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On the calculus of variations for multiple integrals

By

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It is known that there is a whole host of theories for multiple variational integrals in the usual form that differ from each other by the use of different independent integrals, and which lead to different sufficient Legendre conditions ($\mathcal{E}$-functions, resp.) (cf., [**12**], [**3**]). In particular, that host of theories includes the DeDonder-Weyl and Carathéodory theories as special cases (cf., [**16**], [**6**], [**2**], [**10**]). The first one has the advantage that it yields very simple (viz., linear) formulas, while one will be inevitably led to the Carathéodory "determinant method" when one seeks a theory that is applicable to problems with moving boundaries.

It seems that analogous investigations of variational integrals in parametric form have still not been proposed up to now. At the urging of H. BOERNER, a theory of variational integrals of the form $\int f(x_i, p_{i_1\cdots i_m})\,dt_1\cdots dt_m$ [the $p_{i_1\cdots i_m}$ are the direction coordinates of the matrix $(\partial x_i / \partial t_\alpha)$, $\alpha = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$; $m < n$] will then be developed in what follows that will lead to the $\mathcal{E}$-functions, sufficient Legendre condition, and Hamilton-Jacobi differential equation, up to the embedding of a given extremal in a geodetic field. It represents an analogue of the Carathéodory theory, so it is applicable to problems with moving boundaries, in particular. The theory in its parametric representation will be especially justified by the fact that it is simpler and clearer than the analogous theory in the usual form because it will be linear in the $p_{i_1\cdots i_m}$.

In order to derive the $\mathcal{E}$-function and the Legendre condition, use will be made of the calculus of alternating differential forms. The Legendre condition was found already for the special case, $m = n - 1$ by RADON [**14**]. The Legendre transformation that is employed reduces to the transformation that HAAR employed in connection with adjoint variational problems for $m = 2$, $n = 3$ ([**9**], [**7**]).

The special case in which the partial derivatives of $f(x, p)$ with respect to $p_{i_1\cdots i_m}$ were themselves once more components of an m-vector (i.e., they could be regarded as the direction coordinates of a matrix) was already treated in a previous work [**15**], to which no reference will be made. In general, the results were less general, insofar as more was required of the "regular surface elements" there than is required here. The method of the embedding proof that is used here is adapted from the one that was used there in an essentially-simplified form.

1. – In what follows, the index i will always run through the values $i = 1, 2, \ldots, n$ $(2 \le n)$ and the index α will run through the values $\alpha = 1, 2, \ldots, m$ $(m < n)$. We shall then consider a function $f(x_i, p_{i_1 \cdots i_m})$ in the space of $n + \binom{n}{m}$ variables x_i and $p_{i_1 \cdots i_m}$ $(i_1 < i_2 < \ldots < i_m)$, which we shall sometimes write more concisely as $f(x, p)$. It shall be continuously differentiable in all variables as often as is necessary in what follows. Furthermore, let $f(x, p)$ be positively-homogeneous of order one in the $p_{i_1 \cdots i_m}$:

$$f(x, c\,p) = c\,f(x, p) \qquad \text{(for } c > 0\text{).} \tag{1.1}$$

With the use of the abbreviations $p_{(i)} = p_{i_1 \cdots i_m}$ and $f_{(i)} = \partial f / \partial p_{(i)}$, one will get from (1.1) in a known way that:

$$f_{(i)}(x, c\,p) = c\,f_{(i)}(x, p) \qquad \text{(for } c > 0\text{),} \tag{1.2}$$

$$f_{(i)}(x, p)\,p_{(i)} = f(x, p)\,, \qquad f_{(i)(k)}(x, p)\,p_{(k)} = 0 \qquad [\det(f_{(i)(k)}) = 0, \text{ resp.}]. \tag{1.3}$$

(Doubled indices are summed over, so $f_{(i)(i)} = \sum_{i_1 < i_2 < \cdots < i_m} f_{i_1 \cdots i_m}\, p_{i_1 \cdots i_m}$, etc.)

The variational problem that belongs to $f(x, p)$ will then read:

$$\int f(x_i, p_{(i)})\, dt_1 \cdots dt_m = \text{extr.}, \tag{1.4}$$

in which the $p_{(i)}$ replace the Grassmann direction coordinates of the matrix $(\partial x_i / \partial t_\alpha)$, i.e., the determinants that are defined by each m rows in it. Due to (1.1), the variational integral will be invariant under parameter transformations with positive functional determinants, and conversely, every variational integral $\int f'\left(x_i, \dfrac{\partial x_i}{\partial t_\alpha}\right) dt_1 \cdots dt_m$ that is invariant under parameter transformations can also be written in the form (1.4) with (1.1) (cf., [**8**] or [**13**]).

Let some properties of the direction coordinates be briefly recalled: They are homogeneous coordinates, i.e., they are fixed independently of the choice of vectors that span the surface element in question and are unique up to a common factor. In general, they are not independent of each other when $2 \le m < n - 2$. Rather, they will then satisfy certain equations that define the Grassmann manifold or the Grassmann cone (cf., say [**1**]):

$$p_{i_1 i_2 i_3 \cdots i_m}\, p_{k_1 k_2 i_3 \cdots i_m} - p_{k_1 i_2 i_3 \cdots i_m}\, p_{i_1 k_2 i_3 \cdots i_m} - p_{k_2 i_2 i_3 \cdots i_m}\, p_{k_1 i_1 i_3 \cdots i_m} = 0 \tag{1.5}$$

$(i_\alpha, k_\beta = 1, 2, \ldots, n)$. Naturally, the $p_{i_1 \cdots i_m}$ are skew-symmetric in their indices as determinants. Conversely, if there exist skew-symmetric quantities $p_{i_1 \cdots i_m}$ that satisfy eq. (1.5) then one can

always give a matrix ($x_{i\alpha}$) whose direction coordinates they are. (Indeed, the $x_{i\alpha}$ can be expressed in terms of the $p_{i_1\cdots i_m}$ rationally. If the $p_{i_1\cdots i_m}$ are k-times continuously-differentiable functions of a parameter t, in particular, then the same thing will be true of the $x_{i\alpha}$.)

If one expresses eq. (1.5) in terms of the skew symmetry of the $p_{i_1\cdots i_m}$ alone by the $p_{(i)}$ with $i_1 < i_2 < \ldots < i_m$ and then divides ([1]) by $\sqrt{p_{(i)}\, p_{(i)}}$ then it can be written in the form:

$$b_\rho(p) = 0 \qquad (\rho = 1, 2, \ldots) \tag{1.6}$$

when numbered consecutively in any fashion, in which the functions $b_\rho(p)$ are positively homogeneous of order one, so they satisfy the equations:

$$b_\rho(p) = \frac{\partial b_\rho}{\partial p_{(i)}}\, p_{(i)}\,. \tag{1.7}$$

Only the values of $f(x, p)$ on (1.6) are of interest to the variational problem. Two basic functions $f(x, p)$ and $f'(x, p)$ (the associated variational integrals, resp.) will be called *congruent* when one has $f(x, p) = f'(x, p)$ on $b_\rho(p) = 0$. It is clear that congruent functions will yield the same variational problem.

In general, the functions $b_\rho(p)$ are not independent of each other, i.e., in general, the number of them is greater than the rank r of the matrix $(\partial b_\rho / \partial p_{(i)})$. Sometimes we will then select a subsystem of r functions $b_{\rho'}(p)$ ($\rho' = \rho_1, \ldots, \rho_r$) of rank $(\partial b_{\rho'} / \partial p_{(i)}) = r$ at the location considered.

