ will have a common side. In all of the other cases, two triangles on the complex curve will have at most common vertices, and by a further subdivision of the triangle, one can arrange that any two of them will have at most one common vertex. With that, we have found the desired triangulation of the complex curve.

It is clear from the construction that any side will lie on precisely two triangles. If we now consider all triangles that have one vertex common E, along with all of their sides that go through E, then one can go from any such triangle to a neighboring triangle over such a side, etc., until one comes back to the initial triangle. In this way, the triangles that border on E will define one or more “wreaths.” If $\Delta_1^{(\nu)}, \Delta_2^{(\nu')}, \ldots, \Delta_h^{(\nu'')}$ is one such wreath then it can clearly be the case that the series of associated triangles Δ_1, Δ_2, … on the sphere will already be closed. Thus, whereas the wreath $\Delta_1^{(\nu)}, \Delta_2^{(\nu')}$, … will be completed once, the corresponding wreath Δ_1, Δ_2, … on the sphere will be completed perhaps k times on many occasions.

One sees that this k-fold completion of the wreath $\Delta_1, \Delta_2, \ldots$ on the sphere tallies completely with the k-fold circumscribing of a critical point, through which, we defined the cycles or branches of the curve in § 14. *Thus, each branch of a critical locus will correspond to a wreath of triangles around a point E on the complex curve.*

The triangles $\Delta^{(\nu)}$ will define a topological "surface" that possesses singular points wherever one vertex carries several wreaths. If one resolves each such point into several points that each carry *one* wreath of triangles then one will obtain a non-singular topological surface that one calls the *Riemann surface* of the curve. It now follows from the previous statements that the points of the RIEMANN surface will be in one-to-one correspondence with the branches of the curve.

Here, we shall go no further into the theory of RIEMANN surfaces, but only refer to the booklet of H. WEYL, *Die Idee der RIEMANNschen Fläche*, Berlin, 1923.

In order to go on to the n-dimensional case, we next prove an algebraic:

Lemma. *If M ($\neq S_n$) is an irreducible algebraic manifold in complex S_n, and one insures, by a linear coordinate transformation, that one of the equations $F(\eta) = 0$ on M is regular in η_n then M will possess a projection M' onto the subspace S_{n-1} with the equation $\eta_n = 0$, in such a way that each point $\eta'(\eta_0, \ldots, \eta_{n-1}, 0)$ of M' will correspond to at least one point $\eta(\eta_0, \ldots, \eta_{n-1}, \eta_n)$ of M. M' is again an algebraic manifold. If one selects a particular proper submanifold that belongs to N' or M' then for a given η' the coordinates η_n of the associated point η of M will be found by a solving an algebraic equation $e(\eta', \eta_n) = 0$ that is rational in η' and integral in η_n, and which will have nothing but distinct roots for all η' on $M' - N'$.*

Proof. The equations of the projection M' are obtained by eliminating η_n from the equations of M. The irreducibility of M' follows from the first irreducibility criterion (§ 28); if it is true that when a product $f((\eta_0, \ldots, \eta_{n-1})\, g(\eta_0, \ldots, \eta_{n-1})$ is zero for all points of M' then it will also be zero for all points of M, then it will be true that if a factor is zero on M then it will also be zero on M'. (In the case of $d = n - 1$, M' fills all of S_{n-1}.)

A general point ξ' of M' corresponds to a finite number of points ξ of M. The coordinates ξ_n of this point are solutions of an algebraic equation that one finds in the following way: One substitutes the coordinates $\xi_0, \ldots, \xi_{n-1}$ for the $\eta_0, \ldots, \eta_{n-1}$, resp., and an indeterminate z for η_n in the equations $f_\nu = 0$ for M, and finds the greatest common divisor of $d(\xi, z)$ of the polynomial $f_\nu(\xi, z)$ thus obtained. One will then have:

$$(1) \qquad \begin{cases} f_\nu(\xi, z) = g_\nu(\xi, z) e(\xi, z), \\ d(\xi, z) = \sum h_\nu(\xi, z)\, f_\nu(\xi, z), \end{cases}$$

in which d, g, and h_ν are rational in $\xi_0, \ldots, \xi_{n-1}$ and integral in z. It follows from (1) that the intersection of the zero loci ξ_ν of the polynomials $f_\nu(\xi, z)$ will be precisely the zero locus of $d(\xi, z)$.

We now rid $d(\xi, z)$ of multiple factors by taking the greatest common divisor of $d(\xi, z)$ and the derivative $d'(\xi, z)$ and dividing $d(\xi, z)$ by this greatest common divisor. The resulting polynomial, which can be assumed to be integral in $\xi_0, \ldots, \xi_{n-1}$, will be called

$e(\xi, z)$; its degree will be h. $e(\xi, z)$ will be then a divisor of $d(\xi, z)$, or a power of $e(\xi, z)$ will be divisible by $d(\xi, z)$. Thus, it will follow from (1) that:

$$\begin{cases} f_\nu(\xi, z) = a_\nu(\xi, z) e(\xi, z), \\ e(\xi, z)^\rho = \sum b_\nu(\xi, z) f_\nu(\xi, z). \end{cases} \tag{2}$$

One infers from (2) that the intersection of the zero loci ξ_ν of $f_\nu(\xi, z)$ is precisely the zero locus of $e(\xi, z)$. This will remain true for any specialization of ξ, as long as the denominators in $g_\nu(\xi, z)$ and $h_\nu(\xi, z)$ are not zero.

Now, let $p(\xi)$ be the product of these denominators, multiplied by the discriminant of $e(\xi, z)$ and the coefficient of the highest power of z in $e(\xi, z)$. Then, by a specialization $\xi_0 \to \eta_0, \ldots, \xi_{n-1} \to \eta_{n-1}$ the polynomial $e(x, z)$ will always have h distinct roots, and indeed precisely the same roots as all of the $f_\nu(\xi, z)$, as long as $p(\eta)$ remains $\neq 0$. Instead of $e(\xi, z)$ and $p(\eta)$ we can also write $e(\eta', z)$ and $p(\eta')$, since neither of them depend upon η_n .

The equation $p(\eta') = 0$, together with the equations of M', will define a proper submanifold N' of M'. Thus, if η' is a point of $M' - N'$ then one will have $p(\eta') \neq 0$, and the associated points η on M will be exactly the solutions of the equation $e(\eta', \eta_n) = 0$. With that, the lemma is proved.

If a system of several manifolds M of the highest dimension r is given then one can apply the lemma to all of the r-dimensional irreducible components M_i of this manifold. The associated projections M_i' will all have dimension r, and therefore the N_i' will have dimensions $< r$. The intersections D_{ih} of any two irreducible manifolds M_i and M_h will likewise all have projections D_{ih}' of dimensions $< r$. If one now selects from the points of N_i', the point η', which belongs to one of the D_{ih}', then the roots of the equation $e_i(\eta', \eta_n) = 0$ will not only be distinct from each other, but also from the roots of the remaining equations $e_h(\eta', \eta_n) = 0$; usually, a point η must belong to M_i as well as M_h , hence, to D_{ih}, and therefore η' must belong to D_{ih}'.

The union of all the D_{ih}' and N_i' will be called V'. It then follows that:

If one selects from the manifold M those points whose projections η' belong to a manifold V' of dimension $< r$ then all of the remaining points will be found to be solutions of equations $e_i(\eta', \eta_n) = 0$ with nothing but distinct roots, while $\eta'(\eta_0, \ldots, \eta_{n-1}, 0)$ will range over a manifold M_i' in S_{n-1} .

Meanwhile, the points η of the manifold M whose projections η' belong to V' define a submanifold Q of dimension $< r$. If one applies the same theorem to the manifold Q once again and repeats the process until one arrives at a manifold of dimension zero then one will ultimately obtain a *complete decomposition of M into pieces of varying dimensions such that any piece is determined by an equation $e(\eta', \eta_n) = 0$ in the aforementioned way, where η' meanwhile ranges over a piece of the projection M'. The* *pieces of the projection are, at the same time, differences $U' - V'$, where U' and V' are algebraic manifolds.*

We now go from complex projective to real Euclidian spaces.

A *simplex* X_r in A_n will be defined thus: An X_0 is a point, an X_1 is a line segment, an X_2 is a triangle. An X_{r+1} consists of all of the points of an X_r that are connected by line segments to a fixed point outside of the linear space that X_r belongs to. An X_r has $r + 1$ vertices, and any $s + 1$ of them define a *side* X_s of X_r when $s \leq r$. A topological image of a simplex is called a *curvilinear simplex*, and will likewise be denoted by X_r. A union of finitely many (rectilinear or curvilinear) simplexes X_r, any two of which have either nothing in common or an entire side (along with its sides), is called a (rectilinear or curvilinear) *r-dimensional polyhedron.* A triangulation of a region of space is a subdivision of this region into curvilinear simplexes, any two of which have either nothing in common or exactly one side in common.

Theorem 1. *Let there be given finitely many algebraic manifolds M and a ball K:*

$$\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2 \leq a^2$$

in real A_n. There is then a triangulation of the ball by which the manifolds M, as long as they lie in the ball, consist entirely of sides of the triangulation.

Proof. 1. For $n = 1$, the ball is a line segment, and any manifold M ($\neq A_1$) consists of finitely many points. These points decompose the line segment into sub-segments. Thus, the desired triangulation has already been found.

2. The theorem may thus be assumed to be true for the space A_{n-1}. We begin with the sphere for the manifolds M. By an orthogonal transformation, one can arrange that each manifold M possesses an equation $F(\eta_1, \ldots, \eta_n) = 0$ that is regular in η_n .On the basis of the lemma, we then define the projections M' of the M onto the subspace A_{n-1} and decompose them, as above, into pieces $U' - V'$. We apply the induction hypothesis to the algebraic manifolds U' and V' and to the ball $\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2 \leq a^2$. There is a triangulation of this ball by which each of the U' and V' (as long as they lie in the ball) will consist of simplexes of the triangulation. Any point set $U' - V'$ will thus be obtained when one discards the simplexes that comprise V' from the simplexes that comprise U'. What remain will be the interior points of certain simplexes (of varying dimensions) of the triangulation.

The logic of the following proof can be sketched out in the following way: The points η of the ball whose projections η' belong to a simplex X'_r of the triangulation define a cylindrical point set. They will be subdivided into "blocks" by means of the various manifolds M that they consist of, which prove to be curvilinear polyhedra. If one decomposes them into curvilinear simplexes then one will obtain the desired triangulation of the entire ball.

3. In order to carry out this manner of proof, we consider the interior points η' of a simplex X'_r that belongs completely to $U' - V'$. Certain points η of the manifold M lie over η', whose coordinates η_n will be found by solving the equation $e(\eta', \eta_n) = 0$. This equation will have the same degree h for any η' in X'_r, as well as the same number of distinct (complex) roots. However, the number of real roots of the equation must also be constant. Thus, by a continuous change of the points η' one can go from a pair of real roots only to a complex conjugate pair if the pair coincides from time to time.

Therefore, let the real roots of the equation $e(\eta', \eta_n) = 0$ be ordered by increasing magnitude:

(3) $$\eta_n^{(1)} < \eta_n^{(2)} < \cdots < \eta_n^{(l)} .$$

From the theorem of the continuity of the roots of algebraic equations, the $\eta_n^{(1)}$, …, $\eta_n^{(l)}$ are continuous functions of η' inside of X'_r.

We now examine the behavior of the functions $\eta_n^{(1)}$, …, $\eta_n^{(l)}$ in the vicinity of the boundary of the simplex. If η' is close to a boundary point ζ' of X'_r then the $\eta_n^{(1)}$, …, $\eta_n^{(l)}$ will be, in any case, restricted as roots of the equation $F(\eta_1, …, \eta_n) = 0$. Now, if $\eta_n^{(k)}$ were not close to a boundary value ζ_n then one could choose two convergent sequences with different limiting values:

$$\begin{aligned} &\eta'(\nu) \to \zeta', &\qquad &\eta_n^{(k)}(\nu) \to \zeta_n , \\ &\tilde{\eta}'(\nu) \to \zeta', &\qquad &\tilde{\eta}_n^{(k)}(\nu) \to \tilde{\zeta}_n \neq \zeta_n . \end{aligned}$$

Now, one can connect $\eta'(\nu)$ to $\tilde{\eta}'(\nu)$ by a line segment in the neighborhood of the limit point ζ'. If one then moves η' on this line segment $\mathfrak{S}(\nu)$ then the associated $\eta_n^{(k)}$ will vary continuously from $\eta_n^{(k)}(\nu)$ to $\tilde{\eta}_n^{(k)}(\nu)$. By a suitable choice of points $\overset{*}{\eta}'(\nu)$ on this line segment $\mathfrak{S}(\nu)$, one can obtain a third sequence $\overset{*}{\eta}_n(\nu)$ that converges to an intermediate value between ζ_n and $\tilde{\zeta}_n$. Hence, there will be finitely-many points $\overset{*}{\zeta}$ with the same projection ζ' that all lie on the manifold M. However, this does not agree with the equation $F(\zeta_1, …, \zeta_n) = 0$, which is regular in ζ_n , and which all points of M must satisfy.

Therefore, the points $\eta^{(1)}$, …, $\eta^{(l)}$ are continuous functions of η' in the interior and on the boundary of X'_r.

Remark. When the functions $\eta_n^{(1)}$, …, $\eta_n^{(l)}$ take on the boundary values $\tilde{\zeta}_n^{(1)}$, …, $\tilde{\zeta}_n^{(l)}$ on one side X'_s of X'_r ($s < r$), the points $\tilde{\zeta}^{(1)}$, …, $\tilde{\zeta}^{(l)}$ thus defined will again belong to the manifold M. The point ζ of the manifold M that lies over the point ζ' of X'_s will, however, be given by continuous functions $\tilde{\zeta}_n^{(1)}, \ldots, \tilde{\zeta}_n^{(m)}$ on X'_s (exactly as was the case for X'_r). Thus, the boundary values $\tilde{\zeta}_n^{(1)}, \ldots, \tilde{\zeta}_n^{(l)}$ are found among the continuous functions $\zeta_n^{(1)}, \ldots, \zeta_n^{(m)}$, and will share their properties. It follows from this, e.g., that any two functions $\tilde{\zeta}_n^{(\mu)}$ and $\zeta_n^{(\nu)}$ will either agree in all of X'_s or will be different in the entire interior of X'_s.

4. Since the hypersurface of the sphere K appears among the manifolds M, both of the points of the spherical hypersurface that lie over η' must appear among the points $\eta^{(1)}, \ldots, \eta^{(l)}$, and indeed, due to the ordering (3), the first and last points must be $\eta^{(1)}$ and $\eta^{(l)}$.

We now subdivide the ball into "blocks." A block consists of all points x that satisfy one of the following conditions:

a) η' is in X'_r, $\eta_n = \eta_n^{(\nu)}$;

b) η' is in X'_r, $\eta_n^{(\nu)} < \eta_n < \eta_n^{(\nu+1)}$.

Naturally, the blocks b) no longer appear on the boundary of the projection of the ball, where $\eta^{(l)} = \eta^{(1)}$.

It is clear that any point x of the ball will belong to one and only one block. Furthermore, it is clear that the closed hull of a block will again consists of similarly-defined blocks. In Fig. 1, the subdivision of the plane is indicated in the case where the only manifold M is a conic section. In Fig. 2, (left) the form of a block of type b) is indicated in the case of three-dimensional space. The upper and lower surfaces of this block (the upper one is shaded) are blocks of type a).

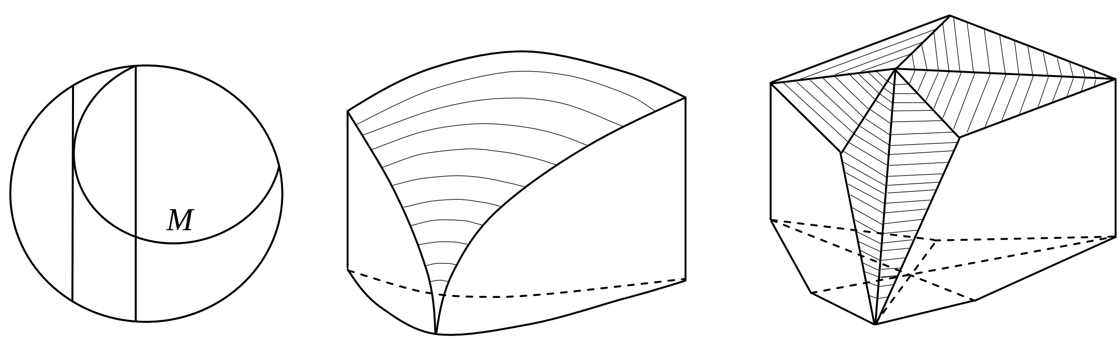


Fig. 1. Fig. 2

5. We now have to show that any block, along with its boundary, can be mapped topologically to a rectilinear polyhedron; hence, it is itself a curvilinear polyhedron.

This is very easy for the blocks of type a): The projection $\eta \to \eta'$ maps the block a), along with its boundary, topologically onto the curved simplex X'_r, along with its boundary; hence, the block is itself a curved simplex.

We map a block of type b) in two steps: In the first step, the coordinates η_n of the point η of the block will be left unchanged, while $\eta_1, \ldots, \eta_{n-1}$ will be transformed such that the simplex X'_r over which the block lies will be mapped onto a rectilinear simplex X_r. After the map, we will thus have a block whose points will be defined by:

$$\eta' \text{ in } X_r , \qquad \eta_n^{(\nu)} < \eta_r < \eta_n^{(\nu+1)} ,$$

where X_r will be a rectilinear simplex, while $\eta_n^{(\nu)}$ and $\eta_n^{(\nu+1)}$ will be continuous functions of η' in the closed simplex X_r . As we have seen, we will have $\eta_n^{(\nu)} < \eta_n^{(\nu+1)}$ in the interior of X_r, while we will have $\eta_n^{(\nu)} \leq \eta_n^{(\nu+1)}$ on the boundary, and indeed, in each boundary simplex X_s of X_r, one will have either $\eta_n^{(\nu)} = \eta_n^{(\nu+1)}$ without exception or $\eta_n^{(\nu)} < \eta_n^{(\nu+1)}$ everywhere in the interior of X_s .

The simplex X_r will now be subdivided "barycentrically." The barycentric subdivision of a simplex is defined recursively: An X_1 is subdivided into two line segments by a division point J_1, and when all of the X_{r-1} on the boundary of X_r have already been barycentrically subdivided then each simplex of this subdivision that has an interior point J_r of X_r will be linked to new simplexes, which will then define the subdivision of X_r . The vertices of such a simplex will thus be $J_0, J_1, \ldots, J_r$, where J_k is an interior point of X_r , while X_{k-1} is always a side of X_k ($k = 1, 2, \ldots, r$).

We would now like to approximate the continuous functions $\eta_n^{(\nu)}$ and $\eta_n^{(\nu+1)}$ by piecewise-linear functions. We remark that a linear function of the coordinates in a rectilinear simplex is established completely whenever the values of the function are known at the vertices of the simplex. We accordingly define two linear functions $\bar{\eta}_n^{(\nu)}$ and $\bar{\eta}_n^{(\nu+1)}$ on the simplex ($J_0 \, J_1 \ldots, J_r$), whose values at the vertices J_k ($k = 0, \ldots, r$) agree with the given values $\eta_n^{(\nu)}(J_k)$ and $\eta_n^{(\nu+1)}(J_k)$.

If two different simplexes ($J_0 \, J_1 \ldots, J_r$) have a common side then the functions $\eta_n^{(\nu)}$ that are defined on them will agree on the common side. Hence, the functions $\bar{\eta}_n^{(\nu)}$ that are defined on the sub-simplexes will merge together into a continuous, piecewise-linear function $\bar{\eta}_n^{(\nu)}$ that is defined on all of X_r, together with its boundary, and the same will be true for $\eta_n^{(\nu+1)}$.

If a linear function is greater than another one at all or some of the vertices of a simplex, but equal at the remaining vertices, then it will also be greater in the interior. Thus, one will have:

$$\bar{\eta}_n^{(\nu)} < \eta_n^{(\nu+1)}$$

in the interior of X_r .

However, one will correspondingly also have on each side X_s of X_r: If one has $\eta_n^{(\nu)} < \eta_n^{(\nu+1)}$ for the interior points of one such side then the same will be true for $\bar{\eta}_n^{(\nu)}$ and $\bar{\eta}_n^{(\nu+1)}$; however, if $\eta_n^{(\nu)} = \eta_n^{(\nu+1)}$ on X_s then one will also have $\bar{\eta}_n^{(\nu)} = \bar{\eta}_n^{(\nu+1)}$ there.

Now, one can map the block that is defined by:

$$\eta' \text{ in } X_r, \qquad \eta_n^{(\nu)} < \eta_n < \eta_n^{(\nu+1)} ,$$

along with its boundary, topologically onto the block that is bounded by linear spaces:

$$\eta' \text{ in } X_s, \qquad \bar\eta_n^{(\nu)} < \bar\eta_n < \bar\eta_n^{(\nu+1)},$$

along with its boundary, in such a way that one leaves the coordinates $\eta_1, \ldots, \eta_{n-1}$ of a point η unchanged, but replaces the coordinates η_n with $\bar\eta_n$, where η_n and $\bar\eta_n$ are coupled to each other by the formulas:

$$\eta_n = \eta_n^{(\nu)} + \lambda(\eta_n^{(\nu+1)} - \eta_n^{(\nu)}) \qquad (0 \leq \lambda < 1),$$
$$\bar\eta_n = \bar\eta_n^{(\nu)} + \lambda(\bar\eta_n^{(\nu+1)} - \bar\eta_n^{(\nu)}).$$

One easily proves that the map thus defined is one-to-one and continuous in both directions. However, the image block may be decomposed into rectilinear simplexes with no further constructions (e.g., barycentric subdivision). Hence, any block b) can be topologically mapped onto a rectilinear polyhedron. This ends the proof.

Remark. If one goes through the proof of part 5 again then one will see that the map of the curvilinear simplexes of the triangulation onto rectilinear ones can be arranged in such a way that the coordinates of the point of a curvilinear simplex are *continuous, differentiable functions* of the coordinates in the interior of the rectilinear image simplex. One must naturally include the continuous differentiability of the mapping functions in the induction hypothesis, and assume that the map of the block b) in the first step is differentiable. Since the algebraic functions $\eta_n^{(\nu)}$ are also differentiable outside of their critical loci the second step of the map will also lead to only differentiable functions.

The next question that we have to examine is the transition from complex algebraic manifolds to real ones. Here, we employ a map of the complex projective space onto a real algebraic manifold that is given by the following formulas:

$$(4) \qquad \begin{cases} \zeta_j \bar\zeta_j = \sigma_{jj}, \\ \zeta_j \bar\zeta_k = \sigma_{jk} + i\tau_{jk} \quad (j < k), \\ \zeta_k \bar\zeta_j = \sigma_{jk} - i\tau_{jk} \quad (j < k). \end{cases}$$

Thus, $\zeta_0, \ldots, \zeta_n$ are the homogeneous coordinates in complex S_n, while the σ_{jk} ($0 \leq j \leq k \leq n$) and τ_{jk} ($0 \leq j < k \leq n$) are homogeneous coordinates in a real S_N. The $\bar\zeta_j$ are complex conjugates of the ζ_j. One immediately sees from equations (4) that the σ_{jk} and τ_{jk} have to be real. If one sets $\sigma_{kj} = \sigma_{jk}$, $\tau_{kj} = -\tau_{jk}$, $\tau_{jj} = 0$ then one can write (4) more concisely as:

$$(5) \qquad \zeta_j \zeta_k = \sigma_{jk} + \tau_{jk} \qquad (j, k = 0, 1, \ldots, n).$$

Similarly, as in § 4, the σ_{jk} and τ_{jk} will be coupled to each other by the relations:

$$(6) \qquad (\sigma_{jk} + i\,\tau_{jk})\,(\sigma_{hl} + i\,\tau_{hl}) - (\sigma_{jl} + i\,\tau_{jl})\,(\sigma_{hk} + i\,\tau_{hk}) = 0\,.$$

The relations (6) are necessary and sufficient for a real point of S_n to be the image point of a point ζ in complex S_n. Equations (6) define an algebraic manifold in real S_N, viz., the SEGRE *manifold* $\mathfrak{S}$. As in § 4, one sees that the map of the space S_n onto the manifold $\mathfrak{S}$ is one-to-one. Naturally, it is also continuous, and therefore topological.

The SEGRE manifold $\mathfrak{S}$ has no point in common with the hyperplane:

$$\sum \sigma_{jj} = \sigma_{00} + \sigma_{11} + \ldots + \sigma_{nn} = 0\,, \tag{7}$$

so $\sum \zeta_j \bar{\zeta}_j$ is nowhere zero when not all $\zeta_j = 0$. Moreover, one has:

$$|\,\sigma_{jk} + i\,\tau_{jk}\,| = |\,\zeta_j\,\bar{\zeta}_k\,| = \sqrt{\zeta_j\,\bar{\zeta}_j}\cdot\sqrt{\zeta_k\,\bar{\zeta}_k} = \sqrt{\sigma_{jj}}\cdot\sqrt{\sigma_{kk}} \leq \sigma_{jj} + \sigma_{kk} \leq \sum\sigma_{jj}$$

everywhere on $\mathfrak{S}$.

If one thus regards the hyperplane (7) as an ideal hyperplane and introduces inhomogeneous coordinates by the normalization $\sum\sigma_{jj} = 1$ then all of the coordinates σ_{jk} and τ_{jk} under consideration will be ≤ 1. Thus, the manifold $\mathfrak{S}$ lies in a restricted subset of Euclidian space (e.g., in the ball $\sum\sigma_{jk}^2 + \sum\tau_{jk}^2 \leq n + 1$).

An algebraic manifold in S_n with the equations:

$$f_\nu(\zeta) = 0 \tag{8}$$

will correspond to an image manifold in $\mathfrak{S}$ whose equations will be found when one multiplies equations (8) by their complex conjugates:

$$f_\nu(\zeta)\,\bar{f}_\nu(\bar{\zeta}) = 0$$

and then substitutes $\sigma_{jk} + i\,\tau_{jk}$ for the products $\zeta_j\,\bar{\zeta}_k$.

Now, if finitely-many algebraic manifolds M in S_n are given then they will likewise correspond to finitely-many real submanifolds of $\mathfrak{S}$. From theorem 1, there will be a triangulation of $\mathfrak{S}$ for which all of these submanifolds consist of simplexes of the triangulation. We have proved:

Theorem 2. *There is a triangulation of complex S_n for which finitely many given manifolds M in S_n consist of nothing but simplexes of the triangulation.*

Up to now, we have not worried about the dimensions of the simplexes of the triangulation. It is, however, clear from the proof of theorem 1 that only simplexes of dimension at most d will be appear in the triangulation of a d-dimensional manifold M in real S_n. The example of a plane, cubic curve with one isolated point shows that simplexes of dimension $< d$ can also enter into the triangulation, and indeed not only as sides of simplexes X_d.

One will double the dimension of an irreducible manifold M by going from complex S_n to the SEGRE manifold $\mathfrak{S}$, since the real and imaginary coordinates of the points of M

will then take the form of independent variables. Hence, the simplexes of the triangulation of M will have dimension at most $2d$. However, one can prove even more, namely:

Theorem 3. *Only $2d$-dimensional simplexes X_{2d} and their sides will appear in the triangulation of a d-dimensional algebraic manifold M in complex S_n.*

Proof. As in § 31, we choose the coordinate system in such a way that the coordinates $\xi_{d+1}, \ldots, \xi_n$ of a general point of M are complete algebraic functions of $\xi_0, \ldots, \xi_d$. Then, from theorem 4 (§ 31), each system of values of the coordinates $\xi_0, \ldots, \xi_d$ will be associated with certain points $\overset{\mu}{\zeta}$ of M ($m = 1, \ldots, k$), whose coordinates $\overset{\mu}{\zeta}_0, \ldots, \overset{\mu}{\zeta}_n$ will be found by factoring the polynomials:

$$h(u_0, \ldots, u_n, \zeta_0, \ldots, \zeta_d, z) = \prod_1^k (z - \zeta_\mu), \tag{9}$$

$$\zeta_\mu = u_0 \overset{\mu}{\zeta}_0 + u_1 \overset{\mu}{\zeta}_1 + \cdots + u_n \overset{\mu}{\zeta}_n .$$

We have seen that the coordinates of a point of a curvilinear simplex X_r in the triangulation of M are continuous, differentiable functions of r real parameters. If we now project X_r onto a subspace of S_d by replacing the coordinates $\zeta_{d+1}, \ldots, \zeta_n$ with zero then the projection of X_r will be a point set whose points will again depend continuously and differentiably upon r real parameters. Such a point set is, however, nowhere dense in S_d when $r < 2d$. If one carries out the projection for all simplexes X_r ($r = 0, 1, \ldots, 2d + 1$) of the triangulation then one will obtain a nowhere-dense point set W in S_d for the union of all of the projections. Any point ζ' of W will thus be the limit of a sequence of points $\zeta'(\nu)$ that do not belong to W.

As we remarked above, the projection of ζ' is associated with a system of k points $\overset{1}{\zeta}, \ldots, \overset{k}{\zeta}$ of M, and likewise any $\zeta'(\nu)$ is associated with a system of k points $\overset{1}{\zeta}(\nu), \ldots, \overset{k}{\zeta}(\nu)$ of M that will each be determined by the factorization (9). If one normalizes the coordinates by way of $\zeta_0 = \zeta_0(\nu) = 1$ then all of the coordinates $\zeta_i(\nu)$ will be restricted simultaneously. Thus, one can select a convergent subsequence from the sequence of the system of k points. One will then have:

$$\overset{1}{\zeta}(\nu) \to \overset{1}{\eta}, \qquad \overset{2}{\zeta}(\nu) \to \overset{2}{\eta}, \qquad \ldots, \overset{k}{\zeta}(\nu) \to \overset{k}{\eta} \qquad (n \to \infty)$$

for this subsequence. Since equation (9) will remain true under passing to the limit, but, on the other hand, the factorization of a polynomial is unique, the limit points $\overset{1}{\eta}, \ldots, \overset{k}{\eta}$ of any sequence must coincide with $\overset{1}{\zeta}, \ldots, \overset{k}{\zeta}$. However, any of the points $\overset{1}{\zeta}, \ldots, \overset{k}{\zeta}$ will be limit points of points of M whose projections do not belong to only the point set W.

However, this means: Any point of a simplex X_r ($r < 2d$) of the triangulation of M is the limit point of points of M that do not belong to any X_r with $r < 2d$, and which can therefore only be interior points of simplexes X_{2d} . It will follow from this that any such X_r ($r < 2d$) will be a side of an X_{2d} of M.

One can also obtain the triangulation of the complex manifold M in a manner that will be similar to the triangulation of a plane curve that was given at the start of this section from a triangulation of the space S_d , when one assumes that any point ζ' of S_d is the projection of k points ζ of M. One thus has to triangulate S_d in such a way that the branched manifold that originates in the zero locus of the discriminant of the polynomial (9) will be triangulated along with it. WIRTINGER and BRAUER ([1]) have examined algebraic functions of two variables in this way.

([1]) BRAUER, K.: Abh. Math. Inst. Hamburg, v. 5.

CHAPTER FIVE

Algebraic correspondences and their applications

Algebraic correspondences are almost as old as algebraic geometry itself is nowadays. A theorem of CHASLES on the number of fixed points of a correspondence between points of a straight line (cf., § 32) was generalized by BRILL ([1]) to correspondences between the points of algebraic curves, carried over by SCHUBERT ([2]), with great success, to systems of ∞^1 point-pairs in space, and further refined and applied to many things by ZEUTHEN ([3]).

However, it was the Italian geometers, namely, SEVERI and ENRIQUES, who first recognized the general significance of the notion of a correspondence as one of the foundations of algebraic geometry. In any case where geometric structures were related to each other in such a way that this relation could be expressed by algebraic equations, the notion of a correspondence found an application. Here, we shall mainly discuss this general and fundamental interpretation of the notion of correspondence. For the aforementioned investigations of numbers of fixed points of correspondences, the reader must refer to the literature cited ([4]).

From now on, x, y, … will no longer mean indeterminates exclusively, but also complex numbers or algebraic function, as the situation dictates.

§ 32. Algebraic correspondences. CHASLES's correspondence principle

Let S_m and S_n be two projective spaces, which may also be the same one. An algebraic manifold of point-pairs (x, y), in which x belongs to S_m and y belongs to S_n, is called an *algebraic correspondence* $\mathfrak{K}$. The correspondence will be given by a system of homogeneous equations (homogeneous in x, as well as in y):

$$f_x(x_0, \dots, x_m, y_0, \dots, y_n) = 0. \tag{1}$$

We will say that the points x in the correspondence *correspond to* or are *associated with* the points y; an associated point y will also be called an *image point* of x under the correspondence, while conversely x will be called a *source point* of y.

Examples of correspondences are correlations (especially polar systems and null systems), which will be given by a bilinear equation:

$$\sum a_{jk}\, x_j\, y_k = 0,$$

and finally, the projections are further given by the projective transformations:

([1]) BRILL, A. v.; Math. Ann. Bd. 6 (1873), pp. 33-65 and Bd. 7 (1874), pp. 607-622.

([2]) SCHUBERT, H.; *Kalkül der abzählenden Geometrie*. Leipzig, 1879.

([3]) ZEUTHEN, H. G.; *Lehrbuch der abzählenden Methoden der Geometrie*. Leipzig, 1914.

([4]) On this, confer S. LEFSCHETZ; Trans. Amer. Math. Soc., **28** (1928), 1-49, and various notes of F. Severi in Rendiconti Accad., Lincei 1936 and 1937.

$$y_j = \sum a_{jk}\, x_k \qquad \text{or} \qquad y_i(\sum a_{jk}\, x_k) - y_j(\sum a_{ik}\, x_k) = 0$$

(y is the projection of x onto a subspace S_n of the space S_m, while x belongs to an arbitrary manifold M).

The notion of correspondence can thus be generalized by saying that other geometrical structures can be chosen – e.g., point-pairs, linear spaces, hypersurfaces – in place of the points x and y, as long as these structures are given by one or more sequences of homogeneous coordinates. Equations (1) must then be homogeneous in each individual sequence of coordinates. All of the following considerations will be valid with no further restrictions for this general case, which is extremely important for the applications. For the formulation of the theorems themselves, we will, however, restrict ourselves to the case where x and y are points; we thus do not speak of the "structure" x and the "structure" y, but simply of the points x and y.

If one eliminates y from equations (1) then one will obtain a homogeneous resultant system:

(2) $$g_\mu(x_0, \ldots, x_m) = 0$$

with the property that for any solution x of (2) at least one point pair (x, y) will belong to the correspondence. Likewise, the elimination of x will yield a homogeneous system of equations:

(3) $$h_\nu(y_0, \ldots, y_n) = 0.$$

Equations (2) define an algebraic manifold M in S_m : viz., the *source manifold* of the correspondence $\mathfrak{K}$; likewise, (3) defines a manifold N in S_n: viz., the *image manifold* of the correspondence. One also speaks of a correspondence $\mathfrak{K}$ *between M and N*. If (x, y) is a point-pair of the correspondence then x will belong to M, and y will belong to N, and each point x of M (or y of N) will correspond to at least one point y of N (x of M, resp.).

If one fixes the point x then equations (1) will define an algebraic manifold in the space S_n, and indeed a submanifold N_x of N. N_x is totality of all points x that correspond to the point y. Conversely, each point y of N will correspond to an algebraic manifold M_y of points x of M.

If M and N are irreducible (the correspondence $\mathfrak{K}$ can be reducible or irreducible) and each general point of M corresponds to β points of N, while conversely every general point y of N corresponds to α points of M then one will speak of an (α, β) correspondence between M and N. A particular point of M can therefore correspond to finitely or infinitely many points of N; later, we shall have to deal with the transition from general points to particular points more extensively.

If the manifold $\mathfrak{K}$ is irreducible then one will call it an *irreducible correspondence*. In that case, M and N will also be irreducible, so when a product of two forms $F(x) \cdot G(x)$ becomes zero at all points of M then it will be zero for all point-pairs (x, y) of the correspondence $\mathfrak{K}$; hence, the factor F or G will become zero for all point-pairs (x, y) of $\mathfrak{K}$, and thus, for all points x of M.

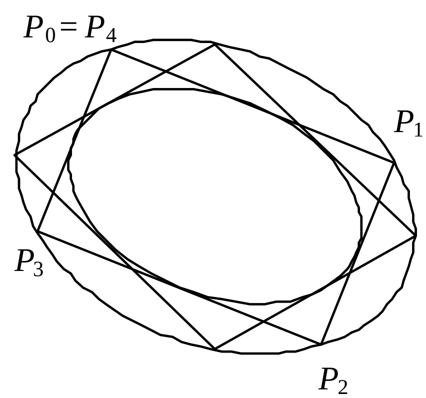
As the simplest, but most important, case we first consider an (α, β)-correspondence between the points x and y of a line S_1 . The correspondence is purely one-dimensional; it is then a hypersurface in the doubly projective space $S_{1,1}$ and (like any hypersurface) will be given by a single equation:

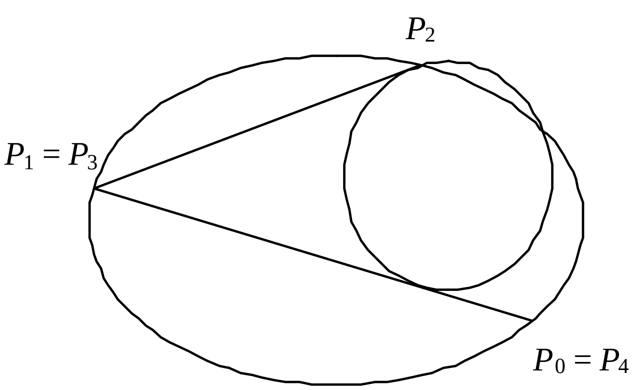
$$f(x, y) = 0, \tag{4}$$

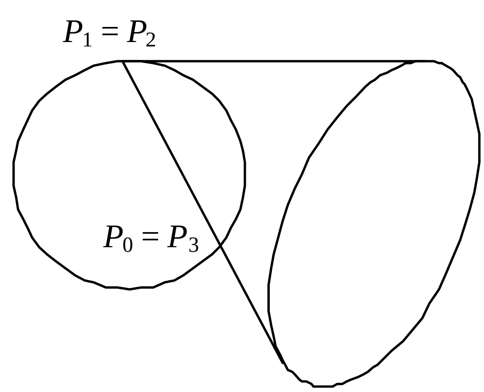
which we will assume is free of multiple factors. The equation is homogeneous in both of the coordinates x_0 , x_1 of the point x, and likewise in those y_0 , y_1 of the point y. If α is its degree in the x and β is its degree in the y then each general point x will obviously correspond to β different points y, and a general point y will likewise correspond to α different points x.

