CHAPTER VIII

CONGRUENCES OF LINES AND CORRESPONDENCES BETWEEN TWO SURFACES

New representation of congruences

1. – In the foregoing, we defined a congruence by giving its support and the direction of the line or lines (D) that pass through each point of the support. More generally (and this would be preferable from the projective standpoint), one can consider two support surfaces that correspond point-by-point, while the lines of the congruence are the ones that join the homologous points of two surfaces. In reality, the contact elements of the two surfaces correspond to each other, and at the same time that one considers the congruence of lines that join the homologous points, one can consider the congruence of intersections of the homologous tangents planes.

It is natural then to employ homogeneous coordinates. Let M (x, y, z, t) and M_1 (x_1, y_1, z_1, t_1) be the homologous points on the two surfaces; the congruence will defined by the equations:

$$X = x + \rho\, x_1, \quad Y = y + \rho\, y_1, \quad Z = z + \rho\, z_1, \quad T = t + \rho\, t_1 .$$

Similarly, let u, v, w, r be the tangential coordinates of a tangent plane to the first surface, and let u_1, v_1, w_1, r_1 be those of the homologous tangent plane to the second surface. The congruence will be defined from the tangential viewpoint by the equations:

$$U = u + \rho\, u_1, \quad V = v + \rho\, v_1, \quad W = w + \rho\, w_1, \quad R = r + \rho\, r_1 .$$

Let (S), (S_1) be the two support surfaces. Since the conjugate systems on those surfaces are invariant under any projective transformation, from their very definition, we will be led to study the relations that exist between them. Let:

$$(S) \qquad x = f(\lambda, \mu), \quad y = g(\lambda, \mu), \quad z = h(\lambda, \mu), \quad t = k(\lambda, \mu),$$
$$(S_1) \qquad x_1 = f_1(\lambda, \mu), \quad y_1 = g_1(\lambda, \mu), \quad z_1 = h_1(\lambda, \mu), \quad t_1 = k_1(\lambda, \mu)$$

be the coordinates of the current and homologous points of the two surfaces, respectively.

The choice of parameters λ, μ is fixed by the following theorem:

When two surfaces (S), (S_1) correspond point-by-point, there will exist a conjugate net on (S) that corresponds to a conjugate net on (S_1), and in general, there will exist only one of them.

Indeed, let $\delta\lambda$, $\delta\mu$, and $\delta'\lambda$, $\delta'\mu$ be the infinitesimal variations of the parameters that correspond to the directions of the two curves of a conjugate net that cross at a point (λ, μ) of (S). Those directions are harmonic conjugates with respect to the asymptotic directions that are defined by the variations $d\lambda$, $d\mu$ that satisfy the equation:

(1) $$E'd\lambda^2 + 2F'd\lambda \cdot d\mu + G'd\mu^2 = 0.$$

Therefore, upon interpreting the variations $d\lambda$, $d\mu$; $\delta\lambda$, $\delta\mu$; $\delta'\lambda$, $\delta'\mu$ as the homogeneous coordinates of the various points of a line, the condition that expresses the idea that the directions that are defined on (S) by $\delta\lambda$, $\delta\mu$; $\delta'\lambda$, $\delta'\mu$ are conjugate will be interpreted thus: The two points ($\delta\lambda$, $\delta\mu$), ($\delta'\lambda$, $\delta'\mu$) are harmonic conjugates with respect to the pair of points that is defined by equation (1).

Similarly, two conjugate directions on (S_1) are harmonic conjugates with respect to the directions:

(2) $$E_1'd\lambda^2 + 2F_1'd\lambda \cdot d\mu + G_1'd\mu^2 = 0,$$

and in order for $\delta\lambda$, $\delta\mu$; $\delta'\lambda$, $\delta'\mu$ to define two such directions, from the preceding interpretation, it is necessary and sufficient that the two points ($\delta\lambda$, $\delta\mu$), ($\delta'\lambda$, $\delta'\mu$) must be harmonic conjugates with respect to the pair of points that are defined by equation (2).

Looking for a common conjugate system then amounts to looking for a pair of points that are harmonic conjugate with respect to the two pairs that are given by two quadratic equations (1) and (2). If the two quadratic forms have no common factor then there will be one and only one pair that answers the question, which will be the pair of double points of the involution that is defined by the two pairs (1) and (2). Now, the preceding two equations define asymptotic lines on the two surfaces. Therefore, if two surfaces correspond point-by-point in such a fashion that one does not have a family of asymptotes on (S) that corresponds to a family of asymptotes on (S_1) then there will exist one and only one conjugate system on (S) that corresponds to a conjugate system (S_1), and it will be defined by the equation:

$$\begin{vmatrix} E'd\lambda + F'd\mu & F'd\lambda + G'd\mu \\ E_1'd\lambda + F_1'd\mu & F_1'd\lambda + G_1'd\mu \end{vmatrix} = 0.$$

Its existence will be impossible if the forms (1) and (2) have one common factor, and it will be indeterminate if two factors are common; i.e., if the asymptotic lines correspond on the two surfaces. Discarding that exceptional case, we suppose that the parameters λ, μ correspond to the common conjugate system.

Use of homogeneous coordinates

2. – We shall recall the usual formulas and see what they will become in homogeneous coordinates.

A *curve* is defined by four equations in homogeneous coordinates:

$$x = f(\lambda), \qquad y = g(\lambda), \qquad z = h(\lambda), \qquad t = k(\lambda).$$

The tangent to the point M (x, y, z, t) joins the point M to the point M' whose coordinates are dx, dy, dz, dt, because the point at infinity whose homogeneous coordinates are:

$$d\left(\frac{x}{t}\right) = \frac{1}{t}dx + x\,d\frac{1}{t}, \quad d\left(\frac{y}{t}\right) = \frac{1}{t}dy + y\,d\frac{1}{t}, \quad d\left(\frac{z}{t}\right) = \frac{1}{t}dz + z\,d\frac{1}{t}, \quad 0 = \frac{1}{t}dt + t\,d\frac{1}{t}$$

is indeed on the line thus-defined. The osculating plane passes through the line *MM′* and the point *M″* with coordinates: d^2x, d^2y, d^2z, d^2t, because the point at infinity whose homogeneous coordinates are:

$$d^2\left(\frac{x}{t}\right) = \frac{1}{t}d^2x + 2dx\cdot d\frac{1}{t} + x\,d^2\frac{1}{t}, \quad d^2\left(\frac{y}{t}\right) = \ldots, \quad d^2\left(\frac{z}{t}\right) = \ldots,$$
$$0 = \frac{1}{t}dt^2 + 2dt\cdot d\frac{1}{t} + t\cdot d^2\frac{1}{t}$$

is indeed in the plane thus-defined.

Correlatively, it results from the classical theory of envelopes that the *developable* that is enveloped by the plane (*P*) with coordinates:

$$u = f(\lambda), \qquad v = g(\lambda), \qquad w = h(\lambda), \qquad r = k(\lambda)$$

will have the intersection of the plane (*P*) and the plane (*P′*) with coordinates *du*, *dv*, *dw*, *dr* for its generator. The contact point with the edge of regression will be in the plane (*P″*) with coordinates d^2u, d^2v, d^2w, d^2r.

An arbitrary *surface* will be defined point-wise by the equations:

(1) $$x = f(\lambda, \mu), \quad y = g(\lambda, \mu), \quad z = h(\lambda, \mu), \quad t = k(\lambda, \mu),$$

and from the tangential viewpoint by the equations:

(2) $$u = F(\lambda, \mu), \quad v = G(\lambda, \mu), \quad w = H(\lambda, \mu), \quad r = K(\lambda, \mu).$$

We seek to define the *tangent plane* by starting with the point-wise equations (1). That plane contains the point, so:

$$\sum u\,x = 0.$$

It contains the tangents to the curves λ = const., μ = const., and thus, the points $\left(\frac{\partial x}{\partial \mu}, \frac{\partial y}{\partial \mu}, \frac{\partial z}{\partial \mu}, \frac{\partial t}{\partial \mu}\right)$ and $\left(\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}, \frac{\partial z}{\partial \lambda}, \frac{\partial t}{\partial \lambda}\right)$; hence, one has the conditions:

$$\sum u\frac{\partial x}{\partial \lambda} = 0, \quad \sum u\frac{\partial x}{\partial \mu} = 0.$$

We then have three equations that define quantities that are proportional to *u*, *v*, *w*, *r*. The point-wise equation for the tangent plane at the point (*x*, *y*, *z*, *t*) will then be:

$$\begin{vmatrix} X & Y & Z & T \\ x & y & z & t \\ \dfrac{\partial x}{\partial \lambda} & \dfrac{\partial y}{\partial \lambda} & \dfrac{\partial z}{\partial \lambda} & \dfrac{\partial t}{\partial \lambda} \\ \dfrac{\partial x}{\partial \mu} & \dfrac{\partial y}{\partial \mu} & \dfrac{\partial z}{\partial \mu} & \dfrac{\partial t}{\partial \mu} \end{vmatrix} = 0.$$

Correlatively, one will define a *point* of the surface upon starting from the tangential equations (2) by means of the conditions:

$$\sum u\,x = 0, \qquad \sum x\frac{\partial u}{\partial \lambda} = 0, \qquad \sum x\frac{\partial u}{\partial \mu} = 0.$$

By definition, *one defines one of the elements – viz., point or tangent plane – as a function of the other one by means of the formulas:*

(3) $$\sum u\,x = 0, \qquad \sum u\,dx = 0, \qquad \sum x\,du = 0.$$

We now propose to *express the idea that the two directions MT* ($d\lambda$, $d\mu$) and *MS* ($\delta\lambda$, $\delta\mu$) *are conjugate.* Those directions will be conjugates if the line *MS* is the characteristic of the tangent plane as the contact point of the tangent plane displaces. Now, that characteristic is defined by the equations:

$$\sum u\,X = 0, \qquad \sum X\,du = 0,$$

while the line *MS* is defined by the point (x, y, z, t) and the point (δx, δy, δz, δt). In order to express the idea that *MS* is the characteristic, one must express the idea that those two points are on the characteristic, which will give:

$$\begin{aligned} &\sum u\,x = 0, && \sum x\,du = 0, \\ &\sum u \cdot \delta x = 0, && \sum du \cdot \delta x = 0. \end{aligned}$$

From formulas (3), the first three equations are verified for any tangent direction ($\delta\lambda$, $\delta\mu$) and for any characteristic direction ($d\lambda$, $d\mu$); *we will then get the single condition:*

(4) $$\sum du \cdot \delta x = 0$$

or the equivalent symmetric condition:

(4′) $$\sum \delta u \cdot dx = 0,$$

which one will obtain by an analogous calculation upon changing the role of the two directions. In particular, we will find the condition for a direction to be conjugate to itself; i.e., for it to be an *asymptotic direction:*

(5) $$\sum du \cdot dx = 0.$$

Having said that, *we express the idea that the curves* λ = const, μ = const. *define a conjugate net.* Here, the equivalent conditions (4), (4′) will give:

(6) $$\sum \frac{\partial u}{\partial \lambda} \cdot \frac{\partial x}{\partial \mu} = 0,$$

(6′) $$\sum \frac{\partial u}{\partial \mu} \cdot \frac{\partial x}{\partial \lambda} = 0.$$

Those conditions can be transformed. When the identity equation:

$$\sum u \frac{\partial x}{\partial \mu} = 0$$

is differentiated with respect to λ, it will give, in fact:

$$\sum \frac{\partial u}{\partial \lambda} \cdot \frac{\partial x}{\partial \mu} + \sum u \frac{\partial^2 x}{\partial \lambda \partial \mu} = 0,$$

and (6) will be written:

(7) $$\sum u \frac{\partial^2 x}{\partial \lambda \partial \mu} = 0.$$

Upon starting with one of the relations:

$$x \frac{\partial u}{\partial \lambda} = 0, \qquad \sum x \frac{\partial u}{\partial \mu} = 0,$$

one will likewise get the necessary and sufficient conditional relation that:

(7′) $$\sum x \frac{\partial^2 u}{\partial \lambda \partial \mu} = 0.$$

Equations (7), (7′) depend upon point-like and tangential elements simultaneously. Upon expressing u, v, w, r as functions of x, y, z, t, and their derivatives, one will obtain the condition in point-like coordinates:

(8) $$\left| x \quad \frac{\partial x}{\partial \lambda} \quad \frac{\partial x}{\partial \mu} \quad \frac{\partial^2 x}{\partial \lambda \partial \mu} \right| = 0.$$

In the relation (8), the left-hand side represents an abbreviation for the determinant whose first row is the row that is written between the two vertical lines, and whose other three rows are deduced from it by replacing x with y, z, t, respectively. *That notation will be currently employed in what follows.*

When t = const., the condition (5) will reduce to the known condition:

$$\begin{vmatrix} \dfrac{\partial x}{\partial \lambda} & \dfrac{\partial x}{\partial \mu} & \dfrac{\partial^2 x}{\partial \lambda \partial \mu} \end{vmatrix} = F' = 0.$$

The condition (8) can be interpreted thus: There exists the same homogeneous, linear relationship between the corresponding elements of the rows, so there will exist functions L, M, N of λ and μ, such that one will have identically:

$$\frac{\partial^2 x}{\partial \lambda \partial \mu} = L\frac{\partial x}{\partial \lambda} + M\frac{\partial x}{\partial \mu} + Nx,$$

$$\frac{\partial^2 y}{\partial \lambda \partial \mu} = \ldots,$$

$$\frac{\partial^2 z}{\partial \lambda \partial \mu} = \ldots,$$

$$\frac{\partial^2 t}{\partial \lambda \partial \mu} = \ldots;$$

i.e.: *the four homogeneous coordinates x, y, z, t satisfy the same linear partial differential equation of the form:*

$$\frac{\partial^2 \varphi}{\partial \lambda \partial \mu} = L\frac{\partial \varphi}{\partial \lambda} + M\frac{\partial \varphi}{\partial \mu} + N\varphi.$$

Upon operating from the tangential viewpoint, one will likewise see that *the condition* (7′), *which can be written:*

$$\begin{vmatrix} u & \dfrac{\partial u}{\partial \lambda} & \dfrac{\partial u}{\partial \mu} & \dfrac{\partial^2 u}{\partial \lambda \partial \mu} \end{vmatrix} = 0,$$

with a notation that is analogous to the one that was just introduced, expresses the idea that u, v, w, r are integrals of the same partial differential equation of the form:

$$\frac{\partial^2 \Phi}{\partial \lambda \partial \mu} = P\frac{\partial \Phi}{\partial \lambda} + Q\frac{\partial \Phi}{\partial \mu} + R\varphi.$$

One can effortlessly show that if x, y, z, t or u, v, w, r satisfy an equation of the preceding form then they will satisfy only one such equation.

Remark. – In Cartesian coordinates, one must suppose that $t = k(\lambda, \mu) \equiv 1$, and the preceding result will apply to point-like coordinates x, y, z upon setting $N = 0$.

Now consider a *ruled surface*. The equations of a generator that joins the point M (x, y, z, t) to the point M_1 (x_1, y_1, z_1, t_1) are:

$$X = x + \rho x_1, \quad Y = y + \rho y_1, \quad Z = z + \rho z_1, \quad T = t + \rho t_1.$$

Suppose that the surface is *developable*. The tangent planes to the points (x, y, z, t) and (x_1, y_1, z_1, t_1) are the same. Now, the tangent plane at M that passes through the generator and the tangent to the curve $\rho = 0$ will contain the point (dx, dy, dz, dt). Similarly, the tangent plane at M_1 will contain the point (dx_1, dy_1, dz_1, dt_1). The condition for the planes to coincide will then be:

$$|\, x \;\; x_1 \;\; dx \;\; dx_1 \,| = 0.$$

If we define the surface in tangential coordinates then we will likewise arrive at the condition:

$$|\, u \;\; u_1 \;\; du \;\; du_1 \,| = 0.$$

Finally, we pass on to the *congruences:* A congruence will once more be represented by the equations:

$$X = x + \rho x_1 , \quad Y = y + \rho y_1 , \quad Z = z + \rho z_1 , \quad T = t + \rho t_1 .$$

However, x, y, z, t and x_1, y_1, z_1, t_1, are functions of two arbitrary parameters (λ, μ) here. Let us look for its *focal elements.* Let F be one focus of a line (D) with parameters (λ, μ). Let ρ be the value that will give the coordinates of that point when it is substituted in the preceding equations. All of the ruled surfaces of the congruence that contain the line (D) will have the same tangent plane at the point F. In particular, consider the surfaces λ = const., μ = const. The tangent planes to the surfaces contain the points (x, y, z, t), (x_1, y_1, z_1, t_1), $\left(\frac{\partial x}{\partial \mu} + \rho \frac{\partial x_1}{\partial \mu}, \cdots\right)$ and (x, y, z, t), (x_1, y_1, z_1, t_1), $\left(\frac{\partial x}{\partial \lambda} + \rho \frac{\partial x_1}{\partial \lambda}, \cdots\right)$, respectively. The condition for those planes to coincide – i.e., the *equation of the focal points* – will then be:

$$\left| \; x \quad x_1 \quad \frac{\partial x}{\partial \lambda} + \rho \frac{\partial x_1}{\partial \lambda} \quad \frac{\partial x}{\partial \mu} + \rho \frac{\partial x_1}{\partial \mu} \; \right| = 0.$$

One will likewise find the *equation of the focal planes:*

$$\left| \; u \quad u_1 \quad \frac{\partial u}{\partial \lambda} + \rho \frac{\partial u_1}{\partial \lambda} \quad \frac{\partial u}{\partial \mu} + \rho \frac{\partial u_1}{\partial \mu} \; \right| = 0.$$

In the foregoing, we have supposed that the homogeneous coordinates are defined by the condition that the ratios x/t, y/t, z/t must be the corresponding Cartesian coordinates. One effortlessly verifies that the results obtained will apply to the more general coordinates that one deduces from them by an arbitrary homogeneous, linear transformation.

