"Sur certaines propriétés des systèmes d'équations différentielles," Ann. sci. de l'E.N.S. (3) **27** (1910), 543-563.

On certain properties of systems of differential equations

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If one is given a system of differential equations then knowing one or several integrals will not, in general, permit one to write down a new integral that is distinct from the former. Nonetheless, that is possible in certain cases: For example, if the equations are canonical then one knows that knowing two distinct integrals will permit one to write down a third one (viz., Poisson's theorem).

Those properties, and some analogous ones, define the topic of this study. In the first section, I will especially study the canonical equations. In the second section, an arbitrary system of first-order differential equations will be in question. Moreover, several of the results that are established directly for canonical systems are recovered as particular cases of the ones that are established for the general systems.

I. – Canonical equations. Canonical changes of variables.

1. – Consider a function $S(x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n)$ that depends upon two sequences of n variables, and set:

$$(a) \qquad y_1 = \frac{\partial S}{\partial x_1}, \qquad y_2 = \frac{\partial S}{\partial x_2}, \qquad \ldots, \qquad y_n = \frac{\partial S}{\partial x_n},$$

$$(b) \qquad \beta_1 = \frac{\partial S}{\partial \alpha_1}, \qquad \beta_2 = \frac{\partial S}{\partial \alpha_2}, \qquad \ldots, \qquad \beta_n = \frac{\partial S}{\partial \alpha_n}.$$

1. – Those $2n$ equations (a), (b) permit one to express $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ as functions of $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$. As a result, the partial derivatives, which are $4n^2$ in number, will take one of the following forms:

$$(c) \qquad \frac{\partial x_i}{\partial \alpha_k}, \quad \frac{\partial x_i}{\partial \beta_k}, \quad \frac{\partial y_i}{\partial \alpha_k}, \quad \frac{\partial y_i}{\partial \beta_k}.$$

2. – The same $2n$ equations (a), (b) permit one to express $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ as functions of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. That will result in a second group of $4n^2$ partial derivatives that take of the following forms:

(d) $$\frac{\partial \alpha_k}{\partial x_i}, \quad \frac{\partial \alpha_k}{\partial y_i}, \quad \frac{\partial \beta_k}{\partial x_i}, \quad \frac{\partial \beta_k}{\partial y_i}.$$

The following relations, which are due to Jacobi, permit one to express ([1]) any one of the derivatives (c) in terms of the derivatives (d):

(e) $$\frac{\partial y_i}{\partial \alpha_k} = \frac{\partial \beta_k}{\partial x_i},$$

(f) $$\frac{\partial y_i}{\partial \beta_k} = -\frac{\partial \alpha_k}{\partial x_i},$$

(g) $$\frac{\partial x_i}{\partial \beta_k} = \frac{\partial \alpha_k}{\partial y_i},$$

(h) $$\frac{\partial x_i}{\partial \alpha_k} = -\frac{\partial \beta_k}{\partial y_i}.$$

We shall make use of those formulas.

Formulas (a), (b) express the idea that the following expressions:

$$\sum_i y_i \, dx_i + \sum_i \beta_i \, d\alpha_i = dS,$$

$$\sum_i \beta_i \, d\alpha_i - \sum_i x_i \, dy_i = d\left(S - \sum_i x_i \, y_i\right)$$

are *exact differentials.*

2. – I suppose that the x_i and y_i are required to satisfy a system of $2n$ canonical equations (in the Hamilton sense):

(1) $$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

in which F denotes a function of the x_i and y_i that can also depend upon t explicitly.

Make the change of variables that was indicated by the formulas (a), (b), which is a change of variables that I will denote by the notation:

([1]) TISSERAND, *Traité de Mécanique céleste*, t. I, pp. 20.

$$(x, y) \to (\beta, \alpha) .$$

The Jacobi relations (*e*), (*f*), (*g*), (*h*) show immediately that the new variables β_k , α_k satisfy the equations:

(2) $$\frac{d\beta_k}{dt} = \frac{\partial F}{\partial \alpha_k}, \qquad \frac{d\alpha_k}{dt} = -\frac{\partial F}{\partial \beta_k},$$

i.e., that the change of variables $(x, y) \to (\beta, \alpha)$ has not altered the canonical form of the differential equations ([1]).

With Poincaré, we say that such a change of variables is *canonical*.

3. – An arbitrary function S $(x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n)$ then permits one to perform a canonical change of variables $(x, y) \to (\beta, \alpha)$. In particular, take S to be the function:

$$S = \alpha_1\, q_1 + \alpha_2\, q_2 + \ldots + \alpha_n\, q_n ,$$

which is linear in the α, while the q are distinct, but arbitrary, functions of the x.

Formulas (*a*), (*b*) become:

(a_1) $$y_1 = \alpha_1 \frac{\partial q_1}{\partial x_i} + \alpha_2 \frac{\partial q_2}{\partial x_i} + \cdots + \alpha_n \frac{\partial q_n}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

(b_1) $$\alpha_i = q_i ,$$

which define a canonical change of variables $(x, y) \to (\beta, \alpha)$ for which the β are *arbitrary* functions of the x and the α are given by the *n linear* equations (a_1). In the case of dynamics, the x_i represent the coordinates of n / 3 material points, the y_i represent the components of their quantities of motion, F is the total energy, and equations (1) are the equations of motion.

Therefore, the change of variables that is defined by formulas (a_1), (b_1) is nothing but the classical Poisson-Hamilton change that defines the passage from Cartesian coordinates to curvilinear coordinates.