The fixed points of the correspondence will be found when one sets $x = y$ in (4). That will yield an equation of degree $\alpha + \beta$ in y that will either be fulfilled identically or will possess precisely $\alpha + \beta$ roots (each counted with its multiplicity). *The correspondence* (4) *will thus either include the identity as a component or have precisely* $\alpha + \beta$ *fixed points that one obtains from the equation* $f(x, x) = 0$, *when one counts the fixed points with their multiplicities*. This is **CHASLES's correspondence principle.**

In order to give a simple application of CHASLES's correspondence principle, we consider two conic sections K, K' that do not touch. From a point P_0 of K, we draw a tangent to K that intersects K' a second time at P_1 . A second tangent to K' goes through P_1 that intersects K a second time at P_2 . One thus proceeds to construct the chain $P_0, P_1, P_2, \ldots, P_n$. We now assert: *If the chain once concludes with* $P_n = P_0$ *in a non-trivial way then it will always conclude this way, no matter how one chooses* P_0 *on* K. We then say that the sequence $P_0, P_1, \ldots, P_n$ concludes *in a trivial way* when either (for even n) the middle term $P_{\frac{1}{2}n}$ is an intersection point of K and K', or when (for odd n) both of the middle terms $P_{\frac{n-1}{2}}$, $P_{\frac{n+1}{2}}$ coincide, and their connecting line is a common tangent to both conic sections (cf., the second and third figures). In both cases, the second half of the chain will be equal to the first one in the opposite sequence; hence, $P_n = P_0$. This trivial case arises (whether n is even or odd) many times, since there are four intersection points and four common tangents. The correspondence between P_0 and P_n will therefore be a (2, 2)-







correspondence that always has four trivial fixed points. If it had one more fixed point, moreover, then, from CHASLES's correspondence principle, it would contain the identity as a component. One can thus produce a closed chain with $P_n = P_0$ that begins with any point P_0 . The same chain, when traversed in the opposite direction, will yield a second closed chain with the same starting point P_0 . Hence, both chains that begin at P_0 will terminate, no matter how P_0 was chosen.

§ 33. Irreducible correspondences. The principle of constant count.

An irreducible correspondence (like any irreducible manifold) is determined by its general point-pair (ξ, η). The characteristic property of this general point-pair is that all homogeneous algebraic relations $F(\xi, \eta) = 0$ that are valid for the general point-pair will be valid for all point-pairs (x, y) of the correspondence. In other words: All point-pairs of the correspondence will arise from relation-preserving specializations of the general point-pair (ξ, η). If one wishes to define an irreducible correspondence then one will start with a suitable (arbitrarily-defined) general point-pair. The totality of pairs (x, y) that arise from this general point-pair by a relation-preserving specialization will then always be an irreducible correspondence.

For example, let M be a given irreducible manifold, and let ξ be its general point. Now, if φ_0 , φ_1 , …, φ_n are forms of equal degree that are not all zero then a second point η will be given by means of:

$$\eta_0 : \eta_1 : \ldots : \eta_n = \varphi_1(\xi) : \varphi_1(\xi) : \ldots : \varphi_n(\xi) \tag{1}$$

that will depend upon ξ rationally. The point-pair (ξ, η) will be the general point-pair of an irreducible correspondence whose point-pairs originate from relation-preserving specializations of it. Such a correspondence is called a *rational map* of M. – On the grounds of the relation-preserving specialization, the relations:

$$\eta_i \, \varphi_j(\xi) - \eta_j \, \varphi_i(\xi) = 0,$$

which are equivalent to (1), must also be valid for each particular point-pair of $\mathfrak{K}$:

$$y_i \, \varphi_j(x) - y_j \, \varphi_i(x) = 0. \tag{2}$$

If not all $\varphi_i(x) = 0$ then the behavior of y will be determined uniquely by (2). However, if all $\varphi_i(x) = 0$ for a point x of M then formula (2) says nothing more than which point y that the point x is associated with. One must then resort to other methods; e.g., by passing to the limit while one approaches the point x of M from both sides and thus watching which limit point the image point y will go to. Due to the continuity of the forms that define the correspondence, each pair (x, y) thus obtained will belong to the correspondence; on the other hand, it follows from Theorem 3 of the Appendix to Chapter 4 that all pairs (x, y) of the correspondence can be obtained in this way.

The dimension q of an irreducible correspondence $\mathfrak{K}$ is the number of algebraically-independent coordinate ratios of the general point-pair (ξ, η). If – say – $\xi_0 \neq 0$ and $\eta_0 \neq 0$ then we can assume that $\xi_0 = \eta_0 = 1$; q will then be the number of algebraically independent quantities amongst $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n$. Now, if a is the number of algebraic independents amongst the ξ, relative to the ground field $\mathbb{K}$ and b is the number of algebraic independents amongst the η, relative to the field $\mathbb{K}(\xi_1, \ldots, \xi_m)$, then one will obviously have:

(3) $$q = a + b.$$

Likewise, when c is the number of algebraic independents amongst the η and d is the number of algebraic independents amongst the ξ by the adjunction of η, one will have:

(4) $$q = c + d.$$

Geometrically, the numbers a, b, c, d mean the dimensions of the manifolds. The x define a general point of M when each homogeneous relation $F(\xi) = 0$ that is true for the point ξ is also true for all points x of M, and conversely. Therefore, a is the dimension of M, and likewise c is that of N. It will now be further asserted that *the submanifold N_ξ of N that corresponds to the general point ξ of M is irreducible relative to the field $\mathbb{K}(\xi_1, \ldots, \xi_m)$ and has dimension b.*

N_ξ exists for all points y such that the point-pair (ξ, y) belongs to the correspondence; i.e., such that any homogeneous algebraic relation that is true for (ξ, η) will also be true for (ξ, y). Under the substitution $\xi = 1$, these relations will lose their homogeneity in the ξ, but will retain it in the η. One can then regard them as homogeneous relations in the ξ with coefficients in the field $\mathbb{K}(\xi) = \mathbb{K}(\xi_1, \ldots, \xi_m)$. Therefore, all homogeneous algebraic relations with coefficients in $\mathbb{K}(\xi)$ that are true for the point η will also be true for all points y of N_ξ, and conversely; however, that says that η will be a general point of N_ξ. It follows that N_ξ will be irreducible with respect to the field $\mathbb{K}(\xi)$ and will have dimension b. The manifold N_ξ can be decomposed completely by an extension of the field $\mathbb{K}(\xi)$, but its absolutely irreducible components will all have the same dimension b (cf., § 31, Theorem 5).

Moreover, the *principle of constant count* follows from (3) and (4):

If a general point ξ of an a-dimensional source manifold M corresponds to a b-dimensional manifold of points in N under a q-dimensional irreducible correspondence between M and N, and conversely, a general point η of the c-dimensional image manifold N corresponds to a d-dimensional manifold of points in M then one will have:

(5) $$q = a + b = c + d.$$

Thus, one must remark that all general point-pairs (ξ, η) of the correspondence will be equivalent to each other; the same will usually be true of the general points of M and N. It is therefore irrelevant whether one starts with a general point x of M and then seeks a general associated point of N, or conversely starts with a general point of N; one will always find the same numbers a, b, c, d and the same properties of the general point-pairs (ξ, η) of the correspondence.

In most applications, one employs formula (5) in order to determine the dimension c of the image manifold N when a, b, and c are given. If one thus finds that $c = n$ then one can conclude that the image manifold is the entire space S_n .

Examples and applications. 1. Let the question be posed of how many parameters a third-order plane curve with a cusp will depend upon; in other words, what is the dimension of the manifold of cubic curves with cusps?

We define a correspondence $\mathfrak{K}$ between points x and cubic curves y in which we associate a point x with all curves y that have a cusp at x. One can provide a general element pair (ξ, η) of this correspondence in the following way: One takes a general point ξ and draws the most general line u in the plane through it. Thus, a cubic curve y that has a cusp at ξ with a tangent u must have coefficients that satisfy a system of five linearly independent equations ([1]). Since ten coefficients appear in the equation of the general cubic curve, one of which can be equal to one, the general solution of the system of equations will depend upon 9 – 5 = 4 arbitrary parameters. If one then counts the one arbitrary constant upon which the tangent u depends (for a given point ξ) then one will obtain five parameters. If one lets all of these parameters be indeterminates then one will obtain a general point-pair (ξ, η), from which, all pairs (x, y) will arise by parameter specialization (hence, by the simplest relation-preserving specialization). The correspondence will therefore be irreducible. The principle of constant count will yield:

$$2 + 5 = c + 0; \quad c = 7.$$

With this, the desired dimension is equal to 7.

One may also express the result thus found as: There are ∞^7 plane cubic curves with cusps. One can carry out the most diverse determinations of dimension in a precisely analogous way (cf., example 3 below, as well as problem 1).

([1]) If one puts the coordinate origin at the point ξ, chooses the line u to be the x_1-axis, and poses the equation of the cubic curve in inhomogeneous coordinates in the following way:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_2 x_1 x_2 + a_5 x_2^2 + \ldots = 0$$

then the conditions for a cusp at ξ with tangent u will read:

$$a_0 = a_1 = a_2 = a_3 = a_4 = 0.$$

If one subsequently transforms the equation of the curve to any other coordinate system then naturally these linear equations will also be transformed; however, five linearly-independent equations will remain.

Example 2. Given a cubic space curve C, prove that a chord or a tangent of the curve goes through each point of space. (In § 11, the proof was carried out by calculation.)

A chord goes through two general point of the curve. One will obtain all chords and tangents from this chord (the latter come about when the two points coincide) by a relation-preserving specialization. Thus, the chords, together with the tangents, will define an irreducible two-dimensional manifold. Furthermore, we define a correspondence between the chords x and their points y, under which we will associate each chord with all of the points y that lie on it; the correspondence will again be irreducible. One sets $a = 2$ and $b = 1$ in formula (5). In order to determine d, we remark that at most one chord goes through each point y outside the curve. Two intersecting chords would then determine a plane that had four points in common with the curve, which is impossible; therefore, $d = 0$. Moreover, it follows from (5) that $c = 3$; i.e., the manifold of points y is the entire space, which was to be proved.

Example 3. The subspace S_m of a space S_n will be mapped onto points y of an image space by means of its PLÜCKER coordinates. We would like to show that the image points define an irreducible manifold of dimension $(m+1)(n-m)$. In other words, there are $\infty^{(m+1)(n-m)}$ subspaces S_m in S_n .

Proof. $(m + 1)$ generally-chosen points in S_n determine a subspace S_m . One obtains any arbitrary system of $(m + 1)$ linearly independent points from these points by a relation-preserving specialization, and therefore, one will obtain an arbitrary S_m from the points of S_n . It was already proved that the subspace S_m defines an irreducible manifold. If we call the general system of $(m + 1)$ points ξ and the subspace that is determined by them η then the pair (ξ, η) will determine an irreducible correspondence whose general element will be just this pair. Since a system of $(m + 1)$ general points in S_n depends upon $(m + 1)\, n$ parameters, but a system of $(m + 1)$ general points in a given S_m depends upon $(m + 1)\, m$ parameters, one will then have:

$$a = (m + 1)\, n; \qquad b = 0; \qquad d = (m + 1)\, m.$$

Moreover, it follows from (5) that $c = (m + 1)(n - m)$.

Problem. 1. There are ∞^{13} plane curves of order four with one double point, ∞^{12} with two of them, and ∞^{11} with three double points. The totality of fourth-order curves with one or two double points is irreducible; that of the curves with three double points decomposes into two irreducible submanifolds of equal dimension 11.

In conclusion, this allows us to mention about a very specialized – but still often-used – criterion for the irreducibility of a correspondence:

Lemma. *The equations of a correspondence $\mathfrak{K}$ may be decomposed into equations in the x alone that define an irreducible source manifold M and equations in x and y that are linear in y and always have the same rank, which therefore associate each point x of M with a linear space N_x that always has the same dimension b. Such a correspondence is irreducible.*

Proof. A general point-pair (ξ, η) of the correspondence will be obtained in the following way: Let ξ be a general point of M, and let η be an intersection point of the linear space N_ξ with b general hyperplanes $\overset{1}{u}, \ldots, \overset{b}{u}$. We now have to show that each pair (x, y) of the correspondence is a relation-preserving specialization of (ξ, η). Thus, let $F(\xi, \eta) = 0$ be any homogeneous relation; we have to show that $F(x, y) = 0$ is also true.

We likewise pass b hyperplanes $\overset{1}{v}, \ldots, \overset{b}{v}$ through the point y that intersect N_x at precisely the point y. One can compute coordinate ratios in determinant form:

$$y_0 : y_1 : \ldots : y_n = D_0(x, v) : D_1(x, v) : \ldots : D_n(x, v) \tag{6}$$

from the y equations of the correspondence and the equations of the hyperplanes $\overset{1}{v}, \ldots, \overset{b}{v}$. Since the determinants $D_0, \ldots, D_n$ are not all zero for the for the particular point x and the particular hyperplanes v, they will also be non-zero for the general point x of M and the general hyperplanes $\overset{1}{u}, \ldots, \overset{b}{u}$. Thus, the solution of the linear system of equations by determinants will also be true when x and v are replaced by ξ and u:

$$\eta_0 : \eta_1 : \ldots : \eta_n = D_0(\xi, u) : D_1(\xi, u) : \ldots : D_n(\xi, u). \tag{7}$$

Due to $F(\xi, \eta) = 0$, it now follows from (7) that:

$$F(\xi, D_\nu(\xi, \eta)) = 0;$$

hence, since x is a general point of M:

$$F(x, D_\nu(x, u)) = 0,$$

and further, by replacing the indeterminates u with v:

$$F(x, D_\nu(x, v)) = 0,$$

or, due to (6):

$$F(x, y) = 0.$$

Thus, (ξ, η) is a general pair of the correspondence, which is irreducible.

§ 34. Intersection of manifolds with general linear spaces and general hypersurfaces

Theorem 1: *The intersection of an irreducible a-dimensional manifold M ($a > 0$) with a general hyperplane $(u\,x) = 0$ is an irreducible manifold of dimension $a - 1$ relative to the field $K(u^0, \ldots, u^n)$.*

Proof. If one associates the points x of the manifold M with the hyperplane y that goes through x then one will obtain an algebraic correspondence $\mathfrak{K}$. The equations of the correspondence will be the equations of M and the equation $(x\,y) = 0$, which expresses the idea that x lies in the hyperplane y. From the Lemma of § 33, $\mathfrak{K}$ will be irreducible and will contain a general point-pair (ξ, η) of $\mathfrak{K}$ when one passes the most general hyperplane η through the general point ξ of M. When c and d have the usual meaning for the correspondence, the principle of constant sum will yield:

$$a + (n - 1) = c + d. \tag{1}$$

Since the general hyperplane u that passes through a general point x will not contain a second arbitrary, but fixed, point x' of the manifold M, its intersection with the manifold M will be at most $(a - 1)$-dimensional; thus, $d \leq a - 1$. It now follows from (1) that $c \geq n$; hence, the image manifold N will be the entire dual space (i.e., the totality of all hyperplanes in the space S_n). Furthermore, one finds that in the inequality $d \leq a - 1$ only the equality sign can be true, since otherwise it would follow that $c > n$, which is impossible. Therefore, under the correspondence, a general hyperplane u will correspond to a manifold of points x of dimension $a - 1$ that is irreducible relative to the field $K(u_0, \ldots, u_n)$.

In precisely the same way, one proves that in general:

Theorem 2. *The intersection of an irreducible a-dimensional manifold M ($a > 0$) with a general hypersurface of degree g is a hypersurface of dimension $a - 1$ that is irreducible relative to the field of coefficients.*

If one applies this theorem a times then it will follow that:

Theorem 3. *The intersection of an irreducible a-dimensional manifold with a general hypersurface of arbitrary degree is a system of finitely many conjugate points.*

In particular:

Theorem 4. *A general linear subspace S_{n-a} of S_n intersects an irreducible a-dimensional manifold M in finitely-many conjugate points.* – The number of these intersection points is called the *degree* of M.

One can prove this latter theorem directly when one considers the correspondence that associates each point of M with all of the spaces S_{n-a} that go through this point. One obtains a general pair of this correspondence when one passes the most general space of dimension $n - a$ through a general point ξ of M, perhaps when one links ξ with $n - a$ general points of the space S_n. As in the proof of the Lemma in § 33, one shows that all pairs (x, y) of the correspondence are relation-preserving specializations of (ξ, η). (One can, in fact, apply the Lemma directly by using the PLÜCKERian coordinates of η.) The irreducibility of the correspondence follows from this. Applying the principle of constant count then easily yields Theorem 4.

Theorem 5. *An a-dimensional manifold M in S_n has no points in common with a general linear subspace S_m as long as $a + m \leq n$.*

Proof. A general linear space S_m will be given by $n - m = a + k$ general linear equations. From the previous Theorem, a of these equations will define finitely-many conjugate points. However, they do not satisfy the resulting k equations, whose new coefficients will be indeterminates that are independent of the previous ones.

An important Theorem on correspondences then follows from Theorem 5:

Theorem 6. *If a general point of the irreducible manifold M corresponds to a b-dimensional manifold of image points under a correspondence $\mathfrak{K}$ then each individual point of M will correspond to a manifold of image points that is at least b-dimensional.*

Proof. The image manifold of M may belong to a projective space S_n . If one adds b general linear equations for the image point to the equations of the correspondence then a new correspondence will come about, in which a general point of M will always be associated with at least one image point. A general point of M will thus belong to the source manifold of this new correspondence. Thus, all points of M will belong to this source manifold; i.e., each point of M will also be associated with at least one image point under this correspondence. That means, in turn, that the image point of each point of M will have at least one point in common with a general linear subspace S_{n-b} under this correspondence. The dimension of this image space must therefore amount to b. (Here, one must understand the word "dimension" to mean the highest dimension of the manifolds in its decomposition.)

When the image manifold does not belong to a projective space but to a multiply-projective space (e.g., a manifold of point-pairs, point-triples, …), one needs only to embed this multiply-projective space in a projective one (§ 4) in order to return from the general case to the already-dealt-with projective case.

One might also seek to carry over Theorems 1-4 of this section to multiply-projective spaces, but one will then encounter occasional exceptions. E.g., in the multiply-projective case, Theorem 1 will read: *The intersection of an irreducible a-dimensional manifold of point-pairs* (x, y) *with a general hyperplane* $(u\ x) = 0$ *of the x-space is an irreducible manifold of dimension* $a - 1$ *relative to the field* $K(u_0 , \ldots, u_m)$, *excluding the case in which the ratios of all of the x-coordinates of the general point-pairs of M are constant, in which case the intersection is empty.*

Theorem 2 will then be true in the doubly-projective case without exceptions only when the equation of the hypersurface considered has a positive degree in the x, as well as in the y. The argument in the singly-projective case will be left to the reader.

Problem. With the help of Theorem 6, one shows: If a correspondence $\mathfrak{K}$ associates each point x of an irreducible source manifold M with an irreducible image manifold M_x that always has the same dimension b then $\mathfrak{K}$ will be irreducible.

Analogues to Theorems 1-4 are also true in line geometry. From § 33 (example 3), there are ∞^4 lines in the space S_3 . One calls a purely three-dimensional line manifold a *line complex*, a purely two-dimensional one a *line congruence*, and a purely one-dimensional one, a *ruled family*. By the same method with which we proved Theorem 1 above, one can now show:

An irreducible line complex has ∞^1 *lines in common with a general* (i.e., determined by a general point) *star of lines, which define a (relatively) irreducible cone: viz., the cone complex of this point. Likewise, the complex has* ∞^1 *lines in common with a general* (i.e., determined by a general plane) *field of lines, which define an irreducible dual curve in the plane: viz., the curve complex of the plane. The degree of the cone complex and the class of the curve complex are both equal to the number of lines that the complex has in common with general pencil of lines.* This number is called the *degree* of the complex.

Something more complicated is true for a congruence:

An irreducible line congruence has finitely many points in common with a general star of lines, excluding the case where the congruence exists only as some (algebraically conjugate) field of lines, in which case, it will naturally have nothing in common with a general star of lines. Dually, the congruence thus has finitely-many lines in common with a general star of lines, excluding the case in which it exists only as some (conjugate) star of lines. The number of lines that the congruence has in common with a general star of lines (field of lines, resp.) is called the *bundle degree* (*field degree*, resp.) of the congruence.

Proof. We define an algebraic correspondence by associating each line of the given irreducible congruence with all of its points. From the Lemma of § 33, the correspondence will be irreducible. Furthermore, $a = 2$, $b = 1$, and thus $a + b = c + d = 3$. Since the image manifold (viz., the totality of all points of all lines of the congruence) must be at least two-dimensional, only two cases are possible:

1. $c = 2$, $d = 1$;
2. $c = 3$, $d = 0$.

We now still have to show that in the first case the congruence only exists as finitely many (conjugate) fields of lines. In case 1, one has $d = 1$, i.e., if one chooses a general point of a general line congruence then ∞^1 rays of the congruence go through this point. We think of the congruence as being decomposed into absolutely irreducible congruences; we then have to prove that such an absolutely irreducible congruence is a plane field.

If g is a general line congruence then the lines of the congruence that intersect g will define an algebraic submanifold of the congruence. The dimension of this submanifold will be, however, equal to that of the entire congruence, namely, two; ∞^1 lines of the submanifold will then go through each point of the line. Now, since the congruence was absolutely irreducible, it will be identical with the submanifold. We thus see that a general line of the congruence will be intersected by all of the lines of the congruence.

Now, let g and h be two general lines of the congruence. Since they intersect, they will determine a plane. A third general line l of the congruence that is chosen independently of g and h, will intersect g, as well as h, but will not, however, go through the intersection point of g and h (only ∞^1 lines of the congruence will go through this intersection point then). Therefore, l will lie in the plane that is determined by g and h. All lines of the congruence will come about by a relation-preserving specialization of l; thus, they will all lie in the one plane. The total congruence will therefore be contained in a plane field, so, due to the dimensional equality, it will be identical with it.

§ 35. The 27 lines on a third-degree surface

As an application of the methods of this chapter, we examine the question of how many straight lines lie on a general surface of n^{th} degree in the space S_3 .

Let p_{ij} be the PLÜCKERian coordinates of line, and let $f(x) = 0$ be the equation of a surface of n^{th} degree. The line lies on the surface if and only if the intersection point of the line with an arbitrary plane always lies on the surface. The coordinates of this intersection point are:

$$x = \sum_k p_{jk} u^k ,$$

and the desired condition will thus be given by:

$$f\left(\sum p_{jk} u^k\right) = 0 \tag{1}$$

identically in the u^k. Thus emerges the PLÜCKERian relation:

$$p_{01}\, p_{23} + p_{02}\, p_{31} + p_{03}\, p_{12} = 0. \tag{2}$$

Equations (1) and (2) define an algebraic correspondence between the line g, on the one hand, and the surface f that contains them, on the other hand. The irreducibility of this correspondence follows from the Lemma of § 33 if the equations (1) are linear in the coefficients of f, and always have the same rank $n + 1$. (They indeed express the fact that the surface f shall contain a prescribed line, and for this to be true it will be sufficient that it contains $n + 1$ different points of the line.)

The lines g define a four-dimensional manifold. The surfaces f define a space S_N of dimension N, when $N + 1$ is the number of coefficients in the equations of a general surface of n^{th} degree. The surfaces that contain a given line define a linear subspace of dimension $N - (n + 1)$. If we thus apply the principle of constant count to our irreducible correspondence then it will follow that:

$$4 + N - (n + 1) = N - n + 3 = c + d. \tag{3}$$

In this, c means the dimension of the image manifold, i.e., the manifold of those surfaces f that generally contain lines, and each such surface contains at least ∞^4 lines (cf., § 34, Theorem 6).

Now, if $n > 3$ then it will follow that $c + d < N$, hence, $c < N$; i.e., *a general surface of n^{th} degree* ($n > 3$) *will contain no lines.* The cases $n = 1, 2, 3$ then remain. It is well-known that a plane contains ∞^2 lines and a quadratic surface ∞^1, in accordance with formula (3). In the case $n = 3$, from (3), one will have:

$$c + d = N.$$

If we could now show that $d = 0$ then it would follow that $c = N$; i.e., the image manifold would be the entire manifold. Each third-degree surface would thus contain at least one line, and, in general, only finitely many of them.

Were $d > 0$ then this would say that each surface of third degree, which generally contains lines, likewise contains infinitely many of them, namely, ∞^d. Thus, if we can give a single example of a cubic surface that indeed contains lines, but only finitely many of them, then we must have $d = 0$.

That example is easy to give now: We consider a cubic surface with a double point at the coordinate origin. The equation of this surface reads:

$$x_0 f_2(x_1, x_2, x_3) + f_3(x_1, x_2, x_3) = 0,$$

in which f_2 and f_3 shall be relatively prime forms of degree 2 (3, resp.). We first examine whether a line through the origin lies on the surface. If one introduces the parameter representation of the line:

$$x_0 = \lambda_0, \qquad x_1 = \lambda_1 y_1, \qquad x_2 = \lambda_1 y_2, \qquad x_3 = \lambda_1 y_3,$$

into the equation of the surface then one will find the conditions:

$$f_2(x_1, x_2, x_3) = 0 \qquad \text{and} \qquad f_3(y_1, y_2, y_3) = 0.$$

These two equations represent a quadratic cone and a cubic cone with a common vertex. We assume that they have precisely six different generators in common, which is indeed the case, in general. There will thus be six lines on the surface through the origin.

We then examine which lines that lie on the surface do not go through the origin O. If h is such a line then the connecting plane of h with the origin will intersect the given surface in a third-order curve whose one component will be the line h, while the other component will be a conic section that must have a double point at O, and will thus decompose into two lines through O. These lines g_1 and g_2 must be among the six previously-found lines through O ([1]). There are 15 such pairs, and each pair determines a plane that intersects the given surface in a line outside of this pair. There are thus (at most) 15 lines h on the surface that do not go through O. In all, the surface contains (at most) 6 + 15 = 21 lines.

It is thus proved: *There are finitely-many lines on a general surface of degree three, and each particular surface contains at least one.*

([1]) One easily concludes that g_1 and g_2 cannot coincide.

We would now like to determine the number of these lines and their mutual positions, and indeed not only for the general third-degree surface, but also for any cubic surface with no double points.

The equation of the surface might read:

$$f(x_0, x_1, x_2, x_3) = c_{000}x_0^3 + c_{001}x_0^2 x_1 + \cdots + c_{333}x_3^3 = 0. \tag{4}$$

In any case, there will be a line l on the surface; we choose the coordinate system in such a way that this line obeys the equations $x_0 = x_1 = 0$. We would now like to first look for those lines on the surface that intersect the line l. To that end, we pass an arbitrary plane $\lambda_1 x_0 = \lambda_0 x_1$ through the line l; for the points of this plane, we can then set:

$$x_0 = \lambda_0 t, \qquad x_1 = \lambda_1 t. \tag{5}$$

Any point in the plane will then be determined by the homogeneous coordinates t, x_2, x_3. The intersection point of a surface with the plane will be found when one substitutes (5) in (4):

$$f(\lambda_0 t, \lambda_1 t, x_2, x_3) = 0. \tag{6}$$

This homogeneous equation in t, x_2, x_3 represents a curve of degree three. Since the line $t = 0$ (or $x_0 = x_1 = 0$) lies on the surface, this third-degree curve will decompose into the line $t = 0$ and a conic section whose equation might read:

$$a_{11} t^2 + 2a_{12}\, t\, x_2 + 2a_{13}\, t\, x_3 + a_{22}x_2^2 + 2a_{23}\, x_2\, x_3 + a_{33}x_3^2 = 0. \tag{7}$$

Equation (7) will be found from (6) by splitting the factor t. The a_{ik} are thus forms in λ_1, λ_2, and indeed one has:

$$\left\{\begin{aligned} a_{11} &= c_{000}\lambda_0^3 + c_{001}\lambda_0^2\lambda_1 + c_{011}\lambda_0\lambda_1^2 + c_{111}\lambda_1^3, \\ 2a_{11} &= c_{002}\lambda_0^2 + c_{012}\lambda_0\lambda_1 + c_{112}\lambda_1^2, \\ 2a_{13} &= c_{003}\lambda_0^2 + c_{013}\lambda_0\lambda_1 + c_{113}\lambda_1^2, \\ a_{22} &= c_{022}\lambda_0 + c_{122}\lambda_1, \\ 2a_{23} &= c_{023}\lambda_0 + c_{123}\lambda_1, \\ a_{23} &= c_{033}\lambda_0 + c_{133}\lambda_1. \end{aligned}\right. \tag{8}$$

Now, in order for the plane to contain a line in addition to the line l, the conic section (7) must decompose; the condition for this is:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \tag{9}$$

On the basis of (8), the determinant Δ will be a form of fifth degree in λ_0 and λ_1. If it does not vanish identically then (9) will be an equation of fifth degree in the ratio $\lambda_0 : \lambda_1$, which will then possess five roots. One will thus find five planes, each of which has two lines in common with the surface $f = 0$ besides the line l. Under the assumption that the surface is free of double points, ee now show:

1. In each plane, the three lines are actually different from each other.
2. The determinant Δ is not identically zero, and its five roots are all different from each other.

Proof of 1. We assume that the surface has two overlapping lines g and a further line h in common with a plane e.

At each point of g, e is then the tangent plane of the surface; all lines in e through such a point P will then have two overlapping intersection points with the surface. We now pass any other plane e' through the g. e' intersects the surface in not only g, but also in any conic section, which must have at least one point in common with g. We again call such a point P. Each line through P in e' has two overlapping intersection points with the surface at P; hence, e' will be the tangent plane of the surface at P. However, this property is already attached to the plane e. Since there is only one tangent plane to each point of a surface that is free of double points, we have arrived at a contradiction.

Proof of 2. Assuming that $\lambda_0 : \lambda_1$ are double roots of the fifth-degree equation, we then choose an associated plane through l:

$$\lambda_1 x_0 = \lambda_0 x_1$$

to be the coordinate plane $x_0 = 0$. The parameter ratio that is associated with the plane is then $0 : 1$ ($\lambda_0 = 0$), and Δ is divisible by λ_0^2. We will thus derive a contradiction, and one will see, with no further assumptions, that the same contradiction will also appear when Δ is identically zero.

As we have already proved, the plane x_0 has three different lines in common with the surface $f = 0$. We thus have two cases to distinguish:

a) The three lines define a triangle.
b) They go through one point.

In case a), we choose the triangle of the lines to be the coordinate triangle in the plane $x_0 = 0$, and in the case b), let the intersection point of the three lines be a corner point of the coordinate triangle. The intersection point of the two lines that are different from l is called D in both cases; in case a), $D = (0, 1, 0, 0)$, and in case b), $D = (0, 0, 1, 0)$. In any case, D is a double point of the conic section (7) whose coefficient matrix, from (8), is given by:

$$\begin{pmatrix} c_{111} & \frac{1}{2}c_{112} & \frac{1}{2}c_{113} \\ \frac{1}{2}c_{112} & c_{122} & \frac{1}{2}c_{123} \\ \frac{1}{2}c_{113} & \frac{1}{2}c_{123} & c_{133} \end{pmatrix}$$

for $\lambda_0 = 0$. In this matrix, since D is a double point, one must have in case a) that the first row and column vanish, while in case b), the second row and column must vanish:

a) $c_{111} = c_{112} = c_{113} = 0$.
b) $c_{112} = c_{122} = c_{123} = 0$.

In order to now express the condition that Δ is divisible by λ_0^2, we develop Δ [equation 9] in the first row in case a), and in the second row in case b). In case a), the elements of the first row and column are divisible by λ_0, so the terms in a_{12} and a_{13} will be divisible by λ_0^2. Thus, the term:

$$a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$$

will be divisible by λ_0^2. The second factor is $\neq 0$ for $\lambda_0 = 0$, since otherwise the two lines into which the conic section (7) decomposes would have to coincide, which, from 1, would be impossible. Thus, a_{11} must be divisible by λ_0^2; i.e., one must have:

$$c_{011} = 0.$$

Likewise, in case b), a_{22} must be divisible by λ_0^2, from which, one will obtain:

$$c_{022} = 0.$$

Furthermore, in each case, one will have $c_{222} = c_{223} = 0$, since the line $x_0 = x_1 = 0$ lies completely on the surface. Thus, in case a) the equation of the surface will lack terms in:

$$x_1^2 x_0, \quad x_1^3, \quad x_1^2 x_2, \quad x_1^2 x_3,$$

and, in case b), terms in:

$$x_2^2 x_0, \quad x_2^2 x_1, \quad x_2^3, \quad x_2^2 x_3.$$

However, this means that the point D will be a double point of the surface in both cases. Now, since the surface was assumed to be free of double points, the assumption that Δ was divisible by λ_0^2 will lead to a contradiction. With that, the assertion is proved.

We thus see that there are precisely five planes through each line of the surface that contain two other lines of the surface. Consequently, *any line of the surface will intersect ten other lines of the surface.* Let π be a plane that intersects the surface in three lines l, m, n. Any further line g of the surface will intersect the plane π in a point S that lies on the surface, as well as on the plane π; hence, its intersection curve, and therefore, one of the three lines l, m, n belongs to that plane. Now, S cannot lie on – say – l and m simultaneously, since three tangents l, m, g that do not lie in a plane would then go through S, and S would be a double point of the surface. All lines of the surface that are different from l, m, n will thus intersect precisely one of the lines l, m, n. In addition to m and n, there will be eight more lines that intersect l, and likewise eight that intersect m and eight that intersect n. If one adds these l, m, and n to these 24 lines then one will obtain 27 lines. Hence:

A double-point-free surface of third order in S_3 *contains precisely* 27 *different lines.*

These 27 lines, each of which will intersect ten others, define a very interesting configuration, about which an extensive body of literature exists ([1])

§ 36. The associated form of a manifold M

In § 7, we learned how to determine the linear subspaces of a space S_n by their PLÜCKERian coordinates. We will now learn how to also represent arbitrary purely r-dimensional manifolds M in S_n by coordinates in the same way.

It is best for us to start with the zero-dimensional manifolds. An irreducible zero-dimensional manifold is a system of finitely many conjugate points:

$$\overset{i}{p} = (\overset{i}{p}_0, \overset{i}{p}_1, \cdots, \overset{i}{p}_n).$$

Thus, it can be assumed that perhaps $\overset{i}{p}_0 = 1$. Now, if $u_0, u_1, \ldots, u_n$ are indeterminates then the quantity:

$$\vartheta_1 = -\overset{1}{p}_1 u_1 - \overset{1}{p}_2 u_2 - \cdots - \overset{1}{p}_n u_n$$

will be algebraic over $K(u_1, \ldots, u_n)$, hence, it will be a zero locus of an irreducible polynomial $f(u_0)$ with coefficients in $K(u_1, \ldots, u_n)$. The remaining zero loci of this polynomial will be the ϑ_i conjugate quantities:

$$\vartheta_i = -\overset{i}{p}_1 u_1 - \overset{i}{p}_2 u_2 - \cdots - \overset{i}{p}_n u_n;$$

hence, one will have the factoring:

$$\begin{aligned} f(u_0) &= \rho \prod_i (u_0 - \vartheta_i) \\ &= \rho \prod_i \left(\overset{i}{p}_0 u_0 + \overset{i}{p}_1 u_1 + \cdots + \overset{i}{p}_n u_n \right). \end{aligned}$$

Thus, $f(u_0)$ will be completely rational in $u_1, \ldots, u_n$; we can thus write:

([1]) See A. HENDERSON: "The twenty-seven lines upon the cubic surface." Cambridge Tracts, v. 13 (1911).

$$f(u_0) = F(u_0, u_1, \ldots, u_n).$$

Since $f(u_0)$ is irreducible in u_0, and since $F(u_0, u_1, \ldots, u_n)$ contains no factors that depend upon only $u_1, \ldots, u_n$, $F(u_0, u_1, \ldots, u_n)$ will be an irreducible form in $u_0, \ldots, u_n$ with coefficients in K. It is called the *associated form* of the point system. With the well-known abbreviation:

$$(u\,p) = (p\,u) = \sum_j p_j u_j\,,$$

we can therefore write:

$$F(u) = \rho \prod_i \left(u \overset{i}{p} \right). \tag{1}$$

A reducible zero-dimensional manifold exists as various systems of conjugate points. We now understand the phrase "associated form" of a reducible manifold to mean the product of the associated forms of the conjugate points of the individual systems:

$$F = F_1\, F_2\, \ldots\, F_h\,.$$

One can also visualize the conjugate points of the individual systems as having arbitrary multiplicities ρ_k and refer to the product:

$$F(u) = F_1^{\rho_1} F_2^{\rho_2} \cdots F_h^{\rho_h}$$

as the associated form of the system of points, including multiplicities. The form $F(u)$ always has the form (1) and determines the irreducible system of points, counting their multiplicities, uniquely.

Now, let $F(u_0, u_1, \ldots, u_n)$ be any form of degree g. We would now like to exhibit the condition for this form to be the associated form of a zero-dimensional manifold. For this to be true, it is necessary and sufficient that the form be completely decomposable into linear factors:

$$F(u_0, u_1, \ldots, u_n) = \rho \prod_{i=1}^{g} \left(\overset{i}{p}_0\, u_0 + \overset{i}{p}_1\, u_1 + \cdots + \overset{i}{p}_n\, u_n \right). \tag{2}$$

If one compares both sides of (2) with the coefficients of the corresponding products of powers of $u_1, \ldots, u_n$ then one will obtain the conditions:

$$a_\nu = \rho \Psi_\nu \left(\overset{1}{p}, \cdots \overset{g}{p} \right), \tag{3}$$

where Ψ_ν is a homogeneous form in each individual sequence of coordinates $\overset{i}{p}$. Eliminating ρ from (3) will yield the homogeneous equations in the $\overset{i}{p}$:

$$a_\mu \Psi_\nu - a_\nu \Psi_\mu = 0. \tag{4}$$

From § 15, the conditions for solubility of this system of equations are found by setting the resultant systems in $\overset{1}{p}, \ldots, \overset{g}{p}$ equal to zero. One thus obtains a homogeneous system of equations:

$$R(a_\mu) = 0 \tag{5}$$

whose satisfaction is necessary and sufficient for the form F with the coefficients a_μ to be an associated form.

Now, let an irreducible r-dimensional manifold M be given. We intersect M with a general linear subspace S_{n-r}, which is the intersection of r general hyperplanes $\overset{0}{u}, \ldots, \overset{r}{u}$. Each symbol $\overset{i}{u}$ thus stands for a sequence of $n + 1$ indeterminates $\overset{i}{u}_0, \overset{i}{u}_1, \ldots, \overset{i}{u}_n$. The intersection points $\overset{1}{p}, \ldots, \overset{g}{p}$ are conjugate to each other over $K\left(\overset{1}{u}, \cdots, \overset{r}{u}\right)$. The associated form of this point system is the product:

$$\prod_{i=1}^{g}\left(\overset{0}{u}\, p_i\right),$$

in which $\overset{0}{u}$ means a new sequence of indeterminates $\overset{0}{u}_0, \overset{0}{u}_1, \ldots, \overset{0}{u}_n$. It is completely irrational in $\overset{0}{u}$ and rational in the $\overset{1}{u}, \ldots, \overset{r}{u}$. If one makes it completely rational and primitive relative to the $\overset{1}{u}, \ldots, \overset{r}{u}$ by multiplying with a polynomial in the $\overset{1}{u}, \ldots, \overset{r}{u}$ then one will obtain an irreducible polynomial that is completely rational in all indeterminates $\overset{1}{u}, \ldots, \overset{r}{u}$:

$$F\left(\overset{0}{u}, \cdots, \overset{r}{u}\right) = \rho \prod_{i=1}^{g}\left(\overset{0}{u}\, \overset{i}{p}\right), \tag{6}$$

namely, the *associated form* of the manifold M. Its degree in $\overset{0}{u}$ will be equal to the degree g of the manifold M.

It is clear that two different irreducible manifolds cannot have the same associated form. One could then obtain a general point $\overset{0}{u}$ of the manifold M from the associated form by factorizing (6), and these general points would thus establish the manifold M. The associated form F thus determines the manifold M uniquely, and the coefficients of F can be taken to be the *coordinates of the manifold*.