Special correspondences

3. – We shall study the *correspondence between two points M, M_1 of two surfaces, such that the developables of the congruence of lines MM_1 cut the two surfaces along the two conjugate nets that they correspond to.* For example, there are the ones that relate to the conjugate nets that are formed by their lines of curvature, so the correspondence is determined on two parallel surfaces by the congruence of their common normals. We suppose that the parameters λ, μ that fix the position of a point on each of the surfaces are precisely the ones that will make the homologous conjugate curves be λ = const. and μ = const. The curves λ = const. and μ = const. are conjugate on the first surface (S). Therefore, x, y, z, t satisfy [§ 2] the same partial differential equation:

$$\frac{\partial^2\varphi}{\partial\lambda\,\partial\mu} = P\frac{\partial\varphi}{\partial\lambda} + Q\frac{\partial\varphi}{\partial\mu} + R\varphi. \tag{1}$$

Similarly, the curves λ = const. and μ = const. are conjugate on the second surface (S_1), x_1, y_1, z_1, t_1 satisfy the same partial differential equation:

$$\frac{\partial^2\varphi}{\partial\lambda\,\partial\mu} = P_1\frac{\partial\varphi}{\partial\lambda} + Q_1\frac{\partial\varphi}{\partial\mu} + R_1\varphi. \tag{2}$$

Now, express the idea that the developables of the congruence correspond to λ = const. and μ = const. If we represent the congruence by the equations:

$$X = x + \rho\, x_1, \quad Y = y + \rho\, y_1, \quad Z = z + \rho\, z_1, \quad T = t + \rho\, t_1$$

then the developables will be given [§ 2] by the equation:

$$|\, x \quad x_1 \quad dx \quad dx_1 \,| = 0.$$

Now:

$$dx = \frac{\partial x}{\partial\lambda}d\lambda + \frac{\partial x}{\partial\mu}d\mu, \qquad dy = \ldots, \qquad dz = \ldots, \qquad dt = \ldots,$$

$$dx_1 = \frac{\partial x_1}{\partial\lambda}d\lambda + \frac{\partial x_1}{\partial\mu}d\mu, \qquad dy_1 = \ldots, \qquad dz_1 = \ldots, \qquad dt_1 = \ldots,$$

and the preceding equation must be verified for $d\lambda = 0$, $d\mu = 0$, so we get the conditions:

$$\left| \; x \quad x_1 \quad \frac{\partial x}{\partial\lambda} \quad \frac{\partial x_1}{\partial\lambda} \; \right| = 0, \tag{3}$$

$$\left| \; x \quad x_1 \quad \frac{\partial x}{\partial\mu} \quad \frac{\partial x_1}{\partial\mu} \; \right| = 0. \tag{4}$$

They express the idea that there exists the same linear, homogeneous relation between the line elements, so there exist factors A, B, A_1, B_1 ; C, D, C_1, D_1 such that one has the identities:

(5) $$Ax + B\frac{\partial x}{\partial \lambda} = A_1 x_1 + B_1\frac{\partial x_1}{\partial \lambda}, \text{ and analogous ones,}$$

(6) $$Cx + D\frac{\partial x}{\partial \mu} = C_1 x_1 + C_1\frac{\partial x_1}{\partial \mu}, \text{and analogous ones.}$$

First case. – Let us first see what happens if one of the four coefficients B, B_1, D, D_1 is zero. For example, let $B_1 = 0$. Equations (5) then express the idea that the point M_1 (x_1, y_1, z_1, t_1) is on the line that joins the points M (x, y, z, t) and $M'\left(\frac{\partial x}{\partial \lambda},\frac{\partial y}{\partial \lambda},\frac{\partial z}{\partial \lambda},\frac{\partial t}{\partial \lambda}\right)$. The line MM_1 is tangent to the curve μ = const. that is traced on the surface (S). All of the lines MM_1 are then tangent to the surface (S), which is one of the sheets of the focal surface of the congruence. The curves μ = const. on that focal surface (S) are the edges of regression of one of the families of developables of the congruence, and in turn, the curves λ = const., which are conjugate to the preceding ones, are the contact curves of the developables of the second family. We seek how one must define (S_1) in order for that surface to be cut along a conjugate net by the developables of the congruence. In the case in question, if one supposes that $A_1 = 0$ (as is legitimate) then equations (5) can be written:

$$x_1 = Ax + B\frac{\partial x}{\partial \lambda}, \qquad y_1 = Ay + B\frac{\partial y}{\partial \lambda}, \qquad z_1 = Az + B\frac{\partial z}{\partial \lambda}, \qquad t_1 = At + B\frac{\partial t}{\partial \lambda}.$$

Since homogeneous coordinates can be replaced with proportional quantities:

$$x = \theta X, \qquad y = \theta Y, \qquad z = \theta Z, \qquad t = \theta T,$$

if θ is a function of λ, μ then the preceding formulas will become:

$$x_1 = A\theta X + B\left(\theta\frac{\partial X}{\partial \lambda} + X\frac{\partial \theta}{\partial \lambda}\right), \qquad y_1 = \ldots, \qquad z_1 = \ldots, \qquad t_1 = \ldots$$

Determine the function θ by the condition:

$$A\theta + B\frac{\partial \theta}{\partial \lambda} = 0,$$

which is always possible. Hence:

$$x_1 = B\theta\frac{\partial X}{\partial \lambda}, \quad y_1 = B\theta\frac{\partial Y}{\partial \lambda}, \quad z_1 = B\theta\frac{\partial Z}{\partial \lambda}, \quad t_1 = B\theta\frac{\partial T}{\partial \lambda},$$

and since the homogeneous coordinates are defined only up to a factor, if we substitute x, y, z, t for X, Y, Z, T then we can write:

$$(7) \qquad x_1 = \frac{\partial x}{\partial \lambda}, \qquad y_1 = \frac{\partial y}{\partial \lambda}, \qquad z_1 = \frac{\partial z}{\partial \lambda}, \qquad t_1 = \frac{\partial t}{\partial \lambda}.$$

From these relations, the differential equation (1), which is verified for $\varphi = x, y, z, t$, will then give:

$$(8) \qquad \frac{\partial x_1}{\partial \mu} = P\, x_1 + Q \frac{\partial x}{\partial \mu} + Rx, \qquad \text{and analogous ones,}$$

which are conditions of the form (6). Equations (3) and (4) will then be verified.

Differentiating the relation (8) with respect to λ will give:

$$\frac{\partial^2 x_1}{\partial \lambda \partial \mu} = \frac{\partial P}{\partial \lambda} x_1 + P \frac{\partial x_1}{\partial \lambda} + \frac{\partial Q}{\partial \lambda} \cdot \frac{\partial x}{\partial \mu} + Q \frac{\partial^2 x}{\partial \lambda \partial \mu} + \frac{\partial R}{\partial \lambda} x + R \frac{\partial x}{\partial \lambda}.$$

However, x_1 satisfies equation (2); i.e., one will have:

$$\frac{\partial^2 x_1}{\partial \lambda \partial \mu} = P_1 \frac{\partial x_1}{\partial \lambda} + Q_1 \frac{\partial x_1}{\partial \mu} + R_1 x_1,$$

and upon also taking (7) into account, the preceding identity will become:

$$(9) \qquad P_1 \frac{\partial x_1}{\partial \lambda} + Q_1 \frac{\partial x_1}{\partial \mu} + R_1 x_1 = \frac{\partial P_1}{\partial \lambda} x_1 + P \frac{\partial x_1}{\partial \lambda} + Q \frac{\partial x_1}{\partial \mu} + R_1 x_1 + \frac{\partial Q}{\partial \lambda} \frac{\partial x}{\partial \mu} + \frac{\partial R}{\partial \lambda} x.$$

Equations (8), (9) are two equations in x and $\partial x / \partial \mu$. If one can solve them then one can infer x, in particular, as a linear function of x_1, $\partial x_1 / \partial \lambda$, and $\partial x_1 / \partial \mu$. Hence, the point $M(x, y, z, t)$ will be found in the plane of the three points (x_1, y_1, z_1, t_1), $\left(\frac{\partial x_1}{\partial \lambda}, \cdots\right), \left(\frac{\partial x_1}{\partial \mu}, \cdots\right)$; i.e., in the tangent plane to the surface (S_1) at M_1. The line MM_1 will also be tangent to (S_1) then, and (S_1) will be the second sheet of the focal surface. *Therefore, we have established the point-by-point correspondence between the two sheets of the focal surface in this case by means of rays of the congruence.*

We discard this case, which was studied in Chapter VI. One must then suppose that the equations (8), (9) can be solved for x and $\partial x / \partial \mu$, which demands that:

$$\begin{vmatrix} Q & R \\ \frac{\partial Q}{\partial \lambda} & \frac{\partial R}{\partial \lambda} \end{vmatrix} = 0;$$

i.e., one will have an identity of the form:

$$R = Q \cdot \Psi(\mu).$$

Recall the relation (8) then and multiply the coordinates x, y, z, t; x_1, y_1, z_1, t_1 by a factor ω, which is a function of μ, in such a fashion that the relation (8) will simplify, which will be written:

$$\frac{\partial x_1}{\partial \mu} = P\, x_1 + Q\left[\frac{\partial x}{\partial \mu} + x\,\Psi(\mu)\right].$$

We choose the factor ω in such a manner that the expression in brackets reduces to $\omega\, \partial x / \partial \mu$. Since the factor ω does not depend upon λ, equations (7) will persist, and we will get relations of the form:

$$\frac{\partial x_1}{\partial \mu} = P' x_1 + Q' \frac{\partial x}{\partial \mu}, \qquad \frac{\partial y_1}{\partial \mu} = \ldots, \qquad \frac{\partial z_1}{\partial \mu} = \ldots, \qquad \frac{\partial t_1}{\partial \mu} = \ldots$$

That amounts to supposing that $R = 0$ in equations (1), which will finally give:

$$(10) \qquad \frac{\partial^2 x}{\partial \lambda \partial \mu} = P \frac{\partial x}{\partial \lambda} + Q \frac{\partial x}{\partial \mu}, \qquad \frac{\partial^2 y}{\partial \lambda \partial \mu} = \ldots, \quad \frac{\partial^2 z}{\partial \lambda \partial \mu} = \ldots, \quad \frac{\partial^2 t}{\partial \lambda \partial \mu} = \ldots$$

Conversely, it is easy to see that if x, y, z, t satisfy (10) then if equations (7) were verified, the conditions (1), (2), (3), (4) would be satisfied. (3) and (1) are, to begin with. Equations (10) can be written:

$$\frac{\partial x_1}{\partial \mu} = P\, x_1 + Q \frac{\partial x}{\partial \mu}, \ \ldots, \ \ldots, \ \ldots,$$

in such a way that the condition (4) will also be verified. One finally infers from this, upon differentiation, that:

$$\frac{\partial^2 x_1}{\partial \lambda \partial \mu} = \frac{\partial P}{\partial \lambda} x_1 + P \frac{\partial x_1}{\partial \lambda} + Q \frac{\partial x_1}{\partial \mu} + \frac{1}{Q}\left(\frac{\partial x_1}{\partial \mu} - P x_1\right)\frac{\partial Q}{\partial \lambda}, \text{ and the analogous ones,}$$

which indeed gives equations of the form (2).

Second case. – We now suppose that $B, D, B_1, D_1 \neq 0$. Recall equations (5), (6). Upon multiplying x, y, z, t, and x_1, y_1, z_1, t_1 by convenient factors, one can make the term in x and the term in x_1 disappear, in such a way:

$$(11) \qquad \frac{\partial x_1}{\partial \lambda} = L \frac{\partial x}{\partial \lambda}, \quad \frac{\partial y_1}{\partial \lambda} = L \frac{\partial y}{\partial \lambda}, \quad \frac{\partial z_1}{\partial \lambda} = L \frac{\partial z}{\partial \lambda}, \quad \frac{\partial t_1}{\partial \lambda} = L \frac{\partial t}{\partial \lambda}.$$

Equation (6) is then written:

(12) $$\frac{\partial x_1}{\partial \mu} = M\frac{\partial x}{\partial \mu} + N\,x + S\,x_1\,;$$

differentiate with respect to λ, while taking (11) into account:

$$\frac{\partial}{\partial \mu}\left(L\frac{\partial x}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}\left(M\frac{\partial x}{\partial \mu}\right) + \frac{\partial}{\partial \lambda}(Nx) + \frac{\partial}{\partial \lambda}(Sx_1)\,.$$

(1) expresses $\frac{\partial^2 x}{\partial \lambda\,\partial \mu}$ as a function of x, $\frac{\partial x}{\partial \lambda}$, and $\frac{\partial x}{\partial \mu}$, and the preceding relation can be written:

$$\frac{\partial}{\partial \lambda}(S\,x_1) = F\left(x, \frac{\partial x}{\partial \lambda}, \frac{\partial x}{\partial \mu}\right),$$

in which F is a linear function, or furthermore:

$$\frac{\partial S}{\partial \lambda}x_1 + SL\frac{\partial x}{\partial \lambda} = F\left(x, \frac{\partial x}{\partial \lambda}, \frac{\partial x}{\partial \mu}\right).$$

If $\partial S / \partial \lambda \neq 0$ then x_1 will be a linear function of $x, \frac{\partial x}{\partial \lambda}, \frac{\partial x}{\partial \mu}$. The point M is in the tangent plane to the surface (S) at M, which will then be one of the sheets of the focal surface, which is a case that was examined previously. One must then suppose that $\partial S / \partial \lambda = 0$, so S will be a function of only μ. Hence, if we recall equation (12) then we can multiply x_1, y_1, z_1, t_1 by a function of μ such that the term in x_1 will disappear, so the relations (11) will keep the same form, and we will convert (12) into the form:

$$\frac{\partial x_1}{\partial \mu} = H\frac{\partial x}{\partial \mu} + K\,x\,.$$

The same argument will show that K is independent of λ, and that one can, in turn, make the term in x disappear. Finally, equations (12) can be reduced to the form:

(13) $$\frac{\partial x_1}{\partial \mu} = M\frac{\partial x}{\partial \mu}, \quad \frac{\partial y_1}{\partial \mu} = M\frac{\partial y}{\partial \mu}, \quad \frac{\partial z_1}{\partial \mu} = M\frac{\partial z}{\partial \mu}, \quad \frac{\partial t_1}{\partial \mu} = M\frac{\partial t}{\partial \mu}\,.$$

The relations (11) and (13) are sufficient, moreover, since one can conclude that:

$$\frac{\partial^2 x_1}{\partial \lambda\,\partial \mu} = \frac{\partial}{\partial \mu}\left(L\frac{\partial x}{\partial \lambda}\right),$$

$$\frac{\partial^2 x_1}{\partial\lambda\,\partial\mu} = \frac{\partial}{\partial\lambda}\left(M\frac{\partial x}{\partial\mu}\right).$$

Hence:

(14) $$\frac{\partial}{\partial\lambda}\left(M\frac{\partial x}{\partial\mu}\right) = \frac{\partial}{\partial\mu}\left(L\frac{\partial x}{\partial\lambda}\right),$$

which is an equation of the form (1) with $R = 0$. One will likewise obtain:

(15) $$\frac{\partial}{\partial\lambda}\left(\frac{1}{M}\cdot\frac{\partial x_1}{\partial\mu}\right) = \frac{\partial}{\partial\mu}\left(\frac{1}{M}\frac{\partial x_1}{\partial\lambda}\right),$$

which is an equation of the form (2) with $R_1 = 0$.

Conclusions. – In the *first case,* in which the surface (S) is one of the focal surfaces of the congruence, which is assumed to be given, we were led to make the term in x disappear in the equation:

(16) $$\frac{\partial^2 x}{\partial\lambda\,\partial\mu} = P\frac{\partial x}{\partial\lambda} + Q\frac{\partial x}{\partial\mu} + Rx,$$

which relates to that focal surface, by means of two transformations that are equivalent to a unique transformation of the form:

$$x = \varpi X.$$

In order to determine the factor ϖ, one directly finds the condition:

$$\frac{\partial^2 \varpi}{\partial\lambda\,\partial\mu} = P\frac{\partial \varpi}{\partial\lambda} + Q\frac{\partial \varpi}{\partial\mu} + R\varpi,$$

in such a way that equations (7) will show that *any surface* (S_1) *that is cut along a conjugate net by the developables of the congruence is defined by the equations:*

$$x_1 = \frac{\partial}{\partial\lambda}\left(\frac{x}{\varpi}\right), \quad y_1 = \frac{\partial}{\partial\lambda}\left(\frac{y}{\varpi}\right), \quad y_1 = \frac{\partial}{\partial\lambda}\left(\frac{z}{\varpi}\right), \quad t_1 = \frac{\partial}{\partial\lambda}\left(\frac{t}{\varpi}\right),$$

in which ϖ is an integral of equation (1).