Finally, let it be recalled that the extremals of (1.4) are explained in a way that is independent of the choice of parameters, namely, by the vanishing of the first variation for an arbitrary family of comparison surfaces $x_i = x_i(\tau, t_\alpha)$ with a common boundary, whereby the function $I(\tau) = \int_G f(x(\tau, t_\alpha), p(\tau, t_\alpha))\, dt_1 \cdots dt_m$, and therefore $dI / d\tau$, as well, does not depend upon the choice of parameters t_α. In what follows, only those extremals that are given as twice-differentiable functions of the parameters will be considered, and therefore ones that satisfy the Euler differential equations:

$$f_{x_i} = \frac{\partial f_{x_{i\alpha}}}{\partial t_\alpha} \qquad \left(\text{with } f_{x_i} = \frac{\partial f}{\partial x_i},\ f_{x_{i\alpha}} = \frac{\partial f}{\partial x_{i\alpha}}\right). \tag{1.8}$$

2. Geodetic fields. $\mathcal{E}$-function. – We consider the family of congruent functions:

$$f(x, p) + B(x, p)\,. \tag{2.1}$$

In it, $B(x, p)$ is an arbitrary function with $B(x, c\,p) = c\,B(x, p)$ (for $c > 0$) and:

([1]) The point $p = 0$, i.e., the vertex of the cone, is excluded from all considerations.

(2.2) $$B(x, p) = 0 \qquad \text{on} \qquad b_\rho(p) = 0 .$$

It is clear that every function $f'(x, p)$ that is congruent to $f(x, p)$ can be written in the form (2.1): One needs only to set $B(x, p) = f'(x, p) - f(x, p)$.

Naturally, the functions of the family (2.1) are again positively-homogeneous of order one in the $p_{(i)}$. In particular, one has:

(2.3) $$f(x, p) + B(x, p) = (f_{(i)}(x, p) + B_{(i)}(x, p))\, p_{(i)} .$$

We now consider the exterior differential form:

(2.4) $$\Omega = \sum_{i_1 < i_2 < \cdots < i_m} \left(f_{i_1 \cdots i_m}(x, p) + B_{i_1 \cdots i_m}(x, p) \right) dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

on the space of $n + \binom{n}{m}$ variables x_i and $p_{(i)}$. Furthermore, let $(x, \pi(x))$ be a field of m-dimensional surface elements, in which the $\pi(x)$ are understood to mean continuously-differentiable functions in a certain region of the space of x_i . If one substitutes $p = \pi(x)$ in (2.4) then Ω will go to a form $[\Omega]$ that is defined on the space of the x_i :

(2.5) $$[\Omega] = [f_{(i)} + B_{(i)}]\; dx_{i_1} \wedge \cdots \wedge dx_{i_m} = \left(f_{i_1 \cdots i_m}(x, \pi(x)) + B_{i_1 \cdots i_m}(x, \pi(x)) \right) dx_{i_1} \wedge \cdots \wedge dx_{i_m} .$$

Definition: The field $(x, \pi(x))$ is called *geodetic* relative to $f(x, p) + B(x, p)$ when $d[\Omega] = 0$.

If V is an $(m + 1)$-dimensional manifold that lies completely within the geodetic field and if R is its n-dimensional boundary then from Stokes's theorem:

$$\int_R [\Omega] = \pm \int_V d[\Omega] = 0 .$$

(The sign, i.e., the orientation, does not matter here.) It then follows It then follows from this that $\int [\Omega]$ is independent of the path, i.e., when $\bar{E}$ and E are two surfaces with common boundaries that both run through the geodetic field then:

(2.6) $$\int_{\bar{E}} [\Omega] = \int_E [\Omega] ,$$

or, with the use of parameters:

$$\int_{\bar{E}} [f_{(i)} + B_{(i)}]\, p_{(i)}\, dt_1 \cdots dt_m = \int_E [f_{(i)} + B_{(i)}]\, p_{(i)}\, dt_1 \cdots dt_m .$$

Now let E be embedded in the geodetic field, i.e., let the surface elements that are tangent to E be likewise surface elements of the field. One will then have:

$$I_E = \int_E \big(f(x,p) + B(x,p) \big) dt_1 \cdots dt_m = \int_E [\, f_{(i)} + B_{(i)} \,]\, p_{(i)}\, dt_1 \cdots dt_m \,.$$

It will then follow that:

$$I_{\bar{E}} - I_E = \int_{\bar{E}} \big(f(x,p) + B(x,p) \big) dt_1 \cdots dt_m - \int_{\bar{E}} [\, f_{(i)} + B_{(i)} \,]\, p_{(i)}\, dt_1 \cdots dt_m = \int_{\bar{E}} \mathcal{E}\, dt_1 \cdots dt_m$$

with

$$\mathcal{E}(x, \bar{p}, \pi) = f(x, \bar{p}) + B(x, \bar{p}) - \Big(f_{(i)}(x, \pi(x)) + B_{(i)}(x, \pi(x)) \Big)\, \bar{p}_{(i)}\,, \tag{2.7}$$

in which $\mathcal{E}(x, \bar{p}, \pi)$ is nothing but the *Weierstrass $\mathcal{E}$-function.* That immediately implies that:

Theorem 2.1:

It is sufficient for a strong minimum that one can embed the surface patch in question in a geodetic field and that the associated $\mathcal{E}$-function is nonnegative for $b_\rho(\bar{p}) = 0$. *(The only thing that one demands of the comparison surfaces is that they must run through the field.)*

3. Legendre condition. –In order to arrive at sufficient conditions for a weak minimum, one proceeds as usual: A Taylor development of $f(x, \bar{p}) + B(x, \bar{p})$ at the location π yields the $\mathcal{E}$-function:

$$\left\{ \begin{aligned} \mathcal{E}(x, \bar{p}, \pi) &= f(x, \bar{p}) + B(x, \bar{p}) - f(x, \pi) - B(x, \pi) - [\, f_{(i)} + B_{(i)}](\bar{p}_{(i)} - \pi_{(i)}) \\ &= \tfrac{1}{2} \{ \tilde{f}_{(i)(k)} + \tilde{B}_{(i)(k)} \} (\bar{p}_{(i)} - \pi_{(i)})(\bar{p}_{(k)} - \pi_{(k)}), \end{aligned} \right. \tag{3.1}$$

and the conditions $b_\rho(\bar{p}) = b_\rho(\pi) = 0$ can be written in the form:

$$\frac{\partial b_\rho}{\partial p_{(i)}} (\bar{p}_{(i)} - \pi_{(i)}) = 0 \,. \tag{3.2}$$

(The tilde generally denotes taking a mean value.) That leads to the consideration of the family of quadratic forms:

$$Q(B) = \{ f_{(i)(k)}(x, p) + B_{(i)(k)}(x, p) \}\, \xi_{(i)}\, \xi_{(k)}\,, \tag{3.3}$$

with the auxiliary conditions:

(3.4) $$\frac{\partial b_\rho}{\partial p_{(i)}} = 0 .$$

Those quadratic forms are also never definite under the auxiliary conditions (3.4) because one needs only to set $\xi_{(i)} = p_{(i)}$ with $b_\rho(p) = 0$ in order to fulfill $\frac{\partial b_\rho}{\partial p_{(i)}} \xi_{(i)} = b_\rho(p) = 0$, i.e., (3.4), and on the other hand, due to the homogeneity of the function $f(x, p) + B(x, p)$ in the $p_{(i)}$, to also have:

$$\{f_{(i)(k)}(x, p) + B_{(i)(k)}(x, p)\}\, p_{(i)}\, p_{(k)} = 0 .$$

Definition. – A surface element (x, p) is called *positive (negative, resp.) regular* with respect to $f(x, p) + B(x, p)$ when the associated quadratic form (3.3) is positive (negative, resp.) definite under the auxiliary conditions (3.4) and the supplementary condition:

(3.5) $$p_{(i)}\, \xi_{(k)} = 0 .$$

Theorem 3.1:

It is sufficient for a weak minimum that one can embed the surface patch in question in a geodetic field and, in addition, that all surface elements of the surface patch are positive-regular, i.e., that the associated Legendre condition:

$$Q(B) = \{f_{(i)(k)}(x, p) + B_{(i)(k)}(x, p)\} > 0 \qquad \textit{for} \qquad \frac{\partial b_\rho}{\partial p_{(i)}} \xi_{(i)} = p_{(i)}\, \xi_{(k)} = 0$$

is fulfilled.