Example. Let M be a line that is determined by the points y and z. We write u and v, instead of $\overset{0}{u}$ and $\overset{1}{u}$. The intersection point of the line with the hyperplane $\overset{1}{u} = v$ will be found from:

$$(v, \lambda_1 y + \lambda_2 z) = \lambda_1 (v\, y) + \lambda_2 (v\, z) = 0.$$

One solution of this equation is:

$$\lambda_1 = (v\,z), \qquad \lambda_2 = -(v\,y).$$

The intersection point is then:

$$p = (v\,z)\,y - (v\,y)\,z,$$

and the associated linear form is:

$$\begin{aligned} F(u, v) &= (p\,u) = (v\,z)(y\,u) - (v\,y)(z\,u) \\ &= \sum_j \sum_k (y_i z_k - y_k z_j)\, u_j v_k\,. \end{aligned}$$

The coefficients of this form are the PLÜCKERian coordinates:

$$\pi_{jk} = y_j\,z_k - y_k\,z_j\,.$$

Problems. 1. If M is a linear subspace S_r then the coefficients of the associated form will be the PLÜCKERian coordinates of M.

2. If M is a hypersurface $f = 0$ then the associated form of M will arise from the form f when one replaces the variables x_0 , …, x_n in f with the n-rowed determinant of the matrix $\overset{i}{u}_k$ ($j = 0, \dots, n - 1$; $k = 0, \dots, n$), with the usual alternating signs.

One can also define the associated form in another way: We define a correspondence between the points y of M, on the one hand, and the sequences of $r + 1$ hyperplanes $\overset{0}{v}, \overset{1}{v}, \dots, \overset{r}{v}$ that go through them, on the other. The equations of the correspondence express the ideas that y belongs to M, and that $\overset{0}{v}, \overset{1}{v}, \dots, \overset{r}{v}$ goes through y. One obtains a general pair of the correspondence when one replaces y with a general point ξ on M and $\overset{0}{v}, \overset{1}{v}, \dots, \overset{r}{v}$ with $r + 1$ general hyperplanes $\overset{0}{w}, \overset{1}{w}, \dots, \overset{r}{w}$ that contain ξ. The correspondence is therefore irreducible.

In the formula:

$$a + b = c + d,$$

which expresses the principle of constant count, one has:

$$\begin{aligned} a &= r, \\ b &= (r - 1)(n - 1), \\ c &= 0; \end{aligned}$$

hence:

$$d = (r + 1)\,n - 1.$$

Thus, the image manifold of the correspondence will be a hypersurface in the $(r + 1)$-fold projective space of hyperplanes $\overset{0}{v}, \overset{1}{v}, \dots, \overset{r}{v}$. There is thus a single irreducible equation:

$$F_0\left(\overset{0}{v}, \overset{1}{v}, \cdots, \overset{r}{v}\right) = 0 \tag{7}$$

whose existence is necessary and sufficient for the hyperplanes $\overset{0}{v}, \overset{1}{v}, \ldots, \overset{r}{v}$ to have a point in common with M.

If one takes the $\overset{1}{v}, \ldots, \overset{r}{v}$ in (7) to be general hyperplanes $\overset{1}{u}, \ldots, \overset{r}{u}$ that intersect M at g points $\overset{1}{p}, \ldots, \overset{r}{p}$ then $F_0\left(\overset{0}{v}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$ will be zero if and only if one of the linear factors satisfies $\left(\overset{i}{p}\overset{0}{v}\right) = 0$; hence, $F_0\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$ will be divisible by the product of the forms $\left(\overset{i}{p}\overset{0}{u}\right)$, i.e., by the previously-defined associated form $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$. However, since F_0 is irreducible, it will then follow that:

$$F_0(u) = F(u);$$

i.e., the form $F_0(u)$ will be the associated form of the manifold M exactly.

An important property will follow from this new definition of the associated form, since $\overset{0}{v}, \overset{1}{v}, \ldots, \overset{r}{v}$ are on an equal footing in it:

The associated form $F(u)$ is homogeneous of degree g, not only in $\overset{0}{u}$, but also in $\overset{1}{u}$, $\ldots, \overset{r}{u}$, and it will go to itself, up to a factor, under the exchange of any two $\overset{i}{u}$.

We now go on to the irreducible, purely r-dimensional manifolds. The associated form of one such will be defined as the product of the associated forms of its irreducible components, when equipped with arbitrarily-chosen positive exponents ρ_i:

$$F = F_1^{\rho_1} F_2^{\rho_2} \cdots F_s^{\rho_s} . \tag{8}$$

If $g_1, \ldots, g_s$ are the degrees of the irreducible components then the degree of the total form F in each of its individual sequences of variables $u_0, u_1, \ldots, u_r$ will equal:

$$g = \sum_i \rho_i g_i .$$

The associated form F will determine the manifold M uniquely, just as its multiplicities ρ_i will determine its irreducible components uniquely. Furthermore, one will have:

The condition $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right) = 0$ is necessary and sufficient for any r hyperplanes $\overset{0}{v}, \overset{1}{v}, \ldots, \overset{r}{v}$ to have a point in common with M.

We already saw above that this theorem is true for irreducible manifolds. By means of the factorization (8), it carries over to decomposed manifolds with no further assumptions.

Problems. 3. If $f_\mu = 0$ are the equations of an irreducible manifold *M* and one further adjoins the $r + 1$ linear forms $\left(\overset{0}{u}\,x\right)$, $\left(\overset{1}{u}\,x\right), \ldots, \left(\overset{r}{u}\,x\right)$ to the forms $f_\mu(x)$, and then constructs the resultant system of all of these forms then the greatest common factor of the resultant system will be a power of the associated form $F(u)$.

4. How does the corresponding theorem for decomposable manifolds read?

§ 37. The totality of all associated forms for all manifolds *M*.

We first ask: How does one find the equations of a manifold *M* when its associated form $F(u)$ is given?

If a point *y* lies on *M* then $r + 1$ arbitrary hyperplanes $\overset{0}{v}, \ldots, \overset{r}{v}$ through *y* will always have a point in common with *M*. However, if *y* does not belong to *M* then one can always pass suitable hyperplanes $\overset{0}{v}, \ldots, \overset{r}{v}$ through *y* that have no point in common with *M*. Namely, one chooses $\overset{r}{v}$ such that they intersect *M* on only a manifold of dimension $r - 1$, and apply complete induction on *r*, since the assertion is clear for $r = 0$. With this, *y* belongs to *M* when and only when $r + 1$ arbitrary hyperplanes that go through *y* always satisfy the condition $F\left(\overset{0}{v}, \overset{1}{v}, \cdots, \overset{r}{v}\right) = 0$.

One obtains an arbitrary hyperplane that goes through *y* most conveniently as the zero locus of *y* relative to an arbitrary null system (which may also be singular):

$$v_j = \sum s_{jl}\, y_l \qquad (s_{jl} = -s_{lj}).$$

We write this briefly as:

$$v = Sy\,.$$

Thus, if $\overset{0}{s}_{jl}, \overset{1}{s}_{jl}, \ldots, \overset{r}{s}_{jl}$ are simple indeterminates with $\overset{i}{s}_{jl} = -\overset{i}{s}_{lj}$ and $S_0, S_1, \ldots, S_r$ are the associated null system then the condition for *y* to lie on *M* will read:

$$F(S_0 y, S_1 y, \ldots, S_r y) = 0 \tag{1}$$

(identically in $\overset{i}{s}_{jl}$). If one sets the coefficients of all powers of products of the $\overset{i}{s}_{jl}$ equal to zero in (1), in which one has replaced the $\overset{i}{s}_{jl}$ with $j > l$ by $-\overset{i}{s}_{lj}$, then one will obtain the equations of *M*.

The main question of this section reads: *What conditions must a form* $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$ *that has the same degree in all sequences of variables* $\overset{1}{u}, \ldots, \overset{r}{u}$ *satisfy in order for it to be the associated form of a manifold?*

Obviously, there are *three necessary conditions:*

1. $F(u)$, when considered as a form in $\overset{0}{u}$, must decompose completely into linear factors:

$$F(u) = \rho \prod_1^g \left(\overset{i}{p} \overset{0}{u} \right) \tag{2}$$

in an extension field of $K\left(\overset{1}{u}, \cdots, \overset{r}{u} \right)$.

2. The points $\overset{i}{p}$ that are defined by (2) must lie in all of the planes $\overset{1}{u}, \ldots, \overset{r}{u}$:

$$\left(\overset{i}{p} \overset{k}{u} \right) = 0 \qquad (i = 1, \ldots, g;\ k = 1, \ldots, r). \tag{3}$$

3. They must further satisfy the equations of M:

$$F\left(S_0 \overset{i}{p}, S_1 \overset{i}{p}, \cdots, S_r \overset{i}{p} \right) = 0. \tag{4}$$

Condition **3** can also be formulated: If the hyperplanes $\overset{0}{v}, \ldots, \overset{r}{v}$ go through one of the points $\overset{i}{p}$ then $F\left(\overset{0}{v}, \overset{1}{v}, \cdots, \overset{r}{v} \right) = 0$.

We now prove that these three conditions are also sufficient.

There is (relative to the ground field K) an irreducible algebraic manifold M_1 that possesses the point $\overset{1}{p}$ as general points. $M_2, \ldots, M_g$ will be defined correspondingly; naturally, they do not all need to be different. The union of all of the irreducible manifolds $M_1, M_2, \ldots, M_g$ is called M.

From **2**, the points $\overset{1}{p}, \ldots, \overset{g}{p}$ lie in the linear space S_{n-r} that is defined by the hyperplanes $\overset{1}{u}, \ldots, \overset{r}{u}$. Condition **3** now states that, outside of $\overset{1}{p}, \ldots, \overset{g}{p}$, S_{n-r} has no further general points in either $M_1, M_2, \ldots,$ or M_g. Namely, if this were the case, then S_{n-r} would contain a further general point q of M_1. Then, on the basis of the uniqueness theorem (§ 29), there would be an isomorphism $K(q) \cong K\left(\overset{1}{p} \right)$ that would take q to $\overset{1}{p}$. That would give rise to an isomorphism $K\left(q, \overset{1}{u}, \cdots, \overset{r}{u} \right) \cong K\left(\overset{1}{p}, \overset{1}{w}, \cdots, \overset{r}{w} \right)$. The relations:

$$\left(q \overset{k}{u} \right) = 0 \qquad (k = 1, \ldots, r),$$

which say that q lies in S_{n-r}, are preserved under isomorphisms; thus, it would follow that:

$$\left(\overset{1}{p}\,\overset{k}{u}\right) = 0 \qquad (k = 1, \ldots, r),$$

Now, if $\overset{0}{w}$ were another arbitrary plane through $\overset{1}{p}$ – hence, $\left(\overset{1}{p}\,\overset{0}{w}\right) = 0$ – then it would follow from condition **3** that:

$$F\left(\overset{0}{w}, \overset{1}{w}, \cdots, \overset{r}{w}\right) = 0.$$

From the STUDY lemma (§ 16), when one replaces the $\overset{0}{w}$ with indeterminates $\overset{0}{u}$, $F\left(\overset{0}{u}, \overset{1}{w}, \cdots, \overset{r}{w}\right)$ would be divisible by $\left(\overset{1}{p}\,\overset{0}{u}\right)$. Applying the isomorphism in the opposite direction would give that $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$ is divisible by $\left(\overset{1}{q}\,\overset{0}{u}\right)$; i.e., due to (2), q would coincides with one of the points $\overset{i}{p}$.

Now, since a general linear space S_{n-r} intersects only finitely many general points of the irreducible manifold M_1 (and indeed at least one point, namely, $\overset{i}{p}$), M_1 will be precisely r-dimensional; the same will be true for $M_2, \ldots, M_g$. The associated form of M_1 will be the product:

$$F_1 = \prod \left(\overset{i}{p}\,\overset{0}{u}\right),$$

which is extended over those $\overset{i}{p}$ that are conjugate to $\overset{1}{p}$.

When the $\overset{i}{p}$ are combined into the groups of conjugate points, the product (2) can, moreover, be written:

$$\begin{aligned} F &= \rho \left\{\left(\overset{1}{p}\,\overset{0}{u}\right) \cdots \left(\overset{e}{p}\,\overset{0}{u}\right)\right\}^{\rho_1} \left\{\left(\overset{e+1}{p}\,\overset{0}{u}\right) \cdots \left(\overset{e+f}{p}\,\overset{0}{u}\right)\right\}^{\rho_{e+1}} \\ &= \rho F_1^{\rho_1} F_{e+1}^{\rho_{e+1}} \ldots \end{aligned}$$

This factorization shows that F is equal to the associated form of a manifold M that is comprised of the components $M_1, M_{e+1}, \ldots$ with the multiplicities $\rho_1, \rho_{e+1}, \ldots$

Conditions **1**, **2**, **3** are thus sufficient.

We will now show that conditions **1**, **2**, **3** can be expressed by homogeneous algebraic relations between the coefficients a_λ of the form $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$.

In order to express condition **1** by homogeneous algebraic equations, we proceed precisely as we did at the start of this section, when we first equated the coefficients of the products of powers of $\overset{0}{u}$ in:

$$F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right) = \rho \prod_{1}^{g} \left(\overset{i}{p}\overset{0}{u}\right),$$

thus:

$$\varphi_\nu\left(\overset{1}{u}, \cdots, \overset{r}{u}\right) = \rho \psi_\nu\left(\overset{1}{p}, \cdots, \overset{g}{p}\right),$$

and then eliminated ρ:

(5) $$\varphi_\mu \, \psi_\nu - \varphi_\nu \, \psi_\mu = 0.$$

Condition **2** reads:

(6) $$\left(\overset{i}{p}\overset{k}{u}\right) = 0 \qquad (i = 1, \ldots, g;\ k = 1, \ldots, r).$$

Condition **3** will be evaluated when one sets the coefficients of the products of the powers of the indeterminates $\overset{i}{s}_{jl}$ in (4) equal to zero:

(7) $$\chi_\mu\left(a_\lambda, \overset{i}{p}\right) = 0 \qquad (i = 1, \ldots, g).$$

One eliminates the $\overset{i}{p}$ from the homogeneous conditions (5), (6), (7), by constructing the resultant system:

$$R_\chi\left(a_\lambda, \overset{1}{u}, \cdots, \overset{r}{u}\right) = 0.$$

These equations must be satisfied identically in $\overset{1}{u}, \ldots, \overset{r}{u}$. If one then sets the coefficients of the products of powers of these $\overset{k}{u}$ equal to zero then one will obtain the desired system of equations:

(8) $$T_\omega(a_\lambda) = 0.$$

The fulfillment of (8) *is necessary and sufficient for a form* $F\left(\overset{0}{u}, \overset{1}{u}, \cdots, \overset{r}{u}\right)$ *of degree g with the coefficients* a_λ *to be the associated form of an r-dimensional manifold M of degree g.*

By a small amendment to the above proof, one can also exhibit the conditions for the manifold *M* to lie on another manifold *N*.

The equations of N might read $g_\nu = 0$. If M lies on N then the general points $\overset{1}{p}, \ldots, \overset{g}{p}$ of the irreducible components of M must lie on N. This gives the conditions:

(9) $$g_\nu\left(\overset{i}{p}\right) = 0 \qquad (i = 1, \ldots, g).$$

We add these equations to (5), (6), (7), and again eliminate the $\overset{i}{p}$. This yields a system of equations that is completely analogous to (8) and is necessary and sufficient for M to lie on N. If N is given by its coordinates b_μ, or – what amounts to the same thing – by its associated form, then one can derive the equations $g_\nu = 0$ by the method that was given at the beginning of this section and then obtain the conditions for M to lie on N in the form of a doubly-homogeneous system of equations:

(10) $$T_\omega(a_\lambda, b_\mu) = 0.$$

Example. We would like to actually present the conditions (8) in the simplest case $r = 1$, $g = 1$. If we write u and v, instead of $\overset{0}{u}$ and $\overset{1}{u}$, then any form of degree 1 in the u and v will have the form:

$$F = \sum\sum a_{jk} u_j v_k .$$

In this case, when we write p instead of $\overset{1}{p}$, condition **1**:

(11) $$p_j = \sum a_{jk} v_k .$$

We can do without making these equations homogeneous, since the elimination of the p_j can come about simply by the substitution (11). Condition **2** yields:

$$\sum p_j v_j = 0,$$

or, when (11) is substituted and the coefficient of $v_j\, v_k$ is set equal to zero:

(12) $$a_{jk} + a_{kj} = 0.$$

When one sets $\overset{0}{s}_{ij} = s_{ij}$ and $\overset{1}{s}_{ij} = t_{ij}$ condition **3** will yield:

$$\sum\sum a_{ik}\left(\sum s_{ij} p_j\right)\left(\sum t_{kl} p_l\right) = 0$$

or, when (11) is substituted and the summation sign is simply omitted, one will have:

$$a_{ik}\, s_{ij}\, a_{jr}\, v_r\, t_{kl}\, a_{ls}\, v_s = 0$$

identically in the s_{ij}, t_{kl}, and v_r. Equating the product powers yields:

$$(1 - P_{ij})(1 - P_{kl})\,(1 + P_{rs})\,a_{ik}\,a_{jr}\,a_{ls} = 0, \tag{13}$$

if P_{ij} means the permutation of the indices i and j. Equations (12) and (13) are thus necessary and sufficient for the form F with the coefficients a_{jk} to be the associated form of a line, or for the a_{ik} to be the PLÜCKERian coordinates of a line. The cubic equations (13) must be equivalent to the previously derived quadratic relations (cf., § 7):

$$a_{ij}\,a_{kl} + a_{ik}\,a_{lj} + a_{il}\,a_{jk} = 0$$

The meaning of the results obtained up to now lies not in the concrete form of the equations of condition obtained, since the example above shows that they are already very complicated in even the simplest case. Rather, it lies in the fact we can now consider the totality of the purely r-dimensional algebraic manifolds of given degree as an algebraic manifold in which the points represent individual manifolds.

The associated form of a manifold will then be given by its coefficients a_λ. If one regards them as the coordinates of a point a in a projective space $\mathfrak{B}$ then each manifold M of a given degree and dimension will correspond to a *structure point A*, and conversely, M will be determined uniquely by A. $\mathfrak{B}$ is called the *structure space* of the manifolds M of degree g and dimension r. We have already presented the totality of all structure points A as an algebraic manifold in $\mathfrak{B}$ whose equations are:

$$T_\omega(a) = 0.$$

One understands the term *algebraic system of manifolds M* to mean a set of manifolds M whose structure set in $\mathfrak{B}$ is an algebraic manifold. For example, the totality of *all* manifolds M (of given degree and dimension) is an algebraic manifold, and likewise for the totality of all M that lie on a given manifold N or contain a given manifold L. These relations will then be expressed by the algebraic equations (10).

One can carry over the ideas and theorems that pertain to algebraic manifolds in this structure space to algebraic systems of manifolds with no further assumptions by means of the one-to-one map of the manifolds M to the points of a structure space $\mathfrak{B}$. One can, e.g., decompose any algebraic system into irreducible systems, one can speak of the *dimension* and *general element* of an algebraic system; one has the theorem that an irreducible system of manifolds is determined uniquely by its general element, etc. One can also consider correspondences between algebraic manifolds and other geometric objects and apply the principle of constant count. For a closer treatment of these ideas, we refer to a paper of CHOW and VAN DER WAERDEN ([1]), and for applications, to other paper of the author ([2]).

([1]) CHOW, W.-L and B. L. v. d. WAERDEN: "Zur algebraischen Geometrie IX," Math. Ann.,, Bd. 113 (1937).

([2]) WAERDEN, B. L. v. d.: "Zur algebraischen Geometrie XI and XIV," Math. Ann.,, Bd. 114 and 115.

CHAPTER SIX

The concept of multiplicity

§ 38. The concept of multiplicity and the principle of the conservation of count

We would like to examine the question: What happens to the solutions to a geometric problem under a specialization of the data of the problem?

Let the data of the problem be given by (homogeneous or inhomogeneous) coordinates x_μ . Let the desired geometric structure be given by one or more sequences of homogeneous coordinates y_μ . In order to have something specific in mind, we imagine that there is *one* sequence of homogeneous coordinates for the x, as well as the y, and correspondingly speak of the "point" x and "point" y. These assumptions are not essential. However, what is essential is another one that we shall now make: *Let the geometric problem be given by a system of equations:*

$$f_\mu(x, y) = 0 \tag{1}$$

that are homogeneous (at least, in the y-coordinates). We shall call such problems *normal problems.*

Equations (1) define an algebraic correspondence between the points x and y. One can thus also define the normal problem by means of an algebraic correspondence; that definition will be equivalent to the previous one.

The point x may run through an *irreducible* manifold M. *For a general point ξ of this manifold, the problem may have at least one solution η with* $f_\mu(x, y) = 0$. The problem will then also have at least one solution y for each point x of M; if the resultant system that comes about from (1) by eliminating y is fulfilled for a general point of M then it will be fulfilled for each point of M. Secondly, we assume that *the problem has only finitely-many solutions for a special point x of M.* Then, from Theorem 6 (§ 34), the problem will also have only finitely-many solutions for the general point ξ of M. Let these finitely-many different solutions be $\eta^{(1)}, \ldots, \eta^{(h)}$.

From a general theorem on relation-preserving specializations (§ 27), one can establish the relation-preserving specialization $\xi \to x$ by means of a relation-preserving specialization of the total system:

$$\xi \to x,\ \eta^{(1)} \to y^{(1)},\ \ldots,\ \eta^{(h)} \to y^{(h)}. \tag{2}$$

We also express this as: "*$\eta^{(1)}, \ldots, \eta^{(h)}$ will go to $y^{(1)}, \ldots, y^{(h)}$ under the specialization $\xi \to x$,*" All $y^{(k)}$ will be solutions of the equations (1); the relations $f(\xi, \eta^{(k)}) = 0$ must then remain preserved under any relation-preserving specialization. However, it is not clear at this point whether one will obtain all solutions of the system of equations (1) in this way.

The points $y^{(1)}, \ldots, y^{(h)}$ do not need to be all different; i.e., there can very well be some "overlapping" solutions under the specialization $\xi \to x,\ \eta^{(k)}, \ldots, y^{(k)}$. The number that gives how often a particular solution y of the problem (1) will occur among the solutions

$y^{(1)}, \ldots, y^{(h)}$ is called the *multiplicity* of this solution y under the relation-preserving specialization (2). The sum of the multiplicities of all the solutions of the problem (1) is obviously equal to h; i.e., it is equal to the number of solutions of the problem for a general point ξ of M. We thus obtain the *principle of the conservation of count: The number of solutions of a normal problem is preserved under the specialization $\xi \to x$, assuming that under this specialization one counts each solution as many times as its multiplicity.*

In order for this principle to actually bear fruit, two conditions must generally be satisfied: First, the multiplicities must be determined *uniquely* through the specialization $\xi \to x$ alone (i.e., independently of how one specializes the solutions $\eta^{(h)}$), and second, one must be certain that one obtains *all* solutions of the problem by the specialization (2); in other words, that no solution will take on multiplicity zero. Now, the requirements are in no way fulfilled by themselves: One can very well give examples of normal problems in which the multiplicity is not unique or in which the solutions come about with multiplicity zero. However, the following theorem gives a sufficient condition under which these untoward circumstances cannot happen.

Main theorem on multiplicities:

1. *If the (normalized) coordinates of the point ξ are rational functions of some algebraic independents under it, and these rational functions remain meaningful under the specialization $\xi \to x$, then the specialized solutions $y^{(1)}, \ldots, y^{(h)}$ will be determined uniquely by the specialization $\xi \to x$, up to the sequence.*

2. *If, in addition, the correspondence defined by* (1) *is irreducible then each solution y of the problem* (1) *will appear among the solutions $y^{(1)}, \ldots, y^{(h)}$ at least once.*

Proof of 1: The solutions $y^{(1)}, \ldots, y^{(h)}$ that belong to a general x subdivide into a system of algebraically conjugate points. It suffices to consider such a system $\eta^{(1)}, \ldots, \eta^{(h)}$ and to prove that the relation-preserving specialization of this system is determined uniquely for $\xi \to x$. For the point $\eta^{(1)}$, there will be at least one coordinate – say $\eta_0^{(1)}$ – that is non-zero; this coordinate is then non-zero for all conjugate points and can be set equal to one: $\eta_0^{(\nu)} = 1$. The coordinates $\eta_0^{(\nu)}$ will then be algebraic quantities over the field $K(\xi)$. Let the coordinates $\xi^1, \ldots, \xi^m$ be indeterminates, while the remaining ones are algebraic functions of them.

Furthermore, let $u, u_1, \ldots, u_n$ be more indeterminates. The quantity $- u_1\eta_1^{(1)} - \ldots - u_n\eta_n^{(1)}$ will be algebraic over the field $K(\xi, u)$ and will therefore be a zero locus of an irreducible polynomial $G(u_0)$ with coefficients in $K(\xi_1, \ldots, \xi_m, u_1, \ldots, u_n)$. In a suitable extension field, this polynomial will decompose completely into linear factors that are all conjugate to $u_0 + u_1\eta_1^{(1)} + \ldots + u_n\eta_n^{(1)}$, and will thus have the form $u_0 + u_1\eta_1^{(\nu)} + \ldots + u_n\eta_n^{(\nu)}$:

(3) $$G(u_0) = h(\xi)\cdot\prod_\nu (u_0 + u_1\eta_1^{(\nu)} + \cdots + u_n\eta_n^{(\nu)}) = h(\xi)\prod_\nu (u\,\eta^{(\nu)}).$$

We think of the arbitrary factor $h(\xi)$ as being determined in such a way that the polynomial $G(u_0)$ is not just rational, but integer rational in $\xi^1, \ldots, \xi^m$, and none of the ξ include factors that depend upon it alone; we then call it $G(\xi, u)$. $G(\xi, u)$ is an indecomposable polynomial in $\xi_1, \ldots, \xi_m$, $u_1, \ldots, u_n$, and, from § 36, it is called the *associated form* of the point system $\eta^{(1)}, \ldots, \eta^{(k)}$. This associated form will now give the means of establishing the relation-preserving specialization of the point systems uniquely.

If we develop both sides of the identity (3) in products of powers of the u and equate the coefficients of this product of powers then one will obtain a system of relations:

(4) $$a_\lambda(\xi) = h(\xi)\, b_\lambda(\eta),$$

from which, we will derive the homogeneous relations:

(5) $$a_\lambda(\xi)\, b_\mu(\eta) - a_\mu(\xi)\, b_\lambda(\eta) = 0.$$

These homogeneous relations must remain valid under any relation-preserving specialization $\xi \to x$, $\eta^{(\nu)} \to y^{(\nu)}$. It will then follow that:

(6) $$a_\lambda(x)\, b_\mu(y) - a_\mu(x)\, b_\lambda(y) = 0.$$

However, these relations state that the $a_\lambda(x)$ will be proportional to the $b_\lambda(y)$. The $b_\lambda(y)$ are the coefficients of the form $\prod_\nu (u\, y^{(\nu)})$; therefore, they do not all vanish. Thus, it follows from (6) that:

(7) $$a_\lambda(x) = \rho\, b_\lambda(y).$$

However, this says that the $a_\lambda(x)$ will be coefficients of the form $G(x, u)$:

(8) $$G(x, u) = \rho\prod_\nu (u\, y^{(\nu)}).$$

If we can still confirm that the form $G(x, u)$ does not vanish identically then one must have $\rho \neq 0$ in (8). On the basis of the theorem of unique factorization, the linear factors on the right-hand side, and thus also the points $y^{(1)}, \ldots, y^{(h)}$, are then determined uniquely, up to their sequence.

In order to confirm the non-vanishing of the form $G(x, u)$, we replace the unknowns $y_0, \ldots, y_n$ with indeterminates $Y_0, \ldots, Y_n$ and construct the resultant system of the forms $f_\mu(\xi, Y)$ and the linear form $(u\, Y)$ in the Y. For special values of u, the forms $R(\xi, u)$ of this resultant system will be zero if and only if the plane u goes through one of the points $\eta^{(\nu)}$. Therefore, the forms $R_\lambda(\xi, u)$ will then be divisible by the linear forms $(\eta^{(\nu)}\, u)$, and therefore their product, as well, and thus the form (3). One sets $\xi_0 = 1$ in $R(\xi, u)$ and, on the grounds of assumption 1, replaces the $\xi_{m+1}, \ldots, \xi_n$ with rational functions of $\xi_1, \ldots, \xi_m$. One then multiplies with a principal denominator (*Hauptnenner*) $N_\lambda(\xi_1, \ldots, \xi_m)$ such that the product $N_\lambda(\xi)\, R_\lambda(\xi, u)$ becomes integer rational in $\xi_1, \ldots, \xi_n$. Since $N_\lambda\, R_\lambda(\xi, u)$ is divisible by $G(\xi, u)$, and since $G(\xi, u)$ possesses no factor that depends upon just the ξ, the stated divisibility will also be valid in the domain of the polynomial in the ξ and u:

$$N_\lambda(\xi)\, R_\lambda(\xi, u) = A_\lambda(\xi, u)\, G(x, u).$$

This identity will remain valid under the replacement of ξ with x. Now, if $G(x, u) = 0$ were the case then, since $N_\lambda(x) \neq 0$, it would follow that $R_\lambda(x, u) = 0$. However, that is not the case, so the $R_\lambda(x, u)$ will define the resultant system of the forms $f_\mu(x, Y)$ and a linear form $(u\, Y)$, and this will vanish for special values of u only when the plane u goes through one of the infinitely-many points y that satisfy the equations (1). In fact, one will thus have $G(x, u) \neq 0$, from which, the proof will be concluded.

Proof of 2: When the correspondence (1) is irreducible, any point-pair (x, y) will be a relation-preserving specialization of the general point-pair (ξ, η). ξ will then be an arbitrary general point of M and η will be any of the associated points $\eta^{(\nu)}$ – say, $\eta = \eta^{(1)}$. From § 27, the relation-preserving specialization $(\xi, \eta) \to (x, y)$ can be extended to a relation-preserving specialization $(\xi, \eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(h)}) \to (x, y, y', \ldots, y'')$. From the previously-proved uniqueness theorem (Part I of this proof), $y, y', \ldots, y''$ must agree with $y^{(1)}, \ldots, y^{(h)}$ in some sequence. Therefore, y must be one of the points $y^{(1)}, \ldots, y^{(h)}$, which was to be proved.

It will follow from the theorem that was proved that the multiplicities of the individual solutions y of the problem (1) will be determined uniquely and positive, under the given assumptions.

Assumption 1 is fulfilled – e.g. – when M is the entire projective or multiply-projective space; $\xi_1, \ldots, \xi_m$ will then be simply the inhomogeneous coordinates of points ξ. However, it will also be fulfilled when M is the totality of all subspaces S_d in S_n. Then, from § 7, all of the PLÜCKERIAN coordinates of such an S_d will then be rational functions of $d(n - d - 1)$ of them.

The given assumptions may indeed be weakened, but not omitted completely. In place of assumption 1, one might satisfy – e.g. – the weaker assumption that point x is a simple point of M ([1]). Likewise, as the proof shows, one might satisfy, in place of assumption 2, the weaker assumption that the point-pair (x, y) is a relation-preserving specialization of any of the point-pairs $(\xi, \eta^{(\nu)})$. However, if one makes absolutely no assumptions then both of assertions 1 and 2 will become false, as the following examples show:

Example 1. The equations that arise from setting all two-rowed sub-determinants of the matrix:

([1]) For the proof, cf., B. L. v. d. WAERDEN, "Zur algebraischen Geometrie VI," Math. Ann. **110** (1935), pp. 144, § 3.

$$\begin{Vmatrix} x_0 & x_1 & x_2 \\ y_0^2 y_1 & y_0 y_1^2 & y_0^3 + y_1^3 \end{Vmatrix}$$

equal to zero will define an irreducible (1, 1)-correspondence between the plane cubic curve:

$$x_0\, x_1\, x_2 = x_0^3 + x_1^3$$

and the y-lines. A general point ξ of the curve will correspond to a single point η:

$$\eta_0 : \eta_1 = \xi_0 : \xi_1 .$$

The double point (0, 0, 1) of the curve, however, will correspond to two different y-points (0, 1) and (1, 0), which will both be relation-preserving specializations of the general point-pair (ξ, η). The relation-preserving specialization will not be determined uniquely then, and the multiplicity of one of the solutions (0, 1) or (1, 0) can, from preservation, be set equal to zero or one.

Example 2. Let there be given a binary bi-quadratic form:

$$a_0 t_0^4 + a_1 t_0^3 t_1 + a_2 t_0^2 t_1^2 + a_3 t_0 t_1^3 + a_1 t_0^4 \tag{9}$$

(or, geometrically: a system of four points on a line). We ask about all projective transformations:

$$t_i' = \sum e_{ik}\, t_k$$

that transform the form (or the point-quadruple) into itself. The problem may be paraphrased, with no further assumptions, in terms of homogeneous equations for the unknown coefficients e_{00}, e_{01}, e_{10}, e_{11} ; one then needs only to define the coefficients of the transformed form and (by setting the two-rowed determinants equal to zero) express the idea that these should be proportional to the original coefficients a_0, a_1, a_2, a_3, a_4 . It is known that the problem has four solutions for a general form (9): There are four projective transformations that transform a general point-quadruple on the line to itself. (They define the KLEIN Vierergruppe.) However, if the point-quadruple is a harmonic one, in particular (i.e., one with double ratio – 1), then there will be eight such transformations. There will then be an involutory transformation that has two of the four points as fixed points, and one can then multiply it with the transformations of the Vierergruppe. In the case of an equiharmonic quadruple (i.e., one with double ratio $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$), there will indeed be twelve transformations of the quadruple into itself that permute them according to the alternating group. The four (eight, resp.) new solutions to the problem that come about will obviously have multiplicity zero, now; they do not then go to one of the four solutions by a relation-preserving specialization in the general case. The assumption of irreducibility of the correspondence will not actually be fulfilled, here.

After these two unfavorable examples, we now give two other ones for which all of the assumptions for the application of the principle of the conservation of count are valid.

Example 3. Let an irreducible manifold M of dimension d in S_n be intersected with a subspace S_{n-d} . From § 34, a general S_{n-d} will intersect the manifold M in infinitely-many points. Now, when a special S_{n-d} likewise intersects the manifold M in only finitely-many points x, each of them take will take on a well-defined multiplicity (viz., the multiplicity of the intersection point). From the principle of conservation of the count, the sum of the multiplicities of all intersection points will be equal to the number of intersection points of M with the general S_{n-d} , and will thus be equal to the degree of the manifold.

We already saw the irreducibility of the correspondence between x and S_{n-d} in § 34 when we gave a general pair (x, S_{n-d}^*), from which, all of the pairs (x, S_{n-d}) with x on M and x in S_{n-d} would came about by a relation-preserving specialization. The "method of problem inversion" that was used for this will also lead to very many other normal problems as its goal. This is due to the fact that we did not start from a general S_{n-d}, but from a general point x of M, and we then draw the most general space S_{n-d}^* through this point. We thus do not start with the data of the normal problems, but with the solution, and then seek suitable – but as general as possible – data.

Moreover, since all of the assumptions of the main theorem are given, it will then follow that *the multiplicities of the intersection points of M with any S_{n-d} are determined uniquely and positive.*

The concept of the intersection point multiplicity can be carried over, with no further assumptions, to a decomposable, purely d-dimensional manifold M.

Example 4. Let the problem be that of determining the lines on a third-degree surface. We already saw in § 35 that the problem will lead to homogeneous equations in the PLÜCKERIAN coordinates of the lines. We likewise saw that the correspondence that is defined by these equations will be irreducible. A cubic surface is given by 20 unrestricted, variable coefficients. As we saw in § 35, there will be 27 different lines on a general cubic surface. Naturally, under a relation-preserving specialization, intersecting lines will go to intersecting lines; in that sense, the configuration of the lines will remain conserved. When only finitely-many lines lie on a special surface (i.e., when the surface is not a ruled surface), it then follows from the main theorem on multiplicities that each of these lines will preserve a certain positive multiplicity under this specialization and that the sum of these multiplicities will be equal to 27.

In connection with the most important example 3, we present the following definition: A point y of M is called a *k-fold point* of the manifold M when a general linear space S_{n-d} that goes through y intersects the manifold M at y with multiplicity k. If $k = 1$ then y will be called a *simple point* of M.

§ 39. A criterion for multiplicity one.

The main theorem on multiplicities that was proved in the previous paragraphs gave a criterion by which one could decide whether the solutions of a normal problem had positive multiplicities in most of the important cases. By an application of the principle of the conservation of count – in particular, "enumerative geometry" – it is, however, likewise important to have the means in hand to evaluate multiplicities as above. The most important of these means is a theorem that says that under certain conditions the multiplicity of a solution will be ≤ 1. With the help of this criterion and the main theorem on multiplicities, one can conclude that the multiplicity of a solution to a normal problem will be equal to exactly one. From this, it will then follow from the principle of the conservation of count that the number of different solutions to the problem in the general case will be precisely equal to the number of solutions in the desired special case. The latter number is often easier to determine than the former one.

The criterion for multiplicity ≤ 1 rests upon the concept of the *polar* – or *tangential* – *hyperplane* to a hypersurface. If y is a point of a hypersurface $H = 0$ and ∂_k means the partial derivative with respect to y_k then the polar or tangential hyperplane of H at the point y will be given by:

$$z_0\,\partial_0 H(y) + z_1\,\partial_1 H(y) + \ldots + z_n\,\partial_n H(y) = 0. \tag{1}$$

From EULER's theorem, the point y itself will lie in this hyperplane:

$$m \cdot H(y) = z_0\,\partial_0 H(y) + z_1\,\partial_1 H(y) + \ldots + z_n\,\partial_n H(y) = 0. \tag{2}$$

If one introduces inhomogeneous coordinates by means of $y_0 = z_0 = 1$ and one subtracts (2) from (1) then one will obtain the equation of the polar hyperplane in the form:

$$(z_1 - y_1)\,\partial_1 H(y) + \ldots + (z_n - y_n)\,\partial_n H(y) = 0. \tag{3}$$

The lines through y in the tangential hyperplane (1) will be the *tangents* to the hyperplane H at the point y. In contrast to Chap. 3, we would like to also preserve this expression when y is a double point of H and therefore equation (1) is fulfilled in z identically: In this case, *all* of the lines through y should be called tangents to H at the point y.

The desired criterion then yields the following:

Theorem. *If a normal problem is given by the equations:*

$$H_\nu(\xi, \eta) = 0,$$

and two different solutions η', η'' *go to a single solution* y *under the relation-preserving specialization* $\xi \to x$ *then the specialized hypersurfaces:*

$$H_\nu(x, y) = 0 \tag{4}$$

will have a common tangent at the point $z = y$.

The proof rests upon the fact that the connecting line of η' and η'' will go to a tangent under the specialization.

We can assume that $y_0 \neq 0$: One will then also have $\eta_0' \neq 0$ and $\eta_0'' \neq 0$. Thus, one can assume $y_0 = \eta_0' = \eta_0'' = 1$. We set:

$$\eta_k'' - \eta_k' = \tau_k ;$$

one will then have $\tau_0 = 0$, so τ is the ideal point of the connecting line η', η''. The relation-preserving specialization $(\xi, \eta', \eta'') \to (x, y', y'')$ can be extended to $(\xi, \eta', \eta'', \tau) \to (x, y', y'', t)$. This yields the equations:

$$H_\nu(\xi, \eta') = 0,$$
$$H_\nu(\xi, \eta'') = H_\nu(\xi, \eta' + \tau) = 0.$$

The latter equation can be developed in powers of $\tau_1, \ldots, \tau_n$. That will then yield:

(5) $$\sum_{k=1}^{n} \tau_k \partial_k H_\nu(\xi, \eta') \text{ + terms of higher degree} = 0.$$

In the terms of higher order, we can, in any event, keep a factor τ_k and further replace the τ_k in the remaining factors with $\eta_k'' - \eta_k'$. With that, (5) will become homogeneous in $\tau_1, \ldots, \tau_n$. If we also make (5) homogeneous in the η' and η'' by the introduction of η_0' and η_0'' then we will obtain an equation that is preserved under a relation-preserving specialization $(\xi, \eta', \eta'', \tau) \to (x, y, y, t)$. The differences $\eta_k'' - \eta_k'$ (or, homogeneously, $\eta_k''\eta_0' - \eta_k'\eta_0''$), however, will vanish under the specialization since η' and η'' will both go to y. Therefore, all that remains of the entire equation (5) will be the first term:

$$\sum_{k=1}^{n} t_k \partial_k H_\nu(x, y) = 0.$$

The tangential hyperplanes of the specialized hypersurface will then have a common (ideal) point t. Since they will also have all of the (ideal) points y in common, they will have a common tangent, as we asserted.