We pass on to the *second case*, in which neither of the two surfaces is a focal surface of the congruence. One is given one of them – say, the surface (S) – and the conjugate net along which it must be cut by the developables of the desired congruence. One must once more eliminate the term in x in equation (1), which corresponds to that conjugate net on (S). That will again amount to looking for an integral of that equation. The equation will take the form:

(17) $$\frac{\partial^2 x}{\partial\lambda\,\partial\mu} = P\frac{\partial x}{\partial\lambda} + Q\frac{\partial x}{\partial\mu}.$$

In order to then determine the factors L and M in formulas (11) and (13), we identify that equation (17) with equation (14) that we obtained previously. That will give the conditions:

$$\frac{\partial L}{\partial\mu} = P\,(M - L), \qquad \frac{\partial M}{\partial\lambda} = Q\,(L - M).$$

Set:

(18) $$L - M = \psi,$$

and those equations will become:

(19) $$\frac{\partial L}{\partial\mu} = -P\,\psi,$$

(20) $$\frac{\partial M}{\partial\lambda} = Q\,\psi.$$

The first one can be written:

(19′) $$\frac{\partial M}{\partial\mu} = -\frac{\partial\psi}{\partial\mu} - P\,\psi,$$

and the compatibility condition for those equations is that ψ must be an integral of the equation:

(21) $$\frac{\partial^2\psi}{\partial\lambda\,\partial\mu} + \frac{\partial(P\psi)}{\partial\lambda} + \frac{\partial(Q\psi)}{\partial\mu} = 0,$$

which is what one calls the *adjoint* of (17). Having ψ, one can determine L and M by a quadrature, because one has the total differential of M – for example, from (19′) and (20), and equation (18) will then give L. Some new quadratures will succeed in determining the surface (S_1) by means of formulas (11) and (13), and similarly, the congruence.

Properties of the foregoing correspondence

It results from the preceding analysis that equations (11) and (13), viz.:

(11) $$\frac{\partial x_1}{\partial\lambda} = L\frac{\partial x}{\partial\lambda}, \quad \text{and the analogues,}$$

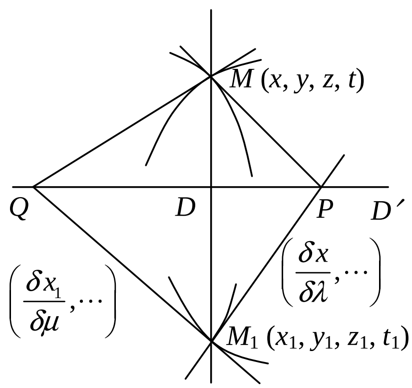
(13) $$\frac{\partial x_1}{\partial\mu} = M\frac{\partial x}{\partial\mu}, \quad \text{and the analogues}$$

completely characterize the special point-by-point correspondence that is determined on the two surfaces (S) and (S_1) by the rays of a congruence whose developables cut each of those two surfaces along a conjugate net. We shall examine the geometric properties that result from those formulas.

Let:

$$M\ (x, y, z, t), \quad M_1\ (x_1, y_1, z_1, t_1)$$

be two homologous points. Let P be the points whose coordinates are $\left(\frac{\partial x}{\partial \lambda}, \cdots\right)$ or $\left(\frac{\partial x_1}{\partial \lambda}, \cdots\right)$, and let Q be the point whose coordinates are $\left(\frac{\partial x}{\partial \mu}, \cdots\right)$ or $\left(\frac{\partial x_1}{\partial \mu}, \cdots\right)$. The line PM is tangent to the curve μ = const. on the surface (S) at M, and PM_1 is tangent to the surface μ = const. on the surface (S_1) at M_1. Similarly, the line QM is tangent to the curve λ = const. on the surface (S) at M, and QM_1 is tangent to the curve λ = const. on the surface (S_1) at M_1 . The tangent planes to the two surfaces (S), (S_1) at the points M, M_1 then cut along the line PQ.



Consider the congruence of those lines PQ. It is defined by the equations:

$$X = \frac{\partial x}{\partial \lambda} + \rho \frac{\partial x}{\partial \mu}, \qquad Y = \frac{\partial y}{\partial \lambda} + \rho \frac{\partial y}{\partial \mu}, \qquad Z = \frac{\partial z}{\partial \lambda} + \rho \frac{\partial z}{\partial \mu}, \qquad T = \frac{\partial t}{\partial \lambda} + \rho \frac{\partial t}{\partial \mu}.$$

The developables of that congruence are defined by the equation:

$$\left| \frac{\partial x}{\partial \lambda} \quad \frac{\partial x}{\partial \mu} \quad \frac{\partial^2 x}{\partial \lambda^2} d\lambda + \frac{\partial^2 x}{\partial \lambda \partial \mu} d\mu \quad \frac{\partial^2 x}{\partial \lambda \partial \mu} d\lambda + \frac{\partial^2 x}{\partial \mu^2} d\mu \right| = 0,$$

but x, y, z, t will satisfy identities of the form:

$$\frac{\partial^2 x}{\partial \lambda \partial \mu} = P \frac{\partial x}{\partial \lambda} + Q \frac{\partial x}{\partial \mu}, \qquad \ldots, \qquad \ldots, \qquad \ldots$$

in such a way that the preceding equation can be written:

$$\Delta \cdot d\lambda \cdot d\mu = 0,$$

in which Δ is a determinant that is non-zero, since the equation is not an identity. *The developables of the congruence of lines PQ, which are intersections of the tangent planes to the surface at two homologous points, will then correspond to the developables of the congruence of lines* MM_1 *that join those homologous points; i.e., to systems of homologous conjugates on the two surface again.*

We now seek the focal points. They are given by the equation:

$$\left| \frac{\partial x}{\partial \lambda} \quad \frac{\partial x}{\partial \mu} \quad \frac{\partial^2 x}{\partial \lambda^2} + \rho \frac{\partial^2 x}{\partial \lambda \partial \mu} \quad \frac{\partial^2 x}{\partial \lambda \partial \mu} + \rho \frac{\partial^2 x}{\partial \mu^2} \right| = 0,$$

which is an equation that will reduce to $\rho = 0$, due to the condition that precedes it; one of its roots will be zero, and the other one will be at infinity. *The focal points are nothing but the points P, Q. They are in the focal planes of the congruence of lines* MM_1 *.* Indeed, those focal planes are the planes MM_1P, MM_1Q, because they must be tangent to the two developables of the congruence that pass through MM_1, and by hypothesis, they will cut the two surfaces (*S*) and (S_1) along the curves μ = const., λ = const., whose tangents are *MP*, M_1P, and *MQ*, M_1Q, respectively.

Consider the point *P*, and suppose that one sets λ = const. The direction of the tangent to the trajectory of the point *P* is defined by a second point whose coordinates are:

$$\frac{\partial}{\partial \mu}\left(\frac{\partial x}{\partial \lambda}\right) = P \frac{\partial x}{\partial \lambda} + Q \frac{\partial x}{\partial \mu}, \text{ and analogous ones.}$$

It is a point of *PQ*. The point *P* then describes a tangent curve to *PQ*. It is the edge of regression of the developable of the congruence of lines *PQ* that corresponds to the value considered λ = const. Likewise, the point *Q* will describe the edge of regression of the developable that corresponds to that value μ = const when μ remains constant.

One sees that the correspondence between the two surfaces (*S*) and (S_1), which is first defined from the point-wise viewpoint by the congruence (*K*) of the lines MM_1, or (*D*), is found to be similarly defined from the tangential viewpoint by the congruence (K') of the lines *PQ*, or (D'). The developables (K') then correspond to the two homologous conjugate nets considered on (*S*) and (S_1). When the pairs of homologous points *M*, M_1 are defined in that way, the congruence (*K*) of lines MM_1 will result as a logical consequence, and the focal planes of the ray (*D*) of that congruence will pass through the foci *P* and *Q* of the homologous ray (D') of the congruence (K').

The properties of the correspondence that we just studied then transform into themselves by duality. Upon choosing the homogeneous tangential coordinates conveniently, one will have, in turn, at the same time as formulas (11) and (13), the identities:

$$\frac{\partial u_1}{\partial \lambda} = H \frac{\partial u}{\partial \lambda}, \text{ and analogous ones,}$$
$$\frac{\partial u_1}{\partial \mu} = K \frac{\partial u}{\partial \mu}, \text{ and analogous ones.}$$

In summary:

If the developables of one congruence (K) *cut two surfaces* (S), (S_1) *along two conjugate nets then the pairs of tangent planes to* (S) and (S_1) *whose contact points are on the same ray* (D) *of* (K) *will cut along the rays* (D') of a *new congruence* (K'), *such that the contact points of the tangent planes that are drawn at* (S) *and* (S_1) *with the generators of the developables of that congruence* (K') *will describe the same two homologous conjugate nets, and conversely. The focal points of the rays* (D'), (K') *are in the focal planes of the rays* (D) *associated with* (K), *and each focal point will be found in the focal plane that it does not correspond to.*

The correspondence between the two surfaces is, in fact, a correspondence between contact elements whose properties will correspond by duality when one passes from the points of those elements to their planes, or conversely.

Correspondence by parallel tangent planes

4. – Consider a point-to-point correspondence between two surfaces (S) and (S_1). On the surface (S), let (C) be one of the curves of the conjugate net that corresponds to a conjugate net on (S_1), and let (C_1) be the corresponding curve on (S_1). Suppose that the tangent planes to the surfaces (S), (S_1) at two arbitrary homologous points are parallel; their characteristics will also be parallel. Hence, *the homologous conjugate directions will be parallel.* If one supposes that the coordinates t and t_1 are equal to 1 here then that parallelism will translate into identities of the form:

$$(1) \qquad \frac{\partial x_1}{\partial \lambda} = L\frac{\partial x}{\partial \lambda}, \qquad \frac{\partial y_1}{\partial \lambda} = L\frac{\partial y}{\partial \lambda}, \qquad \frac{\partial z_1}{\partial \lambda} = L\frac{\partial z}{\partial \lambda}, \qquad \frac{\partial t_1}{\partial \lambda} = L\frac{\partial t}{\partial \lambda} = 0,$$

$$(2) \qquad \frac{\partial x_1}{\partial \mu} = M\frac{\partial x}{\partial \mu}, \qquad \frac{\partial y_1}{\partial \mu} = M\frac{\partial y}{\partial \mu}, \qquad \frac{\partial z_1}{\partial \mu} = M\frac{\partial z}{\partial \mu}, \qquad \frac{\partial t_1}{\partial \mu} = M\frac{\partial t}{\partial \mu} = 0.$$

We can then apply the results that were obtained before. The tangent planes at M, M_1 are parallel, so the line PQ will be at infinity. The lines of the congruence (K') are the lines of the plane at infinity. On each of those lines, the points P, Q will be the points where they are met by the homologous tangents on (S) and (S_1), and the locus of points P, Q is tangent to each line PQ at the points P, Q.

Special case. – In particular, suppose that the surface (S) is arbitrary and the surface (S_1) is a sphere. The congruence of lines MM_1 has developables that cut out conjugate nets on (S) and (S_1) whose homologous tangents are parallel. Now, a conjugate net on a sphere is an orthogonal net. Hence, the conjugate net on (S) is also an orthogonal net. It is the net of the *lines of curvature*, whose study will then be reduced to that of the developables of a congruence. In particular, suppose that the surface (S) has degree two, and consider the congruence of lines PQ of the plane at infinity. The plane at infinity cuts (S), (S_1) along two conics (Γ), (Γ_1). Consider their points of intersection with a line PQ. The points of intersection with (Γ) will correspond to the directions of the generators of (S) that pass through M and which are the asymptotic tangents. The points P, Q, which correspond to the principal directions will then be conjugate with respect to those points of intersection; i.e., conjugate with respect to conic (Γ). They will likewise be conjugate with respect to (Γ_1). The points P, Q are the double points of the involution that is determined on the line PQ by the pencil of conics that has (Γ), (Γ_1) for its bases. The line PQ is tangent at P, Q to two conics of that pencil that are tangent to them, in such a way that the determination of the developables of the congruence (K) – i.e., of the lines of the curvature of the quadric (S) – which amounts to the determination of a pencil of conics, can be done algebraically.

If one takes the parameters to be those of the rectilinear generators that pass through a point of (S) then one will get the integration of the *Euler equation.*

Indeed, consider the hyperboloid of one sheet:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0, \tag{3}$$

which will have the parametric equations:

$$x = a\,\frac{1-w}{u-v}, \quad y = b\,\frac{1+uv}{u-v}, \quad z = c\,\frac{u+v}{u-v}, \tag{4}$$

when it is referred to its rectilinear generators.

The normal at a point will have the direction coefficients:

$$\frac{x}{a^2}, -\frac{y}{b^2}, \frac{z}{c^2},$$

so the differential equation for the lines of curvature, which expresses the idea that the normal will meet the infinitely-close normal, will be:

$$\begin{vmatrix} \dfrac{x}{a^2} & \dfrac{dx}{a^2} & dx \\ -\dfrac{y}{b^2} & -\dfrac{dy}{b^2} & dy \\ \dfrac{z}{c^2} & \dfrac{dz}{c^2} & dz \end{vmatrix} = 0,$$

or

$$(b^2 + c^2)\, x\, dy\, dz + (a^2 - c^2)\, y\, dz\, dx - (a^2 + b^2)\, z\, dx\, dy = 0. \tag{5}$$

The differentiation of the formulas (4) gives:

(6) $$\frac{dx}{a[-(1-v^2)\,du+(1-u^2)\,dv]} = \frac{dy}{b[-(1+v^2)\,du+(1+u^2)\,dv]} = \frac{dz}{2c[-v\,du+u\,dv]} = \frac{1}{(u-v)^2},$$

and after all of the reductions have been made, equation (5) will then become the Euler equation:

(7) $$\frac{du^2}{\Phi(u^2)} = \frac{dv^2}{\Phi(v^2)},$$

upon setting:

(8) $$\Phi(\omega) = \omega^2 + 2k\omega + 1, \qquad k = \frac{b^2+2c^2-a^2}{a^2+b^2}.$$

The points P and Q of the preceding theory are the points at infinity of the tangents to the lines of curvature. Their homogeneous coordinates X, Y, Z will then be given by the denominators of formulas (6), in which du, dv must be replaced by the proportional values $\sqrt{\Phi(u^2)}$, $\pm\sqrt{\Phi(v^2)}$ that are inferred from equation (7).

From the foregoing, the developables of the congruences considered, and consequently, the lines of the curvature of the surface, are obtained by writing that one or the other of the points (X, Y, Z) thus-defined describes one of the conics of the pencil:

$$X^2 + Y^2 + Z^2 + \sigma\left(\frac{X^2}{a^2} - \frac{Y^2}{b^2} + \frac{Z^2}{c^2}\right) = 0$$

in the plane at infinity $T = 0$.

After suppressing the factor $du\,dv$, one will then obtains the general algebraic integral that was asserted:

(9) $$\pm\sqrt{\Phi(u^2)}\sqrt{\Phi(v^2)} - \Phi_0(u^2, v^2) - m\,(u-v)^2 = 0,$$

in which $\Phi_0(\omega, \omega')$ denotes the polar polynomial to the trinomial $\Phi(\omega)$:

$$\Phi(\omega, \omega') = \omega\omega' + k\,(\omega + \omega') + 1,$$

and in which m is an arbitrary constant that is coupled to σ by the equation:

$$m\,(a^2 + b^2) = -2\,(\sigma + c^2).$$

Clear the radical, while taking into account the identity, which is classical in the theory of binary quadratic forms:

$$\Phi(\omega)\,\Phi(\omega') - \Phi_0^2(\omega, \omega') = \Delta^2\,(\omega - \omega')^2,$$

in which Δ is the discriminant of the form. After dividing by $(u - v)^2$, we will get the general rational integral:

$$(9') \qquad (1 - k^2)(u + v)^2 = m^2 (u - v)^2 + 2m\,\Phi_0(u^2, v^2).$$

Now, $2\Phi_0(u^2, v^2)$ is written:

$$2\Phi_0(u^2, v^2) = (1 + uv)^2 + (1 - uv)^2 + k(u + v)^2 + k(u - v)^2.$$

Upon taking formulas (4) into account, one will see that the lines of curvature are the intersections of the hyperboloid with the quadrics:

$$m\frac{x^2}{a^2} + m\frac{y^2}{b^2} + (mk - 1 + k^2)\frac{z^2}{c^2} + mk + m^2 = 0.$$

Replace that equation by the homogeneous combination that is obtained by adding equation (3), when multiplied by $(mk + m^2)$:

$$m(m + k + 1)\frac{x^2}{a^2} - m(m + k - 1)\frac{y^2}{b^2} + (m + k + 1)(m + k - 1)\frac{z^2}{c^2} = 0.$$

That can also be written:

$$\frac{x^2}{a^2(m+k-1)} - \frac{y^2}{b^2(m+k+1)} + \frac{z^2}{c^2 m} = 0,$$

or, due to the value (8) of k:

$$\frac{x^2}{a^2[m(a^2+b^2)+2c^2-2a^2]} - \frac{y^2}{b^2[m(a^2+b^2)+2c^2+2b^2]} + \frac{z^2}{c^2[m(a^2+b^2)]} = 0.$$

Upon then setting:

$$-2s = m(a^2 + b^2) + 2c^2,$$

one will finally write:

$$\frac{x^2}{a^2(s+a^2)} + \frac{y^2}{b^2(s-b^2)} + \frac{z^2}{c^2(s+c^2)} = 0.$$

Moreover, it will suffice to add it to the equation of the hyperboloid, after multiplying it by $(- s)$, in order to obtain the equations of the homofocal quadric:

$$(10) \qquad \frac{x^2}{s+a^2} + \frac{y^2}{s-b^2} + \frac{z^2}{s+c^2} = 0.$$

One then finds the classical result that *the lines of curvature of the hyperboloid* (3) *are the intersections of that surface with the ellipsoids and the hyperboloids with two homofocal sheets* that are represented by equation (10). [Cf., Chap. XII, § 1 and § 6].