Proof:

Let $f + B$ be the basic function relative to which the geodetic field was constructed. We set $f' = f + B$ but drop the prime in what follows for simplicity's sake. The $\mathcal{E}$-function will then have the form:

(3.6) $$\mathcal{E}(x, \bar{p}, \pi) = f(x, \bar{p}) - f_{(i)}(x, \pi)\, \bar{p}_{(i)} = \tfrac{1}{2}\, \tilde{f}_{(i)(k)}(\bar{p}_{(i)} - \pi_{(i)})(\bar{p}_{(k)} - \pi_{(k)}) .$$

However, the $\mathcal{E}$-function is positively homogeneous of order one (zero, resp.) in $\bar{p}$ (π, resp.). If one would merely like to establish the sign of $\mathcal{E}$ then one must restrict oneself to those $\bar{p}$, π that are normalized by means of $\sum \bar{p}_{(i)}^2 = \sum \pi_{(i)}^2 = 1$, i.e., it suffices to consider (3.6) under the auxiliary conditions:

$$b_\rho(\bar{p}) = b_\rho(\pi) = 0 \qquad [\sum \bar{p}_{(i)}^2 - 1 = \sum \pi_{(i)}^2 - 1 = 0, \text{ resp.}]$$

or

$$\frac{\partial b_\rho}{\partial p_{(i)}}(\bar{p}_{(i)} - \pi_{(i)}) = 0\,, \qquad p_{(i)}(\bar{p}_{(i)} - \pi_{(i)}) = 0\,. \tag{3.7}$$

Everything else follows as usual from a theorem on quadratic forms with linear auxiliary conditions that one finds, e.g., in CARATHÉODORY ([2]):

The minimum of the quadratic form:

$$Q = f_{(i)(k)}\,\xi_{(i)}\,\xi_{(k)} \qquad \text{with the conditions} \qquad \frac{\partial b_{\rho'}}{\partial p_{(i)}}\xi_{(i)} = p_{(i)}\,\xi_{(k)} = 0$$

on the point-set $\xi_{(i)}\,\xi_{(i)} = 1$ is equal to the smallest value λ_1 of the characteristic equation:

$$D(\lambda) = \begin{vmatrix} f_{(i)(k)} - \lambda\,\delta_{(i)(k)} & \dfrac{\partial b_{\sigma'}}{\partial p_{(i)}} & p_{(i)} \\ \dfrac{\partial b_{\rho'}}{\partial p_{(k)}} & 0 & 0 \\ p_{(k)} & 0 & 0 \end{vmatrix} = 0\,.$$

(All roots are real.)

For a positive-regular surface element (x^0, π^0), all roots λ_i are positive then, and since the roots are continuous functions of $f_{(i)(k)}$, $\dfrac{\partial b_{\sigma'}}{\partial p_{(i)}}$, and $p_{(i)}$, the roots $\tilde{\lambda}$ of:

$$\tilde{D}(\lambda) = \begin{vmatrix} \tilde{f}_{(i)(k)} - \lambda\,\delta_{(i)(k)} & \dfrac{\partial b_{\sigma'}}{\partial p_{(i)}} & \tilde{p}_{(i)} \\ \dfrac{\partial b_{\rho'}}{\partial p_{(k)}} & 0 & 0 \\ \tilde{p}_{(k)} & 0 & 0 \end{vmatrix} = 0$$

([2]) See [**5**], Chap. 11 (Gewöhnliche Maxima und Minima).

will also be all positive, as long as the $\tilde{f}_{(i)(k)}$, $\dfrac{\partial b_{\sigma'}}{\partial p_{(i)}}$, and $\tilde{p}_{(i)}$ differ sufficiently little from $f_{(i)(k)}(x^0, \pi^0)$, etc., and that is certainly the case for the surface element $(x, \bar{p})$ in a certain neighborhood $\left| x_i - x_i^0 \right| < \varepsilon$, $\left| \bar{p}_{(i)} - \pi_{(i)}^0 \right| < \varepsilon$. One will then have $\mathcal{E}(x, \bar{p}, \pi^0) \geq 0$ for that surface element $(x, \bar{p})$, and also when one subsequently once more waives the normalization of the $\bar{p}$ and π. It will then follow immediately from this that a surface E with nothing but positive-regular surface elements will yield a weak minimum, i.e., a minimum relative to comparison surface that lies in a geodetic field sufficiently close to E and whose direction also does not deviate too strongly from E.

4. The Carathéodory independent integral. – We would now like to turn to the question of whether it is possible to embed a given extremal (i.e., a solution of the Euler differential equation) in a geodetic field. Indeed, the general case will not be treated in the following, but the argument will be applied to the function $B(x, p)$ in a suitable way, and indeed under the assumption that:

$$f(x, p) > 0 \quad \text{on} \quad b_\rho(p) = 0\,, \tag{4.1}$$

the function $B(x, p)$ can be chosen such that the function $f^*(x, p) = f(x, p) + B(x, p)$ possesses the property that:

$$b_\rho(f_{(i)}^*) = 0 \quad \text{and} \quad b_\rho(p) = 0\,,$$

i.e., that $f_{(i)}^* = \partial f^* / \partial p_{(i)}$ belongs to the Grassmann manifold, along with $p_{(i)}$. That is a simple consequence of:

Theorem 4.1 (Kneser):

One can define functions $u_{(i)}(x, p)$ that are positive-homogeneous of order zero in the p and just as many times continuously differentiable in all variables as $f_{(i)}(x, p)$, in such a way that:

$$u_{(i)}\, p_{(i)} = f_{(i)}\, p_{(i)}\,, \qquad u_{(i)}\, dp_{(i)} = f_{(i)}\, dp_{(i)}\,, \qquad b_\rho(u_{(i)}) = 0 \tag{4.4}$$

is always fulfilled on the Grassmann manifold [*so in our notation, on* $b_\rho(p) = 0$, $\left(\dfrac{\partial b_\rho}{\partial p_{(i)}}\, dp_{(i)} = 0, \text{resp.} \right)$] ([3]).

([3]) One finds the proof for only $m = 2$ (n arbitrary) in KNESER [**11**]. For the sake of completeness, we will then give a proof of the theorem at the end of this article.

Geometrically, eq. $(4.4)_2$, viz., $(u_{(i)} - f_{(i)})\, dp_{(i)} = 0$, simply means that in the space of the variables $p_{(i)}$, the vector with the components $(u_{(i)} - f_{(i)})$ is perpendicular to the Grassmann manifold at the point p and can therefore be written in the form ([4]):

$$(u_{(i)} - f_{(i)}) = a_{\rho'} \frac{\partial b_\rho}{\partial p_{(i)}}.$$

Solving that for $a_{\rho'}$ gives $= A_{\rho'}(x, p)$, in which the $A_{\rho'}(x, p)$ are positive homogeneous of order zero in p and just as many times continuously differentiable as the $f_{(i)}(x, p)$, in addition. Due to the fact that $f^*_{(i)} \equiv f_{(i)} + A_{\rho'} \dfrac{\partial b_\rho}{\partial p_{(i)}} \equiv u_{(i)} \pmod{b_\rho(p)}$, the function:

$$f^*(x, p) = f(x, p) + B(x, p) \qquad \text{with} \qquad B(x, p) = A_{\rho'}(x, p) \cdot b_{\rho'}(p)$$

does, in fact, possess the property that $b_\rho(f^*_{(i)}) = 0$ for $b_\rho(p) = 0$.

