“Notice sur la loi que la population suit dans son accroissement,” Corr. math. phys. **10** (1838), 113-121.

Notice on the law that a population follows in its growth

By P.-F. VERHULST

Translated by D. H. Delphenich

One knows that the celebrated *Malthus* has established the principle that the human population *tends* to grow in a geometric progression, in such a manner as to double after a certain period of time – for example, every twenty-five years. That proposition is uncontestable if one abstracts from the difficulty that always increases in regard to procuring the means to subsist when the population has acquired a certain degree of concentration, or the resources that the population draws upon in its growth, even when the society is still nascent, such that there is a greater division of work, the existence of a regular government and the means of defense in order to ensure the public tranquility, etc.

Indeed, *all things being equal, moreover,* if a thousand souls become two thousand after twenty-five years then those two thousand will become four thousand after the same lapse of time.

In our old European societies, where the good lands have been cultivated for a long time, the work that is required to improve a terrain that has already been developed can add only some ever-decreasing quantities to its yield. Upon assuming that one has doubled the yield of the soil in the first twenty-five years, one will have scarcely made it produce perhaps a third more in the second period. The virtual growth of the population thus finds a limit in the area and fertility of the country, and the population will consequently tend to become more and more stationary.

In truth, the same thing is not true in certain, purely exceptional, cases. For example, when a civilized people cultivates a fertile territory that was hitherto uninhabited, or when it practices an industry that gives great temporary benefits. A large family will then become a source of wealth, and the second generation will find that it is easier to get established than the first one, since it does not have to struggle against the obstacles that the savage state of the earth offered the first colonists.

In order to judge the rate at which the population grows in a given country, one must divide the growth in the population over each year by the population that provided it. Since that ratio is independent of the absolute value of the population, it can be regarded as the measure of that rate; if it is increasing then the progression will be more than geometric, and if it is decreasing then it will be less than geometric.

One can make various hypotheses in the resistance or the sum of the obstacles that are opposed to the indefinite development of the population. *Quetelet* supposed that it was proportional *to the square of the rate at which the population tends to grow* ([1]).

([1]) *Essai de physique sociale,* v. 1, pp. 277.

This equates the motion of the population to that of a moving body that falls while traversing a resistant medium. The results of that comparison agree in a satisfactory manner with the ones that are given by statistics and with the ones that I obtained from own formulas when I assumed that there is an indefinitely-increasing density in the layers of the medium being traversed.

The increase in the population necessarily has a limit that is defined by the ground area that is available for the housing of the population. When a nation has consumed all of the fruits of its fields, it can, in truth, procure sustenance from the outside by the exchange of its other products, and thus support a new growth in the population. However, it is obvious that these importations must have limits, and will stop long before the entire surface of the country has been converted into cities. All of the formulas by which one attempts to represent the law of population must then satisfy the condition that they admit a *maximum* that is attained only after an infinitely-extended epoch. That *maximum* will be the population count after it has become stationary.

For some time, I have attempted to determine the most likely law of the population by analysis. However, I have abandoned that kind of research because the observational givens are much too numerous for the formulas to be verified in a manner that leaves no doubt about their exactitude. Nevertheless, since the path that I have followed seems to necessarily lead to the knowledge of the true law when one has enough givens, and the results to which I arrived might offer some interest – at least, as the object of speculation – I believe that I must defer to the invitation of *Quetelet* and make them public.

Let p be the population; represent the infinitely-small increase that it acquires during the infinitely-short time dt by dp. If the population grows in a geometric progression then we will have the equation $dp / dt = mp$. However, since the rate of growth in the population is retarded by the increase itself of the number of inhabitants, we must subtract an unknown function of p from mp, in such a manner that the formula to be integrated will become:

$$\frac{dp}{dt} = mp - \varphi(p).$$

The simplest hypothesis that one can make about the form of the function φ is to suppose that $\varphi(p) = n\,p^2$. One will then find that the integral of the equation above is:

$$t = \frac{1}{m}[\log p - \log(m - np)] + \text{constant},$$

and three observations will suffice to determine the two constant coefficients m and n and the arbitrary constant.