4. – Let us return to the canonical changes, in general, and first of all write the conditions for:

(3) $$\sum_i y_i\, dx_i + \sum_k \beta_k\, d\alpha_k$$

to be an exact differential dS : If we take the independent variables to be the α and β then we will have:

([1]) H. POINCARÉ, *Les Méthodes de la Mécanique céleste*, t. I, Chap. I, *Leçons de Mécanique céleste*, t. I, pp. 3.

$$(4) \qquad \begin{cases} \dfrac{\partial S}{\partial \alpha_k} = \beta_k + \displaystyle\sum_i y_i \frac{\partial x_i}{\partial \alpha_k}, \\ \dfrac{\partial S}{\partial \beta_k} = \displaystyle\sum_i y_i \frac{\partial x_i}{\partial \beta_k}. \end{cases}$$

We write that (3) is an exact differential dS by equating the two different values that (4) gives for the mixed second derivatives $\frac{\partial^2 S}{\partial \alpha_k \, \partial \beta_k}$, $\frac{\partial^2 S}{\partial \alpha_k \, \partial \beta_h}$, $\frac{\partial^2 S}{\partial \alpha_k \, \partial \alpha_h}$, $\frac{\partial^2 S}{\partial \beta_k \, \partial \beta_h}$. We then get the conditions:

$$(5) \qquad \begin{cases} 1 - \displaystyle\sum_i \left(\frac{\partial x_i}{\partial \beta_k} \frac{\partial y_i}{\partial \alpha_k} - \frac{\partial x_i}{\partial \alpha_k} \frac{\partial y_i}{\partial \beta_k} \right) = 0, \\ \displaystyle\sum_i \left(\frac{\partial x_i}{\partial \beta_h} \frac{\partial y_i}{\partial \alpha_k} - \frac{\partial x_i}{\partial \alpha_k} \frac{\partial y_i}{\partial \beta_h} \right) = 0, \\ \displaystyle\sum_i \left(\frac{\partial x_i}{\partial \alpha_h} \frac{\partial y_i}{\partial \alpha_k} - \frac{\partial x_i}{\partial \alpha_k} \frac{\partial y_i}{\partial \alpha_h} \right) = 0, \\ \displaystyle\sum_i \left(\frac{\partial x_i}{\partial \beta_h} \frac{\partial y_i}{\partial \beta_k} - \frac{\partial x_i}{\partial \beta_k} \frac{\partial y_i}{\partial \beta_h} \right) = 0, \end{cases}$$

which are conditions that can be written:

$$(5, \textit{cont.}) \qquad \begin{cases} [\beta_k, \alpha_k] = 1, \\ [\beta_h, \alpha_k] = 0, \\ [\alpha_h, \alpha_k] = 0, \\ [\beta_h, \beta_k] = 0 \end{cases}$$

upon employing the *Lagrange brackets* ([1]).

However, the fact that (3) is an exact differential implies the Jacobi relations (*e*), (*f*), (*g*), (*h*) between the partial derivatives of the (x_i, y_i) with respect to the (α_k, β_k) and those of the (α_k, β_k) with respect to (x_i, y_i). When one replaces the derivatives $\frac{\partial (x, y)}{\partial (\beta, \alpha)}$ with their values that one infers from those relations, the conditions (5) will take the new form:

([1]) H. POINCARÉ, *Leçons de Mécanique céleste*, t. I, pp. 18: In POINCARÉ's definition, the (β, α) denote integration constants. However, one can also just as well suppose that they are new variables.

$$
(6) \qquad \begin{cases}
1 - \displaystyle\sum_i \left(\frac{\partial \beta_k}{\partial x_i} \frac{\partial \alpha_k}{\partial y_i} - \frac{\partial \beta_k}{\partial y_i} \frac{\partial \alpha_k}{\partial x_i} \right) = 0, \\
\displaystyle\sum_i \left(\frac{\partial \beta_k}{\partial x_i} \frac{\partial \alpha_h}{\partial y_i} - \frac{\partial \beta_k}{\partial y_i} \frac{\partial \alpha_h}{\partial x_i} \right) = 0, \\
\displaystyle\sum_i \left(\frac{\partial \beta_k}{\partial x_i} \frac{\partial \beta_h}{\partial y_i} - \frac{\partial \beta_k}{\partial y_i} \frac{\partial \beta_h}{\partial x_i} \right) = 0, \\
\displaystyle\sum_i \left(\frac{\partial \alpha_k}{\partial x_i} \frac{\partial \alpha_h}{\partial y_i} - \frac{\partial \alpha_k}{\partial y_i} \frac{\partial \alpha_h}{\partial x_i} \right) = 0.
\end{cases}
$$

Upon employing the well-known notation of the *Poisson parentheses* (also called the *Jacobi brackets*), those conditions can be written:

$$
(6, \textit{cont.}) \qquad \begin{cases}
(\beta_k, \alpha_k) = 1, \\
(\beta_h, \alpha_k) = 0, \\
(\alpha_h, \alpha_k) = 0, \\
(\beta_h, \beta_k) = 0.
\end{cases}
$$

[Of course, nothing distinguishes the roles of the (x, y) and the (β, α) in all of that. One can make the (x, y) play the role of the (β, α) and conversely, because the change $(x, y) \to (\beta, \alpha)$ is canonical, the change $(\beta, \alpha) \to (x, y)$ will also be so.]