The independent integral that will be introduced in what follows is the parametric representation of the integral in Carathéodory's "determinant method," which has the advantage that it is also applicable to problems with moving boundaries, but we shall not go further into that topic here.

To that end, one starts with some functions $S_\alpha(x_1, \ldots, x_m)$ $(\alpha = 1, 2, \ldots, m)$ that are at least twice continuously differentiable. Let the direction coordinates of the matrix $\left(\dfrac{\partial S_\alpha}{\partial x_i}\right)$ be denoted by $s_{(i)}$. The exterior differential form:

$$\omega = dS_1 \wedge \ldots \wedge dS_m = \sum_{i_1 < \cdots < i_m} s_{i_1 \cdots i_m}\, dx_{i_1} \wedge \cdots \wedge dx_{i_m} \tag{4.5}$$

is obviously a complete differential then, and therefore:

$$\int \omega = \int s_{(i)}\, dx_{i_1} \wedge \cdots \wedge dx_{i_m} \tag{4.6}$$

is independent of the path, i.e., from Stokes's theorem, one will have $\int_{\bar{E}} \omega = \int_E \omega$ when $\bar{E}$ and E are two m-dimensional surface patches with the same boundary. With the use of a parametric representation $x_i = x_i(t_1, \ldots, t_m)$ of the surfaces in question, $\int \omega$ will go to $\int \Delta\, dt_1 \cdots dt_m$ with the Carathéodory determinant:

([4]) Relative to the $b_{\rho'}(p)$, see Section 1.

$$\Delta = \left| \frac{\partial S_\alpha}{\partial t_\beta} \right| = s_{(i)}\,(x)\, p_{(i)}\ .$$

We shall now try to find functions $S_\alpha\,(x_i)$ and a geodetic field $(x,\ \pi\,(x))$ in which the given extremal is embedded such that we will have $\omega = [\Omega]$, i.e., $s_{(i)}\,(x) = f_{(i)}\,(x,\ \pi\,(x))$. That will require some preliminaries.

5. On regularity. – The regular surface elements can also be characterized as follows, under the assumption (4.1):

Theorem 5.1:

A surface element (x, p) *is positive-regular if and only if the quadratic form:*

$$Q_1 = \frac{\partial^2 \frac{1}{2} f^2}{\partial p_{(i)}\, \partial p_{(k)}}\, \xi_{(i)}\, \xi_{(k)} \tag{5.1}$$

is positive-definite under the auxiliary conditions (3.4).

Proof:

One must first replace the auxiliary condition (3.5) $p_{(i)}\ \xi_{(i)} = 0$ with $f_{(i)}\ \xi_{(i)} = 0$ in the definition of a regular surface element. In order to show that, one considers:

$$Q = f_{(i)(k)}\ \xi_{(i)}\ \xi_{(k)} \qquad \text{with} \qquad \frac{\partial b_\rho}{\partial p_{(i)}}\, \xi_{(i)} = 0\ ,\ \ p_{(i)}\ \xi_{(i)} = 0 \tag{5.2}$$

and

$$\bar{Q} = f_{(i)(k)}\ \eta_{(i)}\ \eta_{(k)} \qquad \text{with} \qquad \frac{\partial b_\rho}{\partial p_{(i)}}\, \eta_{(i)} = 0\ ,\ \ f_{(i)}\ \eta_{(i)} = 0\ , \tag{5.3}$$

in succession. In what follows, ξ and η will always denote vectors that satisfy the equations $p_{(i)}\ \xi_{(i)} = 0$ ($f_{(i)}\ \eta_{(i)} = 0$, resp.). Let p now denote an arbitrary vector $|\,p\,| \neq 0$, $b_\rho\,(p) = 0$. p will then span all of space, together with $p_{(i)}\ \xi_{(i)} = 0$ and $f_{(i)}\ \eta_{(i)} = 0$, since one has $p_{(i)}\ p_{(i)} \neq 0$, $f_{(i)}\ p_{(i)} = f \neq 0$. Any vector ξ can then be written in the form $\xi = \alpha\, p + \eta$, and conversely any vector η can be written in the form $\eta = -\,\alpha\, p + \xi$. It follows from $\xi \neq 0$ that $\eta \neq 0$ and conversely. It further follows from $\frac{\partial b_\rho}{\partial p_{(i)}}\xi_{(i)} = 0$ that one also has $\frac{\partial b_\rho}{\partial p_{(i)}}(\alpha\, p_{(i)} + \eta_{(i)}) = \frac{\partial b_\rho}{\partial p_{(i)}}\eta_{(i)} = 0$, and conversely. One can then write:

$$Q = f_{(i)(k)}\,(\alpha\, p_{(i)} + \eta_{(i)})\,(\alpha\, p_{(k)} + \eta_{(k)}) = f_{(i)(k)}\;\eta_{(i)}\;\eta_{(k)} = \bar{Q} \qquad (\text{since } f_{(i)(k)}\, p_{(k)} = 0),$$

i.e., one can replace (5.2) with (5.3), and conversely.

Moreover, one has $Q_1 = f\, f_{(i)(k)}\;\xi_{(i)}\;\xi_{(k)} + (f_{(i)}\;\xi_{(i)})^2 = f\,Q + (f_{(i)}\;\xi_{(i)})^2$. Now, if (x, p) is positive-regular then one will have $Q \geq 0$ under the conditions (3.4), and one will even have $Q > 0$ under the supplementary condition that $f_{(i)}\;\xi_{(i)} = 0$. In each case, one will then have $Q_1 > 0$ under the auxiliary conditions (3.4). Conversely, if (x, p) is not positive-regular then the form $Q_1 = f\,Q$ will not be positive-definite, even with the supplementary condition that $f_{(i)}\;\xi_{(i)} = 0$.

Remark. – For a positive-regular surface element, one then has (cf., [**5**], Chap. 11):

$$D = \begin{vmatrix} \dfrac{\partial^2 \frac{1}{2} f^2}{\partial p_{(i)}\,\partial p_{(k)}} & \dfrac{\partial b_{\sigma'}}{\partial p_{(k)}} \\ \dfrac{\partial b_{\rho'}}{\partial p_{(i)}} & 0 \end{vmatrix} \neq 0\,. \tag{5.4}$$

6. The Legendre transformation. – For the time being, we introduce r further variables $\mu_{\rho'}$ and set $g\,(x, p, \mu) = f\,(x, p) + \mu_{\rho'} b_{\rho'}(p)$ and transform $p_{(i)}$, $\mu_{\rho'}$ into new quantities $P_{(i)}$, $M_{\rho'}$ by means of:

$$P_{(i)} = g\; g_{(i)}\,, \qquad M_{\rho'} = b_{\rho'}(p) \qquad \left(g_{(i)} = \frac{\partial g}{\partial p_{(i)}} \right). \tag{6.1}$$