Upon solving the last equation for p, one will get:

$$p = \frac{mp' e^{mt}}{np' e^{mt} + m - np'}, \tag{1}$$

upon denoting the population that refers to $t = 0$ by p' and the base of the Napierian logarithm by e. If one makes $t = \infty$ then one will see that the corresponding value of p is $P = m / n$. That will then be the *upper limit of the population*.

Instead of assuming that $\varphi(p) = np^2$, one can take $\varphi(p) = np^\alpha$, in which α is arbitrary, or $\varphi(p) = n \log p$. All of these hypotheses likewise satisfy the observed facts quite well; however, they will give very different values for the upper limit of the population.

I supposed, in succession, that:

$$\varphi(p) = np^2, \quad \varphi(p) = np^3, \quad \varphi(p) = np^4, \quad \varphi(p) = n \log p,$$

resp., and that the differences between the calculated populations and the ones that were provided by observation were reasonably the same.

If the population grows in a progression that is more than geometric then the term $-\varphi(p)$ will become $+\varphi(p)$. The differential equation is then integrated as in the preceding case, but one imagines that there can no longer be a *maximum* population.

I have calculated the table that follows from formula (1). The figures for France, Belgium, and the county of Essex have been extracted from official documents. The ones that is concerned with Russia are found in the book by Sadler, *Law of population*, and I cannot guarantee their authenticity, since I am ignorant of the manner by which they were obtained. I was able to extend the tables for France and Belgium up to 1837, and thus verified my formula; however, my occupation does not leave me any time for leisure. My work ended in 1833, and I have not touched it since then.

I shall remark in passing that the table that is concerned with France seems to say that the formula becomes especially exact when the observations pertain to the greatest possible numbers and have been made with the most care. As for the rest, only the future will reveal to us the true mode of action of the retarding force that we have represented by $\varphi(p)$.

Table of population growth in France from 1817 to 1831, from the yearbook for 1834.

Year	From the vital statistics	From the formula	Proportional error	Logarithm of the calculated population
1817	29,981,336 195,902	29,981,336 208,281	0.0000	7.4768490
1818	30,177,238 161,948	30,189,500 204,500	+ 0.0004	7.4798565
1819	30,339,186	30,394,000	0.0018	7.4827875

	199,863	200,500		
1820	30,539,049 188,227	30,594,500 197,300	+ 0.0018	7.4856461
1821	30,727,276 212,144	30,791,800 192,700	+ 0.0021	7.4884310
1822	30,939,420 198,634	30,984,500 189,500	+ 0.0014	7.4911453
1823	31,138,054 221,286	31,174,000 185,223	+ 0.0012	7.4937907
1824	31,359,340 220,546	31,359,340 182,777	0.0000	7.4963719
1825	31,579,886 175,974	31,542,000 178,000	– 0.0012	7.5988859
1826	31,755,860 157,533	31,720,000 175,000	– 0.0011	7.5013366
1827	31,913,393 189,071	31,895,000 172,000	– 0.0005	7.5037257
1828	31,102,464 139,402	32,067,000 168,000	– 0.0011	7.5060547
1829	32,241,866 161,074	32,235,000 164,500	– 0.0002	7.5083251
1830	32,402,940 157,994	32,399,500 161,434	0.0000	7.5105385
1831	32,560,934	32,560,934	0.0000	7.5126965
1 Jan	(most recent data)			