5. – Recall the canonical equations (1), in which we now suppose that the function F does not depend upon t. Perform a canonical change of variables:

$$(x, y) \to (\beta, \alpha)$$

and suppose that one of the new variables – α_1, for example – is the function F itself ([1]): $F = \alpha_1$. The new canonical equations (2) will then be:

$$
\frac{d\beta_1}{dt} = 1, \qquad \frac{d\beta_k}{dt} = 0 \qquad (k \neq 1),
$$

([1]) That is always possible. It suffices to first take the function S to be a complete integral of the Jacobi partial differential equation:

$$F\left(x_i, \frac{\partial S}{\partial x_i} \right) = \alpha_1,$$

which contains $(n - 1)$ other constants $\alpha_2, \alpha_3, \dots, \alpha_n$, none of which is additive, in addition to the constant α_1.

$$\frac{d\alpha_k}{dt} = 0\,.$$

The $2n$ integrals of equations (1) will then be:

$$\begin{array}{llll} \alpha_1 = C_1\,, & \alpha_2 = C_2\,, & \ldots, & \alpha_n = C_n\,, \\ \beta_1 - t = C_{n+1}\,, & \beta_2 = C_{n+2}\,, & \ldots, & \beta_n = C_{2n}\,, \end{array}$$

in which the C are integration constants ([1]).

In order for a function $\varphi(x_i, y_i, t)$ to be an integral of equations (1), it is necessary and sufficient that when it is expressed in terms of the variables (α_k, β_k, t), it will depend upon only the:

$$\begin{array}{llll} \alpha_1\,, & \alpha_2\,, & \ldots, & \alpha_n\,, \\ \beta_1 - t\,, & \beta_2\,, & \ldots, & \beta_n\,. \end{array}$$

6. – Let $\alpha_1, \alpha_2, \ldots$ be integrals of equations (1). I consider an expression:

$$\mathcal{F}\left(\varphi_1, \varphi_2, \ldots, \frac{\partial\varphi_1}{\partial x_i}, \frac{\partial\varphi_1}{\partial y_i}, \ldots, \frac{\partial\varphi_2}{\partial x_i}, \frac{\partial\varphi_2}{\partial y_i}, \ldots\right), \tag{7}$$

which depends upon $\varphi_1, \varphi_2, \ldots$ and their partial derivatives (or arbitrary order). I suppose that this expression remains invariant under all canonical changes of variables, i.e., that the canonical change $(x, y) \to (\beta, \alpha)$ transforms that expression into:

$$\mathcal{F}\left(\varphi_1, \varphi_2, \ldots, \frac{\partial\varphi_1}{\partial \beta_i}, \frac{\partial\varphi_1}{\partial \alpha_i}, \ldots, \frac{\partial\varphi_2}{\partial \beta_i}, \frac{\partial\varphi_2}{\partial \alpha_i}, \ldots\right), \tag{8}$$

in which the $\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\alpha_i}$ simply replace the $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ that enter into (7). I say that under those conditions, the expression (7) will be an integral of equations (1).

Indeed, suppose that the new variables (β, α) are precisely the ones in the preceding section: $\varphi_1, \varphi_2, \ldots$ will then depend upon only:

$$\begin{array}{llll} \alpha_1\,, & \alpha_2\,, & \ldots, & \alpha_n\,, \\ \beta_1 - t\,, & \beta_2\,, & \ldots, & \beta_n\,. \end{array}$$

([1]) That change of variables should be compared with the one that Poincaré made on page 7 of Tome III of *Les Méthodes nouvelles de la Mécanique céleste*.

That is likewise obvious from the derivatives $\frac{\partial \varphi_1}{\partial \beta_i}$, $\frac{\partial \varphi_1}{\partial \alpha_i}$, …, $\frac{\partial \varphi_2}{\partial \beta_i}$, $\frac{\partial \varphi_2}{\partial \alpha_i}$, …, and as a result, the expression (8): Therefore, (8), i.e., (7) is an integral. We then state the following proposition:

Any expression that depends upon integrals of equations (1) *and their partial derivatives, and which remains invariant under a canonical change of variables is itself an integral* ([1]).

7. – It is now easy to recover (and generalize) Poisson's theorem: If φ_1 and φ_2 are two integrals of equations (2) then the Poisson parenthesis (φ_1, φ_2) will be a third. It will suffice to show that the expression (φ_1, φ_2) is invariant under a canonical change of variables $(x, y) \to (\beta, \alpha)$. Now, one has:

$(\varphi_1, \varphi_2) =$

$$\sum_k (\beta_k, \alpha_k) \frac{D(\varphi_1, \varphi_1)}{D(\beta_k, \alpha_k)} + \sum_k (\beta_k, \alpha_h) \frac{D(\varphi_1, \varphi_1)}{D(\beta_k, \alpha_h)} + \sum_k (\beta_k, \beta_h) \frac{D(\varphi_1, \varphi_1)}{D(\beta_k, \beta_h)} + \sum_k (\alpha_k, \alpha_k) \frac{D(\varphi_1, \varphi_1)}{D(\alpha_k, \alpha_k)},$$

and by virtue of the conditions (6, *cont.*), that will reduce to:

$$\frac{D(\varphi_1, \varphi_2)}{D(\beta_k, \alpha_k)},$$

i.e., the parenthesis (φ_1, φ_2) when it is expressed in terms of the new variables (β, α) ([2]).

8. – Poincaré gave the following generalization of Poisson's theorem ([3]): Let φ_1, φ_2, φ_3, φ_4 be four integrals of equations (1). Let Δ_{ik} be their Jacobian with respect to x_i, y_i, x_k, y_k:

$$\Delta_{ik} = \frac{D(\varphi_1, \varphi_2, \varphi_3, \varphi_4)}{D(x_i, y_i, x_k, y_k)}.$$

The expression:

$$\sum_{ik} \Delta_{ik}$$

is once more an integral.

In our way of thinking, that theorem results from the fact that the expression is invariant under a canonical change of variables $(x, y) \to (\beta, \alpha)$. Indeed, the relations (6, *cont.*) show that such a change will transform the expression $\sum_{ik} \Delta_{ik}$ into the following one:

([1]) H. VERGNE, "Sur les changements canoniques de variables," Comptes rendus, 25 April 1910.