The associated functional determinant:

$$\begin{vmatrix} \dfrac{\partial^2 \frac{1}{2} g^2}{\partial p_{(i)}\,\partial p_{(k)}} & g\,\dfrac{\partial b_{\sigma'}}{\partial p_{(k)}} + b_{\rho'}\, g_{(k)} \\ \dfrac{\partial b_{\rho'}}{\partial p_{(i)}} & 0 \end{vmatrix}$$

reduces to:

$$\begin{vmatrix} \dfrac{\partial^2 \frac{1}{2} f^2}{\partial p_{(i)}\,\partial p_{(k)}} & \dfrac{\partial b_{\sigma'}}{\partial p_{(k)}} \\ \dfrac{\partial b_{\rho'}}{\partial p_{(i)}} & 0 \end{vmatrix} \cdot f^{\binom{n}{m}} = D \cdot f^{\binom{n}{m}}$$

for $\mu_{\rho'} = 0$, $b_{\rho'}(p) = 0$. Due to (5.4), (6.1) can then be solved for $p_{(i)}$ ($\mu_{\rho'}$, resp.) in any case in the neighborhood of (x, p, μ) with $\mu_{\rho'} = 0$, (x, p) positive-regular, and one will get:

$$p_{(i)} = \varphi_{(i)}\,(x, P, M)\,, \qquad \mu_{\rho'} = \psi_{\rho'}\,(x, P, M)\,, \tag{6.2}$$

with

(6.3) $$c\, p_{(i)} = \varphi_{(i)}\,(x, c\,P, c\,M)\,, \qquad \mu_{\rho'} = \psi_{\rho'}\,(x, c\,P, c\,M) \qquad \text{for } c > 0.$$

If one now defines functions $G\,(x, P, M)$ and $G_{(i)}\,(x, P, M)$ by means of:

(6.4) $$G = g\,(x, \varphi, \psi)\,, \qquad G_{(i)} = \frac{\varphi_{(i)}}{g\,(x,\varphi,\psi)}$$

then one will next find that one always has $\partial G / \partial P_{(i)} = G_{(i)}$ for $M_{\rho'} = b_{\rho'}(p) = 0$ and that $\partial G / \partial P_{(i)}$ belongs to the Grassmann manifold. In fact, if follows from (6.1) and (6.4) that $p_{(i)}\,P_{(i)} = g\,G$, and after differentiation:

$$(dG - G_{(i)}\,dP_{(i)}) = -\,(dg - g_{(i)}\,dp_{(i)})\,,$$

and since $g_{(i)} = \partial g / \partial p_{(i)}$, one further has:

$$\left[\frac{\partial G}{\partial x_i}dx_i + \frac{\partial G}{\partial M_{\rho'}}dM_{\rho'} + \left(\frac{\partial G}{\partial P_{(i)}} - G_{(i)}\right)dP_{(i)}\right] = -\left(\frac{\partial g}{\partial x_i}dx_i + \frac{\partial g}{\partial M_{\rho'}}d\mu_{\rho'}\right)$$

$$= -\,\frac{\partial f}{\partial x_i}dx_i + b_{\rho'}(p)\,d\mu_{\rho'}\ .$$

For $b_\rho(p) = 0$ (i.e., one also has $M_{\rho'} = dM_{\rho'} = 0$), it will follow from this that $\dfrac{\partial G}{\partial x_i} = -\,\dfrac{\partial f}{\partial x_i}$, $\dfrac{\partial G}{\partial P_{(i)}} = G_{(i)}$, and due to (6.4), $\partial G / \partial P_{(i)}$ will also belong to the Grassmann manifold, along with $p_{(i)}$.

Moreover, one confirms immediately that $G\,(x, c\,P, c\,M) = c\,G\,(x, P, M)$ and $G_{(i)}\,(x, c\,P, c\,M) = G_{(i)}\,(x, P, M)$ for $c > 0$ when one makes use of (6.3).

If one now sets $F\,(x, P) = G\,(x, P, 0)$, $F_{(i)}\,(x, P) = G_{(i)}\,(x, P, 0)$, $F_{x_i} = \dfrac{\partial G}{\partial x_i}\,(x, P, 0)$, as well as $f_{x_i} = \dfrac{\partial f}{\partial x_i}$, then for $\mu_{\rho'} = b_\rho\,(p) = 0$, altogether, one will have the transformation:

(6.5) $$P_{(i)} = f\,f_{(i)}\,, \qquad F_{(i)} = \frac{p_{(i)}}{f}\,, \qquad F = f, \qquad b_\rho\,(p) = b_\rho\,(P) = 0\,,$$

which can be immediately inverted ([5]):

([5]) Namely, it follows from $b_\rho\,(P) = 0$ that conversely $\mu_{\rho'} = 0$: The $g_{(i)}$ are the direction coordinates of a matrix $(\xi_{i\alpha})$. If one sets $x_{i\alpha} = \partial x_i / \partial t_\alpha$, $g_{x_{i\alpha}} = \partial g / \partial x_{i\alpha}$, $c_{\alpha\beta} = \xi_{i\alpha}\,x_{i\beta}$, and defines $g_{\alpha\beta}$ to be the algebraic complement of $c_{\alpha\beta}$

(6.6) $$p_{(i)} = F\, F_{(i)}\,, \quad f_{(i)} = \frac{P_{(i)}}{F}, \qquad f = F, \qquad b_\rho\,(p) = b_\rho\,(P) = 0\,.$$

That transformation then implies a one-to-one correspondence between surface elements (x, p) and (x, P) in the small, and the function $F\,(x, P)$ will possess the following properties on $b_\rho\,(P) = 0$:

(6.7) $$F\,(x, c\,P) = c\,F\,(x, P)\,, \qquad F_{(i)}\,(x, c\,P) = F_{(i)}\,(x, P) \qquad \text{for } c > 0\,,$$

(6.8) $$\frac{\partial F}{\partial P_{(i)}} = F_{(i)}\,, \quad F_{x_i} = -\ f_{x_i}\,.$$

7. Hamilton-Jacobi differential equation. – If one substitutes $P_{(i)} = f\,f_{(i)}$ into $F\,(x, P) = f\,(x, p)$ then one will get the identity $F\,(x_i, f\,f_{(i)}) = f$ (the identity:

$$F\,(x_i, f\,f_{(i)}) = 1,$$

resp., due to the homogeneity of $F\,(x, P)$ in the $P_{(i)}$). Now, if $(x, \pi\,(x))$ is any geodetic field, and if there are functions $S_\alpha\,(x_1, \ldots, x_n)$ with $s_{(i)}\,(x_i) = f_{(i)}\,(x, \pi\,(x))$ then the $S_\alpha\,(x_i)$ must necessarily satisfy the differential equation:

(7.1) $$F\,(x_i, s_{(i)}) = 1\,.$$

(7.1) is then the *Hamilton-Jacobi differential equation* of our variational problem.

In order to embed an extremal in a geodetic field, one then starts from the differential equation (7.1) and seeks to construct a solution $S_\alpha\,(x_i)$ such that $s_{(i)} = f_{(i)}$ on the extremals. If one then sets $P_{(i)} = s_{(i)}\,(x)$ then (6.6) will imply a unique field of surface elements $(x, \pi\,(x))$ with $s_{(i)} = f_{(i)}\,(x, \pi\,(x))$ [since $F\,(x, s\,(x)) = f\,(x, \pi\,(x)) = 1$], i.e., a geodetic field in which the extremal is embedded. In general, the extremal must be continuously differentiable sufficiently often. In order to simply the presentation, no value will be assigned to it, and one tries to make do with the most restrictive assumptions possible.

First, the auxiliary functions $S_\alpha\,(x_i)$ are constructed, for which $s_{(i)} = f_{(i)}$ is fulfilled along the extremals, such that they satisfy eq. (7.1), at least along the extremals. The $S_\alpha\,(x_i)$ are then altered in such a way that they also satisfy the differential equation outside of the extremals. That happens with the help of the theory of characteristics.