Remark 1. – Instead of the plane at infinity, one can consider an arbitrary fixed plane (π). The correspondence will be such that the tangent planes at two homologous points of (S), (S_1) will cut in the plane (π). The results will then be analogous, and similarly, if correlatively, one establishes a correspondence between two surfaces such that the line MM_1 passes through a fixed point.

Remark 2. – Consider two surfaces (S), (S_1) that correspond by parallel tangent planes. Take a fixed point O in space and replace (S_1) with one of its homothetic images with respect to O, namely, (S_1'). Any conjugate net on (S_1) will correspond to a homothetic net on (S_1') that is also conjugate, and the conjugate net on (S), which corresponds to a conjugate net on (S_1), will also correspond to a conjugate net on (S_1'). Imagine that the homothety ratio increases indefinitely: The point M_1' that is homothetic to M_1 will be stretched to infinity, so the line MM_1' will become the parallel to the ray OM_1 that is drawn through M. Hence: *If one has two surfaces* (S), (S_1) *that correspond by parallel tangent planes, and one takes a fixed point O in space and draws the parallel MN to the ray* OM_1 *through the point M on* (S) *then the developables of the congruence of lines MN will cut out the conjugate net on* (S) *that corresponds to a conjugate net on* (S_1). In particular, if we take (S_1) to be a sphere and take O to be its center, then OM_1 will be perpendicular to the tangent plane to (S_1), and consequently to the tangent plane to (S). MN, which is parallel to it, is the normal to (S). *The congruence of normals to a surface will have developables that determine an orthogonal conjugate net on that surface.* One will then recover the fundamental property of the lines of curvature of the surface (S).

We further remark that if the radius of the sphere (S_1) is equal to 1 then the coordinates x_1, y_1, z_1 will be the direction cosines of the normal, and formulas (1), (2) will be nothing but the formulas of Olinde Rodriguez [Chap. V, § 3]: $-L$ and $-M$ are then the principal curvatures.

Isothermal surfaces

5. – One will be led to an important class of surfaces when one looks for the cases in which the correspondence by parallel tangent planes between two surfaces (S) and (S_1) yields a conformal representation of one surface on the other [Chap. II, § 2]. Suppose the two surfaces are referred to homologous conjugate systems, as in the preceding paragraph, in such a way that the correspondence between them satisfies equations (1) and (2) of that paragraph:

$$(1) \qquad \frac{\partial x_1}{\partial \lambda} = L\frac{\partial x}{\partial \lambda}, \qquad \frac{\partial y_1}{\partial \lambda} = L\frac{\partial y}{\partial \lambda}, \qquad \frac{\partial z_1}{\partial \lambda} = L\frac{\partial z}{\partial \lambda},$$

(2) $$\frac{\partial x_1}{\partial \mu} = M\,\frac{\partial x}{\partial \mu}, \qquad \frac{\partial y_1}{\partial \mu} = M\,\frac{\partial y}{\partial \mu}, \qquad \frac{\partial z_1}{\partial \mu} = M\,\frac{\partial z}{\partial \mu},$$

in which the rectangular Cartesian coordinates of the homologous points figure.

Let:

$$ds^2 = E\,d\lambda^2 + 2F\,d\lambda\,d\mu + G\,d\mu^2$$

be the linear element of the surface (S), in such a way that:

(3) $$E = \sum\left(\frac{\partial x}{\partial \lambda}\right)^2, \qquad F = \sum\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu}, \qquad G = \sum\left(\frac{\partial x}{\partial \mu}\right)^2.$$

The condition that expresses the idea that the correspondence considered will realize a conformal representation is that there must exist a function $k(\lambda, \mu)$ such that:

(4) $$dx_1^2 + dy_1^2 + dz_1^2 = k^2\,ds^2.$$

Upon taking formulas (1), (2), (3) into account, it will translate into the equations:

(5) $$(L^2 - k^2)\,E = (LM - k^2)\,F = (M^2 - k^2)\,G = 0.$$

1. Discard the case ($E = 0$, $F = 0$), ($F = 0$, $G = 0$), in which the surface (S) is an isotropic developable [Chap. III, § 4]. We can first suppose that $E = 0$, $G = 0$, in such a way that the coordinate lines will be minimal lines on (S) and (S_1). Since they are conjugate, by hypothesis, the asymptotic directions will be harmonic conjugate with respect to the isotropic directions of the tangent plane, and will be rectangular. Hence, the indicatrix will be an equilateral hyperbola, and the surface (S), like (S_1), will be a minimal surface.

Conversely, the equations that were given in Chapter III, § 6, page 50, to represent an arbitrary minimal surface will imply the formulas:

(6) $$\begin{cases} d(x+iy) = -u^2F'''(u)\,du - v^2G'''(v)\,dv, \\ d(x-iy) = \quad F'''(u)\,du + \quad G'''(v)\,dv, \\ \qquad dz = -u\,F'''(u)\,du - v\,G'''(v)\,dv. \end{cases}$$

Therefore, when two surfaces (S) and (S_1) are represented in that way, with the functions F, F; F_1, G_1, respectively, one will have identities of the form (1), (2), and (5):

$$\frac{\partial x_1}{\partial u} = \frac{F_1'''}{F'''}\cdot\frac{\partial x}{\partial u}, \qquad \frac{\partial y_1}{\partial u} = \frac{F_1'''}{F'''}\cdot\frac{\partial y}{\partial u}, \qquad \frac{\partial z_1}{\partial u} = \frac{F_1'''}{F'''}\cdot\frac{\partial z}{\partial u},$$

$$\frac{\partial x_1}{\partial v} = \frac{G_1'''}{G'''}\cdot\frac{\partial x}{\partial v}, \qquad \frac{\partial y_1}{\partial v} = \frac{G_1'''}{G'''}\cdot\frac{\partial y}{\partial v}, \qquad \frac{\partial z_1}{\partial v} = \frac{G_1'''}{G'''}\cdot\frac{\partial z}{\partial v},$$

$$ds_1^2 = \frac{F_1''' G_1'''}{F''' G'''} \cdot ds^2.$$

Hence: *Two arbitrary minimal surfaces will correspond by parallel tangent planes in such a manner that the correspondence is a conformal representation.*

2. Now suppose that F is not zero, and that E and G are not both zero. When the condition $LM = k^2$ is then combined with one of the conditions $L^2 = k^2$, $M^2 = k^2$, it will imply that:

$$L = M, \qquad k^2 = L^2 = M^2,$$

in which L and M cannot be zero. Upon supposing that $k = L$, which is legitimate, one will then conclude that:

(7) $$dx_1 = k\,dx, \qquad dy_1 = k\,dy, \qquad dz_1 = k\,dz.$$

Now, at least two of the functions x, y, z of λ and μ are independent: Suppose that they are x and y, for example. The first two identities (7) express the idea that the correspondence between (S) and (S_1) will translate into the formulas:

$$x_1 = \varphi(x), \qquad y_1 = \psi(y), \qquad k = \varphi'(x) = \psi'(y),$$

in which x and y can be considered to be independent variables. One then concludes that k is a constant, since it does not depend upon either x or y, and formulas (7) will then give:

$$x_1 = kx + a, \qquad y_1 = ky + b, \qquad z_1 = kz + c,$$

in which a, b, c are three constants. We then find the obvious solutions, in which (S) is an arbitrary surface, and (S_1) is an arbitrary homothetic image of (S).

3. It remains to examine the case in which F is zero, without E or G being so. The conditions (5) will then give:

$$F = 0, \qquad L = -M = k$$

when we discard the hypothesis $L = M$ that we encountered already. We must then examine what the two surfaces (S) and (S_1) would be that are coupled by the conditions:

(8) $$\frac{\partial x_1}{\partial \lambda} = k\frac{\partial x}{\partial \lambda}, \qquad \frac{\partial y_1}{\partial \lambda} = k\frac{\partial y}{\partial \lambda}, \qquad \frac{\partial z_1}{\partial \lambda} = k\frac{\partial z}{\partial \lambda},$$

(9) $$\frac{\partial x_1}{\partial \mu} = -k\frac{\partial x}{\partial \mu}, \qquad \frac{\partial y_1}{\partial \mu} = -k\frac{\partial y}{\partial \mu}, \qquad \frac{\partial z_1}{\partial \mu} = -k\frac{\partial z}{\partial \mu},$$

and

(10) $$F = \frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} = 0.$$

Eliminate x_1, y_1, z_1 by differentiating equations (8) with respect to μ, differentiating equations (9) with respect to λ, and subtracting corresponding sides of the equations. We will then find that x, y, z will satisfy the same equation of the form:

$$(11) \qquad 0 = \frac{\partial}{\partial\mu}\left(k\frac{\partial\omega}{\partial\lambda}\right)+\frac{\partial}{\partial\lambda}\left(k\frac{\partial\omega}{\partial\mu}\right) \equiv 2k\frac{\partial^2\omega}{\partial\lambda\partial\mu}+\frac{\partial k}{\partial\mu}\cdot\frac{\partial\omega}{\partial\lambda}+\frac{\partial k}{\partial\lambda}\cdot\frac{\partial\omega}{\partial\mu}.$$

Now, upon differentiating equations (3), we will get the identities:

$$(12) \qquad 2\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\partial\mu} = \frac{\partial E}{\partial\mu}, \qquad 2\sum\frac{\partial x}{\partial\mu}\frac{\partial^2 x}{\partial\lambda\partial\mu} = \frac{\partial G}{\partial\lambda}.$$

Upon replacing $2\frac{\partial^2 x}{\partial\lambda\partial\mu}$, $2\frac{\partial^2 y}{\partial\lambda\partial\mu}$, $2\frac{\partial^2 z}{\partial\lambda\partial\mu}$ as functions of the first derivatives in this by means of the identities that result from (11) when one replaces ω with x, y, z, and upon taking formulas (3) and the condition (10) into account, those identities (12) will become:

$$-E\frac{\partial k}{\partial\mu} = k\frac{\partial E}{\partial\mu}, \qquad -G\frac{\partial k}{\partial\lambda} = k\frac{\partial G}{\partial\lambda}.$$

Therefore, E and G will have the form:

$$E = \frac{1}{k}\,\varphi(\lambda), \qquad G = \frac{1}{k}\,\psi(\mu),$$

and the linear element of (S) will take the form:

$$(13) \qquad ds^2 = \frac{1}{k}\,[\varphi(\lambda)\,d\lambda^2 + \psi(\mu)\,d\mu^2].$$

We can replace the coordinate λ with a function of λ and the coordinate μ with a function of μ without changing formulas (8) and (9). Moreover, we can arrange for that change of coordinates to reduce formula (13) to the form:

$$(14) \qquad ds^2 = \frac{1}{k}\,[d\lambda^2 + d\mu^2],$$

in which we keep the notations λ, μ in order to denote the new coordinates λ', μ' that are defined by:

$$d\lambda' = \sqrt{\varphi(\lambda)}\cdot d\lambda, \qquad d\mu' = \sqrt{\psi(\mu)}\cdot d\mu.$$

By virtue of formula (4), the linear element of (S_1) will itself reduce to the form:

(15) $$ds_1^2 = k\,(d\lambda^2 + d\mu^2).$$

The systems of coordinate curves that form conjugate nets on (S) and (S_1), by hypothesis, also form orthogonal nets, by virtue of the hypothesis $F = 0$; they will then be systems of lines of curvature on one surface and the other. However, in addition, from formulas (14) and (15), they will form isothermal, orthogonal systems [Chap. IV, § 4]. *The two surfaces can then be divided into infinitely-small squares by their lines of curvature.* In order to express that property, one says that they are *isothermal surfaces. An isothermal surface is then a surface that has a ds^2 of the form* (13):

$$ds^2 = K\,[\varphi(\lambda)\,d\lambda^2 + \psi(\mu)\,d\mu^2]$$

when it is referred to its lines of curvature.

Converse. – Conversely, take an arbitrary isothermal surface (S). Suppose that it is referred to its lines of curvature, in such a way that its ds^2 has the form (14). We have conditions:

(16) $$\sum\left(\frac{\partial x}{\partial\lambda}\right)^2 = \frac{1}{k}, \qquad \sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu} = 0, \qquad \sum\left(\frac{\partial x}{\partial\mu}\right)^2 = \frac{1}{k},$$

at the same time as the condition $F' = 0$, which expresses the idea that the coordinate lines are conjugate:

(17) $$0 = \sum\frac{\partial^2 x}{\partial\lambda\,\partial\mu}\cdot\frac{\partial(y,z)}{\partial(\lambda,\mu)} \equiv \sum A\frac{\partial^2 x}{\partial\lambda\,\partial\mu}.$$

Upon differentiating equations (16), we will get:

(18) $$\sum\frac{\partial^2 x}{\partial\lambda\,\partial\mu}\cdot\frac{\partial x}{\partial\lambda} = \frac{1}{2}\frac{\partial(1/k)}{\partial\mu}, \qquad \sum\frac{\partial^2 x}{\partial\lambda\,\partial\mu}\cdot\frac{\partial x}{\partial\mu} = \frac{1}{2}\frac{\partial(1/k)}{\partial\lambda},$$

and we can infer the values of the second derivatives $\frac{\partial^2 x}{\partial\lambda\,\partial\mu}, \frac{\partial^2 y}{\partial\lambda\,\partial\mu}, \frac{\partial^2 z}{\partial\lambda\,\partial\mu}$ from equations (17) and (18). The three directions:

$$\frac{\partial x}{\partial\lambda}, \frac{\partial y}{\partial\lambda}, \frac{\partial z}{\partial\lambda}; \qquad \frac{\partial x}{\partial\mu}, \frac{\partial y}{\partial\mu}, \frac{\partial z}{\partial\mu}; \qquad A, B, C$$

define a direct tri-rectangular trihedron, so we introduce their direction cosines, which are:

$$\sqrt{k}\,\frac{\partial x}{\partial\lambda}, \sqrt{k}\,\frac{\partial y}{\partial\lambda}, \sqrt{k}\,\frac{\partial z}{\partial\lambda}; \quad \sqrt{k}\,\frac{\partial x}{\partial\mu}, \sqrt{k}\,\frac{\partial y}{\partial\mu}, \sqrt{k}\,\frac{\partial z}{\partial\mu}; \quad kA, kB, kC,$$

since $E = G = 1 / k$, $A^2 + B^2 + C^2 = EG - F^2 = 1 / k^2$. In order to obtain $\frac{\partial^2 x}{\partial\lambda\,\partial\mu}$, for example, it suffices to multiply equations (17) and (18) by k^2A, $k\frac{\partial x}{\partial\lambda}$, $k\frac{\partial x}{\partial\mu}$, respectively, and add them, which will give:

$$\frac{\partial^2 x}{\partial\lambda\,\partial\mu} = \frac{1}{2}k\frac{\partial(1/k)}{\partial\mu}\cdot\frac{\partial x}{\partial\lambda} + \frac{1}{2}k\frac{\partial(1/k)}{\partial\lambda}\cdot\frac{\partial x}{\partial\mu};$$

i.e.:

$$2k\frac{\partial^2 x}{\partial\lambda\,\partial\mu} + \frac{\partial k}{\partial\mu}\frac{\partial x}{\partial\lambda} + \frac{\partial k}{\partial\lambda}\frac{\partial x}{\partial\mu} = 0.$$

Hence, x, y, z indeed satisfy the same equation (11). Now, that is precisely the necessary and sufficient condition for equations (8) and (9), in x_1, y_1, z_1, to be compatible. One can then calculate x_1, y_1, z_1 by quadrature from the total differentials:

$$\text{(19)}\quad dx_1 = k\left(\frac{\partial x}{\partial\lambda}d\lambda - \frac{\partial x}{\partial\mu}d\mu\right),\quad dy_1 = k\left(\frac{\partial y}{\partial\lambda}d\lambda - \frac{\partial y}{\partial\mu}d\mu\right),\quad dz_1 = k\left(\frac{\partial z}{\partial\lambda}d\lambda - \frac{\partial z}{\partial\mu}d\mu\right).$$

The surface (S_1) is then well-defined, and its ds^2 will be given by formulas (15). That is, it is itself isothermal and referred to its lines of curvature, since, from formulas (1), the coordinate lines will be conjugate on the two surfaces, and from (15), they will be orthogonal and isothermal for (S_1).