Population of Belgium

Year	From the vital statistics	From the formula	Error
1815	3,494,985 33,465	3,494,985 35,315	0.000
1816	3,528,450 38,104	3,530,300 35,500	0.000
1817	3,566,554 15,329	3,565,800 35,500	0.000
1818	3,581,883 26,708	3,601,300 35,600	+ 0.005
1819	3,608,591 37,303	3,636,900 35,700	+ 0.008
1820	3,645,894 30,774	3,672,600 35,800	+ 0.007
1821	3,676,668 45,200	3,708,400 35,800	+ 0.008
1822	3,721,868 47,858	3,744,200 36,000	+ 0.006
1823	3,769,726 46,523	3,780,200 36,049	+ 0.003
1824	3,816,249 50,828	3,816,249 36,050	0.000
1825	3,867,077 46,780	3,852,299 36,001	– 0.004
1826	3,913,857 42,661	3,888,300 36,100	– 0.006
1827	3,956,518 38,462	3,924,400 36,200	– 0.008

1828	3,994,980 46,519	3,930,600 36,200	– 0.007
1929	4,041,499 33,213	3,996,800 36,300	– 0.006
1830	4,074,712 22,178	4,033,100 36,300	0.000
1831	4,096,890 33,231	4,069,400 36,400	– 0.008
1832	4,130,121 12,136	4,105,800 36,457	– 0.011
1833	4,142,257	4,142,257	– 0.010
1 Jan.	(most recent data)		

Table of the growth of the population of Essex county in England from 1811 to 1831.

Year	**From the parish registers**	**From the formula**	**Proportional error**
1811	252,473	252,473	0.000
1812	255,410	256,600	+ 0.004
1813	258,393	260,500	+ 0.008
1814	262,705	264,400	+ 0.006
1815	266,143	268,300	+ 0.008
1816	270,270	272,100	+ 0.006
1817	274,088	275,650	+ 0.005

1818	278,513	279,300	+ 0.002
1819	282,232	282,700	+ 0.001
1820	285,797	286,100	+ 0.001
1821	289,424	289,424	0.000
1822	293,085	292,600	– 0.001
1823	296,436	295,750	– 0.002
1824	299,166	298,800	– 0.001
1825	302,302	301,600	– 0.002
1826	304,482	304,570	0.000
1827	306,474	307,300	+ 0.002
1828	308,877	309,800	+ 0.003
1829	311,807	312,400	+ 0.002
1830	314,306	314,900	+ 0.002
1831	317,233	317,233	0.000

Population of Russia (Individuals of the Greek communion)

Year	From the registers of the Greek church	From the formula
1796	29,177,980 461,521	29,177,980
1797	29,639,501 461,525	30,332,000
1798	30,101,026 428,248	31,424,000

1799	30,529,274 432,418	32,456,000
1800	30,961,692 440,000	33,350,000
1801	31,401,692 453,205	34,338,000
1802	31,854,897 616,097	35,191,000
1803	32,470,994 475,372	35,988,000
1804	32,470,994 568,469	36,731,000
1805	33,514,835 542,701	37,423,000
1806	34,057,536 500,662	38,065,000
1807	34,558,198 468,508	38,661,000
1808	35,026,706 462,478	39,213,000
1809	35,489,184 466,712	39,723,000
1810	35,955,896 471,546	40,195,000
1811	36,427,442 369,779	40,630,000
1812	36,797,221 293,033	41,031,000
1813	37,090,253 (dimin.) 2,740	41,401,000

1814	37,087,514 389,255	41,741,000
1815	37,476,769 407,473	42,055,000
1816	37,884,242 637,247	42,342,000
1817	38,521,489 670,045	42,606,000
1818	39,191,534 556,441	42,849,000
1819	39,747,945 603,025	43,071,000
1820	39,747,975 603,025	43,276,000
1821	41,013,719 600,591	43,463,000
1822	41,614,310 562,735	43,634,000
1823	42,177,045 663,345	43,791,000
1824	42,840,390 713,285	43,935,000
1825	43,553,675 633,405	44,067,000
1826	44,187,080 450,386	44,187,000
1827	44,637,466	