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in $(c_{\alpha\beta})$ then one will get $|\,c_{\alpha\beta}\,| = g_{(i)}\,p_{(i)}$, as well as $g_{x_{i\alpha}} = g_{(i)}\,\partial p\,/\,\partial x_{i\alpha} = \partial\,|\,c_{\alpha\beta}\,|\,/\,\partial x_{i\alpha} = \xi_{i\beta}\,\bar{c}_{\beta\alpha}$, i.e., the columns of $(g_{x_{i\alpha}})$ and $(\xi_{i\alpha})$ are linear combinations of each other, and therefore the $g_{(i)}$ are likewise the direction coordinates of $(g_{x_{i\alpha}})$ [up to a factor], and $f_{(i)}$ are those of $(f_{x_{i\alpha}})$, in a completely analogous way. it will then follow from $f_{x_{i\alpha}} = g_{x_{i\alpha}}$ that $f_{(i)} = g_{(i)}$ ($\mu_{\rho'} = 0$, resp.).

8. Proof of embeddability. – First, let us make a remark about the direction coordinates of perpendicular surface elements. We consider the m-dimensional plane in R^n that is spanned by the m column vectors η_α of the matrix $(\eta_{i\alpha})$ ($i = 1, \ldots, n$; $\alpha = 1, \ldots, m$), along with the $(n - m)$-dimensional plane that is perpendicular to it and is spanned by the column vectors of the matrix $(\xi_{i\rho})$ ($\rho = 1, 2, \ldots, n - m$). A simple connection then exists between the direction coordinates $f_{i_1 \cdots i_m}$ ($q_{j_1 \cdots j_{n-m}}$, resp.) of the two matrices (see, perhaps [**1**]):

The $q_{j_1 \cdots j_{n-m}}$ are equal to their algebraic complements in the matrix $(\xi_{i\rho} \mid \eta_{i\alpha})$ of the column vectors $\xi_1, \ldots, \xi_{n-m}, \eta_1, \ldots, \eta_m$. If one lets $[i]$ denote the sequence that arises from $1, 2, \ldots, n$ by omitting $(i) = i_1, \ldots, i_m$ ($i_1 < i_2 < \ldots < i_m$) then one can also write, more briefly:

$$q_{[i]} = k \cdot (-1)^s f_{(i)} \qquad \text{with} \qquad s = \sum_{\alpha=1}^{m} i_\alpha + \tfrac{1}{2}\, m(m+1) \ .$$

Definition: Any m-dimensional surface element (x, p) can be associated with an $(n - m)$-dimensional surface element (x, q) with:

$$q_{[i]} = (-1)^s f_{(i)}\,(x, p) \qquad\qquad [s = \sum_{\alpha=1}^{m} i_\alpha + \tfrac{1}{2}\, m(m+1)\,]$$

by means of $f\,(x, p)$. (x, q) is said to be *transverse* to (x, p).

We now consider the extremals:

$$x_i = x_i\,(t_1, \ldots, t_m)\ , \tag{8.1}$$

and indeed, we assume that the parameters have already been chosen in such a way that we have:

$$f\,(x\,(t_\alpha), p\,(t_\alpha)) = 1 \tag{8.2}$$

along the extremals ([6]). We further assume that the functions (8.1), thus-prepared, are four-times continuously differentiable. The $f_{(i)}\,(x, p)$, as well as the direction coordinates $q_{(j)}$ of the transverse surface element, will then be three-times continuously-differentiable functions of the t_α, and a matrix $(\xi_{i\rho})$ can be found with elements $\xi_{i\rho}\,(t_\alpha)$ that is just as many times continuously-differentiable, and whose direction coordinates are precisely the $q_{(j)}\,(t_\alpha)$. An m-parameter family of $(n - m)$-dimensional planes is then given by:

$$x_i = \hat{x}_i(t_\alpha, u_\rho) = x_i\,(t_\alpha) + \xi_{i\rho}\,(t_\alpha)\, u_\rho\ , \tag{8.3}$$

([6]) If one next has $f\,(x\,(t_\alpha), p\,(t_\alpha)) = \psi\,(t_\alpha)$ then one sets, say, $t_1^* = \int_0^{t_1} \psi(t_\alpha)\, dt_1$, $t_\alpha^* = t_{\alpha'}$ ($\alpha' = 2, 3, \ldots, m$).

whose direction coordinates are precisely the $q_{(j)}$, so they intersect the extremals $x_i = \hat{x}_i(t_\alpha, 0)$ transversally. For the functional determinant:

$$D = \frac{\partial(\hat{x}_1, \ldots, \hat{x}_n)}{\partial(t_1, \ldots, t_m, u_1, \ldots, u_{n-m})} ,$$

if i is the row index and α, ρ are column indices then one will get:

$$D = \det\left(\frac{\partial x_i}{\partial t_\alpha}\,\middle|\,\xi_{i\rho}\right) = p_{(i)}\ f_{(i)} = f \neq 0 \tag{8.4}$$

along the extremals. In other words: Under the assumption that $f \neq 0$, the surface elements (x, p) and (x, q) will have no common direction, i.e., they collectively span all of space. Due to (8.4), one can then solve for t_α, u_ρ uniquely in a certain neighborhood of the point considered along the extremal and obtain (three-times continuously differentiable) functions:

$$t_\alpha = S_\alpha(x_1, \ldots, x_n) , \qquad u_\rho = U_\rho(x_1, \ldots, x_n) .$$

The family of planes can then be written in the form:

$$S_\alpha(x_1, \ldots, x_n) = t_\alpha \tag{8.5}$$

in that neighborhood, and from (8.1), the t_α-parameter curves will be obtained simply from the intersections of the extremals (8.1) with the $m - 1$ hypersurfaces:

$$S_{\alpha'}(x_1, \ldots, x_n) = \text{const.} \qquad (\alpha' = 1, 2, \ldots, \alpha - 1, \alpha + 1, \ldots, m).$$

The m vectors $\left(\frac{\partial S_\alpha}{\partial x_1}, \ldots, \frac{\partial S_\alpha}{\partial x_n}\right)$ span the space that is perpendicular to the $(n - m)$-dimensional plane $S_\alpha(x_1, \ldots, x_n) = \text{const}$. The direction coordinates $s_{(i)}$ of $(\partial S_\alpha / \partial x_i)$ must then agree with the $f_{(i)}$ along the extremals, up to a factor. Due to (8.5), one has $\partial S_\alpha / \partial t_\beta = \delta_{\alpha\beta}$ ($| \partial S_\alpha / \partial t_\beta | = s_{(i)}\, p_{(i)} = 1$, resp.). On the other hand, from (8.2), one also has $f_{(i)}\, p_{(i)} = 1$, i.e., one has $s_{(i)} = f_{(i)}$ along the extremals.

An algebraic identity shall now be derived. We set $S_{i\alpha} = \partial S_\alpha / \partial x_i$, $x_{i\alpha} = \partial x_i / \partial t_\alpha$, and $c_{\beta\gamma} = S_{k\beta}\, x_{k\gamma}$. $\bar{c}_{\beta\gamma}$ denotes the algebraic complement of $c_{\beta\gamma}$ in $(c_{\beta\gamma})$. If one considers $S_{i\alpha}$ and $x_{i\alpha}$ to be independent variables then when one differentiates $| c_{\beta\gamma} | = s_{(i)}\, p_{(i)}$ with respect to $S_{i\alpha}$ ($x_{i\alpha}$, resp.), one will get:

$$p_{(i)} \frac{\partial s_{(i)}}{\partial S_{i\alpha}} = x_{i\gamma}\, \delta_{\alpha\beta}\, \overline{c}_{\alpha\gamma} = x_{i\gamma}\, \overline{c}_{\alpha\gamma} \qquad [\, s_{(i)} \frac{\partial p_{(i)}}{\partial x_{i\alpha}} = S_{i\beta}\, \delta_{\alpha\gamma}\, \overline{c}_{\beta\gamma} = S_{i\beta}\, \overline{c}_{\beta\alpha}\text{, resp.}] \,,$$

or, when one makes use of (8.5), i.e., $c_{\alpha\beta} = \overline{c}_{\alpha\beta} = \delta_{\alpha\beta}$ on (8.1):