The criterion for multiplicity one will then follow immediately from the theorem that we just proved:

If a normal problem fulfills the conditions for the main theorem on multiplicities and if the specialized hypersurfaces (4) *have no common tangent at y then the solution y will have a multiplicity of precisely one.*

Namely, two different solutions of the general problem cannot go to the same one under specialization.

The beauty of this criterion is that in order to apply it one needs only to focus one's attention on the specialized problem (which is usually simpler than the general one); as for the general problem, all that one needs to know is that the assumptions for the application of the concept of multiplicity are valid.

§ 40. Tangential spaces.

The concept of tangential space that was explained in § 9 for curves and in § 39 for hypersurfaces shall now be explained for arbitrary purely r-dimensional manifolds M in S_n .

Let y be a point of M. We consider all of the tangential hyperplanes to all of the hypersurfaces through the point y that contain M. In the event that this space has precisely the same dimension r as the manifold M itself, it shall be called a *tangential space* to M at the point y.

We now show that an irreducible manifold M will possess a tangential space at any *general* point ξ. We normalize the point x to have $\xi_0 = 1$ and assume that $\xi_1, \ldots, \xi_r$ are algebraically independent quantities, so the remaining ones $\xi_{r+1}, \ldots, \xi_n$ will represent algebraic functions. These algebraic functions can be differentiated; the derivative of ξ_k with respect to ξ_j will be denoted $\xi_{k,j}$. We now consider the linear space S_r whose equations read (in inhomogeneous coordinates):

$$(1) \qquad \begin{cases} z_{r+1} - \xi_{r+1} = \sum_{j=1}^{r} \xi_{r+1,j} (z_j - \xi_j) \\ \qquad \cdots \\ z_n - \xi_n = \sum_{j=1}^{r} \xi_{n,j} (z_r - \xi_r). \end{cases}$$

We would now like to show that this space S_r is precisely the tangential space, and thus, the intersection of the polar hyperplanes:

$$(2) \qquad (z_1 - \xi_1)\, \partial_1 f(x) + \ldots + (z_n - \xi_n)\, \partial_n f(x) = 0,$$

where $f = 0$ are the equations of all M that include hypersurfaces. We must then show, first, that S_r is included in this intersection, and second, that this intersection is included in S_r .

The fact that the space S_r is included in all hyperplanes (2) will emerge immediately when one substitutes (1) in (2). The left-hand side of (2) will then be:

$$\begin{aligned} &\sum_{j=1}^{r} (z_j - \xi_j) \partial_j f(\xi) + \sum_{k=r+1}^{n} \sum_{j=1}^{r} \xi_{k,j} (z_j - \xi_j) \partial_k f(\xi) \\ &\qquad = \sum_{j=1}^{r} (z_j - \xi_j) \left\{ \partial_j f(\xi) + \sum_{k=r+1}^{n} \partial_k f(\xi)\, \xi_{k,j} \right\}. \end{aligned}$$

By differentiating the equation $f(\xi) = 0$ with respect to ξ_j, one will see, however, that the last bracketed expressions has the value zero.

We will show that the intersection of the hyperplanes (2) is included in S_r when we give $n - r$ particular hyperplanes (2) whose intersection is precisely S_r . To that end, we consider an irreducible equation that links ξ_{r+i} to $\xi_1, \ldots, \xi_r$:

(3) $$f_i(\xi_1, \ldots, \xi_r, \xi_{r+i}) = 0.$$

The equation can be made homogeneous by the introduction of x_0. Since it is true for the general point of M, it will be true for all points of M; thus, $f_i = 0$ will belong to the hypersurfaces that contain M. Its polar hyperplane (2) reads:

(4) $$\sum_{j=1}^{r}(z_j - \xi_j)\,\partial_j f_i(\xi) + (z_{r+i} - \xi_{r+i})\,\partial_{r+i} f_i(x) = 0.$$

If one divides this by $\partial_{r+i} f_i$ then one will get:

$$-\sum_{j=1}^{r}(z_j - \xi_j)\,\xi_{r+i,j} + (z_{r+i} - \xi_{r+i}) = 0,$$

from the definition of $\xi_{r+i,j}$.

However, that is exactly what equations (1) amount to. Therefore, the intersection of the polar hyperplanes (4) will be precisely the space S_r, from which the proof is concluded.

There then exists a tangential space S_r at a general point of M. That is, the linear system of equations:

$$z_0\,\partial_0 f(x) + z_1\,\partial_1 f(x) + \ldots + z_n\,\partial_n f(x) = 0$$

will have the rank $n - r$. The rank cannot get smaller under specialization of ξ (since a sub-determinant that is zero cannot become non-zero). If the rank becomes larger then the intersection of the polar hyperplanes will become a space S_q with $q > r$. However, if the rank remains the same under specialization then the space M will have a tangent space S_r at the point y that will represent a relation-preserving specialization of the tangential space at the general point ξ.

The tangential space can be used for the application of the criterion of § 39 to advantage. With the help of this criterion, we prove, e.g., the theorem:

If M possesses a tangential space S_r at the point y then y will be a simple point of M.

Proof: If one draws a general linear space S_{n-r} through y then it will have only the point y in common with S_r. S_{n-r} will be the intersection of r hyperplanes, and the tangential spaces to these hyperplanes at y will be the hyperplanes themselves; its intersection will then once more be S_{n-r}. The intersection of the tangential hyperplanes of the hypersurfaces that contain M will be the tangential space S_r. If one now regards the determination of the intersection points of S_r and M as a normal problem then the equations of this normal problem will be the equations of S_r and those of M, taken together. The intersection of the polar hyperplanes of y relative to all of these equations will be the intersection of the S_r and S_{n-r}, and therefore only the point y itself. Therefore, the solution y will have multiplicity one; i.e., y will be a simple intersection point of M and S_{n-r}. The assertion will follow from this.

One also has the converse of this theorem:

If y is a simple point of M then M will possess a tangential space at y.

Proof: First, the theorem is true for hypersurfaces. Namely, if y is a simple point of the hypersurface $H = 0$ then the equation $H(y + \lambda z) = 0$ will have a simple root $\lambda = 0$ for a suitable z, so the derivative:

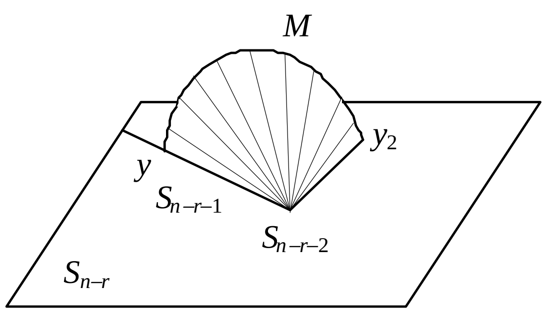
$$\frac{d}{d\lambda} H(y+\lambda z) = \sum_{k=0}^{n} z_k \partial_k H(y+\lambda z)$$

will be non-zero for $\lambda = 0$; i.e., the equation of the polar hyperplane of y:

$$\sum_{k=0}^{n} z_k \partial_k H(y) = 0$$

will not be fulfilled identically in z.

Now, let M be a purely r-dimensional manifold, and let y be a simple point of M. We draw a space S_{n-1} through y that intersects M at y only once. Let its other intersection points with M be $y_2, \ldots, y_g$. We draw a S_{n-r-1} in S_{n-r} that does not contain $y_2, \ldots, y_g$. One will ultimately choose an S_{n-r-2} in S_{n-r-1} that does not go through y. If one now links all of the points of M with all of the points of S_{n-r-2} then one obtains a projected cone K whose dimension equals $n - 1$, from the principle of constant count ([1]).



K is therefore a hypersurface. If a general line S_1 intersects K in just as many points as the connecting space of S_1 and S_{n-r-2} has intersection points with M then the degree of K is equal to the degree of M. If one chooses these lines especially so that the go through y and lie in S_{n-r}, but not in S_{n-r-1}, then one will see that y is a simple point of K. The tangential space of K at y will be a hyperplane through S_{n-r-1} whose intersection with S_{n-r} is precisely S_{n-r-1}.

If one now rotates S_{n-r-1} around y without leaving the space S_{n-r} then the intersection of all of these spaces S_{n-r-1} will be just the point y. Therefore, the tangential hyperplanes of all of the cones K will have only the point y in common with S_{n-r}. Therefore, the intersection of these tangential hyperplanes will be a linear space whose dimension does not amount to more than r, which was to be proved.

([1]) More precisely: One associates every point x of M with all points z of the connecting space of z with S_{n-r-2}. With this, a correspondence will be defined that decomposes into just as many irreducible pieces as the manifold M. From the principle of constant count, the dimension of the image manifold, and thus, the totality of all points z, will be equal to:

$$r + (n - r - 1) = n - 1.$$

By the way, for a suitable choice of coordinates, the cone K will benothing but the one that was used in § 32 (viz., the representation of manifolds as partial intersections of cones and monoids).

§ 41. Intersection of manifolds with special hypersurfaces. BEZOUT's theorem.

Let C be an irreducible curve, H, a general hypersurface of degree g, and H', a special one, where we assume that H' does not contain the curve, so it has only finitely many points in common with it. Let $\eta^{(1)}, \ldots, \eta^{(h)}$ be the intersection points of C and H. Under the specialization $H \to H'$, $\eta^{(1)}, \ldots, \eta^{(h)}$ will go to $y^{(1)}, \ldots, y^{(h)}$ in a relation-preserving manner, and any intersection point y of C and H' will obtain a uniquely determined multiplicity under the specialization that we shall call the *intersection point multiplicity* of y as the intersection point of C and H'.

The intersection point multiplicity is always positive.

Proof. From the criterion of § 38, it will suffice for us to show that the correspondence between the hypersurfaces $H'$and their intersection points y with C is irreducible. That is, however, clear (and was already pointed out in § 34); one then obtains a general pair of this correspondence when one draws the most general hypersurface H through a general point ξ of C.

In precisely the same way, one proves, more generally, that *only points with positive multiplicity will appear in the intersection of a d-dimensional manifold M with d hypersurfaces that intersect M in only finitely many points and are thought of as arising from the specialization of general hypersurfaces.* One also calls these multiplicities *intersection point multiplicities.*

From this fact, follows a:

Dimension theorem. *The intersection of an irreducible d-dimensional manifold M with a hypersurface H′ that does not contain M will contain only components of dimension* $d - 1$.

Proof. Suppose that the intersection D has an irreducible component D_1 of dimension $< d - 1$. Let y be a point of D_1 that does not belong to one of the other irreducible components $D_2 \ldots, D_r$ of D (e.g., a general point of D_1). One draws $d - 1$ hyperplanes $U'_1, \ldots, U'_{d-1}$ through the point y that intersect D in only finitely-many points, moreover. Of these intersection points, now, the point y has multiplicity zero. Then, if H is a general hypersurface, and $U_1, \ldots, U_{d-1}$ are general hyperplanes then one can perform the relation-preserving specialization $H \to H'$, $U_i \to U'_i$ in two steps: First, one specializes $H \to H'$ such that $(U_1, \ldots, U_{d-1}) \to (U'_1, \ldots, U'_{d-1})$. Under the first specialization, the intersection points $\eta^{(1)}, \ldots, \eta^{(h)}$ of M, H, $U_1, \ldots, U_{d-1}$ go to intersection points $\zeta^{(1)}, \ldots, \zeta^{(h)}$ of M, H', $U_1, \ldots, U_{d-1}$, and thus, of D, $U_1, \ldots, U_{d-1}$. On dimensional grounds, when D_1 has no points in common with the general hypersurfaces $U_1, \ldots, U_{d-1}$, none of these points will lie on D_1. Thus, $\zeta^{(1)}, \ldots, \zeta^{(h)}$ will all lie in the union $D_2 + \ldots + D_r$. That will remain true, however, when one specializes $U_1, \ldots, U_{d-1}$ to $U'_1, \ldots, U'_{d-1}$, moreover. $\zeta^{(1)}, \ldots, \zeta^{(h)}$ will then go to points $y^{(1)}, \ldots, y^{(h)}$ that lie on $D_2 + \ldots + D_r$, and therefore y will not appear among them. On the other hand, as we have seen, any intersection y will have a positive multiplicity. The contradiction will prove that our assumption was false.

The dimension theorem that was just proved is a special case of a general theorem on the intersection of two manifolds of dimensions r and s with $r + s > n$, which is, however, essentially more difficult to prove ([1]).

We now turn to the case of a curve that intersects a hypersurface and prove "BEZOUT's theorem," as it relates to this case:

The number of intersection points of an irreducible curve C with a general hypersurface H is equal to the product $g\,\gamma$ of the degrees of C and H.

Proof. Consider the irreducible correspondence that associates each point y of C with all hyperplanes v that go through y. One will obtain a general pair (η, u) of the correspondence when one either draws the most general hyperplane through a general point η of C or when one starts with a general hyperplane u and chooses any of the intersection points of u with C for η. From the *first* way of picturing the general point-pair (η, u), one will learn that the hyperplane u does not contain the tangent to the curve C at the point η, but has only the point y in common with it. It will follow that the same will also be true for the second way of generating a general point-pair, so the algebraic properties of the general pair will always be the same. It then follows that: *A general hyperplane has only one point in common with the curve tangent to its intersection point with the curve C.*

Furthermore, by a relation-preserving specialization, we go from the general hypersurface H to a hypersurface H' that decomposes into mutually independent general hyperplanes $L_1, \ldots, L_\gamma$ at γ. The number of intersection points η of C with H' is obviously equal to $g\,\gamma$. The multiplicities of these intersection points η are, on the one hand, positive, but on the other hand, from the criterion of § 39, also not greater than one, since otherwise the tangential space of the curve at the point η (cf., § 40) would have at least one line in common with the polar hyperplane of η relative to H'. If η is, perhaps, a point of L_1, then the polar hyperplane of η relative to H' will also be L_1, and L_1 will have only one point η in common with the tangent of the curve. The multiplicities of the intersection points η will all be equal to one. From the principle of the conservation of count, the number of intersection points of H and C will now be also equal to $g\,\gamma$ now, which was to be proved.

Generalization. *The intersection of an irreducible manifold M of degree γ with a general hypersurface of degree g has degree $g\,\gamma$.*

Proof. M has the dimension d, so the intersection with H will have dimension $d - 1$. If one intersects M with $d - 1$ general hyperplanes then, from § 33, one will obtain an irreducible curve of degree γ. From BEZOUT's theorem, this will intersect H in $g\,\gamma$

([1]) See B. L. v.d. WAERDEN: "Zur algebraischen Geometrie XII," Math. Ann., Bd. 115, pp. 330.

points. The intersection of M and H will cut the general space S_{n-d+1} in $g\gamma$ points, which was to be proved.

Repeated application yields:

The intersection of an irreducible d-dimensional manifold of reduced degree γ with $k \leq d$ general hypersurfaces of degrees $e_1, \ldots, e_k$ will have degree $\gamma e_1, \ldots, e_k$. In the case where $k = d$, it will then consist of $\gamma e_1, \ldots, e_k$ points.

If one goes from the hypersurfaces $H_1, \ldots, H_k$ to the special hypersurfaces $H_1', \ldots, H_k'$ then the intersection $M \cdot H_1' \ldots H_k'$ might decompose into the irreducible components $I_1, \ldots, I_r$. None of them will have a dimension that is $< d - k$. We will assume that they all have precisely the dimension $d - k$.

We would now like to define the *multiplicity* or intersection multiplicity of an irreducible component I_ν that has dimension $d - k$. To that end, we add $d - k$ general hyperplanes $L_1, \ldots, L_{d-k}$ that cut I_ν in g_ν conjugate points. Any of these intersection might have the multiplicity μ_ν, since they are the intersection points of $M, H_1, \ldots, H_k, L_1, \ldots, L_{d-k}$. We call them the *intersection multiplicities* of I_ν.

The number g_ν is the number of intersection points of I_ν with $L_1, \ldots, L_{d-k}$, so it is the degree of I_ν. The sums of the multiplicities of all conjugate intersections of $I_\nu, L_1, \ldots, L_{d-k}$ is $g_\nu \mu_\nu$ so the sum of the multiplicities of $M, H_1, \ldots, H_k, L_1, \ldots, L_{d-k}$ will be equal $\sum g_\nu \mu_\nu$. On the other hand, this sum is equal to $\gamma e_1 e_2 \ldots e_k$. It will then follow that:

The sum of the degree of the irreducible components of the intersections $M \; H_1' \ldots H_k'$, multiplied by their multiplicities, is equal to the product of the degrees of M and $H_1' \ldots H_k'$:

$$\sum g_\nu \mu_\nu = \gamma e_1 e_2 \ldots e_k .$$

One can generalize from this theorem in two directions. First, one can carry it over to multiply projective spaces, as was done in "Zur algebraischen Geometrie I," Math. Ann., Bd. 108, pp. 121. Second, one can also apply it to manifolds of arbitrary dimensions in projective S_n (cf., Zur algebraischen Geometrie XIV, Math. Ann., Bd. 155, pp. 619).

Problem. Show that the multiplicities of the irreducible intersection components that one obtains when one first intersects M with H_1' and then the individual intersection components with H_2' will be the same as their multiplicities as components of the intersection $M H_1' H_2'$. [The method of proof will be the same as for the dimension theorem: The specialization $(H_1, H_2) \to (H_1', H_2')$ can also be performed in two steps.]

The connection of these paragraphs with the previous one in § 17 can be exhibited by the following theorem:

In the case of two plane curves, the multiplicities for the intersection points that were defined in § 17 *will coincide with the ones that were defined here.*

Proof. First, let one of the two curves be a general curve H of the degree in question. From "BEZOUT's theorem" that we proved in these paragraphs the number of intersection points will then be equal to the product of the degree numbers. The sum of the intersection point multiplicities that was defined in § 17 will be, however, equal to the product of the degrees: These multiplicities must then be equal to one. The resultant $R(p, q)$ that was defined in § 17 will thus have the following factor decomposition with the exponents one:

$$R(p, q) = c\prod_{\nu} (p\,q\,s^{(\nu)})\,. \tag{1}$$

If one now goes from the general curve H to a special curve H' by a relation-preserving specialization then the factor decomposition (1) will remain preserved [cf., the corresponding consideration in § 38, formulas (3) to (8)]. The multiplicities that are defined by the relation-preserving specialization will then agree with the ones that emerge from the factor decomposition of $R(p, q)$, which was to be proved.

In conclusion, we prove the theorem:

If f and g are forms of equal degree, the second of which (but not the first) is zero on the irreducible manifold M, then the intersection of M with the hypersurface $f = 0$ *will agrees precisely with the intersection of M with* $f + g = 0$, *and also as far as the multiplicities of the irreducible components are concerned.*

Proof. M again has the dimension d. If g is zero on M then one will also have $f + g = 0$ an the points of M where $f = 0$, and conversely. By the definition of multiplicities of the irreducible components of the intersection of M with $f = 0$, we must next add $d - 1$ general hyperplanes $L_1, \dots, L_{d-1}$ and then produce the hypersurface $f = 0$ by specializing a general hypersurface $F = 0$; the relation-preserving specialization of the intersection points will then provide the desired multiplicities. We now perform the specialization in two steps: First, we let F go to $f + \lambda g$, where λ is an indeterminate, and then we specialize $\lambda \to 0$, or $\lambda \to 1$, if we would like to have $f + g$ instead of f. As a point set, the intersection of M with $f + \lambda g = 0$ will again be the same as that of M with $f = 0$. The intersection points of M with $L_1, \dots, L_{d-1}, f + \lambda g = 0$ (which will be independent of λ) will have certain multiplicities for undetermined λ, which can be determined as exponents in a certain factor decomposition, and there will then be no other functions of λ besides whole numbers. These multiplicities cannot change under the specializations $\lambda \to 0$ or $\lambda \to 1$, since they do not depend upon λ at all. The assertion follows from this.

CHAPTER SEVEN

Linear families

§ 42. Linear families on an algebraic manifold.

Let M be an irreducible ([1]) algebraic manifold of dimension d in a space S_n. Let a linear family of hypersurfaces:

$$\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_r F_r = 0 \qquad (r \geq 0) \tag{1}$$

be so arranged that no hypersurface of the family contains the entire manifold M. The hypersurfaces (1) will then cut certain submanifolds N_λ of dimension $d - 1$ out of M. From § 41, the irreducible components of N_λ will be endowed with certain multiplicities (viz., intersection multiplicities). If one varies $\lambda_0, \ldots, \lambda_r$ then N_λ will run through a collection of manifolds that one calls a *linear family* of *dimension r*.

The definition above will now be extended to something more convenient by adding an arbitrary fixed (i.e., independent of λ) manifold that lies on M and has the same dimension as it, with arbitrary (positive or negative) multiplicities to the manifolds N_λ, or also by dropping a fixed component of N_λ that is perhaps present.

In order to make that more precise, we define: A sum of irreducible manifolds of the same dimension that are endowed with positive or negative multiplicities is called a *virtual manifold.* If the multiplicities are all positive then one will have an *effective manifold.* Any set of effective manifolds will possess a (possibly empty) *greatest common submanifold* of the same dimension that will consist of the irreducible components that are common to all of the manifolds in the set, each of which will have the lowest multiplicity with which it enters into any manifold of the set.

Let A be the largest common submanifold to all of the intersection manifolds N_λ that are cut out of the M by the hypersurfaces (1). If one then sets $N_\lambda = A + C_\lambda$ then the C_λ will define a *linear family with no fixed component.* Furthermore, if B is an arbitrary, virtual manifold of dimension $d - 1$ in M then the sums $B + C_\lambda$ will define the most general *linear family* with the fixed component B. With this definition, the components with negative multiplicities can therefore only be included in the fixed component B, but not in the varying part C_λ.

Example 1. Let M be a plane, cubic curve with a double point. Let the hypersurfaces (1) be the lines through the double point. The manifolds N_λ consist of the doubly-counted double point and the moving point C_λ. After dropping the doubly-counted double point, we will obtain a linear family with no fixed component whose elements are the single points of the curve. The double point appears twice as an element of the family (corresponding to the double tangents).

([1]) One can also drop the irreducibility condition when explains the concept in a somewhat different way. Cf., on that, F. Severi, "Un nuovo campo di ricerche," Mem. Reale Accad. d'Italia, v. 3 (1932).

Such a linear family of single points is not possible on a double-point-free, plane, cubic curve. If the hypersurface (1) has degree m then and carries $3m - 1$ intersection points with the curve then, from § 24, the $3m^{\text{th}}$ intersection point will then be determined uniquely. In this case, the moving part C_λ of a linear family will then consist of at least two points.

Example 2. Let M be a quadratic surface in S_n. Let the hypersurfaces (1) be planes through a line A that lies on the surface. The manifolds N_λ consist of that line A and some varying lines C_λ. If M is a cone then all generators of the cone will run through C_λ; if M is not a cone then one of the two families of lines of the quadric M will run through C_λ. These two families of lines will then be linear families.

We now drop the assumption that no hypersurface of the family (1) contains the manifold M. Perhaps t linearly-independent forms of the family (1) might contain M; we can assume that they are $F_{r-t+1}, \ldots, F_r$. Any hypersurface (1) will then have precisely the same intersection with M as the hypersurface:

$$\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_{r-1} F_{r-1} = 0, \tag{2}$$

so the rest of the sum on the left-hand side of (1) will indeed become zero on M ([1]). However, the hypersurfaces (2) cut a linear family of dimension $r - 1$ out of M. It follows from this that:

Theorem 1. *A linear family of forms* (1) *of dimension r in which t linearly-independent forms contain M will cut a linear family of dimension $r - 1$ out of M.*

The dimension r of a linear family can be characterized by intrinsic properties of the family; it thus does not depend upon which hypersurfaces cut out the family.

Namely, let P_1 be a point of M that is not a basis point for the family of hypersurfaces (1). If one then wishes to look for those manifolds C_λ in the family that include the points P_1 then one must substitute the point P_1 in equation (1). That will yield a linear equation for the parameters $\lambda_0, \ldots, \lambda_r$, and thus, a linear sub-family of dimension $r - 1$. If one now chooses a second point P_2 that is not a basis point for this sub-family and proceeds in that way up to P_r then one will ultimately obtain a sub-family of dimension 0, and thus, a fixed element C_λ of the original family that includes the points $P_1, \ldots, P_r$. It then follows that:

Theorem 2. *The dimension r of a linear family is equal to the number of arbitrary points through which an element of the family is determined.*

Corollary. *The dimension of a linear family of point-groups on a curve is at most equal to the number of variable points in a point-group of the family.*

([1]) Cf., the last theorem in § 41.

In the sequel, Λ will denote a sequence of undetermined quantities $\Lambda_0, \dots, \Lambda_r$. The associated element C_Λ ($B + C_\Lambda$, resp., when the family contains a fixed component B) will be called the *general element* of the linear family.

Theorem 3. *A linear family is determined by its general elements $B + C_\Lambda$, independently of the family of forms* (1).

Proof. By intersecting M with a general, linear space S_{n-d+1}, the dimension of M can be reduced to one, the dimension of $B + C_\Lambda$, to zero, and the dimension of any special element $B + C_\lambda$ of the family to zero, in any case. However, if the intersection of $B + C_\lambda$ with a general, linear S_{n-d+1} is known then the manifold $B + C_\lambda$ itself will also be known. We can then restrict ourselves completely to the case of a curve ($d = 1$). With that, Theorem 3 then comes down to the following one:

Theorem 4. *Let a linear family be given on a curve M. Its general element $B + C_\Lambda$, just like any special element $B + C_\lambda$, will then be a point-group (i.e., a zero-dimensional manifold) on M. The points of $B + C_\lambda$ will then emerge from the points of $B + C_\Lambda$ through the relation-preserving specialization $\Lambda \to \lambda$.*

Proof. We set:

$$F_\Lambda = \Lambda_0 F_0 + \Lambda_1 F_1 + \dots + \Lambda_r F_r ,$$
$$F_\lambda = \lambda_0 F_0 + \lambda_1 F_1 + \dots + \lambda_r F_r ,$$

and understand F to be a general form of the same degree as F_λ and F_Λ . The multiplicities of the points of N_Λ (viz., the intersection of M with F_Λ) will be defined by the relation-preserving specialization $F \to F_\Lambda$; likewise, the multiplicities of the points of N_λ will be defined by the specialization $F \to F_\lambda$. The last specialization can be performed in two steps: $F \to F_\Lambda$ and $F \to F_\lambda$. Thus, N_Λ will go to N_λ precisely under the relation-preserving specialization $\Lambda \to \lambda$. That will remain valid when the fixed points A are dropped, and if new fixed points B are added then these fixed points will remain simply unchanged under the relation-preserving specialization. Thus, $B + C_\Lambda$ will go to $B + C_\lambda$ under the relation-preserving specialization $\Lambda \to \lambda$.

The linear family whose general element is C_Λ will be denoted by $| C_\Lambda |$.

Problems. 1. A linear family of point-groups on a curve is an irreducible system of zero-dimensional manifolds, in the sense of § 37. (One employs Theorem 4 and the method of § 38.)

2. A linear family of ($d - 1$)-dimensional manifolds on M_d is an irreducible system, in the sense of § 37. (One uses problem 1.)

Any effective, linear family $| B + C_\Lambda |$ is connected with an algebraic correspondence between the parameter values λ and the points η of $B + C_\lambda$. This is seen most easily with the linear family of the complete intersections N_λ of (1) with M; the associated correspondence is, in fact, defined by the equations of M:

$$g_r(\eta) = 0, \tag{3}$$

and through the equation of the hypersurface F_λ :

(4) $$\lambda_0 F_0(\eta) + \lambda_1 F_1(\eta) + \ldots + \lambda_r F_r(\eta) = 0.$$

We now first cease to consider all of the basis points of the hypersurfaces (1), and then seek to get all of the remaining pairs of the correspondence from a general pair (λ^*, ξ). To that end, let ξ be a general point of M, and let λ^* be the general solution of the linear equation:

(5) $$\lambda_0^* F_0(\xi) + \lambda_1^* F_1(\xi) + \cdots + \lambda_r^* F_r(\xi) = 0.$$

We now assert: *All pairs* (λ, η) *of the correspondence that is defined by* (3), (4) *for which not all* $F_\nu(\eta) = 0$ *are relation-preserving specializations of the general pair* (λ^*, ξ).

Proof. Let $F_0(\eta) \neq 0$. If a relation $H(\lambda^*, \xi) = 0$ is true then we set:

(6) $$\lambda_0^* = \frac{\lambda_1^* F_1(\xi) + \cdots + \lambda_r^* F_r(\xi)}{-F_0(\xi)}$$

in it; it will then be fulfilled by $\lambda_1^*, \ldots, \lambda_r^*$ identically. From now on, we replace the general point ξ of M with a special point η. Finally, we replace $\lambda_1^*, \ldots, \lambda_r^*$ with $\lambda_1, \ldots, \lambda_r$. Due to (4), one will have:

$$\frac{\lambda_1 F_1(\eta) + \cdots + \lambda_r F_r(\eta)}{-F_0(\eta)} = \lambda_0,$$

so one can once more subsequently cancel the substitution (6). It will then follow that $H(\lambda, \eta) = 0$. Thus, (λ, η) will be a relation-preserving specialization of (λ^*, ξ).

The general pair (λ^*, ξ) defines an irreducible correspondence $\mathfrak{K}$. Under this correspondence, a general point Λ will correspond to a relatively irreducible manifold of points η of dimension $d - 1$, which, from what was just proved, will include at least all points of $N = A + C_\Lambda$ that are not basis points of the family (1). The irreducible components of N_Λ that consist of nothing but such basis points will be fixed, and will thus be components of A. From what was just said, the remaining irreducible components of N_Λ will all be contained in a single, irreducible manifold of dimension $d - 1$, and will thus be identical with it. Therefore, C_Λ will consist of only a single irreducible component. Furthermore, if Ξ is a general point of C_Λ then (Λ, Ξ) will be a general pair of the correspondence $\mathfrak{K}$ that must agree with the pair (λ^*, ξ) in all of its algebraic properties. With that, we have proved:

Theorem 5. *The general element* C_Λ *of a linear family with no fixed component is irreducible over the field* $K(\Lambda)$. *If* Ξ *is a general point of* C_Λ *then the point pair* (Λ, Ξ) *will agree with the pair* (λ^*, ξ) *in all of its algebraic properties.* It is then completely irrelevant whether one first chooses a general point ξ of M and draws the most general element C_{λ^*} of the linear family $| C_\Lambda |$ through it, or starts with the general element C_Λ of the linear family and chooses a general point Ξ on it.

We now go from the general element C_Λ of the linear family to any special element C_λ and prove:

Theorem 6. *The irreducible correspondence* $\mathfrak{K}$ *that is defined by the general element* (λ^*, ξ) *or* (Λ, Ξ) *associates any value* λ *with the point* C_λ *precisely*. That then means: A pair (λ, η) is a relation-preserving specialization of (Λ, Ξ) if and only if η is a point of C_λ.

Proof.

1. Let η be a point of C_λ, and thus, a point of an irreducible component C_λ^1 of C_λ. Let η^* be a general point of C_λ^1. η will then be a relation-preserving specialization of η^*. It will then suffice to prove that (λ, η^*) is a relation-preserving specialization of (Λ, Ξ).

One can obtain η^* as the intersection point of C_λ^1 and a general linear space S_{n-d+1}. The dimension of M will be reduced by 1 by intersecting it with S_{n-d+1}. M will then go to a curve M, from which a linear family of point-groups will be cut by the hypersurfaces (1). From Theorem 4, any special point-group of this family will emerge from the general point-group of the family by a relation-preserving specialization. Thus, (λ, η^*), and therefore also (λ, η), will be a relation-preserving specialization of (Λ, Ξ).

2. Let (λ, η) be a relation-preserving specialization of (Λ, Ξ). Thus, Ξ can again be described as one of the intersection points of C_Λ with a general linear space S_{n-d+1}. We now also draw a linear space S'_{n-d+1} through η that intersects N_λ only at finitely many points; e.g., when we connect η with $n - d + 1$ general points of the space S_n. As one easily sees, one will then have a relation-preserving specialization:

$$(\Lambda, \Xi, S_{n-d+1}) \to (\lambda, \eta, S'_{n-d+1}).$$

If $\Xi^{(1)}, \ldots, \Xi^{(g)}$ are all intersection points of C_Λ with S_{n-d+1} then one can extend this relation-preserving specialization to a similar specialization of all intersection points:

$$(\Lambda, S_{n-d+1}, \Xi^{(1)}, \ldots, \Xi^{(g)}) \to (\lambda, S'_{n-d+1}, \eta^{(1)}, \ldots, \eta^{(g)}).$$

Thus, $\Xi^{(1)}, \ldots, \Xi^{(g)}$ will be solutions to a normal problem into which Λ and S_{n-d+1} enter as data, and which will also possess only finitely-many solutions under the specialization $(\Lambda, S_{n-d+1}) \to (\lambda, S'_{n-d+1})$, since, in fact, N_λ has only finitely-many intersection points with S'_{n-d+1}. The relation-preserving specialization will be determined from the main theorem of § 38.

We can do this in steps, if we first let Λ go to λ and then S_{n-d+1} to S'_{n-d+1}. From Theorem 4 (when applied to the intersection curve of M with S_{n-d+1}), under the first step, the intersection point of C_Λ and S_{n-d+1} will go to that of C_λ and S_{n-d+1}. Under the second step, the points of C_λ must remain on C_λ, since λ will no longer vary. Thus, $\eta^{(1)}, \ldots, \eta^{(g)}$ will all be points of C_λ ; in particular, η will be a point of C_λ .

§ 43. Linear families and rational maps.

The preeminent importance that linear families possess in algebraic geometry is, above all, based upon the fact that they mediate rational maps.

We first consider a one-dimensional family $| C_\Lambda |$ with no fixed component that might be defined by the family of forms:

$$\lambda_0 F_0 + \lambda_1 F_1 = 0. \tag{1}$$

If η is a point of C_λ that does not belong to the basis manifold $F_0 = F_1 = 0$ then it will follow from (1) that:

$$-\frac{\lambda_1}{\lambda_0} = \frac{F_0(\eta)}{F_1(\eta)}. \tag{2}$$

A *rational function on M:*

$$\varphi(\eta) = \frac{F_0(\eta)}{F_1(\eta)} \tag{3}$$

will then belong to a linear family, which is naturally defined only where its numerator and denominator do no both vanish. In particular, that will be the case for any general point of M. This rational function will bring about a map of M onto a straight line. If the denominator is zero, but not the numerator, then the image point will be the imaginary point of the straight line.

The locus of points η of M at which the function $\varphi(\eta)$ assumes a well-defined value:

$$\lambda = -\frac{\lambda_1}{\lambda_0}$$

(which can also be ∞) will be precisely the manifold C_λ . This locus will then be given by equation (1), in which the points with $F_0(\eta) = F_1(\eta) = 0$ will once more be omitted from consideration.

For example, if M is a curve then $\varphi(\eta)$ will be a rational function on the curve that assumes a well-defined value at every point, with finitely-many exceptions. (Indeed, one can ignore these exceptions by invoking the concept of a branch; the function will assume a well-defined value on any branch). For a fixed λ, the function will have finitely-many λ-points, at which it takes on the value λ, namely, the points of the point-group C_λ. When λ varies, this point-group will run through the linear family $| C_\Lambda |$.

We now go on to the general case of a linear family $| C_\Lambda |$ that is defined by the family of forms:

$$\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_r F_r = 0. \tag{4}$$

We next once more omit all points of M at which all F_r become zero; in particular, the fixed components of the linear family that is defined by (4) will drop out of the analysis.

If one now poses the condition for a point η of M that the element C_λ should contain the point η then one will obtain a linear equation for $\lambda_0, \ldots, \lambda_r$:

$$\lambda_0 F_0(\eta) + \lambda_1 F_1(\eta) + \ldots + \lambda_r F_r(\eta) = 0. \tag{5}$$

The coefficients of this linear equation can be regarded as coordinates of a point η' in the space S_r, which will be:

$$\eta'_j = F_j(\eta) \qquad (j = 0, 1, \ldots, r). \tag{6}$$

Since (6) is especially meaningful when η is a *general* point of M, and since a rational map is determined by the map of a general point of M, to begin with, (6) will define a *rational map of M into S_r*.

In order to determine the map numerically, one must know the forms $F_0, \ldots, F_r$. For the geometric determination of the map, however, it will suffice for one to know the manifold C_λ for each value of λ. One can then pose the linear condition in λ for C_λ to contain the point η for any general point η. However, from § 42, Theorem 3, in order to establish the C_λ, it will suffice to know the general element C_Λ of the linear family. It will then follow that:

Two linear families will define the same map if their general elements C_Λ agree with each other when one omits their fixed components.

The converse of this theorem is also true: *If two linear families define the same map of M into S_r then they will agree with each other, except on their fixed components.*

Proof. Let the two families be given by:

$$\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_r F_r = 0, \tag{7}$$
$$\lambda_0 G_0 + \lambda_1 G_1 + \ldots + \lambda_r G_r = 0. \tag{8}$$

The corresponding maps:

$$\xi'_j = F_j(\xi),$$
$$\xi'_j = G_j(\xi),$$

resp., of a general point ξ of M must then coincide; i.e., one must have:

$$F_0(\xi) : F_1(\xi) : \ldots : F_r(\xi) = G_0(\xi) : G_1(\xi) : \ldots : G_r(\xi),$$

or, what amounts to the same thing:

$$F_0(\xi)\, G_j(\xi) - G_0(\xi)\, F_j(\xi) = 0 \qquad (j = 1, \ldots, r).$$

This equation, which is valid for the general point ξ, must be true for any point of M:

(9) $$F_0\,G_j - G_0\,F_j = 0 \qquad \text{on } M.$$

If one now multiplies equation (7) by G_0, and similarly multiplies (8) by F_0, then only the fixed components of the two linear families will change, and one will get:

(10) $$\lambda_0\,G_0\,F_0 + \lambda_1\,G_0\,F_1 + \ldots + \lambda_r\,G_0\,F_r = 0,$$
(11) $$\lambda_0\,F_0\,G_0 + \lambda_1\,F_0\,G_1 + \ldots + \lambda_r\,F_0\,G_r = 0.$$

On the basis of (9), (10) and (11) will define precisely the same intersection with M. Thus, the two linear families will coincide, up to fixed components.

The forms of equal degree $F_0, \ldots, F_r$ in (4) were entirely arbitrary, up to the condition that no linear combination $\lambda_0\,F_0 + \lambda_1\,F_1 + \ldots + \lambda_r\,F_r$ should be equal to zero on all of M. For the map (6), that would means that no linear equation with constant coefficients should exist between the η'_j; in other words, that the image manifold should not be contained in a proper linear subspace of S_r. We can then summarize what we have proved up to now in the theorem:

Any rational map of M into S_r for which the image manifold M' does not lie in a proper linear subspace of S_r will correspond to a unique linear family on M, and conversely.

This map (6) does not need to be birational; indeed, M can get mapped to a lower-dimensional image manifold M'. If $r = 0$ then the map will become trivial: It will map M to a single point P_0.