([2]) On the subject of the invariance of the Poisson parentheses and the generalized Poisson theorem, *see* two notes by De Donder (*Comptes rendus*, 8 March 1909 and 1 August 1910).

([3]) H. POINCARÉ, *Les Méthodes nouvelles de la Mécanique céleste*, t. III, pp. 43.

$$\sum_{lh} \frac{D(\varphi_1, \varphi_2, \varphi_3, \varphi_4)}{D(\beta_l, \alpha_l, \beta_h, \alpha_h)} .$$

Similarly, if $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6$ are six integrals of equations (1) then the expression:

$$\sum_{ikl} \frac{D(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)}{D(x_i, y_i, x_k, y_k, x_l, y_l)}$$

will also be an integral, and so on. Indeed, such expressions will remain invariant under a canonical change of variables.

9. – Suppose that equations (1) have been integrated, and the x_i and y_i are found to have been expressed as functions of t and $2n$ integration constants $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{2n}$. Consider an expression:

$$\mathcal{F}\left(\frac{\partial x_i}{\partial \mathrm{a}_1}, \frac{\partial x_i}{\partial \mathrm{a}_2}, \ldots, \frac{\partial y_i}{\partial \mathrm{a}_1}, \frac{\partial y_i}{\partial \mathrm{a}_2}, \ldots\right) \tag{9}$$

that depends upon the partial derivatives (of arbitrary order) of the (x_i , y_i) with respect to the integration constants, and suppose that any canonical change $(x, y) \to (\beta, \alpha)$ that is performed on the (x_i , y_i) will transform that expression into:

$$\mathcal{F}\left(\frac{\partial \beta_i}{\partial \mathrm{a}_1}, \frac{\partial \beta_i}{\partial \mathrm{a}_2}, \ldots, \frac{\partial \alpha_i}{\partial \mathrm{a}_1}, \frac{\partial \alpha_i}{\partial \mathrm{a}_2}, \ldots\right) \tag{10}$$

identically, in which the β_i , α_i simply replace the x_i , y_i that enter into (9). I say that under those conditions, *the expression* (9) *will be an integral of equations* (1).

Indeed, suppose that the new variables (β, α) are precisely the ones in no. **5**. One will then have that:

$$\begin{array}{cccc} \alpha_1 , & \alpha_2 , & \ldots, & \alpha_n , \\ \beta_1 - t , & \beta_2 , & \ldots, & \beta_n \end{array}$$

are constants that depend upon only $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{2n}$, and not on t . Obviously, the same thing will be true of:

$$\frac{\partial \beta_i}{\partial \mathrm{a}_1}, \frac{\partial \beta_i}{\partial \mathrm{a}_2}, \ldots, \frac{\partial \alpha_i}{\partial \mathrm{a}_1}, \frac{\partial \alpha_i}{\partial \mathrm{a}_2}, \ldots,$$

and as a result, one will have the expression (10): Therefore (10), i.e., (9), is an integral

10. – Consider the Lagrange bracket:

$$[\mathrm{a}_h\,,\,\mathrm{a}_l] = \sum_i \left(\frac{\partial x_i}{\partial \mathrm{a}_h}\frac{\partial y_i}{\partial \mathrm{a}_l} - \frac{\partial x_i}{\partial \mathrm{a}_l}\frac{\partial y_i}{\partial \mathrm{a}_h} \right).$$

That is an expression of the form (9). Furthermore, a canonical change $(x, y) \to (\beta, \alpha)$ will transform it into:

$$\sum_i \left(\frac{\partial \beta_i}{\partial \mathrm{a}_h}\frac{\partial \alpha_i}{\partial \mathrm{a}_l} - \frac{\partial \beta_i}{\partial \mathrm{a}_l}\frac{\partial \alpha_i}{\partial \mathrm{a}_h} \right),$$

which would result from formulas (5, *cont.*). Therefore, $[\mathrm{a}_h\,,\,\mathrm{a}_l]$ is an integral of equations (1) ([1]).

One can give some generalizations of that theorem that are analogous to the ones that Poincaré gave to Poisson's theorem, and that we recalled in no. **8**. Let D_{ik} denote the Jacobian of x_i, y_i, x_k, y_k with respect to four of our constants of integration – for example, with respect to a_1, a_2, a_3, a_4:

$$D_{ik} = \frac{D(x_i, y_i, x_k, x_k)}{D(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4)}.$$

The expression:

$$\sum_{ik} D_{ik}$$

is an integral of equations (1), and so on. Indeed, such expressions remain invariant under a canonical change of variables. That always results from formulas (5, *cont.*).

11. – One can also propose to form some expressions that depend upon both the derivatives:

$$\frac{\partial \varphi_1}{\partial x_i}, \frac{\partial \varphi_1}{\partial y_i}, \ldots, \frac{\partial \varphi_2}{\partial x_i}, \frac{\partial \varphi_2}{\partial y_i}, \ldots$$

[which are derivatives of the integrals of the system (1) with respect to the variables] and the derivatives:

$$\frac{\partial x_i}{\partial \mathrm{a}_1}, \frac{\partial x_i}{\partial \mathrm{a}_2}, \ldots, \frac{\partial y_i}{\partial \mathrm{a}_1}, \frac{\partial y_i}{\partial \mathrm{a}_2}, \ldots$$

(which are derivatives of the variables with respect to the integration constants), which remain invariant under a canonical change. Such expressions will then be integrals of the system (1) again. For example:

$$\sum_i \left(\frac{\partial \varphi_1}{\partial x_i}\frac{\partial x_i}{\partial \mathrm{a}_h} + \frac{\partial \varphi_1}{\partial y_i}\frac{\partial y_i}{\partial \mathrm{a}_h} \right)$$

([1]) H. POINCARÉ, *Leçons de Mécanique céleste*, t. I, pp. 18.