(8.6) $$x_{i\alpha} = p_{(i)} \frac{\partial s_{(i)}}{\partial S_{i\alpha}} \,,$$

(8.7) $$S_{i\alpha} = s_{(i)} \frac{\partial p_{(i)}}{\partial x_{i\alpha}} \,.$$

In particular, it follows from $f_{x_{i\alpha}} = f_{(i)} \frac{\partial p_{(i)}}{\partial x_{i\alpha}}$ on (8.1) that due to the fact that $s_{(i)} = f_{(i)}$, one will have:

(8.8) $$f_{x_{i\alpha}} = S_{i\alpha} \,.$$

We shall now consider the differential equation (7.1). It is a single differential equation for the m desired functions $S_\alpha\,(x_i)$. We can then specify $m - 1$ of the functions arbitrarily and obtain a differential equation for one desired function. We take advantage of that arbitrariness and choose, once and for all, the (three-times continuously differentiable) functions $S_{\alpha'}\,(x_i)$ ($\alpha' = 2, 3, \ldots, m$) from (8.5) and denote the remaining desired function by $y = S\,(x_i)$. If we further write $y_i = \partial S / \partial x_i$ then (7.1) will go to:

(8.9) $$H\,(x_i\,, y_i) = 1 \,,$$

in which $H\,(x_i\,, y_i)$ is at least twice continuously differentiable in all variables x_i, y_i, such that we can apply the usual theory of characteristics to the differential equation (8.9).

In particular, we make use of the fact that we prescribed the values of the desired function $y = S\,(x_i)$ on an $(n - 1)$-dimensional hypersurface. We then consider the point $x_i^0 = x_i\,(t_\alpha^0)$ on the extremal and define the hypersurface by $x_n = x_n^0$. We prescribe the value of $y = S\,(x_1, \ldots, x_n)$ on that hypersurface by way of $y = S_1(x_1, \ldots, x_{n-1}, x_n^0)$, in wh ich $S_1\,(x_1, \ldots, x_n)$ is the first of the functions in (8.5). We will then have $y_{i'} = \frac{\partial S_1}{\partial x_{i'}}(x_1, \ldots, x_n^0)$ ($i' = 1, 2, \ldots, n - 1$), and the differential equation (8.9) can serve for a unique calculation of y_n in the event that $\partial H / \partial y_n \neq 0$.

A solution along the extremals can be given immediately, namely, $y_n = \frac{\partial S_1}{\partial x_n}(x_1, \ldots, x_n^0)$, since $H\left(x_i, \frac{\partial S_1}{\partial x_i}\right) = F\,(x_i,\, s_{(i)}) = 1$ along the extremals. On the other hand, $H\,(x_i,\, y_i)$ is positively-

homogeneous of order one in the y_i, and therefore $H(x_i, y_i) = \frac{\partial H}{\partial y_i} y_i$. Not all of the partial derivatives $\frac{\partial H}{\partial y_i}$ can vanish at the same time then, since otherwise one would have $H = 0$. One might assume that the notation is chosen in such a way that one has $\partial H / \partial y_n \neq 0$, precisely, such that one can solve for y_n uniquely. The uniquely-determined solution then coincides with $y_i = \frac{\partial S_1}{\partial x_n}(x_1, \ldots, x_{n-1}, x_n^0)$ along the extremals.

One integrates the characteristic differential equations:

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \qquad \dot{y}_i = -\frac{\partial H}{\partial x_i} \tag{8.10}$$

with the fixed initial values, in which the condition $\partial H / \partial y_n = \dot{x}_i \neq 0$ says, geometrically, that the characteristics intersect the surface $x_n = x_n^0$ without contacting it, and therefore simply cover a certain region is space. That will also imply the value of the y_i in space, so one will then get the desired function $y = S(x_1, \ldots, x_n)$ in the known way.

A solution $y = S(x_i)$ of the differential equation (8.9) is thus constructed. The essential point in that is that the $(n - m)$-surfaces $S(x_i) = \text{const.}$, $S_{\alpha'}(x_i) = \text{const.}$ ($\alpha' = 2, 3, \ldots, m$) intersect the extremals transversally, i.e., that their direction coordinates $s^*_{(i)}$ fulfill $s^*_{(i)} = f_{(i)}$ along the extremals. In order to show that, the characteristic differential equations (8.10) must be considered more closely. When written out in full, they read:

$$\dot{x}_i = F_{(i)}\left(x_k, y_k \frac{\partial s_{(k)}}{\partial S_{k1}}\right) \cdot \frac{\partial s_{(k)}}{\partial S_{k1}}, \tag{8.11$_1$}$$

$$\dot{y}_i = -F_{x_i}\left(x_k, y_k \frac{\partial s_{(k)}}{\partial S_{k1}}\right) - F_{(i)}\left(x_k, y_k \frac{\partial s_{(k)}}{\partial S_{k1}}\right) y_j \frac{\partial}{\partial x_i}\left(\frac{\partial s_{(i)}}{\partial S_{j1}}\right). \tag{8.11$_2$}$$

The solution curves of the system (8.11), when interpreted in the space of the variables x_i, y_i, are established uniquely along their entire course by the above choice of initial values, since the right-hand sides of (8.11) are continuously-differentiable functions of x_i and y_i .

An $(m - 1)$-parameter family of solution curves in R^{2n} can even be given explicitly. To that end, we consider the $(m - 1)$-parameter family of t_1-parameter curves on (8.1): $x_i = x_i(t_1, t^0_{\alpha'})$ ($\alpha' = 2, 3, \ldots, m$). We will then have:

Theorem 8.1:

The curves of the $(m - 1)$*-parameter family in* R^{2n}:

(8.12) $$x_i = x_i(t_1, t^0_{\alpha'}) \ , \qquad y_i = \frac{\partial S_1(x(t_1, t^0_{\alpha'}))}{\partial x_i}$$

are solutions of the system (8.11) *that satisfy the initial conditions above.*

In other words: The t_1-curves define a family of characteristics along the extremals that will simply cover the extremals, and the values of the partial derivatives of the constructed solution $y = S(x_1, \ldots, x_n)$ coincide with $y_i = \partial S_1 / \partial x_i$ along the extremals such that when the direction coordinates $s^*_{(i)}$ are constructed with the ultimate functions $S(x_i)$, $S_{\alpha'}(x_i)$ ($\alpha' = 2, 3, \ldots, m$), they will fulfill $s^*_{(i)} = s_{(i)} = f_{(i)}$ along the extremals. When the Legendre transformation is performed, as in no. **7**, it will yield the associated geodetic field in which the extremal is embedded.

In order to prove Theorem 8.1, one must then show that eq. (8.11) must be fulfilled identically by way of (8.12). If one next substitutes (8.12) in $(8.11)_1$ then since $s_{(i)} = f_{(i)}(x, p)$ [$F_{(i)}(x, s) = p_{(i)}$, resp.] along the extremals, one will get:

$$\dot{x}_i = x_{i1} = F_{(i)}(x, s)\frac{\partial s_{(i)}}{\partial S_{i1}} = p_{(i)}\frac{\partial s_{(i)}}{\partial S_{i1}} \ .$$

From (8.6), that is, in fact, an identity.