If two manifolds M_1 and M_2 are mapped to each other birationally then each rational map of M_1 will correspond to a rational map of M_2, and conversely. Now, since rational maps are facilitated by linear families, it will follow that:

Any linear family with no fixed components on M_1 will be in one-to-one correspondence with a similar linear family on M_2.

The one-to-one correspondence does not extend to the fixed components. For example, if M_1 is a cubic curve with a double point, and M_2 is a line onto which M_2 can be mapped birationally by projecting its points from the double point then the double point itself will correspond to two points on M_2. If the double point then enters into a linear family as a fixed point then one will not know which point of M_2 one should make it correspond to. In order to make the one-to-one transformation of individual points possible, one must next decompose the multiple points into their individual branches, and then consequently speak, not of the points of M_1, but of the branches. For d-dimensional manifolds, one can also correspondingly decompose the singular $(d - 1)$-dimensional submanifolds into several "sheets." Therefore, we will first consider this modification of the concept of linear family later. For the time being, we take the manifold M to be such as it is, and we can, as a result, define a birational transformation only for linear families *without* fixed components.

Problem. 1. If M_1 is mapped to M_2 birationally then every linear family without fixed components on M_2 will correspond to a unique family of that kind on M_1 .

2. If the linear family on M_2 in prob. 1 is cut out by the family of forms $\sum \lambda_k F_k$, and the map of M_1 to M_2 is defined by $\xi'_j = \varphi_j(\xi)$ then one will obtain the corresponding linear family on M_1 by replacing the variables in the form $\sum \lambda_k F_k$ with the forms φ_j .

3. What linear family mediates the projection of M from a subspace S_{k-1} to a subspace S_{n-k} of S_n?

4. What map of the plane will be mediated by a net of conic sections with three basis points? (One chooses the basis points to be the corners of the coordinate triangle.)

An element C_λ of the linear family (4) will correspond to the intersection of M' with a hyperplane whose coordinates are $\lambda_0, \ldots, \lambda_r$ under the map (6); it will then follow from (5) and (6) that:

$$\lambda_0\eta'_0 + \lambda_1\eta'_1 + \cdots + \lambda_r\eta'_r = 0. \tag{12}$$

We would now like to examine the extent to which the correspondence between the points of C_λ and the points of the hyperplane λ still remains valid when one adds points for which $F_0(\eta) = F_1(\eta) = \ldots = F_r(\eta) = 0$. Such a point η can correspond to several image points η' under the map. We now assert:

If η lies on C_λ then at least one of the corresponding points η' will lie on the hyperplane (12). *Conversely, if η' lies on the hypersurface* (12) *then η will always lie on C_λ .*

In order to prove this, we consider the irreducible correspondence between the point pairs (η, η') of the map, on the one hand, and the hyperplanes λ that go through η', on the other. The equations of the correspondence express the idea that (η, η') is a point-pair of the map, and that λ goes through η', which is equation (12). One obtains a general pair of elements – or even better, a triple (ξ, ξ', λ^*) – of the correspondence when one starts with the general pair (ξ, ξ') of the rational map and draws the most general hyperplane λ^* through ξ'; λ^* will be defined precisely as in § 42. If (η, η') is a point-pair of the map, and λ is a hyperplane through η' then (η, η', λ) will be a relation-preserving specialization of (ξ, ξ', λ^*), so (η, λ) will be a relation-preserving specialization of (ξ, λ^*). From Theorem 6 (§ 42), it will follow that η will be a point of C_λ . Conversely, if η is a point of C_λ then (η, λ) will be a relation-preserving specialization of (ξ, λ^*) that can be extended to a relation-preserving specialization (η, η', λ) of (ξ, ξ', λ^*). There will then be a point η' that is associated with η by the map and lies in the hyperplane λ. Everything is proved with that.

The theorem that was just proved yields a remarkable corollary. The points η of M that correspond to a manifold of points η' that is at least one-dimensional by a rational map are called *fundamental points* of the map. If η is a fundamental point then any hyperplane in the image space will contain at least one associated point η', so η will lie on all of the manifolds C_λ . Conversely, if η lies on all C_λ then any hyperplane in the image space will contain at least one associated point η', so the points η will define a manifold in image space of dimension at least one. Thus: *The fundamental points of a* *rational map are precisely those points of M that are common to all manifolds of the linear family that mediates the map.*

If M is a curve then it can have no fundamental points when the point-groups C_λ have no common component. In the case of a surface, however, there can be finitely-many fundamental points. For example, the quadratic Cremona transformations that were treated in § 25 have three fundamental points.

The principle of constant count is true for rational maps (as it is for all irreducible correspondences), which reads:

$$d = d' + e$$

in this case, where d and d' are the dimensions of M and M', resp., and e is the dimension of that submanifold of M that is mapped to a general point ξ' of M'. One obtains this submanifold in the following way: One takes a general point ξ of M and looks for those manifolds of the linear family $| C_\Lambda |$ that go through ξ. Let the intersection of these manifolds be E. *E will then consist of a fixed – i.e., independent of ξ – subset E_0 whose points are the fundamental points of the map, and a subset E_ξ that includes ξ and is irreducible over the field $K(\xi')$, whose points possess the common image point ξ'.* The *subset* E_0 can also possibly be absent or be completely or partially contained in E_ξ. By contrast, if E_ξ contains ξ then E_ξ cannot be absent.

Proof. If η belongs to E then all C_λ that go through ξ will also go through η. These C_λ will correspond to the hyperplanes through ξ'. Thus, all of the hyperplanes that go through ξ' will contain at least one image point η' of η. However, that is possible only when either the image point of η' defines at least one curve (i.e., when η is a fundamental point), or when one of the finitely-many image points of η coincides with ξ'. The conclusion can be inverted word-for-word; thus, E will consist of precisely the fundamental points of the map and the points that have ξ' as their image point. However, the fundamental points define a fixed, algebraic manifold E_0, and, from § 33, the points whose image point is ξ' will define a manifold E_ξ that is irreducible over the field $K(\xi')$.

The dimension of E_ξ is the number that was denoted by e above. If it is zero then $d = d'$, and E_ξ will consist of infinitely many points. If its number is β then we will have a $(\beta, 1)$-map of M onto M'. Finally, if $\beta = 1$ then the map will be (1, 1), and thus birational.

If E_ξ consists of only one point – thus, if the elements C_λ of the linear family that contain the given general point ξ have only the basis points of the family in common with each other, besides ξ – then the family $| C_\Lambda |$ will be called *simple*. In the opposite case – and thus, the C_λ that contain the point ξ have points in addition to themselves (not basis points) in common that define a manifold E_ξ – the linear family $| C_\Lambda |$ will be called *composite*, and indeed *composed of the system of irreducible manifolds $| E_\xi |$ whose most general element is E_ξ.*

It then follows that: *A rational map that is mediated by a linear family will be birational if and only if the family is simple.*

Let it be mentioned that in the case $e = 0$ – so the manifolds E_ξ will be point-groups – the irreducible system $| E_\xi |$ will be called an *involution*.

Problems. 5. An involution can also be defined as an algebraic system of zero-dimensional manifolds (non-associated point-groups) on M such that a general point of M belongs to precisely one element of the system.

§ 44. The behavior of linear families at the simple points of *M*.

This paragraph is based entirely upon the following:

Theorem 1. *A k-fold point of M that is not a fundamental point will correspond to at most k image points under a rational map of M.*

Proof. One draws a general linear space S_{n-d} through the k-fold point P. Since the manifold of fundamental points has a dimension $< d$, and since P cannot also be a fundamental point, this manifold will not enter S_{n-d}. The intersection points of S_{n-d} with M will thus not be fundamental points.

Let the intersection points of a general S_{n-d}^* with M be $Q_1, \ldots, Q_g$. They are general points of M, so they will correspond to uniquely-determined image points $Q_1', \ldots, Q_g'$ under the map. The points $Q_1, \ldots, Q_g, Q_1', \ldots, Q_g'$ might go to $P_1, \ldots, P_g, P_1', \ldots, P_g'$ in a relation-preserving manner under the specialization $S_{n-d}^* \to S_{n-d}$. Since (Q_ν, Q_ν') is a pair of the map, (P_ν, P_ν') will be one, as well ($\nu = 1, \ldots, g$). $P_1, \ldots, P_g$ will be the intersection points of S_{n-d} with M, when counted as often as their multiplicities would suggest. Since P is a k-fold point, we can assume that $P_1 = P_2 = \ldots = P_k = P$, while all of the other $P_{k+1}, \ldots, P_g \neq P$. If we can still show that all image points of P appear amongst the points $P_1', \ldots, P_k'$ then it will follow that there are at most k such image points.

The point-pairs (P_ν, P_ν') are solutions of a normal problem, in the sense of § 38. The equations of this normal problem express the ideas that the pair (P_ν, P_ν') belongs to the map and that P_ν lies in S_{n-d}. When the S_{n-d} is replaced with the general S_{n-d}^*, the problem will have precisely the same solutions (Q_ν, Q_ν') ($\nu = 1, \ldots, g$), but the problem will also have only finitely-many solutions under the specialization $S_{n-d}^* \to S_{n-d}$. The intersection points P_ν of S_{n-d} with M will then have only finitely-many image points P_ν'. Therefore, from the main theorem of § 38, the specialized solutions (P_ν, P_ν') will be determined uniquely, up to their sequence. Furthermore, the correspondence between the S_{n-d} and the (P, P') will be irreducible. One will then obtain a general pair of the correspondence when one starts with a general (R, R') of the map and draws the most general S_{n-d} through R. Once more, from the main theorem on multiplicities, it will then happen that any pair (P, P') that fulfills the equations of the normal problem will appear at least once in the sequence (P_ν, P'_ν). That will be true, in particular, when P is already the k-fold point that is denoted by P and P' is its image point. Thus, a P'_ν will equal P'.

We still have to show that ν is one of the numbers 1, 2, …, h, and not one of the numbers $k + 1$, …, g. If the latter were the case then P' would be the image point of one of the points P_{k+1}, …, P_g at which S_{n-d} cuts then manifold M, outside of P. However, that would not be possible when those points of M whose image point is P' define a submanifold of dimension $< d$, and such a submanifold would no longer enter into a general space S_{n-d} that goes through P outside of P.

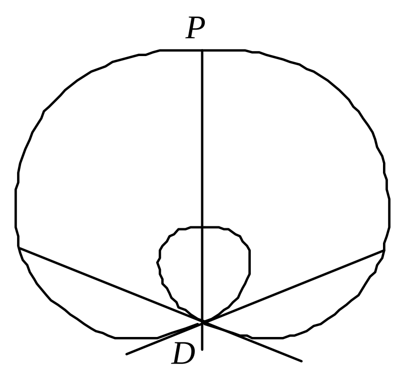
A special case of Theorem 1 is: *A simple point of M that is not a fundamental point of the map will have precisely one image point.*

This special case rests upon:

Theorem 2. *Those manifolds of an effective linear family of dimension r on M that include a given simple point P of M will define a linear sub-family of dimension* $r - 1$ *as long as P does not belong to all manifolds of the family.*

Proof. The linear family, whose fixed component we can omit, mediates a rational map of M into S_ν . From Theorem 1, the point P will correspond to a single image point P' as long as P is not a fundamental point. From § 43, the elements of the linear family will correspond to hyperplanes in S_ν. In particular, the elements that contain the point P will correspond to hyperplanes through P'. However, the phrase "goes through P'" implies a linear condition on the parameters λ_0, …, λ_r of the family. With that, the assertion is already proved.

Remark. The assumption that P is a simple point is essential, as the following example shows: If P is a k-fold point then one can likewise prove that the elements of the family that go through P will define at most k linear sub-families of dimension $r - 1$.



Example. Let M be a fourth-order plane curve with a cusp D. The lines that go through D will cut out a linear family of point pairs outside of the doubly-counted pint D. If one fixes a point P that is different from D then one will obtain a single element of the family, which will contain P. However, if one fixes D itself then one will obtain two different point-pairs that correspond to the two tangents at the point D.

It follows from a k-fold application of Theorem 2 that:

If one fixes any simple points P_1, …, P_k *of M then those elements of an effective linear family on M that contains these points will define a linear sub-family of dimension* r' *with:*

$$r - k \le r' \le r.$$

(In the case $k > r$, the sub-family can also be empty.)

It follows further from this that:

Theorem 3. *If one fixes any irreducible $(d - 1)$-dimensional manifolds $F_1, \ldots, F_k$ on M that do not consist of only multiple points of M then the elements of a linear family on M that contain these $F_1, \ldots, F_k$ with arbitrarily-given multiplicities $s_1, \ldots, s_k$ will define a linear sub-family (which can also be empty).*

Proof, by complete induction on $r + s_1 + \ldots + s_k$. The case $s_1 = \ldots = s_k = 0$ is trivial; let $s_1 > 0$. If F_1 is contained in all elements of the family as a component then we will omit the fixed component, and will obtain a family of the same dimension r and look for the elements in it that contain $F_1, F_2, \ldots, F_k$ with the multiplicities $s_{k-1}, s_g, \ldots, s_k$. From the induction assumption, they will define a linear sub-family. We can now once more add the fixed component F_1 to them.

If F_1 is not a fixed component of the family then we will choose a point P of F_1 that is either the basis point of the family or a multiple point of M. Those elements of the family that contain the point P will define a sub-family of dimension $r - 1$. From the induction assumption, the elements of this sub-family that contain $s_1 F_1 + \ldots + s_k F_k$ as a component will again define a linear sub-family. With that, the assertion is also proved in this case.

Theorem 3 is also valid for linear families of virtual manifolds when they can be made effective by the addition of fixed components, where the given multiplicities $s_1, \ldots, s_k$ are correspondingly raised. In particular, it follows that:

Theorem 4. *The effective manifolds in a linear family of virtual manifolds, when they are present, will define a linear sub-family, assuming that none of the fixed components of negative multiplicity consist of nothing but multiple points.*

Under the same assumption, it also follows that:

If $r + 1$ linearly-independent elements of an r-dimensional linear family are effective then all of the elements of the family will be effective.

From all of these theorems, one sees that linear families exhibit a much more sensible behavior at the simple points of an algebraic manifold than they do at the multiple points. It is therefore of great advantage in the study of linear families to convert algebraic manifolds into ones without multiple points by a birational transformation, when that is possible. In the next paragraphs, we would like to do this, at least in the case of curves.

§ 45. Transformation of curves into ones without multiple points. Places and divisors.

One understands the *degree* of a linear family of point-groups on a curve to mean the number of points that each point-group of the family consists of. If m is the degree, and r

is the dimension of the family then, from § 42, Theorem 3 (corollary), one will have the inequality:

(1) $$r \leq m.$$

The composite families (in the sense of § 43) obey an even sharper inequality. Namely, if one fixes a general point of the composite family then, by the definition of the composite family, k points will simultaneously remain fixed, where $k \geq 2$. If one then fixes a second, … r^{th} general point (cf., § 42, Theorem 2) then k points will remain fixed in each case. One will then have the inequality:

$$r\,k \leq m,$$

from which, since $k \geq 2$, it will follow that:

$$2r \leq m.$$

Although we shall not need it in what follows, we can insert the remark here that the equality sign in (1) can be true only when the curve can be mapped birationally onto a line. Namely, if $r = m$, and one lowers the dimension of the family by $m - 1$, when one fixes $m - 1$ general points in sequence, using the process that was applied in § 42 in the proof of Theorem 2, then one will obtain a linear family of dimension 1 with precisely one variable point. This linear family will map the curve onto a line birationally.

Any algebraic curve can be converted into a plane curve by a birational transformation – namely, by projection (cf., say, § 30). If we pose the problem of transforming all algebraic curves into ones without multiple points birationally then we can restrict ourselves to plane curves.

Let Γ be a plane curve of degree n. The curves of degree $n - 2$ cut out a linear family of dimension:

$$r = \frac{(n-2)(n+1)}{2}$$

and degree:

$$m = (n - 2)\,n$$

from Γ. There are then $\binom{n}{2} = r + 1$ linearly-independent curves of this degree, and, from BEZOUT, each of them will intersect Γ in $(n - 2)\,n$ points. For this linear family, one then has:

(2) $$2r > m.$$

It will then follow from this that the family cannot be composite. It will then map Γ birationally onto an image curve Γ_1 in the space S_r . The point-groups of the family will be cut out of the hyperplanes of S_r along this curve.

Now, if the curve Γ_1 has a multiple point P then we will consider the sub-family of dimension $r - 1$ that is cut out of the hyperplanes that contain P. We then omit the fixed point P from the point-groups of the sub-family as often as it appears in these point-groups. Since P should be a multiple point of Γ_1, it will appear at least twice in the all of the point-groups that contain it. The degree of the family will then diminish by at least two by the omission of the fixed points. The inequality (2) will be preserved under the transition to the sub-family, as long as the left-hand side is reduced by two and the right-hand side by at least two.

We now repeat the same process that always reduces r by 1, as long as it is possible; i.e., as long as the image curve that is each time mediated by linear family still has multiple points. That process must terminate, since m will always become smaller, and the value $m = 0$ cannot be attained; for $m = 0$, it would follow from (2) that $r > 0$, in contradiction to generally-valid inequality (1). Now, when the process terminates, we will have a linear family that maps Γ birationally into an image curve in a projective space that has no multiple points.

With that, it is proved that:

Any algebraic curve can be converted into a curve with no multiple points by a birational transformation.

One calls a curve Γ' with no multiple points onto which Γ is mapped birationally a *singularity-free model* for the curve Γ. Naturally, two such models Γ', Γ'' are also mapped birationally to each other. The latter map is (from § 44, Theorem 1) even one-to-one, *without exception*: Any point of Γ' corresponds to a single point of Γ'', and conversely. The map of Γ to Γ' is, by contrast, single-valued without exception only in the inverse direction. Any point of Γ' will correspond to a single point of Γ, but a multiple point of Γ can correspond to several points of Γ'. One understands the phrase *a place (Stelle) on the line* Γ to mean a point P of Γ, together with an image point P' of P on a fixed singularity-free model Γ'. Which model (i.e., Γ' or Γ'') one then bases things upon will be irrelevant, since the points of Γ' will be in one-to-one correspondence with those of Γ''. For a simple point of Γ, the given of the point P by itself will suffice to determine the place, since a simple point P of Γ will have only one image point P' on Γ'. By contrast, a k-fold point of Γ can correspond to several (and indeed, from § 44, Theorem 1, at most k) places.

The concept of place is (in contrast to that of point) *birationally invariant.* If Γ and Γ_1 are birationally mapped to each other then every place of Γ will be in one-to-one correspondence with a place of Γ_1. One and the same singularity-free model Γ' can then be employed for both Γ and Γ_1. Any place on Γ will correspond to a point of Γ', and any point of Γ' will again correspond to a place on Γ_1.

From now on, in the theory of linear families on an algebraic curve, we will no longer base things upon the *points* of Γ, but only upon its *places.* In that way, the theory will take on an invariant character under birational transformations ([1]). From now on, an element of a linear family will not be a group of points with multiplicities, as before, but a group of places with multiplicities. One also calls such groups of places with arbitrary

([1]) One will obtain an equivalent, likewise invariant, theory when one considers only linear families on the singularity-free model Γ'; therefore, for many purposes, it will be more advantageous to be able to deal with an arbitrary curve Γ. Instead of points, one must then consider just places.

(positive or negative) multiplicities *divisors.* If all multiplicities are positive or zero then one will have an *effective* (or *complete*) *divisor* ([1]).

In order to ascertain the multiplicities of a divisor that enters into a linear family, one proceeds as follows: One goes from the general point-group C_Λ, in the sense that we have been using up to now (thus, with the omission of all fixed points). The points of C_Λ are all general points of Γ, so they are not multiple points; they will therefore correspond to uniquely-determined places. From § 42, any point-group C_λ of the family will arise from the general point-group C_Λ by a relation-preserving specialization. If one carries out this relation-preserving specialization, not just for the points of Γ, but simultaneously for the points of Γ' that they correspond to, as well, then one will obtain a uniquely-determined group of points on Γ with images on Γ', hence, a uniquely-defined divisor that corresponds to the point-group C_λ that we have been considering. One now adds an arbitrary fixed divisor to the divisors thus obtained, and thus obtains the most general *linear family of divisors* on Γ.

The concept of place that was defined here has precisely the same scope as the concept of the branches of a plane curve that was introduced in § 20 in a completely different way. In fact, one has the theorem:

The branches of a plane, algebraic curve Γ *are in one-to-one correspondence with the places on* Γ.

Proof. Let Γ' be a singularity-free model of Γ, and let $\mathfrak{z}$ be a branch of Γ. The branch was defined by series development of a general point ξ of Γ:

$$\begin{cases} \xi_0 = a_0 + a_1\tau + a_2\tau^2 + \cdots \\ \xi_1 = b_0 + b_1\tau + b_2\tau^2 + \cdots \\ \xi_2 = c_0 + c_1\tau + c_2\tau^2 + \cdots \end{cases}$$

The general point ξ corresponds to a point ξ' of Γ' whose homogeneous coordinates are entire, rational functions of ξ_0, ξ_1, ξ_2, so they are once more power series in τ. After bringing out a common power of τ as a factor, they will read like:

$$\xi'_\nu = \tau^h(a'_{\nu 0} + a'_{\nu 1}\tau + a'_{\nu 2}\tau^2 + \cdots) \qquad (\nu = 0, 1, \ldots, n).$$

The coordinates ξ'_ν, when one also drops the factor τ^h, will fulfill the equations of the curve Γ'. However, this will remain correct when one sets $\tau = 0$, and thus specializes the

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([1]) The word "divisor" is taken from the DEDEKIND-WEBER arithmetic theory of algebraic functions. In that theory, which agrees completely with the geometric theory in its results, one calls the entities that were always called the *sum* (*difference*, resp.) of point-groups or divisors here the *product* (*quotient*, resp.). One thus also explains the word "divisor." See the textbook of BLISS, *Algebraic Functions* or the work of M. DEURING on algebraic functions that will appear soon in this collection. The original paper of DEDEKIND and WEBER is also found in J. reine angew. Math., **92** (1882), 181-290.

point ξ' to the point P' with the coordinates $a'_{\nu 0}$ ($\nu = 0, 1, \ldots, n$). Likewise, for $t = 0$, ξ will go to a well-defined point P, which is the starting point of the branch $\mathfrak{z}$. Under the birational map of Γ onto Γ', P' will be an image point of P; the equations of map that are true for (ξ, ξ') will also remain true for $\tau = 0$ then. The pair (P, P') will thus define a place on Γ. Therefore, any branch $\mathfrak{z}$ of Γ will correspond to a uniquely-determined place.

We still have to prove that one will obtain *all* places on Γ in this way, and indeed, each of them precisely once. Therefore, let (P, P') be a well-defined place on Γ. We would now like to take Γ' to a plane curve Γ_1 by projection, and indeed in such a way that the simple point P' will again correspond to a simple point P_1 of Γ_1 under the projection. To that end, we draw a subspace S_{n-1} through P' that intersects the curve Γ' only simply at P'. In S_{n-1}, we draw an S_{n-2} through P' that contains none of the intersection points of S_{n-1} with the curve, except for P'. Finally, in S_{n-2}, we choose an S_{n-3} that does not go through P', and project the curve Γ' from S_{n-3} onto an S_2, with which, a curve Γ_1 will arise. One now sees very easily that the projection will mediate a birational map of Γ' onto Γ_1, and that the point P' will therefore go to a simple point P_1 of Γ_1. This simple point P_1 will carry a single branch $\mathfrak{z}_1$ of Γ_1. Just as every branch $\mathfrak{z}$ of Γ corresponds to a point of Γ', the branch $\mathfrak{z}_1$ of Γ_1 will also correspond to a point of Γ'. It can only be the point P', if P' is the single point of Γ' that goes to P_1 under projection.

Now, the plane curves Γ and Γ_1 are, however, also mapped to each other birationally by means of Γ'; therefore, any branch $\mathfrak{z}_1$ of Γ_1 will correspond to precisely one branch $\mathfrak{z}$ of Γ (cf., § 20). Thus, the point P' of Γ' will correspond to a single branch $\mathfrak{z}$ of Γ, which we wished to prove.

The geometrically-defined concept of place is then suitable for taking over the role that was played by the concept of branch (which is based on series developments), up to now. The advantage is obvious. In place of an infinite series, a rational map appears that is representable by closed formulas. Places are also completely defined by themselves for curves in n-dimensional space. Finally, the restrictions on the characteristic of the ground field that are necessary for the PUISEUX series are completely unneeded here, although we shall not go further into that topic.

The "intersection multiplicity of a branch with a curve" that was explained previously (§ 20) can also be redefined with the help of the concept of a place, and carried over to n dimensions. Let Γ be a curve in S_n, and let H be a hypersurface that cuts the curve at P. The point P can correspond to several places; we choose one of them that is defined by an image point P' on a singularity-free model Γ'. We now embed H in the linear family of all hypersurfaces of equal degree whose general element is, say, H^*. This linear family cuts a linear family $| C_\Lambda |$ out of Γ whose image on Γ' will again be a linear family $| C'_\Lambda |$. A certain number of points of C_Λ will go to the point P under the specialization $H^* \to H$; (by definition) this number will be the intersection multiplicity of H and Γ at the point P. However, it will also take a certain number of points of C'_Λ to P'; this number shall be called the *intersection multiplicity of H with Γ at the place* (P, P'). Obviously, the total intersection multiplicity of H and Γ at the point P will be equal to the sum of the intersection multiplicities of H and Γ at the various places of Γ that belong to P.

The concepts of place and divisor *cannot* be carried over to a d-dimensional manifold M with no further restrictions. Namely, first of all, it is still questionable whether every manifold possesses a singularity-free image for $d > 2$ ([1]). However, secondly, two different singularity-free models are in no way mappable to each other in a one-to-one manner, so the meaning of the concepts of place and divisor would depend upon which model one employed.

From now on, we will therefore employ the concepts of place and divisor for $d > 1$ only in the case of a singularity-free manifold M, and we will then understand a *place* to be a point of M, while a *divisor* will be a virtual $(d - 1)$-dimensional submanifold of M. For $d = 1$, so M is then a singularity-free curve, these concepts go over to the previously-defined ones, while M itself can be chosen to be a singularity-free model of M.

§ 46. Equivalence of divisors. Divisor classes. Complete families.

Theorem 1. *If two linear families on M have an element D_0 in common then both of them will be contained in an enveloping linear family.*

Proof. One family might be given by:

$$\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_r F_r = 0, \tag{1}$$

and the other, by:

$$\mu_0 G_0 + \mu_1 G_1 + \ldots + \mu_r G_r = 0. \tag{2}$$

The fixed virtual manifolds A and B will be added to the complete intersections L_λ and M_λ of (1) [(2), resp.] with M, and will thus contain the elements:

$$D_\lambda = A + L_\lambda\,,$$
$$E_\mu = B + N_\mu$$

of the two linear families.

The two families of common manifolds D_0 might be determined by, say, F_0 and G_0 . We then define the family of forms:

$$\lambda_0 F_0 G_0 + G_0 (\lambda_1 F_1 + \ldots + \lambda_r F_r) + F_0 (\lambda_{r+1} G_1 + \ldots + \lambda_{r+s} G_s), \tag{3}$$

intersect it with M, and add the fixed manifold $A + B - D_0$ to the intersection manifolds. The family of forms (3) includes a sub-family:

$$G_0 (\lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_r F_r)$$

that will cut out precisely the linear family $| D_\Lambda | = | A + L_\Lambda |$ by the addition of $A + B - D_0$, and likewise includes a sub-family:

[1] For a proof in the case $d = 2$, see R. J. WALKER: Ann. of Math. **36** (1935), 336-365.

$$F_0\,(\lambda_0\,F_0 + \lambda_{r+1}\,G_1 + \ldots + \lambda_{r+s}\,G_s)$$

that will yield the linear family $|\,E_\Lambda\,| = |\,B + N_\Lambda\,|$ by the addition of $A + B - D_0$. Theorem 1 is proved with that.

The theorem is certainly true for manifolds M with arbitrary singularities; however, one first learns its true meaning in its application to singularity-free manifolds. Namely, it can happen with manifolds with multiple points that the linear families $|\,D_\Lambda\,|$ and $|\,E_\Lambda\,|$ consist of nothing but effective manifolds, while they include enveloping linear families of fixed components with multiplicities ([1]).

However, if M is free of multiple points then, from § 44, Theorem 4, the effective manifolds that are contained in the enveloping family will define a linear sub-family that envelops the two given linear families. What follows from this is the:

Corollary to Theorem 1. *If two effective linear families on a singularity-free manifold have an element D_0 in common then both of them will be contained in an effective, linear family.*

Two divisors C and D on a singularity-free d-dimensional manifold M are called *equivalent* when there is a linear family that contains C and D as elements. One then writes:

$$C \sim D.$$

Likewise, two divisors on an algebraic curve are called *equivalent* when there is a linear family of divisors that contains both of them. By going to a singularity-free model of the curve, this equivalence will reduce to the former one. If one recalls the interpretation of the one-dimensional linear families that we gave in § 43 then one can also say: *Two divisors on an algebraic curve are equivalent when their difference consists of the zero loci and poles of a rational function on the curve, where the zero loci are understood to have positive multiplicities and the poles, negative multiplicities.*

The concept of equivalence is obviously reflexive and symmetric. However, from Theorem 1, it is also transitive: From $C \sim D$ and $D \sim E$, it will follow that $C \sim E$. One can then combine all of the divisors that are equivalent divisors into a *class of divisors.*

From $C \sim D$, it will obviously follow that $C + E \sim D + E$. One can thus define the *sum of two divisor classes* by choosing a divisor from each class and adding them. The class of the sum $C + E$ is independent of the choice of divisors C and E. *The divisor classes define an Abelian group under addition.*

We now consider the *effective* or complete divisors that are obtained in a class. One can easily show that the dimension of a linear family of effective divisors that contains a

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([1]) **Example.** Let M be a fourth-order cone with a double line D that has separate tangent planes that intersect the curve in the lines A and B, outside of D. The planes through A cut out a linear family of line-triples outside of A, and similarly, for the planes through B. These two linear families have the triple $3D$ (viz., the triply-counted line D) in common. (3) is the family of quadratic cones through A, B, D; its intersection with M, when augmented by $-A-B-3D$, will define a linear family of virtual curves from four lines of positive multiplicity and a line D with multiplicity -1.

given divisor D is restricted ([1]). We will need this theorem as a tool only for the case of a curve; however, in that case it will follow immediately from the inequality (1), § 45. If one now considers a linear family of maximal dimension that contains a given effective divisor D then, on the basis of Theorem 1, this family will envelop all of the effective divisors that are equivalent to D; otherwise, in fact, from Theorem 1, one would be able to define a more-enveloping linear family. Such a maximal linear family is called a *complete family*. It then consists of all effective divisors of a given divisor class. Its dimension is called the *dimension* of the class ([2]). However, it can also happen that a class contains no effective divisors at all; in that case, one will set the dimension of the class equal to – 1.

The complete family that is determined by the effective divisor D will also be denoted by $| D |$.

One understands the *remainder* of a divisor E relative to a complete family $| D |$ to mean the complete family of all complete divisors that are equivalent to $D - E$, if there are any. If F is such a divisor then:

$$D \sim E + F.$$

One can therefore also define the remainder of E relative to $| D |$ to be the totality of those complete divisors F that, together with E, make up a divisor of the complete family.

From the first definition, it will follow that *equivalent divisors will possess the same remainder relative to a complete family* $| D |$.

Problems. 1. Carry out the induction proof that was suggested in footnote ([1]) on the previous page.

2. Two point-groups $P_1, \ldots, P_g$ and $Q_1, \ldots, Q_h$ on a cubic curve with no double points will be equivalent if and only if $g = h$ and the sum of the points P, in the sense of § 24, is equal to the sum of the points Q.

3. Two divisors of the same degree are always equivalent on a line, and therefore also on any birational image of a line. The dimension of a complete family is therefore equal to the degree of a divisor of the complete family, assuming that it is ≥ 0.

§ 47. BERTINI's theorems.

BERTINI's first theorem relates to linear families of point-groups on an algebraic curve and states:

Theorem 1. *The general point-group* $| C_\Lambda |$ *of a linear family with no fixed points consists of nothing but simply-counted points.*

Proof. The point-groups $C_\Lambda + A$ will be cut out by the hypersurface:

([1]) The theorem will follow, e.g., from that fact that the totality of all d-dimensional manifolds of given degree on M is an algebraic system of finite dimension, in the sense of § 37. However, it can also be proved by complete induction on d when one cuts M with a general hyperplane.

([2]) In the arithmetic theory, one prefers to understand the dimension of a family to mean the number of linearly-independent elements of the family; thus, one uses the number $r + 1$. All dimension numbers will thus be increased by 1 when one goes to the arithmetic theory.

(1) $$\Lambda_0 F_0 + \Lambda_1 F_1 + \ldots + \Lambda_r F_r = 0,$$

where $\Lambda_1, \ldots, \Lambda_r$ are indeterminates. The points of C_Λ are algebraic functions of these indeterminates. If ξ is such a point then ξ will be a general point of M, and one will have:

(2) $$\Lambda_0 F_0(\xi) + \Lambda_1 F_1(\xi) + \ldots + \Lambda_r F_r(\xi) = 0.$$

If an algebraic function is equal to the constant zero then its derivatives will also be zero; one can thus differentiate (2) with respect to Λ_j :

(3) $$F_j(\xi) + \sum_k \{\Lambda_0 \partial_k F_0(\xi) + \Lambda_1 \partial_k F_1(\xi) + \cdots + \Lambda_r \partial_k F_r(\xi)\} \frac{\partial \xi_k}{\partial \Lambda_j} = 0.$$

If one now assumes that ξ is a multiple intersection point of the curve M with the hypersurface (1) then, from § 40, tangents to the curve at the point ξ must lie in the polar hyperplane to the hypersurface (1). However, the point:

$$\frac{\partial \xi_0}{\partial \Lambda_j}, \frac{\partial \xi_1}{\partial \Lambda_j}, \ldots, \frac{\partial \xi_n}{\partial \Lambda_j}$$

will always lie on a curve tangent ([1]). One will then have:

(4) $$\sum_k \{\Lambda_0 \partial_k F_0(\xi) + \Lambda_1 \partial_k F_1(\xi) + \cdots + \Lambda_r \partial_k F_r(\xi)\} \frac{\partial \xi_k}{\partial \Lambda_j} = 0.$$

It follows from (3) and (4) that:

$$F_j(\xi) = 0 \qquad (j = 0, 1, \ldots, r);$$

thus, ξ will be a basis point of the family (1), in contradiction to the assumption that ξ should be a point of the point-group C_Λ, which consists of nothing but variable points.

It follows almost immediately from Theorem 1 that:

Theorem 1a. *The general element C_Λ of a linear family of effective $(d - 1)$-dimensional manifolds with no fixed components possesses no multiply-counted component.*

One then comes back to Theorem 1 by intersecting with a general linear space S_{n-d+1}.

([1]) If the hypersurface $f = 0$ contains the curve then it will follow from $f(\xi) = 0$ by differentiation that:

$$\frac{\partial f}{\partial \xi_0}\frac{\partial \xi_0}{\partial \Lambda_j} + \frac{\partial f}{\partial \xi_1}\frac{\partial \xi_1}{\partial \Lambda_j} + \cdots + \frac{\partial f}{\partial \xi_n}\frac{\partial \xi_n}{\partial \Lambda_j} = 0.$$

However, BERTINI's second theorem says a little more, namely, that the general element C_Λ will possess no multiple points at all, besides the basis points of the family and the multiple points of the carrier manifold M. I can prove the theorem here only in the following, somewhat specialized, form:

Theorem 2. *The general hypersurface of a linear family:*

(5) $$\Lambda_0 F_0 + \Lambda_1 F_1 + \ldots + \Lambda_r F_r = 0$$

cuts a manifold C_Λ out of M that possesses no multiple points outside of the basis points of the family (5) *and the multiple points of M.*

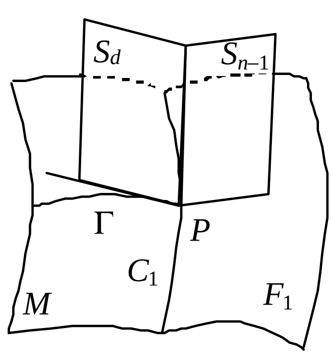
Proof. We first return to the case of an arbitrary, linear family in the case of a bundle:

(6) $$\Lambda F_0 + F = 0 \qquad (F = \Lambda_1 F_1 + \ldots + \Lambda_r F_r),$$

when we append the quantities $\Lambda_1, \ldots, \Lambda_r$ of the basis field in (5), which are then treated as constants. Once the assertion has been proved for the bundle (6), it will then follow that any multiple point P of C_Λ that is not a multiple point of M will necessarily be a basis point of the bundle (6), and will thus satisfy the equation $F_0(P) = 0$. In precisely the same way, it will follow that $F_1(P) = 0, \ldots, F_r(P) = 0$, so P will also be a basis point of the family (5).

It will then suffice to consider the case of a bundle. Call the general hypersurface of the bundle F_Λ; its intersection with M will be C_Λ. Now, let P be a multiple point of C_Λ that is not the basis point of the bundle. The pair (Λ, P) is defined to be the general pair of an irreducible correspondence. Under this correspondence, the general point Λ of the parameter lines of the bundle will correspond to a b-dimensional manifold of points P' that are relation-preserving specializations of P, and thus multiple points of C_Λ. When one cuts this manifold with a linear space S_{n-b}, one can reduce its dimension b to 0, without the property of the points P' that they are double points of C_Λ being thereby lost. In the principle of constant count:

$$a + b = c + d$$



one must now set $a = 1$, $b = 0$. If one had $c = 0$, $d = 1$ then a point P' of the image manifold would correspond to all of the ∞^1 points λ of the parameter line, contrary to assumption. All that remains is the possibility that $c = 1$, $d = 0$. The image manifold of the correspondence will then be a curve Γ.

The hypersurfaces of the bundle cut out a linear family of points groups from Γ, and indeed the general hypersurface F_Λ will cut out the point P¸ among others. From Theorem 1, P will be a simple intersection point of F_Λ and Γ. The proof of Theorem 1 teaches us that, in addition, the tangential space S_{n-1} of F_Λ does not contain the tangents of Γ at P.

If P is now a simple point of M then M will possess a tangential space S_d at P (cf., § 40). The tangent to Γ will lie in S_d . Since S_{n-1} does not contain this tangent, S_{n-1} cannot contain S_d , either; the intersection of S_{n-1} and S_d will then be an S_{d-1} . However, that means that the intersection manifold C_Λ of F_Λ and M will possess a tangential space S_{d-1} at P, so P will be a simple point of C_Λ, which contradicts the assumption. Therefore, P cannot be a simple point of M.

The following two theorems might be proved here for linear families of curves on an algebraic surface M, although they can be extended to linear families of M_{d-1} on M_d with no trouble ([1]). The proofs go back to ENRIQUES.

One understands the *degree* of a linear family of curves to mean the number of intersection points of two general curves of the family besides the basis points.

One understands a *bundle of curves* on M to mean an irreducible one-dimensional system of curves on M that sends precisely one curve through any general point of M. The concept of an irreducible system of curves is thus explained as in § 37. If one is dealing with a one-dimensional linear family, in particular, then one will speak of a *linear bundle* ([2]).