is an integral. However, that obviously is nothing but the integral $\partial\varphi_1 / \partial a_h$. We can likewise say that expressions such as the following ones are integrals:

$$\sum_{ik} \begin{vmatrix} \frac{\partial\varphi_1}{\partial x_i} & \frac{\partial\varphi_1}{\partial y_i} & \frac{\partial\varphi_1}{\partial x_k} & \frac{\partial\varphi_1}{\partial y_k} \\ \frac{\partial\varphi_2}{\partial x_i} & \frac{\partial\varphi_2}{\partial y_i} & \frac{\partial\varphi_2}{\partial x_k} & \frac{\partial\varphi_2}{\partial y_k} \\ -\frac{\partial y_i}{\partial a_h} & \frac{\partial x_i}{\partial a_h} & -\frac{\partial y_k}{\partial a_h} & \frac{\partial x_k}{\partial a_h} \\ -\frac{\partial y_i}{\partial a_l} & \frac{\partial x_i}{\partial a_l} & -\frac{\partial y_k}{\partial a_l} & \frac{\partial x_k}{\partial a_l} \end{vmatrix}, \qquad \sum_{ik} \begin{vmatrix} \frac{\partial\varphi_1}{\partial x_i} & \frac{\partial\varphi_1}{\partial y_i} & \frac{\partial\varphi_1}{\partial x_k} & \frac{\partial\varphi_1}{\partial y_k} \\ \frac{\partial\varphi_2}{\partial x_i} & \frac{\partial\varphi_2}{\partial y_i} & \frac{\partial\varphi_2}{\partial x_k} & \frac{\partial\varphi_2}{\partial y_k} \\ \frac{\partial\varphi_3}{\partial x_i} & \frac{\partial\varphi_3}{\partial y_l} & \frac{\partial\varphi_3}{\partial x_k} & \frac{\partial\varphi_3}{\partial y_k} \\ -\frac{\partial y_i}{\partial a_l} & \frac{\partial x_i}{\partial a_l} & -\frac{\partial y_k}{\partial a_l} & \frac{\partial x_k}{\partial a_l} \end{vmatrix}, \qquad \ldots$$

In order to establish the invariance of those expressions under a canonical change of variables, it is necessary to make use of the two groups of formulas (5, *cont.*) and (6, *cont.*), or one of those two groups, combined with the Jacobi relations (*e*), (*f*), (*g*), (*h*).

It should be pointed out that just as the various generalizations of Poisson's theorem are not essentially distinct form that theorem, similarly, the integrals that I just indicated can be deduced with the combined use of Poisson's theorem and the Lagrange brackets.

12. – The same consideration of invariance under canonical changes of variables will permit us to recover the various integral invariants that Poincaré gave ([1]) for the canonical equations (1).

The differential expression:

$$\sum_i \begin{vmatrix} \delta x_i & \delta y_i \\ \delta' x_i & \delta' y_i \end{vmatrix} = \sum_i (\delta x_i\, \delta' y_i - \delta y_i\, \delta' x_i)$$

(in which δ and δ' represent two different systems of differentials) is invariant under a canonical change [from formulas (5, *cont.*)]. It will then result, and always by the same reasoning, that it is an integral that depends upon three infinitely-close solutions to equations (1), in other words, that the double integral:

$$\iint \sum_i \delta x_i\, \delta y_i$$

is an integral invariant of equations (1).

Similarly, the expression:

$$\sum_{ik} \begin{vmatrix} \delta x_i & \delta y_i & \delta x_k & \delta y_k \\ \delta' x_i & \delta' y_i & \delta' x_k & \delta' y_k \\ \delta'' x_i & \delta'' y_i & \delta'' x_k & \delta'' y_k \\ \delta''' x_i & \delta''' y_i & \delta''' x_k & \delta''' y_k \end{vmatrix}$$

([1]) H. POINCARÉ, *Les Méthodes nouvelles de la Mécanique céleste*, t. III, Chap. II.

is invariant under a canonical change. We deduce the fourth-order integral invariant:

$$\iiint\int \sum_{ik} \delta x_i \, \delta y_i \, \delta x_k \, \delta y_k$$

from it, and so on, up to order $2n$:

$$\iint \cdots \int \delta x_1 \, \delta y_1 \, \delta x_2 \, \delta y_2 \cdots \delta x_n \, \delta y_n \, .$$

II. – General systems of differential equations. Integral invariants.

13. – We shall now abandon the canonical equations in order to occupy ourselves with a general system of differential equations. First of all, we shall establish a preliminary lemma.

Consider n variables:

$$x_1 , \quad x_2 , \quad \ldots, \quad x_n ,$$

and set:

$$(\alpha) \qquad \xi_i = \varphi_i \, (x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) ,$$

in which the φ are n functions that are distinct from the x. We then define a change of variables to be perform on the x :

1. We can regard the ξ as functions of the x . That will then imply partial derivatives of the form:

$$\frac{\partial \xi_i}{\partial x_k} ,$$

which are n^2 in number.

2. We can regard the x as functions of the ξ . A second group of n^2 partial derivatives of the form:

$$\frac{\partial x_k}{\partial \xi_i}$$

will then result.