Therefore:

If one is given an arbitrary isothermal surface that has a ds^2 *that is given by* (14) *when it is referred to its lines of curvature then it will correspond to one and only one other isothermal surface (up to an arbitrary translation), such that the correspondence that is established by parallel tangent planes between the points of those two surfaces is a conformal representation of one surface on the other one. Under that correspondence, the lines of curvature on the two surfaces will correspond, and the* ds^2 *of the second one will be given by formula* (15). *There will be reciprocity between the two surfaces.*

Remark. – The preceding calculations show that in order for a surface to be isothermal, it is necessary and sufficient that the Cartesian coordinates of any point on the surface must satisfy not only the condition that $F = 0$, but also the same partial differential equation of the form (11). That equation will not change in form under a change of variables of the form:

$$\text{(20)}\qquad \lambda' = \varphi(\lambda), \qquad \mu' = \psi(\mu).$$

However, one can simplify this by setting:

$$\text{(21)}\qquad \omega' = \omega\cdot\chi(\lambda, \mu),$$

and suitably determining the factor χ. Indeed, it will become:

$$\frac{\partial^2\omega'}{\partial\lambda\,\partial\mu}+\left[\frac{1}{2k}\frac{\partial k}{\partial\mu}-\frac{1}{\chi}\frac{\partial\chi}{\partial\mu}\right]\frac{\partial\omega'}{\partial\lambda}+\left[\frac{1}{2k}\frac{\partial k}{\partial\lambda}-\frac{1}{\chi}\frac{\partial\chi}{\partial\lambda}\right]\frac{\partial\omega'}{\partial\mu}-\theta\cdot\omega'=0,$$

and it will suffice to take:

(22) $$\chi=\sqrt{k}$$

in order to reduce it to the form:

(23) $$\frac{\partial^2\omega'}{\partial\lambda\,\partial\mu}=\theta\,\omega'.$$

The expression for θ in terms of λ and μ is deduced from the fact that since $\omega = 1$ is a solution of equation (11), $\omega' = \chi = \sqrt{k}$ will be a solution of (23); hence:

(24) $$\theta=\frac{1}{\sqrt{k}}\frac{\partial^2\sqrt{k}}{\partial\lambda\,\partial\mu}.$$

Saying that equation (11) is verified by the Cartesian coordinates x, y, z, 1 is equivalent to saying that equation (23) is verified by the homogeneous coordinates:

$$X = x\sqrt{k}\,,\qquad Y = y\sqrt{k}\,,\qquad Z = z\sqrt{k}\,,\qquad T = \sqrt{k}\,.$$

Therefore: *In order for a surface to be isothermal, it is necessary and sufficient that for a conveniently-chosen system of homogeneous coordinates X, Y, Z, T, the four coordinates of any point on the surface, which is supposed to be referred to its lines of curvature, satisfy the same partial differential equation of the form* (23); *the linear element of the surface will then be:*

$$ds^2=\frac{1}{T^2}(d\lambda^2+d\mu^2).$$

Examples of isothermal surfaces

1. Any *surface of revolution:*

$$x = u\cos\mu,\qquad y = u\sin\mu,\qquad z = \varphi(u)$$

is isothermal because it is then referred to its lines of curvature, and its linear element:

$$ds^2 = [1+\varphi'^2(u)]\,du^2 + u^2\,d\mu^2$$

has the form (13).

The *sphere* is, in turn, isothermal in an infinitude of ways.

2. *Cones* and *cylinders* whose linear elements (1) and (2), which were given in Chap. V, § 4, pp. 91, are referred to their lines of curvature are also isothermal surfaces, from the form of those linear elements:

$$ds^2 = du^2 + dv^2, \qquad ds^2 = u^2\left[\frac{1}{u^2}du^2 + dv^2\right].$$

3. *Second-degree surfaces* are isothermal. We verify this for the hyperboloid of one sheet, while appealing to the formulas of the preceding paragraph. To that effect, formulas [(6), § 4] will give:

$$(u - v)^4\, ds^2 = (a^2 + b^2)\, [\Phi\, (v^2)\, du^2 - 2\Phi_0\, (u^2, v^2)\, du\, dv + \Phi\, (u^2, dv^2)] + 4c^2\, (u - v)^2\, du\, dv.$$

Introduce the parameters of the lines of curvature that were defined by [(7), § 4] by setting:

$$(25) \qquad \frac{du}{\sqrt{\Phi(u^2)}} - \frac{dv}{\sqrt{\Phi(v^2)}} = d\lambda, \qquad \frac{du}{\sqrt{\Phi(u^2)}} + \frac{dv}{\sqrt{\Phi(v^2)}} = d\mu,$$

and the ds^2 will become:

$$(20) \qquad ds^2 = \tfrac{1}{2}(a^2 + b^2)\, \sqrt{\Phi(u^2)}\sqrt{\Phi(v^2)}\; [E_0\, d\lambda^2 + G_0\, d\mu^2]\, (u - v)^{-2},$$

with:

$$(u - v)^2 \cdot E_0 = \sqrt{\Phi(u^2)}\sqrt{\Phi(v^2)} + \Phi_0\, (u^2, v^2) - \frac{2c^2}{a^2 + b^2}\, (u - v)^2,$$

$$(u - v)^2 \cdot G_0 = \sqrt{\Phi(u^2)}\sqrt{\Phi(v^2)} - \Phi_0\, (u^2, v^2) + \frac{2c^2}{a^2 + b^2}\, (u - v)^2.$$

Now, due to the form [(9), § 4] of the integral of the Euler equation [(7), § 4], E_0 = const. defines the same lines of curvature as μ = const. Hence, E_0 is a function of only μ, and similarly, G_0 is a function of only λ. Therefore, the ds^2 in (26) will come down to the form (13), which is characteristic of isothermal surfaces, by using $E_0\, G_0$ as the factor.

4. We will find a new class of isothermal surfaces by looking for the pairs of parallel surfaces (S) and (S_1) on which the common normals determine a conformal correspondence. For that to be true, if l, m, n denote the direction cosines of the normal to S then it will suffice to suppose that:

$$x_1 = x + hl, \qquad y_1 = hm, \qquad z_1 = z + hn,$$

in formulas (8), (9), in which h is a constant length. From the formulas of Olinde Rodriguez [Chap. V, § 3], in which R_1 and R_2 are the radii of principal curvature on (S), one will have:

$$\frac{\partial l}{\partial \lambda} = -\frac{1}{R_1}\cdot\frac{\partial x}{\partial \lambda}, \qquad \frac{\partial m}{\partial \lambda} = -\frac{1}{R_1}\cdot\frac{\partial y}{\partial \lambda}, \qquad \frac{\partial n}{\partial \lambda} = -\frac{1}{R_1}\cdot\frac{\partial z}{\partial \lambda},$$

$$\frac{\partial l}{\partial \mu} = -\frac{1}{R_2}\cdot\frac{\partial x}{\partial \mu}, \qquad \frac{\partial m}{\partial \mu} = -\frac{1}{R_2}\cdot\frac{\partial y}{\partial \mu}, \qquad \frac{\partial n}{\partial \mu} = -\frac{1}{R_2}\cdot\frac{\partial z}{\partial \mu}.$$

Hence:

$$\frac{\partial x_1}{\partial \lambda} = \left(1-\frac{h}{R_1}\right)\frac{\partial x}{\partial \lambda}, \quad \ldots, \qquad \frac{\partial x_1}{\partial \mu} = \left(1-\frac{h}{R_2}\right)\frac{\partial x}{\partial \mu}, \quad \ldots,$$

and in order for one to able to identify those formulas with formulas (8) and (9), it is necessary and sufficient that one must have:

$$\left(1-\frac{h}{R_1}\right)+\left(1-\frac{h}{R_2}\right) = 0$$

or

(26) $$\frac{1}{R_1}+\frac{1}{R_2} = \frac{2}{h};$$

i.e., the mean curvature of (S) must be constant. The same thing is true for the mean curvature of (S_1), which is equal and opposite to that of (S). That is obvious *a priori*, due to the symmetry of the relationship between (S) and (S_1), and one effortlessly confirms that the equality:

$$\frac{1}{R_1-h}+\frac{1}{R_2-h} = -\frac{2}{h}$$

is equivalent to (26). We further remark that the centers of principal curvature that are common to (S) and (S_1) are harmonic conjugates with respect to the feet of the common normal to the two surfaces.

One then finds a means of deducing a surface of constant mean curvature – 1 / h from any surface of mean constant curvature 1 / h.

Therefore: *Any surface of constant mean curvature is isothermal.*

5. The preceding conclusion would no longer be justified if the mean curvature were zero – i.e., if (S) were a minimal surface – because h would have to be infinite then. However, it is easy to verify directly that *any minimal surface is isothermal.*

To that effect, recall formulas (6): The direction l, m, n of the normal is defined by the condition:

$$(l - im)\, d\,(x + iy) + (l + im)\, d\,(x - iy) + 2n\, dz = 0,$$

from which, one will infer:

(27) $$l + in = 2nv, \quad l - im = -2, \quad n = u + v.$$

The condition for the normal to meet the infinitely-close normal is then written:

$$0 = \begin{vmatrix} l & dl & dx \\ m & dm & dy \\ n & dn & dz \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} l+im & d(l+im) & d(x+iy) \\ l-im & d(l-im) & d(x-iy) \\ n & dn & dz \end{vmatrix}.$$

Upon substituting the values (6) and (27), one will then obtain the differential equation of the lines of curvature, which will reduce to:

(28) $$F''' du^2 + G''' dv^2 = 0.$$

On the other hand, from formulas (6), the ds^2 is:

(29) $$ds^2 = d(x + iy)\, d(x - iy) + dz^2 = -(u - v)^2 F''' G''' \, du\, dv.$$

In order to introduce the parameters λ, μ of the lines of curvature, it will suffice to set:

$$\sqrt{F'''} \cdot du - \sqrt{G'''} \cdot dv = d\lambda, \quad \sqrt{F'''} \cdot du + \sqrt{G'''} \cdot dv = d\mu,$$

and ds^2 will become:

$$ds^2 = \frac{(u-v)^2}{4\sqrt{F'''}\sqrt{G'''}} (d\lambda^2 - d\mu^2),$$

which has, in fact, the isothermal form.

Use of penta-spherical coordinates

6. – In order for the equations:

(1) $$x = f(\lambda, \mu), \quad y = g(\lambda, \mu), \quad z = h(\lambda, \mu)$$

to represent a surface that is referred to its lines of curvature, from what we have seen, it is necessary and sufficient that those functions must satisfy the same partial differential equation of the form:

(2) $$\frac{\partial^2 \omega}{\partial\lambda\,\partial\mu} + L \frac{\partial\omega}{\partial\lambda} + M \frac{\partial\omega}{\partial\mu} = 0,$$

at the same time as the orthogonality condition:

(3) $$0 = F = \frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu} + \frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\mu} + \frac{\partial z}{\partial\lambda}\frac{\partial z}{\partial\mu}.$$

One can replace that condition with another one in the following manner. To abbreviate, let $\Omega(\omega)$ denote the left-hand side of (2), and one will get the identity:

$$\tfrac{1}{2}\Omega\,(x^2 + y^2 + z^2) = x\,\Omega\,(x) + y\,\Omega\,(y) + z\,\Omega\,(z) + F.$$

Since $\Omega\,(x)$, $\Omega\,(y)$, $\Omega\,(z)$ are zero, one can then conclude that the condition (3) is equivalent to $\Omega\,(x^2 + y^2 + z^2) = 0$.

Hence:

In order for equations (1) *to represent a surface that is referred to its lines of curvature, it is necessary and sufficient that the four functions x, y, z, and $(x^2 + y^2 + z^2)$ must satisfy the same partial differential equation of the form* (2).

That is obviously equivalent to saying that 1, x, y, z, $(x^2 + y^2 + z^2)$ *must satisfy the same partial differential equation of the more general form:*

$$(4) \qquad \frac{\partial^2\omega}{\partial\lambda\,\partial\mu} + L\frac{\partial\omega}{\partial\lambda} + M\frac{\partial\omega}{\partial\mu} + N\omega = 0.$$

Introduce the combinations:

$$(5) \qquad u = \frac{1 - x^2 - y^2 - z^2}{2}, \quad v = \frac{1 + x^2 + y^2 + z^2}{2i},$$

and let the term *penta-spherical coordinates* of a point with rectangular Cartesian coordinates x, y, z refer to the five quantities:

$$(6) \qquad x_1 = mx, \qquad x_2 = my, \qquad x_3 = mz, \qquad x_4 = mu, \qquad x_5 = mv,$$

in which m is an arbitrary proportional factor. They are related by the relation:

$$(7) \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0.$$

Conversely, if x_1, x_2, x_3, x_4, x_5 are five numbers that are coupled by the condition (7) then one will infer from equations (6), upon noting that $u + iv = 1$, that:

$$(8) \qquad m = x_4 + i\,x_5, \qquad x = \frac{x_1}{m}, \qquad y = \frac{x_2}{m}, \qquad z = \frac{x_3}{m},$$

and the condition (7) will give:

$$x_4 - i\,x_5 = -\,m\,(x^2 + y^2 + z^2) = m\,(u - iv).$$

One will then have:

$$x_4 + i\,x_5 = m\,(u + iv), \qquad x_4 - i\,x_5 = m\,(u - iv),$$

and the latter equations $x_4 = mu$, $x_5 = mv$ will be verified. Hence, five numbers that are linked by equation (7) will be the penta-spherical coordinates of a point.

Having said that, since equation (4) will transform into an equation of the same form when one makes the change of variable $\omega' = \omega \cdot \chi(\lambda, \mu)$, the result that was stated above can be translated as follows:

In order for equations (1) *to represent a surface that is referred to its lines of curvature, it is necessary and sufficient that the five penta-spherical coordinates of a point on that surface will satisfy the same partial differential equation of the form* (4).

Any homogeneous linear combination with constant coefficients of several integrals of (4) will again be an integral. Hence, the same result will persist when one replaces the previously-defined penta-spherical coordinates with the *general penta-spherical coordinates* that one deduces from an arbitrary *orthogonal, homogeneous, linear transformation:*

(9) $$x'_h = \sum_{k=1}^{5} \alpha_{hk} x_k \qquad (h = 1, 2, 3, 4, 6).$$

Saying that this transformation is orthogonal signifies that it leaves the quadratic form $\sum_{h=1}^{5} x_h^2$ invariant; i.e., that equations (9) imply the identity:

(10) $$\sum_{h=1}^{5} x_h'^2 = \sum_{h=1}^{5} x_h^2 .$$

Those orthogonal transformations possess some properties that are quite similar to those of the analogous transformations of three variables; i.e., changes of rectangular coordinates (without displacing the origin).

The identity (10) is equivalent to the *orthogonality conditions:*

(11) $$\sum_{h=1}^{5} \alpha_{hk}^2 = 1, \qquad \sum_{h=1}^{5} \alpha_{hk} \alpha_{hk'} = 1 \qquad (k \neq k' = 1, 2, 3, 4, 5).$$

Hence, one will deduce the equivalent inverse formulas:

(12) $$x_k = \sum_{k=1}^{5} \alpha_{hk} x'_k \qquad (k = 1, 2, 3, 4, 5)$$

by combinations of equations (9), which satisfy orthogonality conditions that are analogous to (11), since the identity (10) does not cease to be true. The orthogonality conditions thus-defined:

(13) $$\sum_{k=1}^{5} \alpha_{hk}^2 = 1, \qquad \sum_{k=1}^{5} \alpha_{hk} \alpha_{h'k} = 1 \qquad (h \neq h' = 1, 2, 3, 4, 5)$$

will then be equivalent to the conditions (11).

Upon squaring the determinant $\Delta = [\alpha_{hk}]$ of the forms (9), one will see that it is equal to ± 1, and upon conveniently choosing the notations (i.e., the order in which those give linear forms are enumerated), one can suppose that it is equal to 1. The identification of formulas (12) with the ones that give the application of Cramer's rule to equations (9) will again give equality between the elements of Δ and the corresponding minors:

$$\alpha_{hk} = \frac{\partial \Delta}{\partial \alpha_{hk}} \qquad (h, k = 1, 2, 3, 4, 5). \tag{14}$$

Interpretation of general penta-spherical coordinates. – It results immediately from the defining formulas (6) that *any homogeneous linear equation:*

$$0 = \sum_{h=1}^{5} a_h x_h \equiv \frac{-m}{2} [(a_4 + i\, a_5)\,(x^2 + y^2 + z^2) - 2a_1\, x - 2a_2\, y - 2a_3\, z - (a_4 - i\, a_5)] \tag{15}$$

represents a sphere, and conversely. We can suppose that the coefficients a_h, which are defined only up to a constant factor, are chosen in such a manner that they satisfy the orthogonality condition:

$$\sum a_h^2 = 1. \tag{16}$$

We then say that a_1, a_2, a_3, a_4, a_5 are the *coordinates of the sphere.*

One confirms immediately that the radius R of that sphere is given by:

$$R^2 = \frac{a_1^2 + a_2^2 + a_3^2 + (a_4 + ia_5)(a_4 - ia_5)}{(a_4 + ia_5)^2} = \frac{1}{(a_4 + ia_5)^2}.$$

For example, and this amounts to choosing the sign ±, which was left arbitrary by the condition (16), one can take:

$$R = \frac{1}{a_4 + ia_5}. \tag{17}$$

The power of the point $(x_1, x_2, x_3, x_4, x_5)$ with respect to the sphere in question then has the expression:

$$P_x = -\frac{2R}{m} \cdot \sum_{h=1}^{5} a_h\, x_h\,. \tag{18}$$

Now consider a second sphere that is similarly defined by its coordinates b_h (h = 1, 2, 3, 4, 5), and radius R'. The angle V between the two spheres is then given by:

$$2RR'\cos V = \frac{2(a_1b_1 + a_2b_2 + a_3b_3) + (a_4 - ia_5)(b_4 + ib_5) + (b_4 - ib_5)(a_4 + ia_5)}{(a_4 + ia_5)(b_4 + ib_5)},$$

so

(19) $$\cos V = \sum_{h=1}^{5} a_h \, x_h \, .$$

This cosine is then defined unambiguously, as long as one is given the signs of the radii of the two spheres. One will note the analogy between formulas (16) and (19) and the ones that are concerned with *directions* in Cartesian geometry with rectangular coordinates.