Theorem 3. *A linear family* $|\ C_\Lambda\ |$ *of degree zero with no fixed components is composed of a bundle whose general curves are absolutely irreducible.*

Proof. The curves of the r-dimensional linear family $|\ C_\Lambda\ |$ that go through a general point P of M will define a linear sub-family of dimension $r - 1$. If one now associates the general point P with a general element-pair C, C' of this sub-family then an irreducible correspondence between the points P and the curve-pairs C, C' will be defined by the general triple P, C, C'. In the principle of constant count:

$$a + b = c + d,$$

one must set $a = 2$ and $b = 2(r - 1)$. Now, if one had $d = 0$ then one would have $c = 2r$; i.e., the pair (C, C') would be a general element-pair of the family. That is, any two general curves C, C' of $|\ C_\Lambda\ |$ would have a (general) point P of the surface in common, in contradiction to the assumption of degree zero. Thus, $d \geq 1$; i.e., if two curves C, C' are drawn through a general point of M then they will have, not just one, but at least ∞^1 points in common. (Naturally, more than ∞^1 is not possible; thus, $d = 1$.)

This will remain true when one chooses C to be a general curve through P, but chooses C' to be a particular one. The common component of C and C' might define a curve K. It will be composed of irreducible components of the fixed curve C', so it cannot depend on the (undetermined) parameters of C at all. We then see that *all curves C of the family $|\ C_\Lambda\ |$ that go through P will have a fixed curve K in common that depends upon only P.*

Now, if P' is another point of K (but not a basis point of the family $|\ C_\Lambda\ |$) then the curves of $|\ C_\Lambda\ |$ that go through P' will once more define a linear family of dimension r –

([1]) See B. L. v. d. WAERDEN, "Zur algebraischen Geometrie X," Math. Ann. **113** (1937), pp. 711.

([2]) There are also nonlinear bundles; e.g., the system of all generators of a cubic cone with no double points. However, on many surfaces – e.g., planes – all bundles will be linear.

1 that will envelop the previous one, so it will be identical to it. *The curves of the family* $|C_\Lambda|$ *that go through any point of K will thus again have the curve K in common.*

The curve K might decompose into absolutely irreducible components K_1, K_2, … None of the components K_ν will stay fixed when P is varied, since otherwise all curves of the family $| C_\Lambda |$ would have these fixed components in common. The irreducible system $|K_\nu|$, whose general element is K_ν, will then have a dimension of at least 1. We would like to show that $| K_\nu |$ is a bundle.

We exhibit an irreducible correspondence between the elements of the system $| K_\nu |$ and the points of M. One will obtain the general pair (K_ν, P_ν) of this correspondence when one chooses a general point P_ν on the curve K_ν. In the principle of constant count:

$$a + b = c + d$$

one will have:	$a \geq 1$, $b = 1$,	so:	$a + b \geq 2$,
but one will also have:	$c \leq 2$, $d = 0$,	so:	$c + d \leq 2$,
and therefore:	$a + b = c + d = 2$, $a = 1$, $c = 2$.		

The system $| K_\nu |$ is therefore one-dimensional, and the image manifold of the correspondence is the entire surface M. It follows that at least one curve K_1' of $| K_1 |$ will go through a general point P of M, at least one curve K_2' of $| K_2 |$, etc., and in all, h different curves K_ν'. All of these curves will be general elements of their systems $| K_\nu |$.

From what was said in the third paragraph of the proof, all of the curves C of the family $| C_\Lambda |$ that go through P must contain all h curves K_ν'. That is, at least h different components of any curve C will go through the point P.

However, we saw in § 42 that one will arrive at the same thing whether one draws the most general curve C through a general point P of M or one first chooses a general curve C of $| C_\Lambda |$ and then a general point P of a component of C. If one does the former then, from the foregoing, P will be at least an h-fold point of C; however, if one does the latter then P will obviously be a simple point of C. Therefore, $h = 1$. That is, there will only be a single system $| K_\nu |$, and only one curve in it will go through the general point P. Therefore, $| K_\nu |$ will be a bundle. Furthermore: If one chooses a general point P on any irreducible component of the general curve C_Λ then it will always lie on a curve K_ν' that is contained in C_Λ; therefore, any irreducible component of C_Λ will be one of the curves K_ν' of the system $| K_\nu |$. That is, $| C_\Lambda |$ will be composed of the bundle $| K_\nu |$.

Theorem 4. *A linear family* $| C_\Lambda |$ *with no fixed components whose general curve* C_Λ *is absolutely irreducible will have degree zero* (and, as a result of Theorem 3, it will be composed of a bundle).

Proof. Let C_1 and C_2 be two general curves of the family $| C_\Lambda |$. If C_1 and C_2 have an intersection point P' besides the basis points of the family then this point P' will never be a double point of the surface or a double point of C_1. C_1 will then contain only finitely many such double points, and the generally-chosen curve C_2 that is independent of C_1 will go through none of these finitely-many points, as long as they are not basis points.

C_1 and C_2 define a bundle in the family $| C_\lambda |$, and since C_1 and C_2 go through P', all curves of the bundle will go through P'. Like any linear bundle, $| C_\lambda |$ will have degree zero, and from Theorem 1 it will therefore be composed of a bundle $| K |$ whose general curve K is absolutely irreducible. The general curve C_λ goes through P', so at least one irreducible component K of C_1 must go through P'. However, if a general curve of the system $| K |$ goes through point P' then all curves of the system $| K |$ will go through P'. In particular, all irreducible components of C_λ will go through P'. By assumption, at least two such components exist; P' will therefore be a multiple point of C_λ. The property of a point that that it is a multiple point will be preserved under the specialization $\lambda \to 0$. Therefore, P' will be a multiple point of C_1, as well, in contradiction to the initial statements. Therefore, the point P' cannot exist.

When taken together, Theorems 3 and 4 will give an exhaustive answer to the question: What is the nature of a linear family whose general element is irreducible? Namely, such a family will either have a fixed component or it will be composed of a (linear or nonlinear) bundle.

An immediate consequence of Theorem 4 is the following theorem: *The intersection of an absolutely irreducible surface with a general hyperplane is an absolutely irreducible curve.* The hyperplanes will then cut a linear family out of the surface whose degree is positive (namely, equal to the degree of the surface); therefore, a general curve of the family cannot be reducible.

We will return to linear families of algebraic curves in the next chapter (§ 49-51). For the detailed theory of linear families of curves on algebraic surfaces, we refer to the report of ZARISKI: *Algebraic Surfaces*, Ergebn. Math., Bd. 3, Heft 5, as well as the literature that is cited there.

CHAPTER EIGHT

NOETHER's fundamental theorem and its consequences.

§ 48. NOETHER's fundamental theorem.

Let $f(x)$ and $g(x)$ be two relatively prime forms in the indeterminates x_0, x_1, x_2 . In order for there to exist an identity of the form:

$$F = Af + Bg \tag{1}$$

for another form $F(x)$, where A and B are again forms, it is, in any case, necessary that all intersection points of the curves $f = 0$ and $g = 0$ must also lie on the curve $F = 0$. However, as we will see, that condition will be sufficient only in the case where all intersections of $f = 0$ and $g = 0$ have multiplicity one. Further conditions must be added for multiple intersection points.

The celebrated "Fundamental theorem of NOETHER," which was first published by MAX NOETHER in Math. Annalen, Bd. 6, gives necessary and sufficient conditions for the identity (1) to exist.

We will call all theorems that give necessary and sufficient conditions for (1) "NOETHER theorems," in the broader sense.

According to P. DUBREIL, all of these theorems can be derived from the following lemma:

VAN DER WOUDE's lemma. *Suppose that the form F contains the term* x_n^2, *and let:*

$$R = Uf + Vg \tag{2}$$

be the resultant of f and g by x_2 (cf., § 16). (1) *will be true if and only if the remainder T of VF under division by f (both of which are considered to be polynomials in* x_2*) is divisible by R.*

Proof. Dividing VF by f yields:

$$VF = Qf + T. \tag{3}$$

It follows from (2) and (3) that:

$$\begin{aligned} RF &= UFf + VFg \\ &= UFf + (Qf + T)\, g \\ &= (UF + Qg)\, f + Tg \end{aligned}$$

or

$$RF = Sf + Tg. \tag{4}$$

Now, if T is divisible by R then Sf will also be divisible by R, so, since f contains no factor that depends upon just x_0 and x_1, S will be divisible by R. One can then cancel R in (4) and obtain (1).

Conversely, if (1) is true then one can always replace A and B in (1) with:

$$A_1 = A + Wg, \qquad\qquad B_1 = B - Wf.$$

If one chooses W especially such that B_1 has a degree $< n$ in x_0 (division with remainder of B by f) then the representation:

$$F = A_1 f + B_1 g$$

will be single-valued ([1]). If one multiples this single-valued representation by R and compares with (4), in which T_1 will have degree $< n$ in x_2, in any case, then it will follow from the single-valuedness of the representation that:

$$S = RA_1 , \qquad T = RB_1 ,$$

so T will, in fact, be divisible by R.

Now, let $\overset{1}{s}, \overset{2}{s}, \ldots, \overset{h}{s}$ be the intersection points of $f = 0$ and $g = 0$, and let $\sigma_1, \ldots, \sigma_h$ be their multiplicities. From § 17, one will then have:

$$R = \prod_{\nu} \left(\overset{\nu}{s}_0 \, x_1 - \overset{\nu}{s}_1 \, x_0 \right)^{\sigma_\nu} . \tag{5}$$

We can arrange the coordinates so that no two intersection points have the same ratio $s_0 : s_1$. The factors $\overset{\nu}{s}_0 \, x_1 - \overset{\nu}{s}_1 \, x_0$ in (5) will be all different then. VF will be divisible by R if and only if all of the individual factors:

$$\prod_{\nu} \left(\overset{\nu}{s}_0 \, x_1 - \overset{\nu}{s}_1 \, x_0 \right)^{\sigma_\nu}$$

are divisible. We then already have a first "NOETHER theorem."

Theorem 1. (1) *is true if and only if for every intersection s of the curves* $f = 0$ *and* $g = 0$ *with multiplicity s, the remainder T that was defined in the lemma above is divisible by:*

$$(s_0 \, x_1 - s_1 \, x_0)^{\sigma}.$$

The proof yields the following:

([1]) It would then follow from $F = A_1 f + B_1 g = A_2 f + B_2 g$ that $(A_1 - A_2) f = (B_2 - B_1) g$, so $B_2 - B_1$ would be divisible by f, which would be not possible when B_1 and B_2 both have degree $< n$.

Corollary. *The coefficients of A and B can be calculated rationally from the coefficients of the given forms f, g, F.*

On the basis of Theorem 1, any individual intersection point s will be associated with certain conditions that express the divisibility of T by $(s_0\, x_1 - s_1\, x_0)^\sigma$, and which will be collectively (i.e., for all intersection points together) necessary and sufficient for (1). We call them the NOETHER *conditions* for the intersection point s in question. The NOETHER conditions are obviously *linear* conditions on the form F: i.e., when F_1 and F_2 fulfill them, $F = F_1 + F_2$ will also fulfill them.

In order to give an application of Theorem 1, we consider the case $\sigma = 1$.

Let, perhaps, $Q = (1, 0, 0)$ be a simple intersection point of the curves $f = 0$ and $g = 0$. We choose the coordinates once and for all such that the line $x_1 = 0$ and the curve $f = 0$ contact nowhere, and intersect only at finite points. It follows from (2) that V must be zero at the $n - 1$ intersections points of $f = 0$ and $x_1 = 0$ that are different from Q; R will then contain the factor x_1, and g will be $\neq 0$ at these points. Now, it follows from (3) that T will also be zero at these points. F and f will vanish at the point Q itself, so from (3), T will, too. Now, if one sets $x_0 = 1$ and $x_1 = 0$ in T then one will obtain a polynomial in x_2 of degree $\leq n - 1$ that will have n different zeroes, so it must vanish identically. That is, T will be divisible by x_1 . One will then have the result:

The NOETHER conditions are already fulfilled at a simple intersection point of $f = 0$ *and* $g = 0$ *when* $F = 0$ *goes through that point.*

We next consider the case in which the point $Q = (1, 0, 0)$ is a simple point of the curve $f = 0$. That curve will then have a single branch $\mathfrak{z}$ at the point Q. For the existence of the identity (1), it is, in any case, necessary that the form F have the same order [1] as the form g on the branch $\mathfrak{z}$. We will now show that this condition is also sufficient in the sense of the NOETHER conditions.

Let T be precisely divisible by x_1^λ :

$$T = x_1^\lambda\, T_1 .$$

If $\lambda \geq \sigma$ then the NOETHER condition (viz., the divisibility of T by x_1^σ) will be fulfilled. Therefore, let $\sigma < \lambda$. If will follow from (4) that the form RF will have the same order as Tg on the branch $\mathfrak{z}$. If one had $T_1 \neq 0$ at the point Q then T would have order λ and R, order σ, so R would have a higher order than T, and furthermore, by assumption, F would have at least the same order as g, which would give RF a higher order than Tg, which is not true. Therefore, T_1 must be zero at the point Q. However, precisely the same conclusion will also be true for the all branches at the remaining $n - 1$ intersection points of $f = 0$ with the lines $x_1 = 0$; g indeed has order zero at these points. Therefore, the polynomial T_1 will have n different zeroes for $x_0 = 1$ and $x_1 = 1$. It will follow from this, as in the previous proof, that T_1 is divisible by x_1 and therefore T is divisible by $x_1^{\lambda+1}$,

[1] The order of F on $\mathfrak{z}$ is the intersection multiplicity of $F = 0$ with the branch $\mathfrak{z}$ (cf., § 20 and § 45) – or, what amounts to the same thing in this case – the intersection multiplicity of $F = 0$ and $f = 0$ at Q.

which is contrary to the assumption that T is precisely divisible by x_1^λ. With that, a theorem of KAPFERER is proved:

Theorem 2. *If all intersection points of the curves* $f = 0$ *and* $g = 0$ *are simple points of* $f = 0$*, and if they are also intersection points of* $f = 0$ *and* $g = 0$ *with at least the same multiplicity then the identity* (1) *will be true.*

The NOETHER conditions cannot be expressed as merely multiplicity conditions at the multiple points of the curves $f = 0$ and $g = 0$. Later (viz., Theorem 4), we will bring the precise necessary and sufficient conditions into a form that is independent of the coordinate system. However, in any case, there will be multiplicity conditions that are sufficient for the identity (1). In that regard, we next treat the case in which the curve $f = 0$ has an r-fold point with r separate tangents at Q. Let the associated branches be $\mathfrak{z}_1, \ldots, \mathfrak{z}_r$; the curve $g = 0$ will intersect these branches with the multiplicities $\sigma_1, \ldots, \sigma_r$. The total intersection multiplicity of the points Q will then be $\sigma = \sigma_1 + \ldots + \sigma_r$. We now prove:

Theorem 3. *If the curve* $F = 0$ *cuts each of the r branches* $\mathfrak{z}_j$ ($j = 1, 2, \ldots, r$) *of the curve* $f = 0$*, which does not contact it, at Q with a multiplicity of at least* $\sigma_j + r - 1$ *then the NOETHER conditions for Q will be fulfilled.*

Proof. As in the proof of Theorem 3, let:

$$T = x_1^\lambda\, T_1$$

and $\lambda < 0$. RF will have the same order as Tg on any branch $\mathfrak{z}_j$. That is, when δ_j is the order of T_1 on $\mathfrak{z}_j$:

$$\sigma + (\sigma_r + r - 1) \leq \lambda + \delta_j + \sigma_j .$$

Since $\lambda \leq \sigma - 1$, it will then follow that:

$$r \leq \delta_j . \tag{6}$$

We would now like to show that the curve T_1 has an at least r-fold point at Q. If that were not the case, so it would have an at most $(r - 1)$-fold point at Q, then it would also have at most $r - 1$ tangents at Q, and since the branches $\mathfrak{z}_1, \ldots, \mathfrak{z}_r$ collectively have r different tangents there would be a branch $\mathfrak{z}_j$ that contacts no branch of the curve $T_1 = 0$. From the rules of § 20, the intersection multiplicity of $T_1 = 0$ with this branch $\mathfrak{z}_j$ would then seem to be at most $r - 1$. However, that would contradict the inequality (6). Therefore, $T_1 = 0$ will have an at least r-fold point at Q.

In addition, the curve $T_1 = 0$ will contain the remaining $n - r$ intersection points of $f = 0$ and $x_1 = 0$, as before. In total, the polynomial T_1 will have an n-fold zero for $x_0 = 1$, x_1

= 0. As above, it will follow from this that T_1 is divisible by x_1, so T is divisible by $x_1^{\lambda+1}$, which is contrary to the assumption that T is divisible by x_1^{λ}, precisely.

Remark. The last part of the proof can also be carried out in such a way that the assumption that the line $x_1 = 0$ cuts the curve at $n - r$ different points is not used in it, but only the assumption that $x_1 = 0$ is not a tangent to the point Q, and does not go through any other intersection points of $f = 0$ and $g = 0$, and that its imaginary point (0, 0, 1) does not lie on the on the curve $f = 0$. One then concludes: R and T are divisible by x_1^{λ} in (4), so S must also be divisible by x_1^{λ}. If one drops x_1^{λ} then it will follow that:

$$R_1\, F = S_1\, f + T_1\, g.$$

If one sets $x_1 = 0$ here then S_1, T_1, f, g will go to S_1^0, T_1^0, $f^{\,0}$, g^0, while R_1 will be divisible by x_1; it will then follow that:

$$-\, S_1^0 f^{\,0} = T_1^0 g^0.$$

$f^{\,0}$ contains the factor x_2^r, which also comes out of T_1^0; $f = 0$ and $T_1 = 0$ will both have an r-fold point at Q then. The remaining factors of $f^{\,0}$ are relatively prime to g^0, since the line $x_1 = 0$ contains no other intersection points of $f = 0$ and $g = 0$ than Q. Therefore, these factors must drop out of $f^{\,0}$ and T_1^0. Therefore, T_1^0 will be divisible by $f^{\,0}$. However, T_1^0 has degree $< n$ in x_2, while $f^{\,0}$ has degree n. Thus, $T_1^0 = 0$; i.e., T_1 will be divisible by x_1, etc., as before.

Now that we have cast a glance over the most important special case, we go on to the general case. NOETHER's *fundamental theorem* gives the necessary and sufficient conditions for the existence of the identity (1) in such a form that we avoid the singling out of x_2 that we have been doing. We set $x_0 = 1$ and then go over to inhomogeneous coordinates. In order to be able to formulate the theorem and its proof simply, we introduce the concept of the *order of a polynomial* $f(x_1, x_2)$ *at a point* Q*:* f has order r at Q when the curve $f = 0$ has an r-fold point at Q. If one again has $Q = (1, 0, 0)$, and one develops f in increasing powers of x_1 and x_2 then the development will begin with terms of degree r (in x_1 and x_2 collectively). NOETHER's theorem now reads, in a form that P. DUBREIL gave to it:

Theorem 4. *Let f and g be relatively prime polynomials in x_1, x_2. Let the orders of f and V be r and l, resp., at the point s. Let the intersection multiplicity of $f = 0$ and $g = 0$ be s. If there are then two polynomials A' and B' such that the difference:*

$$\Delta = F - A'f - B'g$$

has an order of at least:

$$\sigma + r - 1 - l$$

in s then the NOETHER conditions for F will fulfilled at the point s.

Proof. If Δ and $A'f + B'g$ both fulfill the NOETHER conditions at the point s then their sum F will also fulfill them. From Theorem 1, $A'f + B'g$ will always fulfill the NOETHER conditions. Hence, it will suffice to prove that Δ fulfills them, as long as the order of Δ in s is at least $\sigma + r - 1 - l$.

In order to be able to apply the previous conditions, we again call Δ, F. We again assume that $s = (1, 0, 0)$, and draw the line $x_1 = 0$ through x in such a way that it does not contact the curve at s.

Again, let:

$$T = x_1^\lambda T_1 \qquad \lambda < \sigma.$$

It will then follow from (3) that:

(7) $$V F = Q f + x_1^\lambda T_1 .$$

If we develop both sides of (7) in increasing powers of x_1 and x_2 then the left-hand side will be missing all terms whose degree (in x_1 and x_2 together) is less than $\sigma + r - 1 - l$. Since the last term in (7) is divisible by x_1^λ, any term in Qf whose degree is less than $\sigma + r - 1$ must be divisible by x_1^λ. The developments of Q and f in components of increasing degree might read:

$$Q = Q_0 + Q_1 + Q_2 + \dots,$$
$$f = f_r + f_{r+1} + f_{r+2} + \dots$$

It will then follow that:

$$Qf = Q_0 f_r + (Q_1 f_r + Q_0 f_{r+1}) + (Q_2 f_r + Q_1 f_{r+1} + Q_0 f_{r+2}) + \dots$$
$$+ (Q_{\sigma-2} f_r + Q_{\sigma-3} f_{r+1} + \dots) + \dots$$

On the left-hand side, all components of degree $< r + \sigma - 1$ are divisible by x_1^λ. The same thing must also be true on the right. However, f_r is relatively prime to x_1. One then sees successively that $Q_0, Q_1, \dots, Q_{\sigma-2}$ must be divisible by x_1^λ. We can then write:

(8) $$Q = x_1^\lambda C + D,$$

in which D has order $\geq \sigma - 1$ in s.

If one substitutes (8) in (7) then that will give:

$$VF - Df = x_1^\lambda (T_1 - Cf).$$

The left-hand side has order $\geq r + \sigma - 1$ in s. Thus the parenthesis on the right, namely:

$$T_1 - Cf,$$

will have order $\geq r + \sigma - 1 - \lambda \geq r$. Since Cf also has order $\geq r$ in s, T_1 will have order $\geq r$.

From here on, the proof proceeds in exactly the same way that it did in the last part of the proof of Theorem 3.

The importance of NOETHER's fundamental theorem rests upon the following: Assume that one finds that of the $m\ n$ intersection points of the curves $F = 0$ and $f = 0$ (where m is the degree of F and n is the degree of f), a certain number $m' n$ of them lie on a curve $g = 0$ of order $m' < m$. If the NOETHER conditions are fulfilled at these points, in addition, then one can conclude that the remaining $(m - m') \cdot n$ intersection points will lie on a curve $B = 0$ of degree $m - m'$. Namely, it will follow immediately from the identity (1) that the $m\ n$ intersection points of $F = 0$ and $f = 0$ will be the same as those of $Bg = 0$ with $f = 0$, and will thus consist of the $m' n$ intersection points of f with g and the $(m - m') \cdot n$ intersection points of f with B. We saw in § 24 how important theorems of this kind can be: Theorems 1, 2, 6, 7 in that section can be derived immediately from NOETHER's Theorem 2 in the given way.

Problems. 1. If two conic sections intersect two other conic sections at 16 different points, and if 8 of these 16 intersection points lie on another conic section then the same will be true for the other 8.

2. One derives the so-called "*simple case of NOETHER's theorem*" from Theorem 3 or Theorem 4: If, at an intersection point of the curves $f = 0$ and $g = 0$ that is an r-fold point of the first curve and a t-fold point of the second curve, the r tangents to the first curve are different from the t tangents of the second curve at the point s, and if F has order at least $r + s - 1$ at that point, then the NOETHER conditions will be fulfilled at that point.

3. Prove NOETHER's fundamental theorem in the original NOETHER formulation: If an identity:

$$F = P\,f + Q\,g$$

is true at any real intersection point (with the inhomogeneous coordinates s_1, s_2) of the curves $f = 0$ and $g = 0$, where f, g, F are polynomials in x_1, x_2, and P, Q are power series in $x_1 - s_1$, $x_2 - s_2$, then an identity (1) will also be true with polynomials A and B. [One truncates the power series after the terms of degree $(r + \sigma - 1 - l)$ and makes the equation thus-obtained homogeneous.]

§ 49. Adjoint curves. The remainder theorem.

One can just as well base the considerations of this paragraph on the concept of branch that was defined in § 20 as on the concept of place that was defined in § 45 independently of it. We choose the former, because we need the conceptual machinery of Chap. 3, as well. In connection with that, we understand a *place* on a plane curve Γ to mean a branch, together with the starting point of that branch. A *divisor* on the curve Γ is a finite set of places with whole-number multiplicities. The *sum* of two divisors will be defined by combining the places that appear in them and adding their multiplicities. An arbitrary curve $g = 0$ that has no component in common with Γ will cut out a well-defined divisor from Γ. A linear family of forms $\lambda_0\, g_0 + \lambda_1\, g_1 + \ldots + \lambda_r\, g_r$ will cut out a *linear family of divisors* from Γ, to which one can add a fixed divisor (cf., § 42). Two divisors of the same linear family will be called *equivalent*. A *complete family* is a linear family of complete divisors that are all equivalent to a given divisor. The goal of this paragraph is the construction of complete families from a given curve. The adjoint curves will serve to facilitate this construction, which we shall now explain.

Let s be a multiple point of an irreducible plane curve Γ with the equation $f = 0$, and let $\mathfrak{z}$ be a well-defined branch at the point s. The polars of the points y of the plane whose equations reads:

$$y_0 \, \partial_0 f + y_1 \, \partial_1 f + y_2 \, \partial_2 f = 0$$

will all go through s. They will then cut out a linear family of divisors from Γ in which the place $(\mathfrak{z}, s)$ appears as a fixed component with a certain multiplicity v. Naturally, for special y, the intersection multiplicity of the polar with the branch $\mathfrak{z}$ can be higher; v will then be defined to be the smallest value that this intersection multiplicity can assume.

The point s also has a well-defined multiplicity κ on the branch $\mathfrak{z}$ (cf., § 21); κ is the smallest intersection multiplicity of $\mathfrak{z}$ with a line through s.

We will later see that the difference:

$$\delta = v - (k - 1)$$

is always positive. We understand *a curve adjoint to* Γ to means a curve $g = 0$ whose intersection multiplicity with any branch $\mathfrak{z}$ (at each multiple point of Γ) is always $\geq \delta$. The form g is then also called an *adjoint form*.

For a simple point of the curve, one will have $v = 0$, $\kappa = 1$, so $\delta = 0$. Therefore, there is no adjointness condition on it. For an r-fold point with separate tangents, from § 25, one will have:

$$v = r - 1, \qquad \kappa = 1,$$

so $\delta = r - 1$. An adjoint curve will then have to cut all branches of this r-fold point with a multiplicity of at least $r - 1$. In the case of an ordinary vertex, one will have $v = 3$, $\kappa = 2$, so $\delta = 2$. An adjoint curve shall then cut the vertex branch with a multiplicity of at least 2; i.e., it shall go through the vertex at least simply.

Problems. 1. Any adjoint curve must have an at least $(r - 1)$-fold point at an r-fold point with separate tangents.

2. Establish how the adjointness condition will read for a beak and a contact junction.

For the computational evaluation of the adjointness condition, it is convenient to know that is it not necessary to define the polars to *all* points y (as often happens in the definition) in order to succeed in calculating the difference δ, but only the polar to an arbitrary point outside of the curve. Namely, let v' be the intersection multiplicity of the polar to such a fixed point y with the branch $\mathfrak{z}$, and let κ' be the intersection multiplicity of the connecting line ys with the branch $\mathfrak{z}$. We will then show that the difference:

$$\delta' = v' - (\kappa' - 1)$$

is independent of the choice of y and equal to δ.

If we compare two different points y', y'' with each other then we can assume that they do not lie on a line with the point s; otherwise, we could interpolate a third point outside of the line as an intermediate term and compare it with both of them. We can then assume that s, y', y'' are the vertices of a coordinate triangle:

$$\begin{aligned} s &= (1, 0, 0), \\ y' &= (0, 1, 0), \\ y'' &= (0, 0, 1). \end{aligned}$$

The polar of y' is $\partial_1 f = 0$, and that of y'' is $\partial_2 f = 0$. Let the intersection multiplicity of these polars with the branch $\mathfrak{z}$ be ν' (ν'', resp.). Let the intersection multiplicities of the lines $s\,y'$ ($x_2 = 0$) and $s\,y''$ ($x_1 = 0$) with the curve be κ' and κ'', resp. We will then have to prove that:

$$\nu' - (\kappa' - 1) = \nu'' - (\kappa'' - 1).$$

Let a general point of the curve be $\xi = (1, \xi_1, \xi_2)$. We then know that:

$$\frac{d\xi_2}{d\xi_1} = -\frac{\partial_1 f(\xi)}{\partial_2 f(\xi)}. \tag{2}$$

When expressed in terms of the position uniformization of the branch $\mathfrak{z}$, ξ_2 will have order κ', so $d\xi_2$ will have order $\kappa' - 1$, and likewise, $d\xi_1$ will have order $\kappa'' - 1$; furthermore, $\partial_1 f(\xi)$ and $\partial_2 f(\xi)$ have orders ν' and ν''. It will then follow from (2) that:

$$(\kappa' - 1) - (\kappa'' - 1) = \nu' - \nu''$$

or

$$\nu'' - (\kappa'' - 1) = \nu' - (\kappa' - 1).$$

It is thus proved that δ' is independent of the choice of y. If one chooses ν such that κ' is minimal then, since:

$$\delta' = \nu' - (\kappa' - 1),$$

ν' will also be minimal, and δ' will go to δ. One will thus have:

$$\delta = \nu' - (\kappa' - 1),$$

independently of the choice of the point y.

The fact that $\nu' \geq \kappa' - 1$ (with the equality sign only in the case of a simple point) follows immediately from the developments in § 21 (cf., prob. 4 there). It then follows that:

$$\delta \geq 0,$$

with the equality sign only in the case of a simple point.

The adjoint forms of degree $n - 3$ (where n is the degree of the curve) have a special meaning due to their relationship to the differentials of the first kind of the associated algebraic function field. Namely, if g is such a form of degree $n - 3$ then one will define the expression:

$$d\Omega = \frac{g(\xi)(\xi_0 d\xi_1 - \xi_1 d\xi_0)}{\partial_2 f(\xi)}$$

for an arbitrary point ξ of Γ. Since one can also write this expression as:

$$d\Omega = \frac{g(\xi)\cdot\xi_0^2}{\partial_2 f(\xi)}\, d\,\frac{\xi_1}{\xi_2},$$

in which the numerator and denominator in the first fraction have the same degree, it will depend upon only the ratios of the ξ; i.e., $d\Omega$ will be a differential of the field $K(\xi_1 : \xi_0, \xi_2 : \xi_0)$, in the sense of § 26.

$g(\xi)$ has order at least δ on a branch $\mathfrak{z}$ of Γ. Since $\partial_2 f(\xi)$ has order ν' and $\xi_0\, d\xi_1 - \xi_1\, d\xi_0$ has order $\kappa' - 1$, moreover ([1]), $d\Omega$ will have order at least:

$$\delta - \nu' + (\kappa' - 1) = 0$$

on the branch $\mathfrak{z}$. That means that the differential $d\Omega$ has no pole then (i.e., it is "everywhere finite"). One calls such a differential *a differential of the first kind*. More precisely, the calculation that we just performed yields that: *If g has order $\delta + \varepsilon$ on $\mathfrak{z}$ then $d\Omega$ will have order ε.*

The divisor that consists of places that belong to the multiple points of Γ, with the multiplicities δ for branch $\mathfrak{z}$ that were defined above, is called the *double-point divisor* of the curve Γ. Any multiple point will thus contribute to the double-point divisor. An r-fold point with separate tangents will contribute its r places (which correspond to the r branches of the r-fold points), each with the multiplicity $r - 1$. An ordinary cusp will contribute the place of the cusp with multiplicity 2, etc. The double-point divisor will be suggestively denoted by D.

The most important theorem on adjoint curves – viz., the BRILL-NOETHER remainder theorem – can be derived from the following *double-point divisor theorem:*

If a curve $g = 0$ cuts out the divisor D from Γ, and if an adjoint curve $F = 0$ cuts out at least the divisor $D + G$ then there will exist an identity:

(3) $$F = A\,f + B\,g,$$

with an adjoint B.

([1]) If one further assumes $\xi_0 = 1$ then $\xi_0\, d\xi_1 - \xi_1\, d\xi_0$ will go to $d\xi_1$, and we already saw before that $d\xi_1$ has a real place of order $\kappa' - 1$. In the case of an imaginary point, one simply switches the roles of ξ_0 and ξ_1.

Otherwise expressed: *If the intersection multiplicity of* $F = 0$ *with any branch* $\mathfrak{z}$ *of* Γ *contributes at least* $\delta + \sigma$, *where* δ *is defined as above, and* σ *is the intersection multiplicity of* $g = 0$ *with* Γ, *then* (3) *will be true, and the curve* $B = 0$ *will be adjoint to* Γ.

The last part of the statement – viz., the adjointness of B – is a consequence of (3). Then, from (3), $F = 0$ has the same intersection multiplicity with the branch $\mathfrak{z}$ as $Bg = 0$, and since $g = 0$ has only the intersection multiplicity σ, but $F = 0$ has at least $\sigma + \delta$, one must have $B = 0$ for the remaining δ.

In the case where all multiple points of Γ have separate tangents, the double-point divisor theorem will obviously be contained in Theorem 3 (§ 48). Then, if the NOETHER conditions are fulfilled at every intersection point of $f = 0$ and $g = 0$ then (3) will indeed be true. We will resolve the more difficult general case in the next paragraph.

We now come to the BRILL-NOETHER remainder theorem. In its most succinct formulation, it states:

The adjoint curves of any degree m will cut out a complete family from Γ *outside of the double-point divisor D.*

If we recall the definition of a complete family then we can also express the same thing as:

If an adjoint curve φ *cuts out the divisor* $D + E$ *from* Γ, *and if* E' *is a complete divisor that is equivalent to E then there will be a second adjoint curve that cuts the divisor* $D + E'$ *out of* Γ.

Proof. The equivalent divisors E and E' will cut out a linear family of forms through two forms g and g', which might cut out a fixed divisor C, in addition. The form:

$$F = \varphi\, g'$$

cuts out the divisor $D + E + C + E'$, but the form g will cut out the divisor $C + E$. From the theorem of the double-point divisor, one will then have:

$$F = Af + Bg.$$

Thus, F and Bg will cut out the same divisor $D + E + C + E'$. Therefore, B must cut out the divisor $D + E'$, with which the assertion is proved.

The remainder theorem gives one the means to construct any arbitrary complete family. Namely, if G is any complete divisor then one will draw an adjoint curve through $G + D$. It might cut out, in all, the divisor $G + D + F$ from Γ. One then draws all possible adjoint curves of the same degree m through $D + F$; one thus obtains only point-groups $G' + D + F$, where G' is equivalent to G. Conversely, if G' is equivalent to G then $G + F$ will be equivalent $G' + F$, so $G' + F$ will belong to the complete family that is cut out by the adjoint curves of degree m; i.e., there will be an adjoint curve of degree m

that cuts out the divisor $G' + D + F$. *The desired complete family* $| G |$ *will then be cut out from the adjoint curves, which will cut out the fixed divisor* $D + F$*, in addition.* One can also express this as: *The complete family* $| G |$ *is the remainder of* F *relative to the complete family that is cut out by the adjoint curves of a sufficiently high degree* m.

If one would like to decide whether two divisors C, C' are equivalent then one can represent the difference $C - C'$ as the difference of two complete divisors:

$$C - C' = G - G',$$

and observe whether G' belongs to the complete family $| G |$.

Problems. 3. A complete family of degree n will have dimension n on a line.

4. A complete family of degree n will have dimension $n - 1$ on a cubic curve with a double point for $n > 0$.

5. An isolated point or a point-pair on a fourth-order curve with a junction or a cusp will determine a complete family of dimension 0, assuming that the point-pair lies on a line with the double point. A point-triple will determine a complete family of dimension 1, and a point-quadruple, a complete family of dimension 2.

§ 50. The double-point divisor theorem.

In § 49, we proved the double-point divisor theorem for the special case in which the base curve $f = 0$ has no other singularities than multiple points with separate tangents. Here, the general case shall now be solved.

Lemma. *If two power series:*

$$A(t) = a_\mu t^\mu + a_{\mu+1} t^{\mu+1} + \ldots \qquad (a_\mu \neq 0),$$
$$B(t) = b_\nu t^\nu + b_{\nu+1} t^{\nu+1} + \ldots \qquad (b_\nu \neq 0)$$

are such that the first one has at least the same order as the second one − *i.e., if:*

$$\mu \geq \nu,$$

then the first one will be divisible by the second one:

$$A(t) = B(t)\, Q(t). \tag{1}$$

Proof. We assume:

$$Q(t) = c_{\mu-\nu} t^{\mu-\nu} + c_{\mu-\nu+1} t^{\mu-\nu+1} + \ldots,$$

substitute that into (1), and compare the coefficients of t^μ, $t^{\mu+1}$, … on both sides. That will yield the condition equations:

$$b_\nu c_{\mu-\nu} = a_\mu ,$$
$$b_\nu c_{\mu-\nu+1} + b_{\nu+1} c_{\mu-\nu} = a_{\mu+1} ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

from which, one can determine $c_{\mu-\nu}$, $c_{\mu-\nu+1}$, … since $b_\nu \neq 0$.

In the following, $f(t, z)$, $g(t, z)$, etc., will mean polynomials in z whose coefficients are power series in t (with all non-negative exponents). We assume that $f(t, z)$ is double-root-free and regular in z (i.e., that the coefficient of the highest power of z equals 1). Furthermore, $f(t, z)$ might divide completely into linear factors:

$$f(t, z) = (z - \omega_1)(z - \omega_2) \ldots (z - \omega_n) \tag{2}$$

in the domain of the power series.

Under these assumptions, one will have:

Theorem 1. *If $F(t, z)$ and $g(t, z)$ are so arranged that the order of the power series $F(t, \omega_j)$, for $j = 1, 2, \ldots, n$, is equal to at least the order of the product:*

$$(\omega_j - \omega_1) \ldots (\omega_j - \omega_{j-1})\,(\omega_j - \omega_{j+1}) \ldots (\omega_j - \omega_n)\, g(t, \omega_j) \tag{3}$$

then one will have an identity:

$$F(t, z) = L(t, z)\, f(t, z) + M(t, z)\, g(t, z). \tag{4}$$

Proof. From the lemma, $F(t, \omega_j)$ will be divisible by the product (3); in particular, for $j = 1$, one will have:

$$F(t, \omega_1) = (\omega_1 - \omega_2) \ldots (\omega_1 - \omega_n)\, g(t, \omega_1)\, R(t),$$

where $R(t)$ is a power series in t. The difference:

$$F(t, z) - (z - \omega_2) \ldots (z - \omega_n)\, g(t, z)\, R(t),$$

will be zero for $z = \omega_1$, so it will be divisible by $(z - \omega_1)$:

$$F(t, z) = R(t)\,(z - \omega_2) \ldots (z - \omega_n)\, g(t, z) + S(t, z)\,(z - \omega_1). \tag{5}$$

In the case $n = 1$, this equation reads simply:

$$F(t, z) = R(t)\, g(t, z) + S(t, z)\, f(t, z);$$

the assertion (4) is already proved for $n = 1$ with that. It will then be assumed to be true for polynomials of degree $n - 1$.

If one sets $z = \omega_j$ ($j = 2, \ldots, n$) in (5) then the first term on the right will vanish, and one will see that $S(t, \omega_j)(\omega_j - \omega_1)$ has the same order as $F(t, \omega_j)$, so it has the same order as:

$$(\omega_j - \omega_1)(\omega_j - \omega_2) \ldots (\omega_j - \omega_{j-1})(\omega_j - \omega_{j+1}) \ldots (\omega_j - \omega_n)\, g(t, \omega_j).$$

As a result, for $j = 2, \ldots, n$, $S(t, \omega_j)$ will have at least the same order as:

$$(\omega_j - \omega_1)\ldots(\omega_j - \omega_{j-1})(\omega_j - \omega_{j+1})\ldots(\omega_j - \omega_h)\, g(t, \omega_j).$$

When the induction assumption is applied to $f_1 = (z - \omega_2)\ldots(z - \omega_h)$, it will follow from this that:

(6) $$S(t, z) = C(t, z)\,(z - \omega_2)\ldots(z - \omega_h) + D(t, z)\, g(t, z).$$

If one substitutes (6) into (5) then one will obtain the assertion (4) immediately.