First of all, I will write down certain relations that exist between those two types of derivatives. I shall let Δ denote the Jacobian (which is not identically zero, by hypothesis) of the ξ with respect to x :

$$\Delta = \frac{D(\xi_1, \xi_2, \ldots, \xi_n)}{D(x_1, x_2, \ldots, x_n)} \, .$$

The relations between the two types of derivatives that I have in mind are the following ones:

$$(\beta)\qquad \left\{\begin{array}{l}
\dfrac{\partial x_h}{\partial \xi_i} = \dfrac{(-1)^{h+i}}{\Delta}\dfrac{D(\xi_1,\xi_2,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n)}{D(x_1,x_2,\ldots,x_{h-1},x_{h+1},\ldots,x_n)}, \\[2ex]
\dfrac{D(x_h,x_l)}{D(\xi_i,\xi_k)} = \dfrac{(-1)^{h+i+k+l}}{\Delta}\dfrac{D(\xi_1,\xi_2,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_n)}{D(x_1,x_2,\ldots,x_{h-1},x_{h+1},\ldots,x_{l-1},x_{l+1},\ldots,x_n)}, \\[2ex]
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\[2ex]
\dfrac{D(x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_l)}{D(\xi_1,\xi_2,\ldots,\xi_{s-1},\xi_{s+1},\ldots,\xi_n)} = \dfrac{(-1)^{r+s}}{\Delta}\dfrac{\partial \xi_s}{\partial x_r}.
\end{array}\right.$$

The first of those relations is immediate: If one differentiates the n equations (α) with respect to ξ_i then one will get:

$$1 = \frac{\partial \xi_i}{\partial x_1}\frac{\partial x_1}{\partial \xi_i} + \frac{\partial \xi_i}{\partial x_2}\frac{\partial x_2}{\partial \xi_i} + \cdots + \frac{\partial \xi_i}{\partial x_n}\frac{\partial x_n}{\partial \xi_i},$$

$$0 = \frac{\partial \xi_k}{\partial x_1}\frac{\partial x_1}{\partial \xi_i} + \frac{\partial \xi_k}{\partial x_2}\frac{\partial x_2}{\partial \xi_i} + \cdots + \frac{\partial \xi_k}{\partial x_n}\frac{\partial x_n}{\partial \xi_i}$$

$$(k = 0, 1, 2, \ldots, i-1, i+1, \ldots, n),$$

and that system of n linear equations in the $\partial x_h / \partial \xi_i$ immediately gives:

$$\frac{\partial x_h}{\partial \xi_i} = \frac{1}{\Delta}\frac{\partial \Delta}{\partial \dfrac{\partial \xi_i}{\partial x_h}},$$

which is nothing but the first of (β). From that same formula, one will then have:

$$(\gamma)\qquad \frac{\partial x_h}{\partial \xi_i}\frac{\partial x_l}{\partial \xi_k} - \frac{\partial x_h}{\partial \xi_k}\frac{\partial x_l}{\partial \xi_i} = \frac{1}{\Delta^2}\left[\frac{\partial \Delta}{\partial \dfrac{\partial \xi_i}{\partial x_h}}\frac{\partial \Delta}{\partial \dfrac{\partial \xi_k}{\partial x_l}} - \frac{\partial \Delta}{\partial \dfrac{\partial \xi_k}{\partial x_l}}\frac{\partial \Delta}{\partial \dfrac{\partial \xi_i}{\partial x_h}}\right].$$

Now, from a well-known formula that relates to determinants, the latter bracket is equal to:

$$\Delta\frac{\partial^2 \Delta}{\partial \dfrac{\partial \xi_i}{\partial x_h}\,\partial \dfrac{\partial \xi_k}{\partial x_l}},$$

such that the formula (γ) is nothing but the second of (β). All of the formulas in (β) are then established, step-by-step, as consequences of the first one.

14. – Now take the system of n first-order differential equations:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt\,, \tag{11}$$

in which the X are given functions of $x_1, x_2, \ldots, x_n, t$. Furthermore, suppose that when those equations have been solved, one can write down the n integrals of that system:

$$\begin{aligned} \varphi_1 (x_1, x_2, \ldots, x_n, t) &= \xi_1\,, \\ \varphi_2 (x_1, x_2, \ldots, x_n, t) &= \xi_2\,, \\ &\cdots\cdots\cdots \\ \varphi_n (x_1, x_2, \ldots, x_n, t) &= \xi_n\,, \end{aligned}$$

in which the ξ are integration constants. If we regard t as fixed then the last n equations will define n^2 derivatives $\partial\xi_i / \partial x_k$ and n^2 derivatives $\partial x_k / \partial\xi_i$, between which the relations (β) exist.

I say that, first of all, the Jacobian:

$$\Delta = \frac{D(\xi_1, \xi_2, \ldots, \xi_n)}{D(x_1, x_2, \ldots, x_n)}$$

is a *multiplier* (in the Jacobi sense) for equations (11). From a theorem of Poincaré ([1]), it will suffice to see that the integral of order n :

$$\overbrace{\iint\cdots\int}^{n} \Delta\,\delta x_1\,\delta x_2 \cdots \delta x_n$$

is an integral invariant. Now, that is obvious since the element that is placed under the $\int$ sign is nothing but:

$$\delta\xi_1\ \delta\xi_2\ \ldots\ \delta\xi_n\,,$$

and the ξ are constants.

15. – Now suppose that equations (11) admit an integral invariant of order $n - 1$:

$$\overbrace{\iint\cdots\int}^{n-1} \sum_i M_i\,\delta x_1\,\delta x_2 \cdots \delta x_{i-1}\,\delta x_{i+1} \cdots \delta x_n$$

(in which the M_i are functions of $x_1, x_2, \ldots, x_n, t$): That means that the expression:

([1]) H. POINCARÉ, *Les Méthodes nouvelles de la Mécanique céleste*, t. III, Chap. II; P. APPELL, *Traité de Mécanique rationelle*, t. II, pp. 462.