Having said that, the interpretation of the coordinates (9) is immediate. The equations $x_h' = 0$ (h = 1, 2, 3, 4, 5) define five spheres (S_1), (S_2), (S_3), (S_4), (S_5) that have the coefficients of the left-hand sides of the corresponding equations (9) for their coordinates. From the conditions (11), those spheres are pair-wise orthogonal: They constitute what one calls an *orthogonal penta-sphere*, which will serve as the *reference penta-sphere* for the definition of the coordinates (9). *From formula* (18), *the penta-spherical coordinates* (9) *are themselves proportional to the quotients that are obtained by dividing the powers of the point M considered with respect to the five reference spheres by the respective radii of those spheres.*

Here is another interpretation that we will find useful. Let M be the point considered, and suppose that its coordinates x_1' and x_2' are not both zero; i.e., that there is no common point to the spheres (S_1) and (S_2). We can then determine one and only one sphere (S) that passes through M and cuts the spheres (S_3), (S_4), (S_5) at a right angle, because the coordinates b_1, b_2, b_3, b_4, b_5 of (S) will be defined by the conditions:

(20) $$\sum_{h=1}^{5} b_h \, x_h = 0, \quad \sum_{h=1}^{5} b_h \, \alpha_{3h} = 0, \quad \sum_{h=1}^{5} b_h \, \alpha_{4h} = 0, \quad \sum_{h=1}^{5} b_h \, \alpha_{5h} = 0.$$

Those equations in b_1, b_2, b_3, b_4, b_5 are independent, since otherwise one would have:

$$x_h = \lambda_3 \, \alpha_{3h} + \lambda_4 \, \alpha_{4h} + \lambda_5 \, \alpha_{5h} \qquad (h = 1, 2, 3, 4, 5),$$

and in turn, due to the orthogonal conditions, $x_1' = x_2' = 0$, which is contrary to hypothesis.

Let V_1 and V_2 denote the angles that (S) makes with (S_1) and (S_2), resp. They are defined by the formulas:

(21) $$\cos V_1 = \sum_{h=1}^{5} b_h \, \alpha_{1h} \, , \qquad \cos V_2 = \sum_{h=1}^{5} b_h \, \alpha_{2h} \, ,$$

and if one takes into account the fact that $\sum b_h^2 = 1$ and the orthogonality conditions (13) then that will imply the condition:

(22) $$\cos^2 V_1 + \cos^2 V_2 = 1,$$

in such a way that the two angles V_1 and V_2 will be complementary. One relation will then suffice to determine them; one obtains it by eliminating the b_h from equations (20) and (21). Upon leaving aside the first equation in (20), one will infer:

(23) $$b_h = \alpha_{1h} \cos V_1 + \alpha_{2h} \cos V_2 \qquad (h = 1, 2, 3, 4, 5),$$

and upon substituting these values in the equation $\sum b_h x_h = 0$, one will get:

(24) $$x'_1 \cos V_1 + x'_2 \cos V_2 = 0.$$

Equations (22) and (24) determine cos V_1 and cos V_2 up to a factor ($\pm$ 1); that comes from the indeterminacy in the definition of (S) that pertains to the sign of its radius. However, no matter what sign is adopted, formula (24) will unambiguously give the ratio of the coordinates x'_1 and x'_2 as a function of cos V_1 and cos V_2.

One should note that if x'_2, for example, is zero then the solution to equations (20) will be given by $b_h = \alpha_{1h}$; i.e., the sphere (S) will then be the sphere (S_1). As a result, cos V_1 = 1, cos V_2 = 0, and formula (24) will give $x'_1 / x'_2 = 0$, since x'_2 is non-zero, by hypothesis.

We then conclude that *the penta-spherical coordinates of a point*, which are defined only up to the same factor, *are determined completely by the cosines of the angles that the spheres that pass through that point and are orthogonal to three spheres of the reference penta-sphere make with the other two spheres of that penta-sphere.*

Remark 1. – From formulas (12), the sphere that has the coordinates b_1, b_2, b_3, b_4, b_5 in the initial system of penta-spherical coordinates x_1, x_2, x_3, x_4, x_5 has the equation:

$$\sum_{h=1}^{5} \cdot \left(\sum_{k=1}^{5} b_k \, \alpha_{hk} \right) \cdot x'_h = 0$$

in the general system of coordinates (9).

One will then say that the quantities:

(29) $$b'_h = \sum_{k=1}^{5} b_k \, \alpha_{hk} \qquad (h = 1, 2, 3, 4, 5)$$

are the coordinates of the sphere in the new system. It results from the orthogonality conditions (11) that such coordinates will further satisfy the orthogonality condition that is analogous to (16):

$$\sum_{h=1}^{5} b'^2_h = 1.$$

The coordinate transformation of the spheres is then performed like that of the coordinates of points.

It further results from the orthogonality conditions (11) that formula (19), which gives the angle between two spheres, will keep the same form in general penta-spherical coordinates.

Remark 2. – Let x_0, y_0, z_0 be the coordinates of the center of a sphere (S), let R be its radius, and let P be the power of the origin with respect to (S): The five penta-spherical coordinates of (S) that are defined by equation (15) and the condition $\sum a_k^2 = 1$ are:

$$(30) \qquad a_1 = \frac{x_0}{R}, \qquad a_2 = \frac{y_0}{R}, \qquad a_3 = \frac{z_0}{R}, \qquad a_4 = \frac{1-P}{2R}, \qquad a_1 = \frac{1+P}{2iR}.$$

One can replace them with the six *homogeneous coordinates* c_1, c_2, c_3, c_4, c_5, c_6 , which are coupled with the symmetric relation:

$$(31) \qquad \sum_{k=1}^{6} c_k^2 = 0.$$

To that effect, we set:

$$(32) \qquad \begin{aligned} &c_1 = \rho\,(1-P), \qquad && c_2 = -\rho\, i\,(1-P), \qquad && c_3 = 2\rho\, x_0\,, \\ &c_4 = 2\rho\, y_0\,, && c_5 = 2\rho\, z_0\,, && c_6 = -2i\rho R, \end{aligned}$$

in which ρ is an arbitrary factor.

For $c_6 = 0$, the sphere has radius zero, and c_1 , c_2 , c_3 , c_4 , c_5 are the penta-spherical coordinates of its center. For $c_6 \neq 0$, formulas (32) are equivalent to the following ones:

$$(33) \qquad a_1 = \frac{c_3}{ic_6}, \qquad a_2 = \frac{c_4}{ic_6}, \qquad a_3 = \frac{c_5}{ic_6}, \qquad a_4 = \frac{c_1}{ic_6}, \qquad a_5 = \frac{c_2}{ic_6}.$$

One can employ formulas that are analogous to the latter in order to pass from the general penta-spherical coordinates of a sphere that is defined by equations (29) to the homogeneous coordinates that satisfy the condition (31).

Formula (19) shows that a linear, homogeneous relation:

$$(34) \qquad \sum_{k=1}^{6} C_k\, c_k = 0,$$

in which the C_k are arbitrary constants, expresses the idea that the sphere (S) cuts the sphere (S') with the homogeneous coordinates:

$$(35) \qquad c'_h = C_h (h = 1, 2, 3, 4, 5), \qquad c'_6 = i\sqrt{C_1^2 + C_2^2 + \cdots + C_5^2}$$

at a constant angle V that is given by the formula:

$$(36) \qquad \cos V = \frac{C_6}{c'_6}.$$

In the case where the constants C_k verify the condition $\sum_{k=1}^{6} C_k^2 = 1$, one will have $c_6' = C_6$. The constants C_k are themselves the coordinates of the sphere (S) then, and the condition (34) will express the idea that the two spheres (S) and (S') are tangents.

Remark 3. – In the case where the sphere (S) reduces to the plane $\lambda x + \mu y + \nu z - \delta = 0$, where λ, μ, ν are the direction cosines of a direction that is normal to the plane, the coordinates a_h will be:

$$(37) \qquad a_1 = \lambda, \quad a_2 = \mu, \quad a_3 = \nu, \quad a_4 = -\delta, \quad a_5 = -i\delta \qquad (a_4 + i\, a_5 = 0).$$

Remark 4. – One can pass directly from the coordinate system x_h' that relates to an orthogonal penta-sphere (II) to the coordinate system x_h'' that relates to another orthogonal penta-sphere (II′). From the formulas:

$$x_k = \sum_{h=1}^{6} \alpha_{hk}\, x_h', \qquad x_l'' = \sum_{h=1}^{6} \beta_{lk}\, x_k \qquad (k, l = 1, 2, 3, 4, 5),$$

one concludes, in fact, that:

$$(38) \qquad x_l'' = \sum_{h=1}^{6} \cdot \left(\sum_{k=1}^{6} \alpha_{lk}\, \beta_{hk} \right) x_h' = \sum_{h=1}^{6} \beta_{lk}'\, x_k' \qquad (l = 1, 2, 3, 4, 5).$$

In that expression for the coordinates x_l'', the coefficients:

$$\beta_{lh}' = \sum_{k=1}^{6} \beta_{lk}\, \alpha_{hk} \qquad (h = 1, 2, 3, 4, 5)$$

are again the coordinates of the new reference sphere (S_l') with respect to the first penta-sphere (II). The analogy with the formulas for the transformation of rectangular Cartesian coordinates (without displacing the origin) is obvious.

Condition for a surface to be isothermal. – From what we saw in § 5, in order for the surface considered to be isothermal, it is necessary and sufficient that equation (4) can be reduced to the form of equation (23) of § 5 by a transformation $\omega' = \omega \cdot \chi(\lambda, \mu)$. Therefore:

In order for equations (1) *to represent an isothermal surface that is referred to its lines of curvature, it is necessary and sufficient that the five penta-spherical coordinates of a point satisfy the same partial differential equation of the form:*

$$\frac{\partial^2 \omega}{\partial \lambda \partial \mu} = \theta(\lambda, \mu) \cdot \omega \tag{39}$$

for a convenient choice of proportionality factor that figures in those coordinates.

Remark. – An argument that is similar to the one at the beginning of this paragraph can be made for the coordinates of a tangent plane to the surface, which one assumes can be written in the form:

$$ax + by + cz + 1 = 0.$$

The coefficients are functions of λ and μ, and in order for the surface that is defined to be envelope of those planes to have the lines λ = const., μ = const. for its lines of curvature, it is necessary and sufficient that 1, a¸ b, c $(a^2 + b^2 + c^2)$ satisfy the same equation of the form (4).

Application to the cyclides

7. – Let x_1, x_2, x_3, x_4, x_5 be the five penta-spherical coordinate of a point in an arbitrary system of such coordinates. A surface will be represented by a homogeneous equation between those coordinates. We have seen that the case in which that equation has degree one corresponds to the sphere. The surfaces that are represented by second-degree equations will be called *cyclides.*

It results from the theory of quadratic forms that if:

$$\Phi(x_1, x_2, x_3, x_4, x_5)$$

is a homogeneous second-degree polynomial then one can always find a homogeneous linear transformation:

$$x'_k = \sum_{h=1}^{5} \alpha_{kh} \, x_h \qquad (k = 1, 2, \ldots, 5)$$

that leaves the form $\sum x_h^2$ invariant and transforms Φ into:

$$\Phi(x_1, x_2, x_3, x_4, x_5) = \sum_{h=1}^{5} s_h \, x'^2_h .$$

There will then exist a change of penta-spherical coordinates that will reduce the equation of any cyclide to the form:

$$\sum_{h=1}^{5} s_h \, x_h^2 = 0.$$

If one discards the special case in which one or more of the s_h (which are roots of the equation in s that is obtained by equating the discriminant of $\Phi - s\sum_{h=1}^{5} x_h^2$ to zero) are zero then one can take the equation of the cyclide in the form:

$$\sum_{h=1}^{5}\frac{x_h^2}{a_h} = 0,$$

and consider it to belong to the family of cyclides that is represented by the equation:

$$\sum_{h=1}^{5}\frac{x_h^2}{a_h-\sigma} = 0, \tag{1}$$

in which σ is an arbitrary parameter.

By hypothesis, the coordinates x_h are coupled by the condition $\sum_{h=1}^{5} x_h^2 = 0$, so equation (1) will be an equation of degree three in σ, in such a way that three cyclides of the family will pass through each point of space. The parameters σ_1, σ_2, σ_3 of those three cyclides will then be the curvilinear coordinates for the points of space. One calculates the x_h as functions of σ_1, σ_2, σ_3 by the same mode of calculation that served for the analogous problem that related to the families of homofocal quadrics. Set:

$$\varphi(\sigma) = \prod_{h=1}^{5}(\sigma - a_h),$$

and we can write down the identity:

$$\sum_{h=1}^{5}\frac{x_h^2}{\sigma-a_h} = \frac{(\sigma-\sigma_1)(\sigma-\sigma_2)(\sigma-\sigma_3)}{\varphi(\sigma)},$$

upon neglecting the identification factor on the right-hand side, since the x_h can be calculated up to the same factor. Here, one has the identity for decomposing the right-hand side, which is a rational fraction in s, into simple elements, so:

$$x_h^2 = \frac{(a_h-\sigma_1)(a_h-\sigma_2)(a_h-\sigma_3)}{\varphi'(a_h)} \qquad (h = 1, 2, \ldots, 5). \tag{2}$$

If one supposes that σ_3 = const. then one will have the parametric representation of any of the surfaces (1).

Now, if one sets, in general:

$$\omega = \sqrt{(a-\sigma_1)(a-\sigma_2)}$$

then one will have:

$$\frac{\partial\omega}{\partial\sigma_1} = \frac{\sigma_2 - a}{2\omega}, \qquad \frac{\partial\omega}{\partial\sigma_2} = \frac{\sigma_1 - a}{2\omega},$$

$$\frac{\partial^2\omega}{\partial\sigma_1\,\partial\sigma_2} = \frac{1}{2\omega} - \frac{(\sigma_1 - a)(\sigma_2 - a)}{4\omega^3} = \frac{1}{4\omega},$$

so

$$(3) \qquad 2\,(\sigma_1 - \sigma_2)\,\frac{\partial^2\omega}{\partial\sigma_1\,\partial\sigma_2} + \frac{\partial\omega}{\partial\sigma_1} - \frac{\partial\omega}{\partial\sigma_2} = 0.$$

This is an equation of the form (11), § 5: Indeed, one needs only to set $k = (\sigma_1 - \sigma_2)^{-1}$ in equation (11) to recover present equation (3). It will then be reducible to the form (39), § 6, by a transformation $\omega' = \omega\chi$.

Therefore, the penta-spherical coordinates (2) of any cyclide (1) indeed satisfy the condition that was stated above, and *the cyclides are isothermal surfaces.*

Remark 1. – It is then proved that the three cyclides of the system (1) that pass through a point will intersect pair-wise along common lines of curvature: They will then cut at a right angle, and as a result, any two of the cyclides will cut at a right angle all along their intersection.

Remark 2. – An analogous calculation applies to the homofocal quadrics:

$$\sum_{h=1}^{3} \frac{x_h^2}{a_h - \sigma} - 1 = 0,$$

in which x_1, x_2, x_3 are rectangular coordinates. One finds that:

$$x_h^2 = \frac{(a_h - \sigma_1)(a_h - \sigma_2)(a_h - \sigma_3)}{\varphi'(a_h)}, \qquad \varphi(\sigma) = (\sigma - a_1)(\sigma - a_2)(\sigma - a_3).$$

Therefore, x_1, x_2, x_3 satisfy equation (13). It remains to verify that $\sum_{h=1}^{3} x_h^2$ also satisfies it. Now, the substitution of that function in the left-hand side of (13) will give:

$$(\sigma_1 - \sigma_2) \cdot \sum_{h=1}^{3} \frac{a_h - \sigma_3}{\varphi'(a_h)},$$

and when one identifies that with the left-hand side, and expresses the idea that there is no term in σ^2 in the right-hand side, the identity:

$$\frac{\sigma - \sigma_3}{\varphi(\sigma)} = \sum_{h=1}^{3} \frac{a_h - \sigma_3}{\varphi'(a_h)(\sigma - a_h)}$$

will give:

$$\sum_{h=1}^{3} \frac{a_h - \sigma_3}{\varphi'(a_h)} = 0.$$

Application to the conformal transformations

8. – *Definitions.* – Consider a *point-like* transformation; i.e., one that makes any point of M correspond to a homologous point M' (as displacements, homotheties, and inversions do, for example). It is defined by equations:

$$(1) \qquad x' = f(x, y, z), \qquad y' = g(x, y, z), \qquad z' = h(x, y, z)$$

that give the coordinates (x', y', z') of M' as functions of the coordinates (x, y, z) of M. Inversely, one supposes that each point M' corresponds to a point M; i.e., that equations (1) define the implicit functions:

$$(2) \qquad x = F(x', y', z'), \qquad y = G(x', y', z'), \qquad z = H(x', y', z').$$

In order to do that, as one knows, it will suffice that f, g, h must have continuous partial derivatives and that the functional determinant $\frac{\partial(f, g, h)}{\partial(x, y, z)}$ must not be identically zero.

The transformation makes any locus of points M correspond to a homologous locus of points M': e.g., a curve will go to a curve, and a surface, to a surface. Two curves that intersection at M_0 will correspond to two curves at the point M_0' that is the homologue of M_0, and two curves that are tangent at M_0 will go to two curves that are tangent at M_0'.