The derivation of $f(t, z)$ with respect to z is:

$$\partial_2 f(t, z) = \sum_{j=1}^{n} (z - \omega_1)\ldots(z - \omega_{j-1})(z - \omega_{j+1})\ldots(z - \omega_h).$$

The assumption of Theorem 1 can thus be also formulated as: $F(t, \omega_j)$ *shall have at least the same order as* $\partial_2 f(t, \omega_j)\, g(t, \omega_j)$ *for* $j = 1, \ldots, r$.

Now, let $f(u, z)$ be a polynomial in u and z that is regular in z and free of multiple factors. From § 14, $f(u, z)$ will divide into linear factors:

$$f(u, z) = (z - \omega_1)\ldots(z - \omega_h),$$

in which $\omega_1, \ldots, \omega_h$ are power series in fractional powers of u. In each case, κ_j might define a power series ω_j, together with a branch $\mathfrak{z}$. ω will then be a power series in the position uniformization τ_j that is defined by:

$$u = \tau_j^{u_j}.$$

If h is the smallest common multiplicity of all κ_j then we can set:

$$u = t^h,$$

and write all of the $\omega_1, \ldots, \omega_h$ as power series in t.

Let $F(u, z)$ and $g(u, z)$ be further polynomials in u and z. Let the orders of $g(u, \omega_j)$, $\partial_2 f(t, \omega_j)$, and $F(u, \omega_j)$ as power series in τ_j be σ_j, ν_j, and ρ_j, resp. Corresponding to the assumptions of the double-point divisor theorem, let:

$$\rho_j \geq \delta_j + \sigma_j = \nu_j - (\kappa_j - 1) + \sigma_j,$$

or

$$\rho_j - (\kappa_j - 1) \geq \nu_j + \sigma_j .$$

$F(u, \omega_j) \cdot \tau_j^{\kappa_j - 1}$ will then have a larger order than $\partial_2 f(t, \omega_j)\, g(u, \omega_j)$. That will first be true when $\tau_j^{\kappa_j - 1}$ is replaced with t^{h-1}, when one will have:

$$\tau_j^{\kappa_j - 1} = t_j^{\frac{h}{\kappa_j}(\kappa_j - 1)} = t_j^{h - \frac{h}{\kappa_j}},$$

$$h - \frac{h}{\kappa_j} \leq h - 1.$$

Thus, $F(t^h, \omega_j)\, t^{h-1}$ will have at least the same order as a power series in t as $\partial_2 f(t^h, \omega_j)$ $g(t^h, \omega_j)$. It will then follow from Theorem 1 that:

(7) $$F(t^h, \omega_j)\, t^{h-1} = L(t, z)\, f(t^h, z) + M(t, z)\, g(t^h, z).$$

If we order both sides of (7) in powers of t then only powers whose exponents are congruent to – 1 (mod h) will appear on the left-hand side. One can then drop all terms t^λ whose exponents λ are not $\equiv -1 \pmod{h}$ from $L(t, z)$ and $M(t, z)$, without perturbing the validity of (7). One can then drop t^{h-1} from both sides of (7) and replace t^h with u. One will then get:

(8) $$F(u, z) = P(u, z)\, f(u, z) + Q(u, z)\, g(u, z),$$

in which P and Q are polynomials in z and power series in u.

In the original formulation of the double-point divisor theorem, we were not dealing with polynomials $f(u, z)$, but with forms $f(x_0, x_1, x_2)$. However, for the examination of the NOETHER conditions at a well-defined point $O = (1, 0, 0)$, we can set $x_0 = 1$. Correspondingly, we now write $f(1, u, z)$, instead of $f(u, z)$, and combine what was proved up to now:

Under the assumptions of the double-point divisor theorem, one will have an identity:

(9) $$F(1, u, z) = P(u, z)\, f(1, u, z) + Q(u, z)\, g(1, u, z),$$

in which P and Q are polynomials in s whose coefficients are power series in u.

If one truncates all of these power series at a sufficiently high power of u then it will follow immediately from Theorem 4 (§ 48) that the NOETHER conditions will be fulfilled at the point O. However, we would like to avoid the application of Theorem 4, in order to come to a shortest-possible proof of the double-point divisor theorem, and will then employ Theorem 1 of the same paragraph directly.

As was shown in § 48, one can always assume that the degree of $Q(u, z)$ in z is $< n$ in any identity of the form (9). The representation will then be single-valued. If one multiplies this single-valued representation on both sides by the resultant R of f and g with respect to z, and then compares it with (4), § 48 then, due to the single-valuedness of the representation, it will follow that:

$$S = R\, P, \qquad T = R\, Q.$$

R will then be a polynomial in just u that contains the factor u^σ (where σ is the intersection multiplicity of O as the intersection point of $f = 0$ and $g = 0$), while Q will be a power series in u whose coefficients are polynomials in z. If one now arranges both sides of the last equation $T = R\,Q$ in increasing powers of u then one will see that T is divisible by u^σ. However, those are precisely the NOETHER conditions at the point O.

Since the same thing is true for any arbitrary intersection of $f = 0$ and $g = 0$, it will then follow from Theorem 1 (§ 48) that there exists an identity:

$$F = A\,f + B\,g$$

in the domain of the forms. The double-point divisor theorem is proved with that.

Problem. Present the proof that was given here in such a way that no power series appear in it anymore by truncating all of the power series that appear at a sufficiently high power of t (u, resp.).

§ 51. The RIEMANN-ROCH theorem.

The question that is answered by the RIEMANN-ROCH theorem reads: How large is the dimension of a complete family – or, what amounts to the same thing, the dimension of a divisor class of given degree – on an algebraic curve?

Since the concept of a complete family is birationally invariant, we can replace Γ with any birational image of Γ. We can thus assume that Γ is a plane curve with only normal singularities (which are multiple points with separate tangents). Let the degree of this curve be m, the "number of double points," d, and the genus, p. One will then have:

$$p = \frac{(m-1)(m-2)}{2} - d,$$

and

$$d = \sum \frac{r(r-1)}{2},$$

which is summed over all multiple (viz., r-fold) points of the curve.

A special role is played by one divisor class, namely, the *differential class*, or *canonical class*. The zero locus and poles of a differential, in the sense of § 26:

$$f(u, \omega)\, du,$$

will define a divisor when one computes the zeroes with positive multiplicities and the poles with negative ones. Since all differentials will arise from the differential du upon multiplying by a function $f(u, w)$, all associated divisors will be equivalent. They will therefore define a class, namely, the *differential class*.

The degree of the differential class – i.e., the number of zeroes minus the number of poles of a differential – is, from § 26, equal to:

$$2p - 2.$$

We now ask what the dimension of the differential family is – i.e., the dimension of the complete family that consists of effective divisors of the differential class. These effective divisors belong to differentials without poles (i.e., differentials of the first kind). From § 49, these differentials have a close relationship to the adjoint curves of degree $m - 3$, which one also calls *canonical curves.* Namely, if such a canonical curve cuts a divisor C out of Γ, outside of the double-point divisor D, then C will be an effective divisor of the differential class, and since the canonical curves always cut out a complete family, outside of D, one will also obtain all effective divisors of the differential class in this way.

In the sequel, when we say that an adjoint curve φ cuts out the divisor C, we will always mean that the curve cuts out the divisor, outside of the double-point divisor. Likewise, we will say that φ goes through the divisor C' when φ cuts out at least the divisor $D + C'$, so when C is contained as a subset of the previously-considered divisor C.

In the case $p = 0$, $2p - 2$ is negative, so there can be no effective divisors in the differential class. From the convention that was used in § 46, the dimension of the differential class is to be set to $- 1$ in this case.

Thus, let $p \geq 1$, and therefore $m \geq 3$. The number of linearly-independent curves of degree $m - 3$ in the plane is:

$$\frac{(m-1)(m-2)}{2}.$$

Should such a curve be adjoint, its coefficients would then have to fulfill:

$$\frac{r(r-1)}{2}$$

at every r-fold point. The number of linearly-independent adjoint curves of degree $m - 3$ would then be equal to at least:

$$\frac{(m-1)(m-2)}{2} - \sum \frac{r(r-1)}{2} = \frac{(m-1)(m-2)}{2} - d = p.$$

Therefore, there will always be canonical curves for $p \geq 1$ ([1]), *and the dimension of the complete family that they cut out will be at least* $p - 1$.

If we determine the dimension of the complete family that is cut out by the adjoint curves of degree $m - 1$ in the same way then we will find at least the value:

$$\frac{(m-1)(m-2)}{2} - d - 1 = p + 2m - 2.$$

The degree of the complete family will be equal to:

([1]) Except that in the case $m = 3$, one cannot actually speak of adjoint "curves" of degree $m - 3$; of course, there are adjoint forms of degree 0 for a double-point-free cubic curve, namely, the constants. The complete family (of dimension 0) that is cut out by them will consist of only the zero divisor, as is always true in the case $p = 1$, moreover.

$$m(m-1) - 2d = 2p + 2m - 2.$$

These calculations are also true for $p = 0$.

Corollary. *If the divisor C consists of* $p + 1$ *points then the complete family* $|\,C\,|$ *will have dimension at least* 1.

Proof. One can draw an adjoint curve of degree $m - 1$ through the $p + 1$ points; the dimension that is achieved above will then be $\geq p + 1$, if the trivial case of $m = 1$ is excluded. Outside of C, this curve cuts out a remainder C' that consists of:

$$(2p + 2m - 2) - (p + 1) = p + 2m - 3$$

points. The remainder of C' relative to the adjoint curves of order $m - 1$ is, moreover, a complete family that contains the divisor C and has dimension at least:

$$(2p + 2m - 2) - (p + 2m - 3) = 1.$$

The assertion is then proved.

In particular, if $p = 0$ then it will follow that any isolated point belongs to a complete family of dimension 1. The complete family will map the curve Γ onto a line birationally. *Any curve of genus* 0 *will then be equivalent to a line*. Such curves are called *rational* or *unicursal curves*.

In order to prove the RIEMANN-ROCH theorem, BRILL and NOETHER presented the following **Reduction theorem:**

Let C be an effective divisor, and let P be a simple point of Γ. *If there is a canonical curve* φ *that goes through C, but not through C + P, then P will be a fixed point of the complete family* $|\,C + P\,|$.

Proof. One draws a line g through P that cuts P in m different points $P, P_2, \ldots, P_m$. g and φ collectively define an adjoint curve of degree $m - 2$ that goes through $C + P$, and in addition, cuts out a remainder E from Γ to which the points $P_2, \ldots, P_m$ certainly belong, but not the point P. Now, in order to obtain the complete family $|\,C + P\,|$, from § 49, one must draw all possible adjoint curves of order $m - 2$ through E. All of them will have the $m - 1$ points $P_2, \ldots, P_m$ in common with the lines g; they will therefore contain the line, and therefore also the point P. Hence, P will be a fixed point of the complete family.

One understands the *specialty index i* of an effective divisor to mean the number of linearly-independent canonical curves that go through C. If there are no such curves then one will set $i = 0$. If $i > 0$ then C will be called a *special divisor*, and the complete family $|\,C\,|$ will be called a *special family*.

A special family $|\,C\,|$ can always be obtained as the remainder of a second special divisor C' relative to the canonical family $|\,W\,|$. Namely, if one draws a canonical curve through C then it will cut out a divisor $C + C' = W$, and, from § 49, the complete family $|C|$ will be the remainder of C' relative to the canonical complete family $|\,W\,|$.

A divisor whose degree is $> 2p - 2$ will certainly not be special when W has degree $2p - 2$. On the other hand, a divisor whose degree is $< p$ will certainly be special; one can then always draw a divisor of the complete family $| W |$ through $p - 1$ points, since it will have a dimension of at least $p - 1$.

The **RIEMANN-ROCH theorem** (in the BRILL-NOETHER formulation) now says:

If n is the degree and i is the specialty index of an effective divisor C, and if r is the dimension of the complete family $| C |$ then one will have:

$$r = m - p + i. \tag{1}$$

Proof. Case 1: $i = 0$. If $r > 0$ then we will fix a point P that is not a fixed point for all divisors of the complete family from the outset, and define the remainder $| C_1 |$ of P relative to $| C |$. The specialty index of C_1 will then be once more zero. Hence, if there were adjoint curves that went through C_1 then, from the reduction theorem, P would be a fixed point of $| C_1 + P | = | C |$, which is not the case. Upon going from C to C_1, one will reduce both the dimension r and the degree n by 1, while p and i ($= 0$) do not change; thus, (1) will be true for C, as long as (1) is true for C_1.

One proceeds in this way, while always holding a point fixed, until the dimension of the complete family becomes zero. One will then have to prove that formula (1) is true for this case ($r = i = 0$); i.e., that $n = p$ in this case. In any case, n cannot be $< p$, since, from a previously-made remark, the divisor would then be special, and thus one would have $i > 0$. Now, if one had $n > p$ then one could choose $p + 1$ points of C, and embed this divisor in a linear family of dimension > 0 (cf., the "corollary" above). If one thenw added the remaining points of C as fixed points then one would obtain a linear family that contained C and had a dimension > 0, which would contradict the assumption that $r = 0$. Therefore, only the possibility that $n = p$ would remain, with which (1) is proved for this case.

Case 2: $i > 0$. (Complete induction on i) Let formula (1) be true for divisors of specialty index $i = 1$. Draw a canonical curve through C – which is possible, since $i > 0$ – and choose a simple point P of Γ outside of that curve. From the reduction theorem, P will then be a fixed point of the complete family $| C + P |$. This complete family will thus have the same dimension r as the original complete family $| C |$, so it will again have degree $n + 1$ and specialty index $i - 1$; the condition of obtaining P outside of C will come down to a linear condition equation for the coefficients of a canonical curve. From the induction assumption, one will then have:

$$r = (n + 1) - p + (i - 1) = n - p + i.$$

With that, the proof is concluded. It consisted in simply the fact that one defines $| C - P |$ in the first case and $| C + P |$ in the second case, and applies the reduction theorem both times, by which r and i were reduced until they both became zero.

1. Consequence. One will always have $r \geq n - p$, with the equality sign for non-special divisors.

2. Consequence. The dimension of the canonical family is equal to precisely $p - 1$. Its degree is then $n = 2p - 2$, and its specialty index is $i = 1$.

The RIEMANN-ROCH theorem can also be formulated in another way. If $\{C\}$ means the dimension of the complete family $|\ C\ |$ then one will obviously have:

$$\{W - C\} = i - 1,$$

so formula (1) will assume the form:

$$\{C\} = n - p + 1 + \{W - C\}. \tag{2}$$

If one introduces the order of $|\ W - C\ |$:

$$n' = (2p - 2) - n$$

then one can bring (2) into the symmetric form:

$$\{C\} - \frac{n}{2} = \{W - C\} - \frac{n'}{2}. \tag{3}$$

Formula (3) was proved for the case in which C is an effective divisor, or at least equivalent to one. However, since one can switch the roles of C and $W - C$, (3), and therefore (2), will also be true when $W - C$ is equivalent to an effective divisor. However, it is easy to show that (2) is even true when either C or $W - C$ is equivalent to a complete divisor, so when $\{C\} = \{W - C\} = -1$.

Let C be the difference between two complete divisors: $C = A - B$. Let the degree of B be b, so that of A will be $n + b$. If one had $n \geq p$ then, from Consequence 1, the dimension of the complete family would be:

$$\geq (n + b) - p \geq b,$$

so one could find an effective divisor A' that would be equivalent to A and would contain B as a component, and $C = A - B \sim A' - B$ would be equivalent to an effective divisor, contrary to the assumption. Therefore, $n \leq p - 1$. However, one will likewise also have:

$$n' = (2p - 2) - n \leq p - 1, \qquad \text{so } n \geq p - 1.$$

It follows that $n = p - 1$; thus, both sides of (2) will have the value $- 1$.

Therefore, formula (2) *is true for any divisor C of degree n.* This statement is the **generalized RIEMANN-ROCH theorem.**

Problems. 1. If $C = A - B$ is the difference between two complete divisors then the specialty index:

$$i = \{W - C] + 1$$

will be equal to the number of linearly-independent differentials that have zeroes at the points of A whose orders are at least equal to the multiplicities of the points, and only poles at the points of B whose orders are at most equal to the multiplicity.

2. On the basis of Problem 1, show: There are exactly p linearly-independent differentials without poles. There is no differential with precisely one pole of first order. The number of linearly-independent differentials with two poles of first order or one pole of second order is one greater than the number of differentials without poles. If one adds another pole, or if one raises the order of a pole by 1 then one will raise the number of linearly-independent differentials by 1 in each case.

3. A curve of genus 1 (viz., an "elliptic curve") is always birationally equivalent to a third-order plane curve with no double point. (The rational map will be mediated by a complete family of dimension 2 and order 3.)

4. A curve of genus 3 is either birationally equivalent to a fourth-order curve with no double point or a fifth-order curve with a triple point, according to whether its canonical family is simple or composite, resp.

§ 52. NOETHER's theorem for space.

Let f and g be two relatively prime forms in x_0, x_1, x_2, x_3. We ask what the conditions would be for a third form F to be represented in the form:

$$F = A\,f + B\,g. \tag{1}$$

The answer is given by the following theorem:

If a general plane cuts the surfaces $f = 0$, $g = 0$, *and* $F = 0$ *in curves such that the third curve fulfills the first two of the NOETHER conditions (cf.,* § 48) *then* (1) *will be true.*

Proof. Let the general plane be determined by three general points p, q, r: its parameter representation reads:

$$y_k = \lambda_1\, p_k + \lambda_2\, q_k + \lambda_3\, r_k\,. \tag{2}$$

The equations for the intersection curves are obtained by substituting (2) into the equations $f = 0$, $g = 0$, $F = 0$. From the NOETHER theorem for the plane, one will have, since the NOETHER conditions are fulfilled:

$$\begin{cases} F(\lambda_1 p + \lambda_2 q + \lambda_3 r) \\ = A(\lambda)\, f(\lambda_1 p + \lambda_2 q + \lambda_3 r) + B(\lambda)\, g(\lambda_1 p + \lambda_2 q + \lambda_3 r) \end{cases} \tag{3}$$

identically in $\lambda_1, \lambda_2, \lambda_3$. From the corollary to Theorem 1 (§ 48), the coefficients of the forms $A(\lambda)$ and $B(\lambda)$ will be rational functions of p, q, r.

The points p, q, r can be specialized so these rational functions will remain meaningful. We choose fixed points for p and q, in particular, and choose r to be the general point of a fixed line:

$$r = s + \mu\, t.$$

If we substitute this into (3) then we will obtain:

$$(4)\qquad \begin{cases} F(\lambda_1 p + \lambda_2 q + \lambda_3 r) \\ = A(\lambda)\, f(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_3 \mu t) + B(\lambda)\, g(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_3 \mu t). \end{cases}$$

We briefly denote the left-hand side by $F_1(\lambda, \mu)$ and correspondingly employ the notations f_1 and g_1. The forms $A(\lambda)$ and $B(\lambda)$ depend upon μ rationally. We multiply both sides of (4) with a polynomial in μ such that the right-hand side becomes completely rational in μ:

$$(5)\qquad h(\mu)\, F_1(\lambda, \mu) = A_1(\lambda, \mu)\, f_1(\lambda, \mu) + B_1(\lambda, \mu)\, g_1(\lambda, \mu).$$

We decompose $h(\mu)$ into linear factors:

$$h(\mu) = (\mu - \alpha_1)(\mu - \alpha_2) \dots (\mu - \alpha_s),$$

and seek to convert (5) step-wise in such a way that these linear factors can be sequentially dropped. If we set $\mu = \alpha_1$ in (5) then the left-hand side will vanish, and one will come to:

$$(6)\qquad A_1(\lambda, \alpha_1)\, f_1(\lambda, \alpha_1) + B_1(\lambda, \alpha_1)\, g_1(\lambda, \alpha_1) = 0.$$

In case the intersection curve of the surfaces $f = 0$ and $g = 0$ contains the plane curve Γ_e as a component, we can always choose p and q such that they do not both lie in a plane, together with a curve Γ_e. This means that the forms in $\lambda_1, \lambda_2, \lambda_3$:

$$f_1(\lambda, \alpha) = f(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_3 \alpha t),$$
$$g_1(\lambda, \alpha) = g(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_3 \alpha t)$$

will be relatively prime for any value of α. If will then follow from (6) that $A_1(\lambda, \alpha_1)$ is divisible by $g_1(\lambda, \alpha_1)$, and $B_1(\lambda, \alpha_1)$ is divisible by $f_1(\lambda, \alpha_1)$:

$$A_1(\lambda, \alpha_1) = C_1(\lambda)\, g_1(\lambda, \alpha_1),$$
$$B_1(\lambda, \alpha_1) = -\, C_1(\lambda)\, f_1(\lambda, \alpha_1).$$

The differences:

$$A_1(\lambda, \alpha_1) - C_1(\lambda)\, g_1(\lambda, \alpha_1),$$
$$B_1(\lambda, \alpha_1) + C_1(\lambda)\, f_1(\lambda, \alpha_1)$$

will both be zero for $\mu = \alpha_1$, and will thus be divisible by $\mu - \alpha_1$:

$$A_1(\lambda, \alpha_1) = \quad C_1(\lambda)\, g_1(\lambda, \mu) + (\mu - \alpha_1)\, A_2(\lambda, \mu),$$
$$B_1(\lambda, \alpha_1) = -\, C_1(\lambda)\, f_1(\lambda, \mu) + (\mu - \alpha_1)\, B_2(\lambda, \mu).$$

If one substitutes this into (5) then the terms in $C_1(\lambda)$ will drop out, and it will follow that:

$$h(\mu)\, F_1(\lambda, \mu) = (\mu - \alpha_1)\, A_2(\lambda, \mu)\, f_1(\lambda, \mu) + (\mu - \alpha_1)\, B_2(\lambda, \mu)\, g_1(\lambda, \mu).$$

One can now cancel $\mu - \alpha_1$ from both sides, and repeat the process until all factors $(\mu - \alpha_1)\ldots(\mu - \alpha_s)$ have been canceled. It will then follow that:

$$F_1(\lambda, \mu) = A_2(\lambda, \mu)\, f_1(\lambda, \mu) + B_2(\lambda, \mu)\, g_1(\lambda, \mu).$$

Here, we substitute:

$$\mu = \frac{\lambda_4}{\lambda_3}$$

on the left and right, where λ_4 is a new indeterminate, multiply the left and right-hand sides by a factor of λ_3 such that everything becomes completely rational again, and again cancel the factors λ_2 by the process that was just described. We then obtain:

$$(7) \qquad \begin{cases} F(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_3 t) \\ = A'(\lambda)\, f(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_4 t) + B'(\lambda) g(\lambda_1 p + \lambda_2 q + \lambda_3 s + \lambda_4 t). \end{cases}$$

Finally, one solves the equations:

$$\lambda_1\, p_k + \lambda_2\, q_k + \lambda_3\, s_k + \lambda_4\, t_k = x_k$$

for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which is always possible when p, q, r, s are linearly-independent points, and substitute the λ-values thus found into (7). (7) will then become the desired identity (1).

It follows from the proof that instead of posing the requirement that the NOETHER conditions should be fulfilled on a *general* plane, one can also demand that they should be fulfilled on a general plane of a certain bundle, where one must assume only that no plane of the bundle contains a component of the intersection curve of the surfaces $f = 0$ and $g = 0$.

The conditions of NOETHER's theorem for space are fulfilled, in particular, when any component of the intersection curve of $f = 0$ and $g = 0$ has the multiplicity one, and when $F = 0$ contains the entire intersection curve, or also when the intersection points of a general plane with the intersection curve of $f = 0$ and $g = 0$ are simple points of $f = 0$, and every irreducible component of this intersection curve with at least the same multiplicity also enters into the intersection curve of $F = 0$ and $f = 0$ (cf., § 48, Theorem 2).

NOETHER's theorem can be carried over from the space S_n to the space S_{n+1} in precisely the same way that it was carried over from the plane to space here. *NOETHER's theorem for space S_n* then follows by complete induction on n:

If a general plane S_2 in S_n cuts the hypersurfaces $f = 0$, $g = 0$, *and* $F = 0$ *(where f and g are relatively prime forms) in curves such that the third curve fulfills the NOETHER conditions at any intersection point with the first two then there will be a identity:*

$$F = Af + Bg.$$

As an application, we prove the following theorem:

An algebraic manifold M of dimension $n - 2$ *on a double-point-free quadric Q in the space* S_n *will always be the intersection of A with another hypersurface for* $n > 3$.

Proof. We project M onto a point O of the quadric Q that lies outside of M. The projecting cone K is a hypersurface of the space S_n . The intersection of Q and K consists of the points A of Q whose connecting line with O enters the manifold M. If such a point A does not lie in the tangential hyperplane of Q at O then OA will not lie on Q, and will therefore meet Q only at O and A_l now, since O does not belong to M, so A must belong to M. The complete intersection of Q and K will then consist of all points of M, and possibly certain points of the tangential hyperplane S_{n-1} to Q at O.

Now, S_{n-1} intersects the quadric Q in a quadratic cone K_{n-2} whose intersection with an arbitrary S_{n-2} in S_{n-1} is, from § 9, a double-point-free quadric Q_{n-2} in S_{n-2} . Such a thing will always be irreducible for $n > 3$; thus, the cone K_{n-2} will also be irreducible (and of dimension $n - 2$).

From § 41, all irreducible components of the intersection of Q and K have dimension $n - 2$. The irreducible components of M belong to these components, *a priori*. In case there are more irreducible components, as we know, they will be contained in the irreducible cone K_{n-2}, so since it is irreducible and has the same dimension $n - 2$, it will be identical with it. The intersection of Q and K will then consist of M and the cone K_{n-2} with a certain multiplicity μ that can also be zero.

If $\mu = 0$ then we will be finished. Thus, let $\mu > 0$. The μ-times counted plane S_{n-1} might have the equation $L^\mu = 0$. Furthermore, K and Q might have the equations $K = 0$ and $Q = 0$. The intersection of L^μ and Q will then be contained in K. The NOETHER conditions will be fulfilled if one cuts L^μ, Q, and K with a general plane, if Q had no multiple points and K cuts Q in K_{n-2} with the same multiplicity μ as L^μ. There will thus exist an identity:

$$K = A\,Q + B\,L^\mu.$$

The intersection of $K = 0$ and $Q = 0$ is the same as the intersection of $Q = 0$ and $B\,L^\mu = 0$. It decomposes into the μ-times counted cone K_{n-2} and the manifold M. Thus, M is the complete intersection of the hypersurfaces $Q = 0$ and $B = 0$. With that, the theorem is proved.

In the special case $n = 5$, if we map the points of the quadric Q to the lines in the space S_3, according to § 7, then we will get the following theorem of FELIX KLEIN:

Any line complex in S_3 *is given by two equations in PLÜCKER coordinates, of which, the first one is the identity:*

$$\pi_{01}\,\pi_{23} + \pi_{02}\,\pi_{31} + \pi_{03}\,\pi_{12} = 0.$$

§ 53. Space curves up to fourth order.

In this paragraph, we would like to enumerate the irreducible space curves of lowest orders 1, 2, 3, and 4, and examine them.

A space curve of order 1 is a line.

Namely, if one draws two planes through its points then both of them will have more than one intersection point with the curve, and will thus contain it.

An irreducible space curve of order 2 is a conic section.

Namely, if one draws a plane through three of its points then it must contain the space curve. However, a plane curve of order 2 is a conic section.

An irreducible space curve of order 3 is either a plane curve or a cubic, space curve, in the sense of § 11.

One can, in fact, always draw two quadratic surfaces through 7 points of the curve. Both of them must contain the curve, since it has more than 6 intersection points with it. If one of these surface decomposes into two planes then the curve will lie in one of these planes, and will be a plane, cubic curve. However, if both surfaces are irreducible then they will have no common component, and their intersection will be a curve of order 4 that contains the given third-order curve, and will therefore decompose into it and a line. From § 11, the intersection of two quadratic surfaces that have a line in common will consist of that line and a cubic, space curve (or will decompose into lines and conic sections).

An irreducible space curve of order 4 is either a plane curve or it lies in at least one irreducible quadratic surface.

In fact, one can always draw a quadric through 9 points of the curve. It must contain the curve, since it has more than 8 intersection points with it. If it decomposes into two planes then the curve will lie in one of these planes; in the other case, it will lie on an irreducible quadric.

We can ignore plane curves of order 4; we then turn to real space curves. If two different (irreducible) quadrics go through such a curve then the space curve will obviously be the complete intersection of these two surfaces. It will then be called a *fourth-order space curve of the first kind,* and will be denoted by C_I^4. By contrast, if it goes through only one quadric then it will be called a *fourth-order space curve of the second kind,* and denoted by C_{II}^4.

We thus have the following theorem:

If a fourth-order space curve lies on a quadratic cone K then it will be of the first kind – i.e., it will be the complete intersection of the cone with a second quadric.

Proof. At least ∞^6 cubic surfaces will go through 13 points of the curve, so, from § 10, the cubic surfaces can be mapped to the points of a linear space S_{19}, in which 13 linear equations will determine an at least 6-dimensional subspace. The ∞^3 decomposed surfaces that contain the cone K as a component will belong to these ∞^6 surfaces. There will thus be at least one cubic surface that contains the curve that does not contain cone K

as a component. This surface F will intersect K in a sixth-order curve that contains the given curve C^4 as a component, so it will consist of C^4 and a conic section or C^4 and two lines. A conic section or line-pair on K will, however, always be a plane section of the cone K ([1]) – perhaps, the intersection of K with a plane E.

We now apply the spatial NOETHER theorem to F, K, and E. F contains the entire intersection of K and E. If it consists of two coincident lines then the intersection of F and K will likewise contain these lines doubled; the NOETHER conditions will then be fulfilled in any event. If $F = 0$, $K = 0$, $E = 0$ are the equations of F, K, E, resp., then it will follow that:

$$F = A\,K + B\,E.$$

The curve C^4 will lie on the surfaces $F = 0$ and $K = 0$, but not in the plane $E = 0$, so it will lie on the quadric $B = 0$. With that, the theorem is proved.

It follows from the theorem that a fourth-order space curve of the second kind does not lie on a cone, but on a double-point-free quadric Q. If one further brings a Q that does not contain the cubic surface F through the curve C^4 then the complete intersection of F and Q will consist of the curve C^4 and two (possibly coincident) lines of *the same* family. Then, when the remainder intersection is an irreducible conic section or consists of two lines, one can conclude, on the basis of the reasoning that was applied in the last proof, with the help of NOETHER's theorem, that C lies on yet a second quadratic surface, which would make it of the first kind.

These two ruled surfaces on the quadric Q might be denoted by I and II, and the two skew or coincident lines that meet F and Q outside of C^4, by g and g', resp. We can assume that g and g' belong to the family I. A general line of the family I cuts the surface F at three points, so it will also cut the curve C^4 at three points. (The fact that all three of them are different would follow, e.g., from BERTINI's first theorem, § 47.) A general line of the family II likewise cuts F in three points, of which, however, two of them are assigned to g and g', such that only one of them will remain for C^4. The curve C^4 will thus be met by any general line of the family I at three points, but at one point by any line of the family II.

With this property, one can essentially distinguish between the curves of the first kind C_I^4 that lie on Q, which one obtains when cuts Q with another quadratic surface. These will then be obviously be cut by all generators of Q in two points. It follows from this that the remainder intersection of Q with a cubic surface F that has two generators of the family I in common with Q can never be a curve of the first kind C_I^4; it then cuts each generator of the family I at three points and each generator of the family II at one point.

We now summarize:

There are two types of bi-quadratic space curves. A curve C_I^4 is, by definition, the complete intersection of two quadrics. A curve C_{II}^4 is the remainder intersection of a quadratic ruled surface Q with a cubic surface F that contains two generators of a ruled

---

([1]) The conclusion is true only for the cone, but not for other quadrics; a line-pair on a quadratic ruled surface can then consist of two skew lines.

surface on Q. Conversely, any such remainder section is a C_{II}^4*, as long it is irreducible. A* C_{II}^4 *cuts any generator of the one ruled family of Q at three points, and each generator of the other family at one point. By contrast, a* C_I^4 *will cut every generator of a quadric that contains each of them at two points.*

The curve C_{II}^4 *is* rational. Namely, if we draw all possible planes through a generator of the family *I* then they will cut the surface *Q* in the generators of the family *II*, so it will cut the curve C_{II}^4 at one point (except for the three fixed intersection points of the curve with the generators of the family *I* that we started with). There is thus a linear family of point-groups of order 1 on C_{II}^4. From § 43, it will map the C_{II}^4 onto a line birationally.

For a closer study of fourth-order curves in a quadratic ruled surface *Q*, we put the equation for *Q* into the form:

$$y_0\, y_0 - y_2\, y_3 = 0,$$

and we introduce two homogeneous parameter pairs λ, μ by:

$$(1) \qquad \begin{cases} y_0 = \lambda_1\mu_1, \\ y_1 = \lambda_2\mu_2, \\ y_2 = \lambda_1\mu_2, \\ y_3 = \lambda_2\mu_1. \end{cases}$$

The parameter lines λ = const. and μ = const. are the generators of the families *I* and *II*. If one intersects *Q* with a second quadratic surface $g = 0$ by substituting (1) into the equation $g = 0$ then one will obtain an equation of degree 2 in the λ, and likewise in the μ:

$$(2) \qquad a_0\lambda_1^2\mu_1^2 + a_1\lambda_1^2\mu_1\mu_2 + \cdots + a_8\lambda_2^2\mu_2^2 = 0,$$

which will therefore represent a curve C_I^4 when its left-hand side is not decomposable. If one intersects *Q* with a cubic surface $F = 0$ in the same way then one will obtain an equation that has degree 3 in the λ, as well as in the μ. If the cubic surface *F* contains two lines λ = const. then the aforementioned equation with the degree numbers 3, 3 must contain two linear factors in just the λ; after dropping them, what remains will be an equation with the degree numbers 1, 3:

$$(3) \qquad a_0\lambda_1\mu_1^2 + a_1\lambda_1\mu_1^2\mu_2 + \cdots + a_7\lambda_2\mu_2^3 = 0.$$

Equation (3) will then represent the curve C_{II}^4.

On the basis of the map (1), the surface *Q* appears to be the image of a double-projective space (cf., § 4). The plane sections of *Q* define the projectivities that transform the points of a λ-line projectively into an μ-line. The cubic space curves on *Q* will be represented by equations in λ and μ with the degree numbers 2, 1 or 1, 2. This brief glimpse into the geometry of curves on a quadratic surface might suffice.

The genus of the curve C_{II}^4 is equal to 0, since the curve is rational. In order to ascertain the genus of the curve C_I^4, one projects it onto a plane from a generally-chosen point O of Q. What comes about is an irreducible, fourth-order plane curve. Two points of C_I^4 lie on each of the two lines of Q through O that go to a point in the plane under projection; the projection has two nodes, in any case. The projection will have further double points if and only if the original curve C_I^4 has them. If one now computes the genus of the projected curve by means of the formula of § 26 then that will yield the value 1 or 0, according to whether the original curve has no double point or one of them, respectively; for more than one double point, it must decompose. On the basis of the invariance of the genus under birational maps, it follows from this that:

The genus of a space curve C_I^4 *is* 1 *when the curve has no double points and* 0 *when it has one of them.*

Problems. 1. If one intersects a quadratic ruled surface Q with three planes and constructs generators on each of them for a family of the fourth harmonic point P to their intersection points with these three planes then the point P will describe a curve C_{II}^4. (Compute the equation of the curve in the parameters λ, μ.).

2. A fourth-order, rational, space curve is either a C_I^4 with double points or a C_{II}^4. In both cases, the coordinates of a general point of the curve will be proportional to four fourth-order forms in two homogeneous parameters λ, μ.

3. Projecting a curve C_I^4 or C_{II}^4 onto a simple point of the curve will yield a third-order, plane curve with or without double points, according to the genus.

4. By computing the equation of the curve, show that the two double points that correspond to the projection of C_I^4 onto a general point of Q are, in fact, ordinary junctions. (Choose the equation of the surface as above, and choose O to be a vertex of the coordinate system.)

5. The genus of a curve on Q that is given by an equation of degrees n and m in the parameters λ and μ, resp., is equal to:

$$p = (n - 1)(m - 1) - d - s,$$

where d is the number of double points, and s is the number of cusps, in the sense of § 26.

CHAPTER NINE

The analysis of singularities of plane curves.

The situation that is treated in this chapter is of fundamental significance for the theory of algebraic surfaces. In the main theorem, one is dealing with the precise definition of the concept of "infinitely-close points," or, as we will say here, the neighboring points that M. NOETHER first coined in connection with his resolution of singularities (cf., § 25), and which F. ENRIQUES ([1]) then developed further.

In order to not expand the scope of this book excessively, it was, unfortunately, necessary to treat this situation in a more cramped manner than was used in the previous chapters; in particular, I have been forced to dispense with explaining the concepts presented with simple examples. The reader will then be encouraged to carefully work through the problems, which contain such examples. One will find a thorough, didactic, and generally excellent presentation with worked examples in the book by ENRIQUES that was already cited in ([1]). One can further refer to an interesting paper of O. ZARISKI ([2]), in which the theory of infinitely-close points is related to evaluation theory and ideal theory.

§ 54. The intersection multiplicity of two branches of a curve.

In this chapter, we employ inhomogeneous coordinates x, y; the coordinate origin (0, 0) will be denoted by O.

A branch of an algebraic curve at the point O whose tangent is not the y-axis will be given by a cycle of conjugate power series that arise from a power series:

$$y = a\,x + a_1 x^{\frac{\nu+\nu'}{\nu}} + a_2 x^{\frac{\nu+\nu'+\nu''}{\nu}} + \cdots + a_s x^{\frac{\nu+\nu'+\cdots+\nu^{(s)}}{\nu}} + \cdots \tag{1}$$

by the substitution:

$$x^{1/\nu} \to \zeta\, x^{1/\nu} \qquad \text{with} \qquad \zeta^\nu = 1.$$

Let a second branch be given in the same way by ([3]):

$$y = a\,x + a_1 x^{\frac{\nu+\nu'}{\nu}} + a_2 x^{\frac{\nu+\nu'+\nu''}{\nu}} + \cdots + a_s x^{\frac{\nu+\nu'+\cdots+\nu^{(s)}}{\nu}} + \cdots \tag{2}$$

If the initial coefficients a, a_1, …, a_s coincide with the initial coefficients of one of the power series $\bar{y}_1$, $\bar{y}_2$, …, $\bar{y}_\mu$ that are conjugate to (2) – so one has:

([1]) F. ENRIQUES, O. CHISINI, *Teoria geometrica delle equazioni e delle funzione algebraiche*, vol. II, Libro Quarto, Bologna, ed. Zanichelli.

([2]) ZARISKI, O., "Polynomial ideals defined by infinitely many base points," Amer. J. Math. **60** (1938), 151-204.