$$(12) \qquad \sum_i M_i \begin{vmatrix} \delta_1 x_1 & \delta_1 x_2 & \cdots & \delta_1 x_{i-1} & \delta_1 x_{i+1} & \cdots & \delta_1 x_n \\ \delta_2 x_1 & \cdots & \cdots & \delta_2 x_{i+1} & \delta_2 x_{i+1} & \cdots & \delta_2 x_n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \delta_{n-1} x_1 & \cdots & \cdots & \delta_{n-1} x_{i-1} & \delta_{n-1} x_{i+1} & \cdots & \delta_{n-1} x_n \end{vmatrix}$$

in which $\delta_1, \delta_2, \ldots, \delta_{n-1}$ are $n - 1$ different symbols for differentials, is an integral that depends upon n neighboring solutions. The differentials δ must be taken without varying t, so we can take:

$$\delta_\alpha x_\beta = \frac{\partial x_\beta}{\partial \xi_\alpha} \, \delta\xi_\alpha .$$

The expression (12) will then become:

$$\sum_i M_i \frac{D(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l)}{D(\xi_1, \xi_2, \ldots, \xi_{n-1})} \delta\xi_1 \, \delta\xi_2 \cdots \delta\xi_{n-1} .$$

When one suppresses the *constant* factor $\delta\xi_1 \, \delta\xi_2 \cdots \delta\xi_{n-1}$ and takes into account the relations (β), one will see that:

$$\sum_i \frac{(-1)^i}{\Delta} M_i \frac{\partial \xi_n}{\partial x_i}$$

[in which $\xi_n = \varphi_n(x_1, x_2, \ldots, x_n, t)$ represents an *arbitrary* integral of the system (11)] is a multiplier. That theorem is due to Koenigs ([1]), who established it upon supposing that the M_i and the X_i are independent of t.

If one sets:

$$B(\theta) = \sum_i (-1)^i M_i \frac{\partial \theta}{\partial x_i}$$

then one can, with Koenigs, deduce that if α and β are two integrals of (11) that do not annul $B(\theta)$ then the function:

$$\frac{B(\beta)}{B(\alpha)}$$

will once more be an integral. Indeed, it is the quotient of two multipliers.

16. – Now suppose that one knows an integral invariant of order $n - 2$ for equations (11):

([1]) G. KOENIGS, "Sur les invariants intégraux," Comptes rendus, 6 January 1896.

$$\overbrace{\iint\cdots\int}^{n-2} \sum_i M_i \, \delta x_1 \, \delta x_2 \cdots \delta x_{i-1} \, \delta x_{i+1} \cdots \delta x_{k-1} \, \delta x_{k+1} \cdots \delta x_n \ .$$

An argument that is entirely similar to the foregoing one will show that the expression:

$$\sum_{ik} (-1)^{i+k} M_{ik} \frac{D(\xi_{n-1}, \xi_n)}{D(x_i, x_k)} \tag{13}$$

[in which ξ_{n-1} and ξ_n are two arbitrary integrals of the system (11)] is a multiplier. If one possesses three integrals then one can form two distinct multipliers whose quotient will give a new integral.

The generalization of that theorem suggests itself immediately: If one possesses an integral invariant of order $n - p$ and p integrals for equations (11) then one can form a multiplier. That theorem was proved in a different way by De Donder (*Circolo di Palermo*, t. XV, 1901).

17. – I now suppose that the X in equations (11) *do not depend upon t* and that the simple integral:

$$\int \mu_1 \, \delta x_1 + \mu_2 \, \delta x_2 + \cdots + \mu_n \, \delta x_n$$

is a first-order integral invariant. That means that the sum:

$$\mu_1 \, \delta x_1 + \mu_2 \, \delta x_2 + \ldots + \mu_n \, \delta x_n$$

is an integral when the δx_i are a system of solutions to the equations *of variation* ([1]) of the system (11). Now, the X that do not depend upon t constitute one such system of solutions; therefore:

$$\mu_1 X_1 + \mu_2 X_2 + \ldots + \mu_n X_n$$

is an integral of equations (11) ([2]).

If $\varphi(x_1, x_2, \ldots, x_n, t)$ is an integral then we will have the first-order invariant:

$$\int \frac{\partial \varphi}{\partial x_1} \delta x_1 + \frac{\partial \varphi}{\partial x_2} \delta x_2 + \cdots + \frac{\partial \varphi}{\partial x_n} \delta x_n \ ,$$

and as a result, the integral:

$$\frac{\partial \varphi}{\partial x_1} X_1 + \frac{\partial \varphi}{\partial x_2} X_2 + \cdots + \frac{\partial \varphi}{\partial x_n} X_n \ .$$

However, since φ is an integral, we will have:

([1]) H. POINCARÉ, *Les Méthodes nouvelles de la Mécanique céleste*, t. I, page 162.
([2]) *Ibid.*, t. III, Chap. I.

(13, *cont.*)
$$\frac{\partial\varphi}{\partial x_1}X_1+\frac{\partial\varphi}{\partial x_2}X_2+\cdots+\frac{\partial\varphi}{\partial x_n}X_n=0$$

identically, and as a result the preceding integral will be nothing but $-\partial\varphi/\partial t$.

Therefore, when the X do not depend upon t, *if* φ *is an integral then* $\partial\varphi/\partial t$ *will be another one* ([1]). Similarly, $\frac{\partial^2\varphi}{\partial t^2}$, $\frac{\partial^3\varphi}{\partial t^3}$, … are integrals. (Cf., P. APPELL, *Traité de Mécanique rationelle*, t. II, pp. 419.)