That will result from what one deduces from equations (1) upon differentiating them:

$$(3) \qquad dx' = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \qquad dy' = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz,$$

$$dz' = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz,$$

in such a way that each *linear element* $(x, y, z\,;\, dx, dy, dz)$ will correspond to a homologous linear element $(x', y', z'\,;\, dx', dy', dz')$, which is the same for any curve that passes through M and to which the former element belongs. One says that the transformation of the linear elements of space, thus-defined, results from a *prolongation* of the transformation (1).

From formula (3), the square ds'^2 of the transformed linear element is a quadratic form in dx, dy, dz whose coefficients are functions of x, y, z, namely:

$$(4) \qquad ds'^2 = \Phi(dx, dy, dz),$$

and the angle between the two linear elements that are homologous to two linear elements at the same point (x, y, z) (which we suppose to correspond to two different differentiations d and δ) is given by the formula:

(5) $$\cos V' = \frac{\sum \frac{\partial \Phi}{\partial(dx)} \delta x}{\sqrt{\Phi(dx, dy, dz)}\sqrt{\Phi(\delta x, \delta y, \delta z)}}.$$

Having said that, one says that *the transformation* (1) *is a conformal transformation if it preserves angles;* i.e., if the homologues of two arbitrary curves that cut at M make an angle at the homologous point M' that is equal to the one that the former two made at M. That amounts to saying that the angle between any two linear elements at the same point M is equal to the angle between the transformed linear elements.

If that were true then a right angle, in particular, would correspond to a right angle, and as a result the equation:

$$\frac{\partial \Phi}{\partial(dx)} \delta x + \frac{\partial \Phi}{\partial(dy)} \delta y + \frac{\partial \Phi}{\partial(dz)} \delta z = 0$$

would be a consequence of the equation:

$$dx\,\delta x + dy\,\delta y + dz\,\delta z = 0$$

for any x, y, z. One would then conclude an identity of the form:

$$\frac{\partial \Phi}{\partial(dx)} \delta x + \frac{\partial \Phi}{\partial(dy)} \delta y + \frac{\partial \Phi}{\partial(dz)} \delta z = 2k^2 (x, y, z)\,(dx\,\delta x + dy\,\delta y + dz\,\delta z),$$

since those two equations are homogeneous and have degree two in the differentials. In the particular case in which $\delta x = dx$, $\delta y = dy$, $\delta z = dz$, that identity will imply the following one:

(6) $$\Phi\,(dx, dy, dz) = 2k^2 (x, y, z)\,(dx^2 + dy^2 + dz^2).$$

Hence: *Any conformal transformation will imply an identity of the form:*

(7) $$ds'^2 = k^2 \cdot ds^2 ;$$

i.e., it will transform all of the linear elements at the same point by a constant ratio, and that ratio k will be a function of the coordinates of the point considered.

Conversely, if such an identity (7) or (6) were valid then formula (5) would reduce to:

$$\cos V' = \frac{\sum dx\,\delta x}{\sqrt{dx^2 + dy^2 + dz^2}\sqrt{\delta x^2 + \delta y^2 + \delta z^2}} = \cos V,$$

and the transformation would be a conformal transformation.

The preceding property can then be taken to be the definition of conformal transformations. [Cf., Chap. II, § 2.]

Search for the conformal transformations. – By virtue of the identity (7), any conformal transformation will change the equation $ds^2 = 0$ into $ds'^2 = 0$. It will then change any minimal curve into a minimal curve, and in turn, any isotropic developable whose minimal curves coincide into a surface with double minimal curves; i.e., an isotropic developable.

Having said that, consider an isotropic line. One can find two isotropic developables that touch along that line in an infinitude of ways. Their transforms will touch along a common minimal line, and therefore along an isotropic line. Hence, any isotropic line will have an isotropic line for its homologue, any isotropic ruled surface will become an isotropic ruled surface, and any sphere that is doubly generated by isotropic lines will change into a doubly-ruled surface with isotropic generators – i.e., a sphere.

Conversely, any point-like transformation that changes spheres into spheres will change any pair of isotropic lines that one can always consider to be the curve of intersection of two tangent spheres into a pair of isotropic lines. It will then change the isotropic lines that pass through a point M into isotropic lines that pass through its homologue M'. As a result, it will change the set of isotropic linear elements at that point, which are characterized by the equation $ds^2 = 0$, into the analogous set that is characterized by the equation $ds'^2 = 0$. It will then give rise to an identity of the form (7), and it will be a conformal transformation.

The conformal transformations of three-dimensional space will then be the transformations that change any sphere into a sphere.

Having said that, let (T) be a conformal transformation, and suppose that the points M are referred to an orthogonal penta-sphere (π). The transformation (T) changes spheres into spheres and preserves angles, so it will change the penta-sphere (π) into another orthogonal penta-sphere (π'). The coordinates of the homologue M' to M, when taken with respect to (π'), are the same as the coordinates of M with respect to (π), because the latter coordinates depend upon only the angles that the spheres that are drawn through M normally to three spheres of (π) will make with the other two spheres of (π). [Cf., page #-9], and since the transformation (T) does not alter angles, it will not alter the coordinates of the point with respect to the penta-sphere, which are supposed to transform at the same time as that point.

Therefore, let x_1, x_2, x_3, x_4, x_5 be the coordinates of M with respect to the penta-sphere (π), and let $b_h = \beta_{kh}$ ($k = 1, 2, \ldots, 5$; $h = 1, 2, \ldots, 5$) be the coordinates with respect to (π) of the spheres that the transformation (T) substitutes for the spheres $x_k = 0$ ($k = 1, 2, \ldots, 5$), respectively. The powers of the point M' with respect to those spheres will be quantities that are proportional to the coordinates x_k of M, multiplied by their radii R'_k, respectively. On the other hand, one will have values that are proportional to the expressions $R'_k \cdot \sum_{h=1}^{5} \beta_{kh}\, x'_h$ for the same products [§ 6, formula (18)]. The formulas of the transformation (T) can then be written:

(8) $$x = \sum_{h=1}^{5} \beta_{kh}\, x'_h\,, \qquad x'_h = \sum_{k=1}^{5} \beta_{kh}\, x_k \qquad (k, h = 1, 2, \ldots, 5).$$

Therefore, the conformal transformations are represented by orthogonal, homogeneous, linear transformations in penta-spherical coordinates. Consequently, they define a group of ∞^{10} *transformations*, since twenty-five coefficients figure in them that are linked by fifteen independent relations. The word *group* indicates that when two of those transformations are performed in succession, that will give another conformal transformation as a final result, which is obvious *a priori*.

One proves that each of those transformations can be decomposed into displacements, homotheties, and inversions.

Remark. – If we compare those formulas (8) with the formulas for the change of penta-spherical coordinates, which are defined by replacing the reference penta-sphere (π) with the penta-sphere (π') that is homologous to (π) under the transformation (T), and those formulas are [§ 6, eq. (38)]:

$$x'_k = \sum_{h=1}^{5} \beta_{kh}\, x_h\,,$$

then we will see that inversions correspond to changes of coordinates in penta-spherical coordinates, just as displacements correspond to changes of rectangular coordinates in Cartesian geometry. The preceding analysis gives the reason for that analogy.

Conformal transformations of the plane. – The point-like transformations of a plane:

(9) $$x' = f(x, y), \qquad y' = g(x, y)$$

define and *prolong* to transformations of linear elements (x, y; dx, dy), just as the point-like transformations of space do. The *conformal transformations* are defined by the invariance of angles, and by reasoning as above, one confirms that this invariance is equivalent to the invariance of ds^2, up to a coefficient k^2. Upon developing that identity:

$$df^2 + dg^2 = k^2\,(dx^2 + dy^2),$$

one will obtain the conditions:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial x}\right)^2 = k^2, \qquad \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\frac{\partial g}{\partial y} = 0, \qquad \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 = k^2.$$

One then concludes from the Lagrange identity that:

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = \varepsilon\, k^2 \qquad (\varepsilon = \pm 1),$$

in which ε is equal to + 1 or – 1, as is easy to verify, according to whether the homologous angles have the same disposition or opposite dispositions.

Be that as it may, one has two linear equations in $\partial f / \partial x$ and $\partial f / \partial y$, so one will infer:

$$\frac{\partial f}{\partial x} = \varepsilon \frac{\partial g}{\partial y}, \qquad \frac{\partial f}{\partial y} = -\varepsilon \frac{\partial g}{\partial x},$$

which is equivalent to saying that $f + ig$ (when $\varepsilon = + 1$) and $g + if$ (when $\varepsilon = - 1$) is an analytic function of $x + iy$.

The study of conformal transformations of the plane is then equivalent to the theory of analytic functions of a complex variable.

Those transformations depend upon an arbitrary function, and no longer upon a certain number of arbitrary constants, as in the case of space. It will no longer be exact to say that any conformal transformation changes any circle into a circle, but one can look for the point-like transformations of the plane that do change any circle into a circle, just as we looked for the transformations of space that changed any sphere into a sphere.

To that effect, one introduces *tetra-cyclic* coordinates, which will be:

$$x_1 = mx, \qquad x_2 = my, \qquad x_1 = m\,\frac{1-x^2-y^2}{2}, \qquad x_4 = m\,\frac{1+x^2+y^2}{2i},$$

and more generally, combinations of them:

$$x'_h = \sum_{k=1}^{4} \alpha_{hk}\, x_k \qquad (h = 1, 2, 3, 4)$$

will define orthogonal, homogeneous, linear transformations in four variables. One will find that in arbitrary tetra-cyclic coordinates, *the transformations that change any circle into a circle are defined by the various orthogonal, homogeneous, linear transformations in four variables.* One will then have a group of ∞^6 transformations, which one calls the *group of reciprocal radius vectors* because its transformations can be decomposed into displacements, homotheties, and inversions (or transformations by reciprocal radius vectors).

Invariance of the lines of curvature and isothermal nets. – We return to the case of space: From a remark that was made already, if the penta-spherical coordinates x_1, x_2, x_3, x_4, x_5 of a point on a surface satisfy an equation of the form (4), § 6, then the variables x'_1, x'_2, x'_3, x'_4, x'_5 that one deduces by an arbitrary homogeneous, linear transformation will satisfy the same equation. Hence, if the equations (1), § 6 represent a surface that is referred to its lines of curvature then the same thing will be true for the equations that are deduced from them by an arbitrary conformal transformation.

In other words, *the conformal transformations leave invariant the property of a curve on a surface that it is a line of curvature.* [Cf., Chap. XI, § 6].

On the other hand, since the conformal transformations multiply the ds^2 at a point M by a function of the coordinates of the point M, they will not at all alter the form ds^2 of a surface that characterizes the isothermal orthogonal coordinates. Hence, *the conformal transformations will leave invariant the property of a net of curves on a surface that it defines an isothermal, orthogonal net.*

One concludes from this that *conformal transformations change every isothermal surface into an isothermal surface.* That will also result from the remark that was made for the lines of curvature; equation (4), § 6 will then be reducible to the form:

$$\frac{\partial^2 \omega}{\partial \lambda \partial \mu} = \omega \cdot \theta(\lambda, \mu).$$

Remark. – The last results can be established and completed without calculation by the following geometric considerations: From the remark that was made in Chap. VI, § 4 in regard to the lines of curvature, any line of curvature is a locus of points M of the surface (S) considered, such that it will be possible to associate each of its points with a sphere that is tangent to (S) in such a manner that the sphere that is tangent to (S) at M will also be tangent to the infinitely-close sphere at that point. It will then result immediately that any point-like transformation that changes any sphere into a sphere will change any line of curvature of (S) into a line of curvature of the homologous surface by that fact itself.

Conversely, any point-like transformation that changes any line of curvature into a line of curvature will change any isotropic ruled surface that is not developable or spherical into a surface of the same nature, because those surfaces are the only ones whose lines of curvature are double [Chap. III, § 7]. Moreover, the lines of curvature of those surfaces are their isotropic generators, and an isotropic line can be considered to be the generator of such a surface in an infinitude of ways, so the transformation will change any isotropic line into an isotropic line, and as a result, as we have seen above, any sphere into a sphere. Therefore, *any point-like transformation that changes any line of curvature into a line of curvature is a conformal transformation.*

On the other hand, any conformal transformation that preserves the angles and the ratios of infinitely-close arc lengths that issue from the same point will transform any net of infinitely-small squares that is traced on a surface into a similar net that is traced on the transformed surface. In other words, *any conformal transformation will change any isothermal, orthogonal net that is traced on a surface into an isothermal, orthogonal net of the homologous surface.*

Upon combining the two results thus-obtained, one will conclude that any conformal transformation will change any isothermal surface into an isothermal surface.

Conversely, *any point-like transformation that changes any isothermal surface into an isothermal surface is a conformal transformation.* Indeed, it must change any sphere into a sphere, because the sphere (the plane being regarded as a special case of a sphere) is the only surface that is isothermal in an infinitude of ways.

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CHAPTER IX

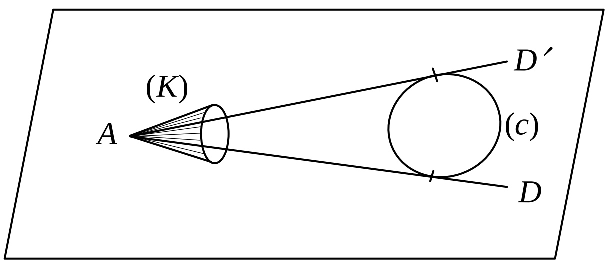
LINE COMPLEXES AND FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Fundamental elements of a line complex

1. – One calls a system of ∞^3 lines – i.e., a family of lines that depend upon three parameters – a *complex.*

Let A be a point of space. There are ∞^1 lines (D) of the complex that pass through that point, and they constitute the *cone of the complex* that is attached to A: We call it the cone (K).

Correlatively: Let (P) be a plan. There are ∞^1 lines of the complex in that plane, and they envelop a curve (C) that is the *curve of the complex* that is associated with (P). The tangent at any point of that curve is a line of the complex.



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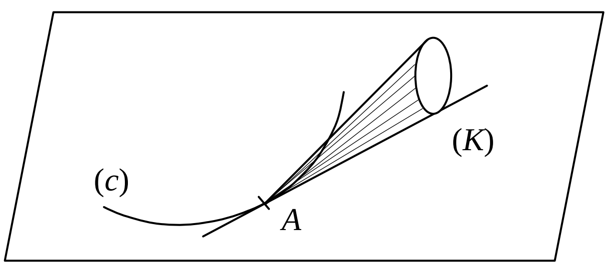

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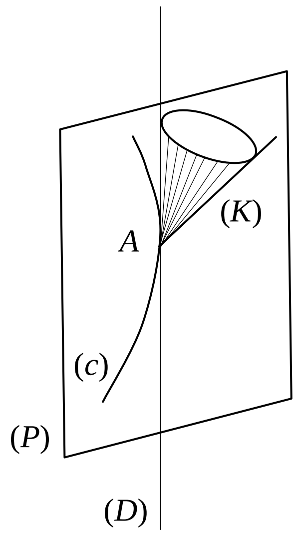
More generally, we call a curve (C) whose tangents all belong to the complex a *curve of the complex*. Consider a point A on such a curve and the cone of the complex (K) that is associated with the point A. That cone is tangent to the curve (C). *A curve of the complex is a curve that is tangent to the cones of the complex that are associated with each of its points.*

Consider a plane (P) and a point A on that plane. We seek the lines of the complex that are situated in the plane (P) and pass through A. One can obtain them in two ways: First, consider the cone of the complex that is associated with the point A. The desired lines are the generators of the cone that are situated in the plane (P). On the other hand, consider the curve of the complex that is associated with the plane (P). The desired lines will also be the tangents to that curve that issue from A. Having said that, we seek the locus of points A in the plane P such that two of the lines of the complex that are situated in the plane (P) and pass through A coincide. From the foregoing, the corresponding points A are the ones for which the corresponding cone of the complex is tangent to the plane (P), and must also be on the curve of the complex. The coincident lines of the complex will coincide with the contact generator of the cone of the complex and with the tangent to the curve of the complex. Hence: *The curve of the complex that is situated on a plane is the locus of the points of that plane for which the cone of the complex is tangent to the plane, and the contact generator at such a point is the tangent to the curve.* The curve of the complex is then defined by points and tangents.