If one substitutes (8.12) in (8.11)2 then one will get:

(8.13) $$\frac{\partial S_{i1}}{\partial t_1} = -F_{x_i}(x, s) - F_{(i)}(x, s)\, S_{k1}\frac{\partial}{\partial x_i}\left(\frac{\partial s_{(i)}}{\partial S_{k1}}\right) \ .$$

On the other hand:

(8.14) $$\frac{\partial S_{i1}}{\partial t_1} = \frac{\partial S_{i1}}{\partial x_k}\frac{\partial x_k}{\partial t_1} = \frac{\partial S_{i1}}{\partial x_k}\frac{\partial s_{(i)}}{\partial S_{k1}} F_{(i)}(x, s) \ .$$

It then suffices to show that the difference D_i between (8.13) and (8.14), namely:

$$D_i = F_{x_i}(x, s) + F_{(i)}(x, s)\left\{\frac{\partial S_{i1}}{\partial x_k}\frac{\partial s_{(i)}}{\partial S_{k1}} + \frac{\partial}{\partial x_i}\left(\frac{\partial s_{(i)}}{\partial S_{k1}}\right) S_{k1}\right\},$$

is zero. Since $S_1(x_i)$ is twice continuously differentiable, one will have:

$$\frac{\partial S_{k1}}{\partial x_i} = \frac{\partial^2 S_1}{\partial x_i\, \partial x_k} = \frac{\partial S_{i1}}{\partial x_k} \ .$$

That implies that:

$$D_i = F_{x_i}(x,s) + F_{(i)}(x,s)\left\{\frac{\partial S_{k1}}{\partial x_i}\frac{\partial s_{(i)}}{\partial S_{k1}} + S_{k1}\frac{\partial}{\partial x_i}\left(\frac{\partial s_{(i)}}{\partial S_{k1}}\right)\right\} = F_{x_i} + F_{(i)}\cdot\frac{\partial s_{(i)}}{\partial x_i} \ .$$

However, from (8.8), one will have $S_{i\alpha} = f_{x_{i\alpha}}$ along the extremals, and therefore:

$$\frac{df_{x_{i\alpha}}}{dt_\alpha} = \frac{dS_{i\alpha}}{dt_\alpha}\frac{\partial^2 S_\alpha}{\partial x_i\,\partial x_k}\cdot\frac{\partial s_{(i)}}{\partial S_{k\alpha}}F_{(i)}(x,s) = F_{(i)}\cdot\frac{\partial s_{(i)}}{\partial x_i} \ .$$

On the other hand, from (8.6), the Legendre transformation will yield $F_{x_i}(x,s) = -\ f_{x_i}(x,p)$, i.e.:

$$D_i = \frac{df_{x_{i\alpha}}}{dt_\alpha} - f_{x_i} = 0 \ ,$$

since the surface is, by assumption, an extremal, so it satisfies the Euler differential equations.

9. Proof of Kneser's theorem. – We shall now consistently consider the $p_{(i)}$ on the Grassmann manifold $b_\rho(p) = 0$, such that they can always be regarded as the direction coordinates of a matrix $(x_{i\alpha})$. If $p = p(t)$ is a continuously-differentiable curve on the Grassmann manifold then (cf., the remark in no. **1**) a continuously-differentiable $x_{i\alpha}(t)$ can also be always given such that the $p_{(i)}$ are the direction coordinates of $(x_{i\alpha}(t))$. Differentiating along the curve yields:

$$\frac{dp_{(i)}}{dt} = \frac{\partial p_{(i)}}{\partial x_{i\alpha}}\frac{dx_{i\alpha}}{dt} \qquad \text{or more briefly} \qquad dp_{(i)} = \frac{\partial p_{(i)}}{\partial x_{i\alpha}}dx_{i\alpha} \ .$$

If one again writes $f_{x_{i\alpha}} = \partial f / \partial x_{i\alpha}$ and denotes the direction coordinates of $\left(f_{x_{i\alpha}}\right)$ by $v_{(i)}$ then one will next get:

$$D = \left|\, f_{x_{i\alpha}} x_{i\beta} \,\right| = v_{(i)}\, p_{(i)} \ . \tag{9.1}$$

On the other hand, since $f_{x_{i\alpha}} = f_{(i)}\dfrac{\partial p_{(i)}}{\partial x_{i\alpha}}$, one will always have $f_{x_{i\alpha}} x_{i\beta} = f\,\delta_{\alpha\beta}$, i.e., $D = f^m$, or:

$$\frac{v_{(i)}}{f^{m-1}}\, p_{(i)} = f_{(i)}\, p_{(i)} \ . \tag{9.2}$$

Moreover, one has:

$$v_{(i)}\, dp_{(i)} = v_{(i)} \frac{\partial p_{(i)}}{\partial x_{i\alpha}} dx_{i\alpha} ,$$

so, e.g., $\dfrac{\partial p_{(i)}}{\partial x_{i1}} dx_{i1}$ are the direction coordinates of the matrix:

$$(d\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_m) ,$$

in which the $\mathfrak{x}_\alpha$ denote the column vectors of the matrix $(x_{i\alpha})$. In order to then calculate the term $v_{(i)} \dfrac{\partial p_{(i)}}{\partial x_{i1}} dx_{i1}$, one simply replaces the column vector $\mathfrak{x}_1$ in (9.1) with $d\mathfrak{x}_1$. That implies that:

$$v_{(i)} \frac{\partial p_{(i)}}{\partial x_{i1}} dx_{i1} = \begin{vmatrix} f_{x_{i1}} dx_{i1} & 0 & \cdots & 0 \\ * & f & & 0 \\ \vdots & & \ddots & \\ * & 0 & & f \end{vmatrix} = f^{m-1} f_{x_{i1}}\, dx_{i1} ,$$

so, in all:

$$v_{(i)}\, dp_{(i)} = f^{m-1} f_{x_{i\alpha}}\, dx_{i\alpha} = f^{m-1} f_{(i)} \frac{\partial p_{(i)}}{\partial x_{i\alpha}} dx_{i\alpha} = f^{m-1} f_{(i)}\, dp_{(i)} ,$$

or

(9.3) $$\frac{v_{(i)}}{f^{m-1}} dp_{(i)} = f_{(i)}\, dp_{(i)} .$$

We now consider the two determinants:

$$v_{(i)} = \left| f_{x_{i_\nu \alpha}} \right| , \quad p_{(k)} = \left| x_{x_{k_\nu \alpha}} \right|$$

and take their product:

$$v_{(i)}\, p_{(i)} = \left| f_{x_{i_\nu \alpha}} x_{x_{k_\nu \alpha}} \right| = \left| f_{(j)} \frac{\partial p_{(j)}}{\partial x_{i_\nu \alpha}} x_{x_{k_\nu \alpha}} \right| .$$

The $\dfrac{\partial p_{(j)}}{\partial x_{i_\nu \alpha}} x_{x_{k_\nu \alpha}}$ in that is again a determinant that is composed of m rows of the matrix $(x_{i\alpha})$, i.e., $v_{(i)}\, p_{(k)}$ can be written as a function of the x_i, $p_{(i)}$, which we suggest by putting them in brackets:

$$w_{(i)(k)}\,(x, p) = [v_{(i)}\, p_{(k)}] .$$

(That expression will also be meaningful when the $p_{(i)}$ do not belong to the Grassmann manifold.)

If one now defines the quantities in Kneser's theorem by means of:

$$u_{(i)}\,(x, p) = \frac{[v_{(i)}\; p_{(k)}]\, p_{(k)}}{f^{\,m-1} p_{(j)}\; p_{(j)}}$$

then when one replaces the $p_{(i)}$ with the direction coordinates of the matrix $(x_{i\alpha})$, one will get:

$$u_{(i)} = \frac{v_{(i)}}{f^{\,m-1}}\ ,$$

and therefore, due to (9.2) and (9.3), one will also have:

$$u_{(i)}\, p_{(i)} = f_{(i)}\, p_{(i)}\ , \qquad u_{(i)}\, dp_{(i)} = f_{(i)}\, dp_{(i)}\ , \qquad b_\rho\,(u_{(i)}) = 0 \qquad \text{on} \qquad b_\rho\,(p) = 0\ .$$

In addition, one confirms the homogeneity and differentiability properties that were stated in Theorem 4.1 directly.

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