([3]) In both power series, only the non-zero terms were written down; however, the initial term ax (bx, resp.) can also be zero.

$$\begin{aligned} a &= b, \\ a_1 &= b_1\, \zeta^{\mu'}, \\ &\ldots\ldots\ldots\ldots, \\ a_s &= b_s \zeta^{\mu' + \cdots + \mu^{(s)}}, \end{aligned}$$

then we will write this briefly as:

$$(a, a_1, \ldots, a_s) = (b, b_1, \ldots, b_s).$$

The intersection multiplicity of the two branches that are given by (1), (2) is defined to be the order of the power series:

$$(y - \bar{y}_1)(y - \bar{y}_2) \cdots (y - \bar{y}_\mu)$$

under the position uniformization $\tau = x^{1/\nu}$ of the first series, where the roles of the two branches have also been switched. The following theorem gives the precise value of this multiplicity:

Theorem 1. *The branch that is given by* (1), (2) *might coincide in the first s* + 1 *terms of the series developments. Therefore, let:*

$$(3) \qquad \begin{cases} \dfrac{\nu'}{\nu} = \dfrac{\mu'}{\mu} = \dfrac{\rho'}{\rho}, & (\rho', \rho) = 1, \\ \dfrac{\nu''}{\nu} = \dfrac{\mu''}{\mu} = \dfrac{\rho''}{\rho\rho_1}, & (\rho'', \rho_1) = 1, \\ \ldots\ldots\ldots\ldots, & \ldots, \\ \dfrac{\nu^{(s)}}{\nu} = \dfrac{\mu^{(s)}}{\mu} = \dfrac{\rho^{(s)}}{\rho\rho_1 \cdots \rho^{s-1}} & (\rho^{(s)}, \rho_{s-1}) = 1, \end{cases}$$

$$(a, a_1, \ldots, a_s) = (b, b_1, \ldots, b_s).$$

Now, if:

$$\frac{\nu^{(s+1)}}{\nu} \neq \frac{\mu^{(s+1)}}{\mu}$$

then the intersection multiplicity of the two branches will be equal to the smaller of the two numbers:

$$\lambda = \mu\,\nu + \mu\,\nu' + \frac{\mu\nu''}{\rho} + \frac{\mu\nu'''}{\rho\rho_1} + \cdots + \frac{\mu\nu^{(s+1)}}{\rho\rho_1 \cdots \rho_{s-1}},$$

$$\lambda' = \nu\, m + \nu\, m' + \frac{\nu\mu''}{\rho} + \frac{\nu\mu'''}{\rho\rho_1} + \cdots + \frac{\nu\mu^{(s+1)}}{\rho\rho_1 \cdots \rho_{s-1}}.$$

By contrast, if:

$$\frac{\nu^{(s+1)}}{\nu} = \frac{\mu^{(s+1)}}{\mu} = \frac{\rho^{(s+1)}}{\rho\rho_1\cdots\rho_s}$$

then $\lambda = \lambda'$ *will represent the intersection multiplicity, as long as one does not have:*

$$(a, a_1, \ldots, a_{s+1}) = (b, b_1, \ldots, b_{s+1}).$$

Preliminary remark. One concludes from formulas (3) that:

$$(\nu, \nu') = \frac{\nu}{\rho}, \qquad (\mu, \mu') = \frac{\mu}{\rho},$$

$$(\nu, \nu', \nu'') = \frac{\nu}{\rho\rho_1}, \qquad (\mu, \mu', \mu'') = \frac{\mu}{\rho\rho_1},$$

……………………………………………………………………..…..,

$$(\nu, \nu', \ldots, \nu^{(s)}) = \frac{\nu}{\rho\rho_1\cdots\rho_{s-1}}, \qquad (\mu, \mu', \ldots, \mu^{(s)}) = \frac{\mu}{\rho\rho_1\cdots\rho_{s-1}}.$$

Proof. Let – say – $\bar{y}_1$ be the power series that is conjugate to $\bar{y}$ whose initial coefficients agree precisely with $a, a_1, \ldots, a_s$:

$$\bar{y}_1 = a\,x + a_1 x^{\frac{\mu+\mu'}{\mu}} + a_2 x^{\frac{\mu+\mu'+\mu''}{\mu}} + \cdots + a_s x^{\frac{\mu+\mu'+\cdots+\mu^{(s)}}{\mu}} + \cdots$$

In the difference $y - \bar{y}_1$, the terms in $a, a_1, \ldots, a_s$ drop out. If we assume, say:

$$\mu\,\nu^{(s+1)} < \nu\,\mu^{(s+1)}, \qquad \text{so} \qquad \lambda < \lambda',$$

then the first term that does not drop out, namely:

$$a_{s+1} x^{\frac{\nu+\nu'+\cdots+\nu^{(s)}}{\nu}} = a_{s+1}\tau^{\nu+\nu'+\cdots+\nu^{(s)}},$$

will have order $\nu + \nu' + \ldots + \nu^{(s+1)}$ in τ. If one now goes from $\bar{y}_1$ to a conjugate power series $\bar{y}_i$ by means of the substitution:

$$x^{1/\mu} \to \zeta x^{1/\mu}, \quad \zeta^{\mu} = 1$$

then some of the initial terms of $\bar{y}$ will remain unchanged, although they will change past a certain point in the coefficients. Let, say:

$$\zeta^{\frac{\mu}{\rho}} = 1, \qquad \zeta^{\frac{\mu}{\rho\rho_1}} = 1, \qquad \ldots, \qquad \zeta^{\frac{\mu}{\rho\rho_1\cdots\rho_{t-1}}} = 1,$$

while $\zeta^{\frac{\mu}{\rho\rho_1\cdots\rho_t}} \neq 1$. $\bar{y}_1 - y$ will then have an initial term with:

$$x^{\frac{\mu+\mu'+\cdots+\mu^{(s)}}{\mu}} = x^{\frac{\nu+\nu'+\cdots+\nu^{(t+1)}}{\nu}} = \tau^{\nu+\nu'+\cdots+\nu^{(t+1)}}$$

being of order $\nu + \nu' + \ldots + \nu^{(t+1)}$.

If one now forms the product $(y-\bar{y}_1)(y-\bar{y}_2)\cdots(y-\bar{y}_\mu)$ then its order will be a sum of expressions $\nu + \nu' + \ldots + \nu^{(s+1)}$ and $\nu + \nu' + \ldots + \nu^{(t+1)}$, and indeed the term $\nu^{(t+1)}$ will appear in this sum as often as there are solutions of the equation:

$$\zeta^{\frac{\mu}{\rho\rho_1\cdots\rho_{t-1}}} = 1;$$

i.e., $\dfrac{\mu}{\rho\rho_1\cdots\rho_{t-1}}$ times. Therefore, the intersection multiplicity will be:

$$\lambda = \mu\,\nu + \mu\,\nu' + \frac{\mu}{\rho}\nu'' + \ldots + \frac{\mu}{\rho\rho_1\cdots\rho_{t-1}}\nu^{(t+1)} + \ldots + \frac{\mu}{\rho\rho_1\cdots\rho_{s-1}}\nu^{(s+1)}.$$

One argues in a completely analogous way in the case of $\lambda > \lambda'$, and also in the case $\lambda = \lambda'$.

We would now like to analyze the expression that was obtained for the intersection multiplicity more closely. In order to have a definite case in mind, we assume that $s = 2$; the series (1) and (2) will then already truncate at the third term. Factorize (ν, ν') and (μ, μ') by the Euclidian algorithm:

$$(4) \qquad \left\{\begin{aligned} \nu' &= h\nu + \nu_1, \\ \nu &= h_1\nu_1 + \nu_2, \\ &\ldots\ldots\ldots\ldots, \\ \nu_{\sigma-1} &= h_\sigma \nu_\sigma, \end{aligned}\right. \qquad \left\{\begin{aligned} \mu' &= h\mu + \mu_1, \\ \mu &= h_1\mu_1 + \mu_2, \\ &\ldots\ldots\ldots\ldots, \\ \mu_{\sigma-1} &= h_\sigma \mu_\sigma. \end{aligned}\right.$$

Since $\dfrac{\nu'}{\nu} = \dfrac{\mu'}{\mu}$, the two developments will run exactly parallel to each other. We now proceed in precisely the way that would determine the greatest common divisor of (ν_σ, ν'') and (μ_σ, μ''). The two developments run perhaps somewhat parallel, but in the case $\dfrac{\nu'}{\nu} \neq \dfrac{\mu'}{\mu}$, they must eventually separate:

$$
(5)\qquad
\begin{cases}
\nu'' = k\nu_\sigma + \nu_{\sigma+1}, \\
\nu_\sigma = k_1\nu_{\sigma+1} + \nu_{\sigma+1}, \\
\dots\dots\dots\dots, \\
\nu_{\sigma+j-1} = k_j\nu_{\sigma+j} + k_j\nu_{\sigma+j+1}, \\
\nu_{\sigma+j} = k_{j+1}\nu_{\sigma+j+1} + k_j\nu_{\sigma+j+2}, \\
\dots\dots\dots\dots\dots\dots\dots,
\end{cases}
\qquad
\begin{cases}
\mu'' = k\mu_\sigma + \mu_{\sigma+1}, \\
\mu_\sigma = k_1\mu_{\sigma+1} + \mu_{\sigma+2}, \\
\dots\dots\dots\dots, \\
\mu_{\sigma+j-1} = k_j\mu_{\sigma+j} + \mu_{\sigma+j+1}, \\
\mu_{\sigma+j} = l_{j+1}\mu_{\sigma+j+1} + \mu_{\sigma+j+2}, \\
\dots\dots\dots\dots\dots\dots\dots,
\end{cases}
$$

with $k_{j+1} \neq l_{j+1}$. It can also happen that $k_{j+1} = l_{j+1}$, but the division with the quotients k_{j+1} can take place on the left-hand side, but not on the right (or conversely). In the second case $\dfrac{\nu''}{\nu} = \dfrac{\mu''}{\mu}$, the developments will run entirely parallel, up to the step:

$$\nu_{\sigma+\sigma'-1} = k_{\sigma'}\,\nu_{\sigma+\sigma'}, \qquad \mu_{\sigma+\sigma'-1} = k_{\sigma'}\,\mu_{\sigma+\sigma'}.$$

In order to once more have something definite in mind, we consider the first case and assume that $l_{j+1} < k_{j+1}$. That means:

a) when j is even, $\dfrac{\mu''}{\mu} > \dfrac{\nu''}{\nu}$, so $\nu\,\mu'' > \mu\,\nu''$,

b) when j is odd, $\dfrac{\mu''}{\mu} < \dfrac{\nu''}{\nu}$, so $\nu\,\mu'' < \mu\,\nu''$.

Since $(\nu, \nu') = \nu_\sigma$, from the preliminary remark, one will have $\nu = \rho\,\nu_\sigma$; likewise, one will have $\mu = \rho\,\mu_\sigma$. From Theorem 1 in case a), the intersection multiplicity will be:

$$\lambda = \mu\,\nu + \mu\,\nu' + \frac{\mu\nu'}{\rho} = \mu\,\nu + \mu\,\nu' + \mu_\sigma\,\nu'',$$

and in the case b):

$$\lambda' = \nu\,\mu + \nu\,\mu' + \frac{\nu\mu''}{\rho} = \nu\,\mu + \nu\,\mu' + \nu_\sigma\,\mu''.$$

We leave the first term $\mu\,\nu$ unchanged. The second term will be developed on the basis of (4):

$$
\begin{aligned}
\mu\,\nu' &= \mu\,(h\nu + \nu_1) &&= h\mu\,\nu + \mu\,\nu_1, \\
\mu\,\nu_1 &= (h_1\,\mu_1 + \mu_1)\,\nu_1 &&= h_1\,\mu_1\,\nu_1 + \mu_2\,\nu_1, \\
\mu_2\,\nu_1 &= \mu_2\,(h_2\,\nu_2 + \nu_3) &&= h_2\,\mu_2\,\nu_2 + \mu_2\,\nu_3, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
\mu_\sigma\,\nu_{\sigma-1} &= \mu_{\sigma-1}\,\nu_\sigma &&= h_\sigma\,\mu_\sigma\,\nu_\sigma.
\end{aligned}
$$

That will yield:

$$\nu\,\mu' = \mu\,\nu' = h\,\mu\,\nu + h_1\,\mu_1\,\nu_1 + h_2\,\mu_2\,\nu_2 + \dots + h_\sigma\,\mu_\sigma\,\nu_\sigma.$$

The third term $\mu_\sigma\, v''$ ($v_\sigma\, \mu''$, resp.) in case a) [b), resp.] can be developed likewise by means of (5); the details of this will be left to the reader. If one now combines the various terms then in both cases a) and b) one will get:

$$
(6) \qquad \left\{ \begin{aligned} \Lambda = \nu\mu\ &+ h\nu\mu + h_1\,\nu_1\mu_1 + \cdots + h_\sigma\,\nu_\sigma\mu_\sigma \\ &+ k\,\nu_\sigma\mu_\sigma + k_1\,\nu_{\sigma+1}\mu_{\sigma+1} + \cdots + k_j\,\nu_{\sigma+j}\mu_{\sigma+j} \\ &+ l_{i+1}\,\nu_{\sigma+j+1}\mu_{\sigma+j+1} + \nu_{\sigma+j+1}\mu_{\sigma+j+2}. \end{aligned} \right.
$$

for the intersection multiplicity Λ.

In order to do the division $\mu_{j+\sigma} : \mu_{j+\sigma+1}$, the last term in (6) must be replaced with zero. If $k_{j+1} < l_{j+1}$ or $k_{j+1} = l_{j+1}$, and one is to do the division $\nu_{j+\sigma} : \nu_{j+\sigma+1}$ then the roles of k and l, as well as those of μ and ν, must be switched. In the case $\mu\, v'' = v\, \mu''$, the final term:

$$\kappa_\sigma \cdot \nu_{\sigma+\sigma'}\, \mu_{\sigma+\sigma'}$$

will enter in place of the last two terms.

Problems. 1. From a certain number m onward, one will have:

$$
\begin{aligned} v_{\sigma+\sigma'+\cdots+\sigma^{(m)}} = (v, v', \ldots, v^{(m+1)}) &= 1, \\ \rho\,\rho_1 \ldots \rho_m &= v. \end{aligned}
$$

2. A branch of order 2:

$$y = a_1\, x + a_2\, x^2 + \ldots + a_s\, x^s + a_{s+1}\, x^{s+\frac{1}{2}} + \ldots + a_{s+1}\, x^{s+1} + \ldots$$

will have, with a linear branch:

$$y = b_1\, x + b_2\, x^2 + \ldots,$$

the intersection multiplicity:

$$2, 4, \ldots, 2s, \text{ or } 2s + 1,$$

when its developments agree up to the terms in:

$$1, x, \ldots, x^{s-1}, \text{ or } x^s,$$

respectively. A higher multiplicity will be excluded.

§ 55. Neighboring points.

One computes the intersection multiplicity of two branches at a point O using formula (6) in precisely the same way as if one had two curves with several intersection points O, O_1, …, O_h, O_{h+1}, … , $O_{h+h_1+\cdots+h_\sigma+k+k_1+\cdots+k_j+l_{j+1}+1}$, instead of the two branches, where the curves take on the following multiplicities at the at those points:

the multiplicities ν and μ at O, O_1, …, O_h,
the multiplicities ν_1 and μ_1 at O_{h+1}, …, O_{h+h_1},

etc., according to formula (6).

In order to justify this situation, one introduces the following terminology: First, let the initial piece of a power series be given, say:

$$y = a\,x + a_1\, x^{\frac{\nu+\nu'}{\nu}}, \tag{7}$$

and second, let a sequence of natural numbers $p, p_1, \ldots, p_j, p_{j+1}$ be given (or possibly just a single natural number p). The *neighboring point* to O that belongs to this defining piece is then defined to be *the totality of all curve branches whose power series development* (1) *begins with the terms* (7), *while the exponent* $\frac{\nu+\nu'+\nu''}{\nu}$ *of the next term is so arranged that the quotients* $k, k_1, \ldots, k_{j+1}$ *in the successive divisions* (5) *satisfy the conditions:*

$$\begin{aligned} &k = p - 1, \\ &k_1 = p_1, \ldots, k_j = p_j, \\ &k_{j+1} > p_{j+1} \text{ or } k_{j+1} = p_{j+1}, \qquad k_{j+2} > 0, \end{aligned}$$

while in the case of a single number p, one will have the condition:

$$k \geq p - 1.$$

Which neighboring point of O belongs to a branch according to this definition, when its series development is given by (1)? At first, it will be the neighboring point that belongs to $a\,x$, and indeed:

O_1	will go with the number sequence 1,
O_2	with the number sequence 2,
…………………………………………….	
O_{h+1}	with the number sequence $h + 1$,
O_{h+2}	with the number sequence $h + 1, 1$,
………………………………………………	
O_{h+h_1+1}	with the number sequence $h + 1, h_1$
O_{h+h_1+2}	with the number sequence $h + 1, h_1, 1$,
…………………………………………………	
$O_{h+h_1+\cdots+h_\sigma}$	with the number sequence $h + 1, h_1, \ldots, h_\sigma - 1$.

One then comes to neighboring points that belong to the final piece:

$$a\,x + a_1\, x^{\frac{\nu+\nu'}{\nu}}, \tag{7}$$

and indeed when one sets $h + h_1 + \ldots + h_\sigma = H$:

O_{H+1}	will go with the number sequence 1,
O_{H+2}	with the number sequence 2,
…………………………………………	
O_{H+k+1}	with the number sequence $k + 1$,
O_{H+k+2}	with the number sequence $k + 1$, 1,
………………………………………………	
O_{H+k+k_1+1}	with the number sequence $k + 1$, k_1
…………………………………………………	
$O_{H+k+k_1+\cdots+k_{\sigma'}}$	with the number sequence $k + 1$, k_1, …, $k_{\sigma'}$.

One then comes to the neighboring point to the initial piece:

$$a\,x + a_1 x^{\frac{\nu+\nu'}{\nu}} + a_2 x^{\frac{\nu+\nu'+\nu''}{\nu}}\,,$$

etc.

We further define that the branch that is defined by (1) should have the following *multiplicities* at the neighboring points $O_1, \ldots, O_H, O_{H+1}, \ldots$:

the multiplicity ν	at $O_1, \ldots, O_h$,
the multiplicity ν_1	at $O_{h+1}, \ldots, O_{h+h_1}$,
………………………………………	
the multiplicity ν_σ	at $O_{h+h_1+\cdots+h_{\sigma-1}+1}, \ldots, O_H$,
likewise ν_σ	at $O_{H+1}, \ldots$:, O_{H+k},
likewise $\nu_{\sigma+1}$	at $O_{H+k+1}, \ldots$:, O_{H+k+k_1},

etc.

Formula (6) of the previous paragraph now yields:

Theorem 2. *The intersection multiplicity of two branches at O is equal to the sum of the products of the multiplicities of the two branches at O and at the neighboring points to O that are common to them.*

The first neighboring point consists of two branches whose power series begins with ax; i.e., the branches with well-defined tangents at O. It will depend upon a continuously-varying parameter a.

Neighboring points like $O_1, \ldots, O_{h+1}$, whose number sequence $(p, p_1, \ldots)$ consists of only one natural number p, are called *free neighboring points*, because any of them can be varied continuously while fixing the neighboring points that precede them. In order to make that clear with an example, we consider the neighboring point O_h (under the assumption that $h > 1$). It will consist of all branches whose developments begin with:

$$a\,x + 0x^h.$$

Here, the coefficient of x^h (which only occasionally has the value zero) is continuously-varying. Corresponding statements are true for all points $O_1, \ldots, O_{h+1}$, as well as for $O_{H+1}, \ldots, O_{H+k+1}$, etc. By contrast, $O_{h+2}, \ldots, O_H$ are not free, since when $O_1, \ldots, O_{h+1}$ are fixed they will be determined exclusively by arithmetic data. They depend upon the existence of the second term in the development (1) and the value of its exponent $\frac{v+v'}{v}$, but not, however, on the value of the coefficients a_1 of this term. Such non-free neighboring points are called *satellite points* of the last preceding free neighboring points.

Problems. 1. Nothing but free, simple neighboring points $O_1, O_2, \ldots$ to O belong to a linear branch.

2. On a quadratic branch:

$$y = a_1 x + a_2 x^2 + \ldots + a_s x^s + a_{s+1} x^{s+\frac{1}{2}} + a_{s+2} x^{s+1} + \ldots,$$

the double point O is first followed by $s - 1$ free two-fold, neighboring points $O_1, \ldots, O_{s-1}$, then a free, simple point O_s, a simple, satellite point O_{s+1}, and finally, nothing but free, simple neighboring points O_{s+2}, $O_{s+3}, \ldots$ For an ordinary cusp, one will have $s = 1$.

3. From a certain number onward, all neighboring points to O and a branch $\mathfrak{z}$ will be free and simple.

4. If $(v, v', \ldots, v^{(s+1)}) > (v, v', \ldots, v^{(s)})$, so $\rho_s > 1$, then the terms in the series (1) that have the exponent $\frac{v+v'+\cdots+v^{(s+1)}}{v}$ will be called *characteristic terms.* There are finitely many of them. The associated free neighboring points are the ones that follow satellite points immediately.

If one considers more closely the multiplicities $v, v_1, \ldots, v_s, \ldots$ of a branch at the neighboring points $O_1, \ldots, O_{h+1}, \ldots, O_{H+1}, \ldots$ that were defined above then one will see that there are two possibilities for a neighboring point O_n with the multiplicity v_i:

Either: The next neighboring point O_{n+1} has the same multiplicity v_s; one then calls O_{n+1} the *follower* to O_n .

Or: O_{n+1} has a smaller multiplicity v_{i+1} ; due to (4) or (5), one of the two equations:

(8a) $$v_i = q\, v_{i+1} + v_{i+2} ,$$
(8b) $$v_i = q\, v_{i+1}$$

will then be true. In these cases, O_n will next be followed by q neighboring points $O_{n+1}, \ldots, O_{n+q}$ with the multiplicity v_{i+1} , and then in the case (8a), another one with the multiplicity v_{i+2} . All of these points will be called the *followers* ([1]) of O_n .

If the first follower O_{n+1} belongs to the number sequence $(p, p_1, \ldots, p_j)$ then the followers of O_n will be given, in any case, by the number sequences:

$$\begin{array}{l} (p, p_1, \ldots, p_j), \\ (p, p_1, \ldots, p_j, 1), \\ (p, p_1, \ldots, p_j, 2), \\ \ldots\ldots\ldots\ldots\ldots; \end{array}$$

the sequence will then be established, until it leaves the branch under scrutiny. Therefore, if O_{n+k} belongs to the followers of O_n on a branch then the same thing will be true on any other branch that goes through $O_n, O_{n+1}, \ldots, O_{n+k}$.

The relations (8) now yield the following theorem, which is also trivially true in the case of a single follower of the same multiplicity:

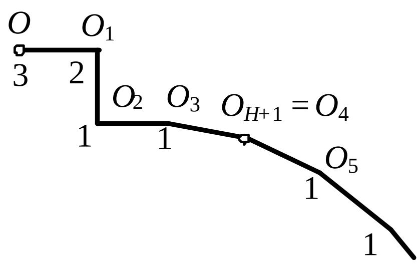
([1]) ENRIQUES: "Punti prossimi." ZARISKI: "proximate points."

Theorem 3. *The multiplicity of O_n on a branch $\mathfrak{z}$ is equal to the sum of the multiplicities of the followers to O_n on $\mathfrak{z}$.*

Theorem 3 is also true when one takes the point O, instead of O_n .

If one considers only those followers that simultaneously belong to a second branch $\mathfrak{z}'$ then the equality sign must be replaced with $\geq$.

Problems. 5. If one represents $O, O_1, O_2, \ldots$ graphically by a succession of points along a piece-wise linear path, where one makes a kink at the point O_{n+1} any time when O_{n+1} has a smaller multiplicity than O_n (see Figure) then the followers of O_n will be the point O_{n+1} , and in the event that the latter is a kink-point, the points that follow it up to the next kink-point (inclusive) or up to the next free point (exclusive). If the characteristic point (cf., Prob. 3) – that is, the free point that immediately follows the satellite points – is specially marked (in the figure, by a circle) then one can immediately realize the graphical representation of the following points, and with the help of Theorem 3, graphically ascertain the multiplicities $v, v_1, \ldots, v_\omega$ by starting with the last one $v_\omega = 1$. The number sequence $(p, p_1, \ldots, p_{j+1})$ that belongs to a neighboring point will give how many steps that one must take in order to arrive at this neighboring point from a point like O or O_H .



6. Complete graphical representations, in the sense of problem 5, for the branches $y = x^{3/2}$ and $y = x^{7/5}$, and indicate the multiplicity of each point.

Theorem 4. *If the sequence of neighboring points $O_1, O_2, \ldots$ on a branch were truncated arbitrarily at O_m then there would always be a curve that possesses only a single branch at O, goes through $O_1, \ldots, O_m$, but not through O_{m+1} , and has multiplicity one at O_m , while the follower of O_m on this branch would be free.*

Proof. One first computes the sequence of multiplicities $v, v_1, \ldots, v_\tau$ so the branch can be defined backwards while starting at $v_\tau = 1$, on the basis of the relations (8). The exponents of the series development of the branch will be established by these numbers. One determines the coefficients such that the required initial piece of the sequence agrees with the given branch. The coefficient of the next term (that belongs to the free followers of O_m) can be chosen freely, but must not be chosen to be equal to the corresponding coefficients of the given branch (or a conjugate one). The sequence will then be truncated with that term. This truncated power series ω_1, together with its conjugate $\omega_2, \ldots, \omega_v$, will determine an algebraic curve:

$$(y - \omega_1)(y - \omega_2)\ldots(y - \omega_v) = 0$$

that will satisfy all of the demands.

We now go on to the consideration of curves that possess several branches at the point O. If we define the multiplicity of such a curve at a neighboring point O_n to O to be the sum of the multiplicities of O_n on the different branches of the curve, as long as they contain O_n, then it will be clear that Theorem 2 is true for the intersection multiplicity of two arbitrary curves at a point O.

We now consider a fixed branch $\mathfrak{z}'$ at O with the neighboring points $O_1, O_2, \ldots$, and pose the question of whether there are curves C that have given multiplicities $r_0, r_1, \ldots, r_s$ at $O, O_1, \ldots, O_s$, resp. The following *follower relations* will be necessary in any case:

$$r_n \geq r_{n+1} + \ldots + r_{n+q}\,, \tag{9}$$

in which the sum extends over all followers $O_{n+1}, \ldots, O_{n+q}$ of O_n on $\mathfrak{z}'$. From Theorem 3, inequality (9) will then be valid for each individual branch $\mathfrak{z}$ of C, and thus also for C itself.

However, the conditions (9) are also sufficient:

Theorem 5. *If the follower relations* (9) *are fulfilled then there will be a curve C that has the multiplicities* $r_0, r_1, \ldots, r_s$ *at* $O, O_1, \ldots, O_s$, *resp.*

Proof, by complete induction on $r_0 + r_1 + \ldots + r_s$. If the sum is zero then everything will be clear. Now, if r_m is the last non-zero number in $r_0, \ldots, r_s$ then we will subtract the multiplicities $\rho_0, \rho_1, \ldots, \rho_s$ from the given ones $r_0, r_1, \ldots, r_s$, resp., for the curve C_m that exists according to Theorem 4, and for which one will have $\rho_m = 1$, $\rho_{m+1} = \ldots = \rho_s = 0$. (For $m = 0$, one chooses C_m to be an arbitrary line through O that does not contact $\mathfrak{z}'$.) The follower relations will, in fact, be true with the equality sign for this curve C_m:

$$\rho_m = \rho_{m+1} + \ldots + \rho_{m+q} \qquad (n < m). \tag{10}$$

It will follow from (9) and (10) by subtraction that:

$$(r_n - \rho_n) \geq (r_{n+1} - \rho_{n+1}) + \ldots + (r_{n+q} - \rho_{n+q}) \qquad (n < m).$$

However, this inequality is also true for $n \geq m$, since the right-hand side will be zero then. Therefore, from the induction assumption, there will be a curve C' with the multiplicities $r_0 - \rho_0, \ldots, r_s - \rho_s$. The curve $C_m = C + C'$ will then fulfill the conditions that were posed.

The multiplicities of the curves C_m that were used in the proof at the points $O, O_1, \ldots, O_m$ might now be computed rigorously with:

$$\rho_{m0}\,,\ \rho_{m1}\,,\ \ldots,\ \rho_{mm}\,.$$

From Theorem 2, a curve with the prescribed multiplicities $r_0, r_1, \ldots, r_s$ will then have the intersection multiplicity with C_m:

$$\sigma_m = r_0\,\rho_{m0} + r_1\,\rho_{m1} + \ldots + r_m\,\rho_{mm} \qquad (m = 0, 1, \ldots, s), \tag{11}$$

in the event that it does not have another neighboring point in common with C_m, other than $O_1, \ldots, O_m$. However, C_m depends upon one free parameter, and for a general choice of that parameter, the expression (11) will represent the precise intersection multiplicity.

Conversely: *If the intersection multiplicity of C with C_m* ($m = 0, 1, \ldots, s$) *is represented by* (11) *for a general choice of the parameters that enter into C_m then C will have the multiplicities $r_0, r_1, \ldots, r_s$ at $O, O_1, \ldots, O_s$, resp.*

One gets the proof with nothing further by complete induction on s. The assertion is clear for $s = 0$, and if $r_0, r_1, \ldots, r_{s-1}$ agree with the multiplicities of C then the same thing will be true from the last of equations (11).

For every m, the curves of a fixed degree that have an intersection multiplicity with C_n that is $\geq \sigma_m$ for a general choice of the parameters that enter into C_m, where σ_m is given by (11), will define a *linear family*. Substituting the series development of the single branch of C_m into the equations for C and setting the coefficients of the terms whose order are $< \sigma_m$ equal to zero will then yield linear conditions for the coefficients of C. Now, if a curve C belongs to this linear family for each $m = 0, 1, \ldots, s$ then all of the linear conditions that were mentioned will be satisfied, and one will say that the curve C has the *virtual multiplicities* $r_0, r_1, \ldots, r_s$ at $O, O_1, \ldots, O_s$. The true – or *effective* – multiplicities $\bar{r}_0, \ldots, \bar{r}_s$ can be in part larger and in part smaller than the virtual ones; however, they must still satisfy the inequalities:

$$\bar{r}_0\, \rho_{m0} + \bar{r}_1\, \rho_{m1} + \ldots + \bar{r}_m\, \rho_{mm} \geq \sigma_m\,.$$

Problems. 7. Show that the follower relations are fulfilled when one prescribes that the v-fold point O has the multiplicity $v - 1$ and that each v_s-fold neighboring point has the multiplicity $v_s - 1$.

8. Explicitly exhibit the linear conditions for the coefficients of the curves C with given virtual multiplicities for the case in which the given branch has an ordinary cusp (say, $y = x^{3/2}$) and the virtual multiplicities are given by:

$$r_0 = 3, \quad r_1 = 2, \quad r_2 = 1.$$

§ 56. The behavior of neighboring points under Cremona transformations.

We would like to investigate how the sequence of neighboring points $O, O_1, O_2, \ldots$ to a point O on a branch $\mathfrak{z}$ behave under the quadratic, Cremona transformation:

$$(1) \qquad \begin{cases} \zeta_0 : \zeta_1 : \zeta_2 = \eta_1\eta_2 : \eta_2\eta_0 : \eta_0\eta_1, \\ \eta_0 : \eta_1 : \eta_2 = \zeta_1\zeta_2 : \zeta_2\zeta_0 : \zeta_0\zeta_1 \end{cases}$$

that was defined in § 25, if O is the vertex (1, 0, 0), and the tangent to the branch is not a side of the coordinate triangle.

As in § 25, any curve $f(\eta) = 0$ will correspond to a curve $g(\zeta) = 0$ under the transformation, where the form g will be defined by:

(2) $$f(z_1\, z_2,\, z_2\, z_0,\, z_0\, z_1) = z_0^r\, z_1^s\, z_2^t\, g(z_0, z_1, z_2)\,.$$

Any branch $\mathfrak{z}$ with the starting point O will also correspond to a branch $\mathfrak{z}'$ whose starting point lies on the opposite side of $\zeta_2 = 0$. If $\zeta_0(\tau)$, $\zeta_1(\tau)$, $\zeta_2(\tau)$ are the power series for the branch $\mathfrak{z}'$ then:

(3) $$\eta_0(\tau) = \zeta_1(\tau)\, \zeta_2(\tau), \qquad \eta_1(\tau) = \zeta_2(\tau)\, \zeta_0(\tau), \qquad \eta_2(\tau) = \zeta_0(\tau)\, \zeta_1(\tau)$$

will be the power series for the branch $\mathfrak{z}$.

We now consider the intersection multiplicity of the branch $\mathfrak{z}$ with the curve $f = 0$ under the assumption that the branch does not lie on the curve. This multiplicity is defined to be the order of the power series $f(\eta_0(\tau), \eta_1(\tau), \eta_2(\tau))$. If one substitutes (3) here and employs (2) then one will obtain the power series:

$$\zeta_0(\tau)^r\, \zeta_1(\tau)^s\, \zeta_2(\tau)^t\, g(\zeta_0(\tau), \zeta_1(\tau), \zeta_2(\tau)).$$

Since $\zeta_1(\tau)$ and $\zeta_2(\tau)$ do not vanish for $\tau = 0$, the order of this power series will be equal to the order of:

$$\zeta_0(\tau)^r\, \zeta_1(\tau)^s\, g(\zeta_0(\tau), \zeta_1(\tau), \zeta_2(\tau)) = \eta_2(\tau)^r\, g(\zeta_0(\tau), \zeta_1(\tau), \zeta_2(\tau)),$$

so it will be equal to the intersection multiplicity of $g = 0$ with $\mathfrak{z}'$, plus r times the intersection multiplicity of the line $\eta_2 = 0$ with $\mathfrak{z}$. The latter multiplicity will be precisely the order of the branch $\mathfrak{z}$ (or the multiplicity of the point O on the branch $\mathfrak{z}$), while r will be the multiplicity of O on the curve $f = 0$. We then have:

Theorem 6. *The intersection multiplicity of the branch $\mathfrak{z}$ with a curve C is equal to the intersection multiplicity of the transformed branch $\mathfrak{z}'$ with the transformed curve C' plus the product of the multiplicities of C and $\mathfrak{z}$ at O.*

Let the successive neighboring points to O on $\mathfrak{z}$ be O_1, O_2, … The multiplicities of $\mathfrak{z}$ at O, O_1, O_2, … might be denoted by r_0, r_1, r_2, …, resp., and the multiplicities of C at these points by ρ_0, ρ_1, ρ_2, …, resp. Let the intersection multiplicity of C and $\mathfrak{z}$ be Λ, and let that of C' and $\mathfrak{z}'$ be Λ'. Theorem 6 then yields the formula:

(4) $$\Lambda = \Lambda' + \rho_0\, r_0\,.$$

With the help of Theorem 6, we now prove:

Theorem 7. *Under the transformation* (1), *the successive neighboring points O_1, O_2, …, O_m, … to O on the branch $\mathfrak{z}$ will go to O', O_1', …, O_{m-1}', resp., where O' is the starting point of the transformed branch $\mathfrak{z}'$, and O_1', …, O_{m-1}' are the successive* *neighboring points to O' on $\mathfrak{z}'$. If $\mathfrak{z}$ has the multiplicities $r_1, r_2, \ldots, r_m$ at $O, O_1, O_2, \ldots, O_m$, resp., then $\mathfrak{z}'$ will have the multiplicities $r_1, r_2, \ldots, r_m$ at $O', O'_1, \ldots, O'_{m-1}$, resp.*

We prove the theorem for O_m under the assumption that it is true for $O_1, \ldots, O_{m-1}$. One will see that the proof is also true for $m = 1$.

We choose C to be the curve C_m that exists according to Theorem 4, which has the multiplicities $\rho_0, \rho_1, \ldots, \rho_m$ at $O, O_1, \ldots, O_m$, resp., with $\rho_m = 1$. From the induction assumption, C' will have the multiplicities $\rho_1, \rho_2, \ldots, \rho_{m-1}$ at $O', O'_1, \ldots, O'_{m-2}$, resp. Let the neighboring point that follows O'_{m-2} on the curve C' be O'_{m-1}, and let the multiplicities of $\mathfrak{z}'$ and C' at O'_{m-1} be r'_m and ρ'_m. From Theorem 2 (§ 55), the intersection multiplicity of $\mathfrak{z}$ and C will be equal to:

$$\Lambda = \rho_0\, r_0 + \rho_1\, r_1 + \ldots + \rho_m\, r_m\,, \tag{5}$$

while, on the other hand, from Theorem 6, it will be equal to:

$$\begin{cases} \Lambda = \rho_0\, r_0 + \Lambda', \\ \quad = \rho_0\, r_0 + \rho_1\, r_1 + \cdots + \rho_{m-1}\, r_{m-1} + \rho'_m\, r'_m + \cdots, \end{cases} \tag{6}$$

where the terms + … refer to possible further neighboring points after O'_{m-1} that $\mathfrak{z}'$ and C' can still have in common.

A comparison of (5) and (6) yields:

$$\rho_m\, r_m = \rho'_m\, r'_m + \ldots \tag{7}$$

Since ρ_m and ρ'_m are positive, it will then follow from (7): $r'_m > 0$ if and only if $r_m > 0$. That means: $\mathfrak{z}'$ goes through the neighboring point O'_{m-1} if and only if $\mathfrak{z}$ goes through O_m. The neighboring point O_m – i.e., the totality of all branches through O_m – then goes to, in fact, the totality of branches through O'_{m-1} under the transformation.

As we know, there exists a certain freedom in the choice of curves C_m, since the follower of O_m on C_m is free. The curves C_m have just one branch. We now choose C to be a curve C_m, and choose $\mathfrak{z}$ to be the single branch on another curve C_m that indeed has $O, O_1, \ldots, O_m$ in common with the first curve, not the follower of O_m ! One will then have $r_m = \rho_m = 1$ in (7). It will then follow that only one term can enter into the right-hand side and that it will have the value one. Therefore: The different C_m contain O'_{m-1} only simply and have no further neighboring point in common with each other past O'_{m-1}.

We now again consider an arbitrary branch $\mathfrak{z}$ through $O, O_1, \ldots, O_m$. A suitably-chosen curve C_m will have only $O, O_1, \ldots, O_m$ in common with $\mathfrak{z}$, and the transformed C'_m will also have only $O', O'_1, \ldots, O'_{m-1}$ in common with $\mathfrak{z}'$. Therefore, the terms + … on the right-hand side of (7) must be dropped; furthermore, one must set $\rho_m = \rho'_m = 1$. It will then follow that $r_m = r'_m$; i.e., the multiplicity of $\mathfrak{z}'$ at O'_{m-1} will be equal to that of $\mathfrak{z}$ at O_m. With that, the induction is complete.

Since the numbers of all neighboring points to O will be reduced by one under the singly-quadratic Cremona transformation (1), one can convert every arbitrary neighboring point O_k into an ordinary point by k-times repeated quadratic transformations. One can even define the neighboring points, as NOETHER originally did, by these repeated transformations.

The same method of investigation can also be applied to arbitrary Cremona transformations (i.e., birational transformations of the plane into itself). Especially simple are the results for the case in which the transformation is one-to-one at the location O, or more precisely, in which the rational forms for the transformation, as well as for its inverse, remain meaningful at the location O (at the corresponding location O', resp.). For this case, in place of Theorem 6, one will have the simple statement that the intersection multiplicity of $\mathfrak{z}$ and C does not change under the transformation; in place of (3), one will then have:

$$\Lambda = \Lambda'.$$

The method of proof that was applied to Theorem 7 will then yield the simple result that the sequence of neighboring points $O_1, O_2, \ldots$ to O on $\mathfrak{z}$ will be transformed into the sequence of neighboring points $O'_1, O'_2, \ldots$ to O' on $\mathfrak{z}'$, while the multiplicities of $\mathfrak{z}$ on O, $O_1, O_2, \ldots$will remain unchanged.

In place of algebraic curves and curve branches, one can bring into consideration general, analytic curves $F(x, y) = 0$ in the vicinity of a fixed point O and analytic curve branches in these investigations. The methods of proof and results will not change essentially. One will obtain – e.g., – the theorem that the concepts of a neighboring point and the multiplicities of branch at the neighboring points to O will remain unchanged under analytic transformations that are single-valued and uniquely, analytically invertible in the neighborhood of O.

Problem. 1. Carry out the proof that was suggested.