18. – We apply the preceding results to a canonical system of equations:

(14)
$$\frac{dx_i}{\dfrac{\partial F}{\partial y_i}}=\frac{dy_i}{-\dfrac{\partial F}{\partial x_i}}=dt\qquad(i=1,2,\ldots,n),$$

in which the function F depends upon the x, the y, and t. One knows, *a priori*, an integral invariant I_{2k} of each even order $2k$ for those equations. Those invariants are:

$$I_2=\iint\sum_i\delta x_i\,\delta y_i\,,$$
$$I_4=\iiiint\sum_{ik}\delta x_i\,\delta y_i\,\delta x_k\,\delta y_k\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$I_{2n}=\iint\cdots\int\delta x_1\,\delta y_1\,\delta x_2\,\delta y_2\cdots\delta x_n\,\delta y_n\,.$$

The invariant I_{2n} tells us that 1 is a multiplier for equations (14), which is well-known. The invariant I_{2n-2}, to which one applies the theorem in no. **16**, give us the integral (13), which is:

([1]) That remark is entirely obvious, moreover. Indeed, upon differentiating the identity (13, *cont.*) with respect to t, one will get:

$$\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial^2\varphi}{\partial x_1\,\partial t}X_1+\frac{\partial^2\varphi}{\partial x_2\,\partial t}X_2+\cdots+\frac{\partial^2\varphi}{\partial x_n\,\partial t}X_n=0\,,$$

which proves that $\partial\varphi/\partial t$ is an integral.

More generally, I suppose that the X depend upon the x and t, but that all of the X satisfy *the same* linear, homogeneous, first-order partial differential equation *with constant coefficients:*

$$\mathcal{F}(X_i)=\frac{\partial X_i}{\partial t}+a_1\frac{\partial X_i}{\partial x_1}+a_2\frac{\partial X_i}{\partial x_2}+\cdots+a_n\frac{\partial X_i}{\partial x_n}=0\qquad(i=1,2,\ldots,n).$$

Hence, if φ = const. is one integral of the system (11) then $\mathcal{F}(\varphi)$ = const. will be another one.

Similarly, if all of the X except X_1 satisfy $\mathcal{F}(X_i)=0$, and if one has an integral φ that does not depend upon x_1, then one will once more have the integral $\mathcal{F}(\varphi)$.

$$\sum_i \frac{D(\varphi_1, \varphi_2)}{D(x_i, y_i)}$$

here, in which φ_1 and φ_2 denote two arbitrary integrals of the system (14): That is Poisson's theorem. Similarly, the invariant I_{2n-4} will tell is that if φ_1, φ_2, φ_3, φ_4 are four integrals then the expression:

$$\sum_{ik} \frac{D(\varphi_1, \varphi_2, \varphi_3, \varphi_4)}{D(x_i, y_i, x_k, y_k)}$$

is once more an integral, and so on. That is the generalization of Poisson's theorem that Poincaré gave (*Méthodes Nouvelles*, t. III, pp. 43) and that we recalled above.

Now suppose that equations (14) have been integrated, so the x_i and y_i are found to have been expressed as functions of t and $2n$ integration constants $a_1, a_2, a_3, \ldots, a_{2n}$. The invariant I_2 tells us that the sum:

$$\sum_i (\delta x_i \, \delta' y_i - \delta' x_i \, \delta y_i)$$

will be an integral when the δx_i, δy_i and the $\delta' x_i$, $\delta' y_i$ are two systems of solutions to the equations of variation of the system (14). If we take, for example:

$$\delta x_i = \frac{\partial x_i}{\partial a_1} \delta a_1, \qquad \delta y_i = \frac{\partial y_i}{\partial a_1} \delta a_1,$$

$$\delta' x_i = \frac{\partial x_i}{\partial a_2} \delta a_2, \qquad \delta' y_i = \frac{\partial y_i}{\partial a_2} \delta a_2$$

then we can say that the expression:

$$[a_1, a_2] = \sum_i \left(\frac{\partial x_i}{\partial a_1} \frac{\partial y_i}{\partial a_2} - \frac{\partial x_i}{\partial a_2} \frac{\partial y_i}{\partial a_1} \right)$$

is an integral: That is the theorem about *Lagrange brackets*. (Cf., no. **10**.)

The invariant I_4 likewise tells us that:

$$\sum_{ik} \frac{D(x_i, y_i, x_k, y_k)}{D(a_1, a_2, a_3, a_4)}$$

is an integral, and so on.

19. – I shall once more indicate a property of the canonical system (14): I suppose that this system admits the first-order integral invariant:

$$\int \sum_i M_i \, \delta x_i + N_i \, \delta y_i \, . \tag{15}$$

One can prove that the (n – 1)-fold integral:

$$\overbrace{\iint\cdots\int}^{n-1}\sum_i N_i\,\delta\omega_{x_i}+M_i\,\delta\omega_{y_i}$$

is also an integral invariant: $\delta\omega_{x_i}$ represents the product of the differentials:

$$\delta x_1\ \delta y_1\ \delta x_2\ \delta y_2\ \ldots\delta x_n\ \delta y_n\ ,$$

in which one has suppressed the factor δx_i , while $\delta\omega_{y_i}$ represents the product, but with the factor δy_i missing.

Apply Koenigs's theorem (no. **15**) to the last invariant of order n – 1 . If φ is an integral then the expression:

$$\sum_i -N_i\,\frac{\partial\varphi}{\partial x_i}+M_i\,\frac{\partial\varphi}{\partial y_i} \tag{16}$$

will be a multiplier, and as a result, it will be an integral (since its quotient with the multiplier 1 is an integral). Therefore, in the case of canonical equations, knowing a first-order invariant and an integral will permit one to write down a new integral.

Suppose that φ_1 is an integral other than φ, so the first-order integral (15) that we start with is:

$$\int\sum_i\frac{\partial\varphi_1}{\partial x_i}\delta x_i+\frac{\partial\varphi_1}{\partial y_i}\delta y_i\ .$$

The integral (16) will then become:

$$\sum_i\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi_1}{\partial y_i}-\frac{\partial\varphi}{\partial y_i}\frac{\partial\varphi_1}{\partial x_i}\,.$$

We thus recover Poisson's theorem.

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