Now consider a line (D) of the complex: Take a point A on that line, and consider the cone (K) of the complex that is associated with the point A. Let (P) be the tangent plane to that cone along the generator (D). Each point A of the line will then correspond to a plane (P). Consider the curve (C) of the complex that is situated in the plane (P), as well. It is tangent to the line (D) at precisely the point A in such a way that a point of that line will correspond to each plane (P) that passes through the line. *There is a homographic correspondence between the points and the planes of a line of the complex.*

Let us specify the nature of that homography. An arbitrary line is represented by two equations of the form:

(1) $$X = aZ + f, \qquad Y = bZ + g.$$

In order for it to belong to a complex, it is necessary and sufficient that there must exist a relation between the parameters a, b, f, g, namely:

(2) $$\varphi(a, b, f, g) = 0.$$

We then seek all of the lines of the complex that are infinitely close to the line (1) and meet that line. Such a line is represented by the equations:

(3) $$X = (a + da)\, Z + (f + df), \qquad Y\, (b + db)\, Z + (g + dg).$$

We express the idea that it meets the line (1). The equations:

(4) $$Z\,da + df = 0, \qquad Z\,db + dg = 0$$

must have a common solution in *Z*, which will give the condition:

(5) $$da \cdot dg - db \cdot df = 0.$$

The point of intersection *M* of the two infinitely-close lines will then have the parameter:

(6) $$Z = -\frac{df}{da} = -\frac{dg}{db}.$$

If we suppose that the point *M* is known then the relations (4), in which *Z* is known, will determine the ratios of the differentials. Moreover, the plane that passes through the two infinitely-close lines (1) and (3) is obtained by multiplying equations (3) by *db* and –*da*, respectively, and adding them, because upon taking (5) into account, that will give the equation of a plane that passes through the line (1):

(7) $$(X - aZ - f)\,db - (Y - bZ - g)\,da = 0.$$

The equation of that plane depends upon only the ratio *da* / *db*. We then conclude that *all of the lines of the complex that are infinitely close to the line* (*D*) *and meet that line at a given point M are in the same plane, and conversely, all of the lines of the complex that are infinitely close to the line* (*D*) *and situated in the same plane that passes through* (*D*) *will meet that line at the same point.* To abbreviate, set:

(8) $$\lambda = \frac{da}{db}.$$

Equation (7) is written:

(9) $$X - a\,Z - f - \lambda\,(Y - b\,Z - g) = 0.$$

We show that there is a homographic relation between λ and *Z*. For that, it will suffice to infer *df*, *dg* from equations (4) and substitute them into the identity:

$$\frac{\partial\varphi}{\partial a}da + \frac{\partial\varphi}{\partial b}db + \frac{\partial\varphi}{\partial f}df + \frac{\partial\varphi}{\partial g}dg = 0,$$

which results from differentiating the equation of the complex (2). One will get:

$$\left(\frac{\partial\varphi}{\partial a} - Z\frac{\partial\varphi}{\partial f}\right)da + \left(\frac{\partial\varphi}{\partial b} - Z\frac{\partial\varphi}{\partial g}\right)db = 0,$$

and from (8), the homographic relation will be:

(10) $$\lambda\left(\frac{\partial\varphi}{\partial a} - Z\frac{\partial\varphi}{\partial f}\right) + \frac{\partial\varphi}{\partial b} - Z\frac{\partial\varphi}{\partial g} = 0.$$

In particular, consider the cone of the complex with its summit at *M*. The infinitely-close generator is a line of the complex that meets (*D*) at *M*. The plane of those two lines is the tangent plane to the cone of the complex, and we recover the homography that was defined before.

Once more, let an arbitrary curve of the complex be tangent to the line (*D*) at the point *A*. Consider an infinitely-close tangent to that curve. In the limit, that tangent will meet (*D*) at the point *A*, and the plane of those two lines will be nothing but the osculating plane to the curve at the point *A*. Hence, that osculating plane will be associated with the point *A* under the preceding homography. Therefore: *All of the curves of the complex that are tangent to a line* (*D*) *at the same point A will have the same osculating plane at that point: It is the tangent plane to the cone of the complex that is associated with the point A.*

Finally, consider a congruence of lines that belongs to the complex. Take a line (*D*) in that congruence and a focal point *A* on that line. The point *A* belongs to one of the sheets of the focal surface of the congruence. It will also belong to the edge of regression that is one of the developables of the congruence, and that edge of regression, which is the envelope of the lines (*D*) that belong to the complex, will be a curve of the complex. Its osculating plane at *A* will be the second focal plane of the congruence. From the foregoing, *all of the congruences of the complex that pass through the line* (*D*) *and have a focus at A will have the same second focal plane that relates to the line* (*D*). There is a homographic correspondence between that second focal plane and the point *A*.

Surfaces of a complex

2. – We seek to find whether there are congruences in a complex that have a double focal surface. On such a surface (Φ), the edges of regression of the developables are the asymptotic lines [Chap. VI, § 1, pp. 127, § 2, pp. 132]; now, they are curves of the complex.

The problem then comes down to finding surfaces such that a family of asymptotic lines is composed of curves of the complex. Consider such an asymptote (*C*) and one of its points *A*. The osculating plane to the curve (*C*) at *A* is the tangent plane to the cone (*K*) of the complex that is associated at the point, and that osculating plane is tangent to the surface (Φ). The desired surfaces are then tangent at each of their points to the cone of the complex that is associated with each point. *Conversely*, let (Φ) be such a surface. Consider the contact generator (*D*) of the cone of the complex at each of its points with the tangent plane. There will exist a family of curves (*C*) on the surface (Φ) that are tangent at each of their points to that line (*D*) of the family that is thus associated with that point [cf., Chap. VI, pp. 126]. Those curves (*C*) are the curves of the complex. Their osculating plane is the tangent plane to the cone of the complex along the line (*D*). It is then the tangent plane to the surface (Φ), and the curves (*C*) are asymptotes of that surface.

Such surfaces that are tangent at each point to the cone of the complex that has that point for its summit are called *surfaces of the complex.*

Consider the equations of a line of the complex:

(1) $$x = az + f, \qquad y = bz + g.$$

a, b, f, g are linked by the equation:

(2) $$\varphi(a, b, f, g) = 0$$

on it.

Transport the origin to the point (x, y, z) and call the new coordinates X, Y, Z. X, Y, Z will then be the direction coefficients of a line that passes through the point x, y, z, and the angular coefficients of that line will be:

$$a = \frac{X}{Z}, \qquad b = \frac{Y}{Z},$$

so the equation of the cone of the complex that is associated with the point (x, y, z) will be:

$$\varphi\left(\frac{X}{Z}, \frac{Y}{Z}, x - \frac{X}{Z}z, y - \frac{Y}{Z}z\right) = 0,$$

or, upon making that homogeneous:

(3) $$\Psi(X, Y, Z, xZ - zX, yZ - zY) = 0.$$

It will then result that *the curves of the complex are defined by the differential equation:*

(4) $$\Psi(dx, dy, dz, x\,dz - z\,dx, y\,dz - z\,dy) = 0,$$

which is homogeneous in dx, dy, dz. One can consider it to be the equation of the complex itself since one can deduce it by replacing dx, dy, dz with X, Y, Z in the general equation (3) of the cones of complex, and one will then get back to equation (2) of the complex by setting:

$$X = a, \quad Y = b, \quad Z = 1, \quad x - az = f, \qquad y - bz = g.$$

Now introduce the tangential equation of the cone of complex:

(5) $$\chi(x, y, z, U, V, W) = 0,$$

which, by definition, will express the idea that the plane:

$$UX + VY + WZ = 0$$

is tangent to the cone (3).

The condition for such a surface $z = G(x, y)$ to be tangent to that cone at each of its points is that equation (5) must be verified by $U = \partial G / \partial x = p$, $V = \partial G / \partial x = q$, $W = -1$. *The surfaces of the complex are then defined by the partial differential equation:*

$$\chi(x, y, z, p, q, -1) = 0, \tag{6}$$

which has the form:

$$F(x, y, z, p, q) = 0. \tag{7}$$

We will then get a first-order partial differential equation that represents the complex from the tangential viewpoint, since one can immediately deduce the tangential equation (5) of the cone of the complex in the form:

$$F\left(x, y, z, -\frac{U}{W}, -\frac{V}{W}\right) = 0. \tag{8}$$

Conversely, any first-order partial differential equation (7) will express the idea that the tangent plane to an integral surface is tangent to the cone (8) that is associated with the point of contact. However, the generators of all those ∞^2 cones will generally fill up all of space and will form a complex only in the exceptional case.

Likewise, an arbitrary *Monge equation* – i.e., any equation of the form:

$$G(x, y, z, dx, dy, dz) = 0 \tag{9}$$

that is homogeneous in dx, dy, dz – will define the curves of a complex only exceptionally, because it will not reduce to the form (4), in general.

On certain partial differential equations

3. – In order to be able to specify those exceptional cases better, we recall some essential notions on the geometric theory of first-order partial differential equations; i.e., ones of the form:

$$F(x, y, z, p, q) = 0. \tag{1}$$

An *integral contact element* is a contact element whose coordinates (x, y, z, p, q) satisfy the given equation (1).

The *elementary cone* that is associated with the point (x, y, z) is the envelope of integral contact elements that belong to that point. With the preceding notations, its tangential equation is the equation:

$$F\left(x, y, z, -\frac{U}{W}, -\frac{V}{W}\right) = 0. \tag{2}$$

Any linear element that is composed of a point and a generator of the elementary cone that is associated with that point is called an *integral linear element.* If dx, dy, dz are the direction coefficients of one such generator then the equation that characterizes the integral linear elements will be obtained by looking for the point-wise equation of the cone that has equation (2) for its tangential equation and replacing the coordinates X, Y, Z with dx, dy, dz. That amounts to eliminating p and q from the equations:

$$(3) \qquad F(x, y, z, p, q) = 0, \quad dz - p\,dx - q\,dy = 0, \quad \frac{\partial F}{\partial p}dy - \frac{\partial F}{\partial q}dx = 0,$$

which will define the linear element along which the elementary cone with summit (x, y, z) touches the integral contact element (x, y, z, p, q).

The equation that one obtains is a *Monge equation:*

$$(4) \qquad G(x, y, z, p, q) = 0,$$

which is said to be associated with the partial differential equation (1).

The *integral curves* are the curves for which all of the linear elements (viz., points and tangent planes) are integral linear elements. They are defined by equation (4).

Conversely, any Monge equation (4) will define integral curves of a partial differential equation that one gets by passing from the point-wise equation:

$$(5) \qquad G(x, y, z, X, Y, Z) = 0$$

to the corresponding tangential equation (2) – i.e., upon eliminating dx, dy, dz from equation (4) and the equations:

$$(6) \qquad \frac{\partial G}{\partial(dx)} + p\frac{\partial G}{\partial(dz)} = 0, \qquad \frac{\partial G}{\partial(dy)} + q\frac{\partial G}{\partial(dz)} = 0,$$

which define the coefficients p, q of the tangent plane to the cone (5) along the generator:

$$\frac{X}{dx} = \frac{Y}{dy} = \frac{Z}{dz}.$$

If one appeals to the principle of duality then one will be led to consider another direction on each integral contact element, in addition to the integral linear element. Indeed, let A and (P) be the point and the plane that constitute the integral contact element (x, y, z, p, q). The elementary cone (K) with its summit at A that is the envelope of the planes that form integral contact elements with A will correspond to the curve (Γ) by duality, which is the locus of points M that will give integral contact elements when they are associated with the plane (P). The contact generator of the elementary cone (K) and the plane (P) that is the intersection of that plane and the infinitely-close tangent plane to (K) will correspond to the tangent to (Γ) at A that joins A to the infinitely-close point of (Γ). It is the direction of that tangent that must then intervene. We call the linear element that it defines with *A the characteristic linear element* of the contact element considered.

We now look for that characteristic element. Let (δx, δy, δz) be an infinitesimal displacement of the point A. If it defines the element considered then the contact element ($x + \delta x$, $y + \delta y$, $z + \delta z$) will be an integral contact element that is expressed by the conditions:

$$F(x + \delta x, y + \delta y, z + \delta z, p, q) = 0, \qquad \delta z - p\,\delta x - q\,\delta y = 0.$$

In the first of these, one must neglect the higher-order infinitesimals, and since equation (1) is verified, by hypothesis, what will remain are the equations:

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0, \qquad \delta z = p\,\delta x + q\,\delta y,$$

which will give the desired direction. One can write them as:

$$\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)\delta x + \left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)\delta y = 0, \qquad \delta z = p\,\delta x + q\,\delta y. \tag{7}$$

We are now in a position to express analytically the idea that the partial differential equation (1) defines the surfaces of the complex. Indeed, it results from § 1 that in this case the curve (Γ), which is the curve of the complex that is situated in the plane (P) then, will have the contact generator of the cone (K) with that plane for its tangent at A. Therefore, the integral linear elements and the characteristic linear element of the contact element [A, (P)] will then coincide. From formulas (3) and (7), one will then have:

$$\frac{\partial F}{\partial p}\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right) + \frac{\partial F}{\partial q}\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right) = 0 \tag{8}$$

for any system of numbers (x, y, z, p, q) that verifies equation (1). In other words, equation (8) is a consequence of equation (1).

That condition is sufficient, because it implies the coincidence of the integral linear element and the characteristic linear element for any integral contact element, and we shall show that this coincidence demands that the elementary cones (K) are the cones of a complex of lines.

Indeed, recall the point-wise equation (5) of the cones (K). Any integral contact element is defined by a point A (x, y, z) and the plane (P) that is tangent to the cone (5) along any of its generators. It is defined by its direction coefficients X, Y, Z, and the six quantities x, y, z; X, Y, Z verify equation (5).

An infinitely-close integral contact element is similarly defined by the six quantities $x + \delta x$, $y + \delta y$, $z + \delta z$; $X + \delta X$, $Y + \delta Y$, $Z + \delta Z$, and the six differentials δy, δy, δz; δX, δY, δZ are coupled by $\delta G = 0$; i.e.:

$$\frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial y}\delta y + \frac{\partial G}{\partial z}\delta z + \frac{\partial G}{\partial X}\delta X + \frac{\partial G}{\partial Y}\delta Y + \frac{\partial G}{\partial Z}\delta Z = 0. \tag{9}$$

If the direction δy, δy, δz is that of the characteristic linear element of the first contact element then it will be parallel to the plane (P), which will give:

$$\frac{\partial G}{\partial X}\delta x+\frac{\partial G}{\partial Y}\delta y+\frac{\partial G}{\partial Z}\delta z = 0, \tag{10}$$

and the direction $X + \delta X$, $Y + \delta Y$, $Z + \delta Z$ of the new contact generator is once more in the plane (P), in such a way that δX, δY, δZ is also a direction in that plane. One will likewise have:

$$\frac{\partial G}{\partial X}\delta X+\frac{\partial G}{\partial Y}\delta Y+\frac{\partial G}{\partial Z}\delta Z = 0$$

then.

Upon comparing this with equation (9), one will conclude that:

$$\frac{\partial G}{\partial x}\delta x+\frac{\partial G}{\partial y}\delta y+\frac{\partial G}{\partial z}\delta z = 0. \tag{11}$$

Equations (10) and (11) then define the characteristic linear element. Moreover, if one expresses the idea that its direction is precisely X, Y, Z then one will deduce the equation:

$$X\frac{\partial G}{\partial X}+Y\frac{\partial G}{\partial Y}+Z\frac{\partial G}{\partial Z} = 0$$

from (10), which is nothing but (5), by virtue of Euler's theorem on homogeneous functions, and one will infer the desired condition from (11):

$$X\frac{\partial G}{\partial x}+Y\frac{\partial G}{\partial y}+Z\frac{\partial G}{\partial z} = 0. \tag{12}$$

We must then express the idea that equation (12) is a consequence of equation (5). To that effect, we take it in the solved form:

$$x-\Gamma\left(y,z,\frac{X}{Z},\frac{Y}{Z}\right)=0$$

and make the change of variable:

$$y=\omega+\frac{Y}{Z}z,$$

in such a way that Γ will be a function ξ of $\omega \equiv y - (Y / Z)\, z$, and of z, X/Z, Y/Z. Equation (5) will then be written:

$$0 = G \equiv x-\xi\left(\omega,z,\frac{X}{Z},\frac{Y}{Z}\right), \qquad \omega\equiv y-\frac{Y}{Z}z, \tag{13}$$

and the condition (12) will become:

$$X - Y\frac{\partial \xi}{\partial \omega} - Z\left(-\frac{Y}{Z}\frac{\partial \xi}{\partial \omega} + \frac{\partial \xi}{\partial z}\right) = 0;$$

i.e.:

$$\frac{\partial \xi}{\partial z} = \frac{X}{Z}.$$

That equation must be a consequence of (13), but, since it does not contain x, that will demand that it must be an identity. Upon integrating, one will then conclude that:

$$\xi = \frac{X}{Z}z + \psi\left(\omega, \frac{X}{Z}, \frac{Y}{Z}\right).$$

Equation (13) for the cones (K) will then be:

$$x - \frac{X}{Z}z = \psi\left(y - \frac{Y}{Z}x, \frac{X}{Z}, \frac{Y}{Z}\right),$$

and from the calculations of § 2, that is the general equation for the cones of the complex:

$$f = \psi(g, a, b).$$

We can then conclude that *the partial differential equations whose integral surfaces are the surfaces of a complex are characterized by the coincidence of the integral linear element and the characteristic linear element of each of their integral contact elements. They are the equations:*

(1) $$F(x, y, z, p, q) = 0,$$

which will imply the equation:

(8) $$\frac{\partial F}{\partial p}\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right) + \frac{\partial F}{\partial q}\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right) = 0$$

as an algebraic consequence.

Characteristics and the surfaces of the complex

4. – The integration of the first-order partial differential equation:

(1) $$F(x, y, z, p, q) = 0$$

and the Monge equations:

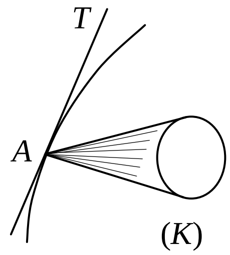
(2) $$G(x, y, z, dx, dy, dz) = 0$$

will result from the following considerations:

One calls the locus of contact elements that belong to the same curve (viz., points and tangent planes), and which are all integral contact elements an *integral band*. It will then be a set of ∞^1 contact elements that satisfy the equations:

(3) $$F(x, y, z, p, q) = 0, \qquad dz - p\,dx - q\,dy = 0.$$

If one takes an arbitrary curve and draws a tangent plane to the elementary cone that is associated with the point of contact through each of its tangents then one will obtain an integral band. Hence, if equation (3) is algebraic in p, q then a limited number of integral bands will pass through any curve. That number will reduce by one unit in the case in which the curve is an integral curve.



Imagine an integral surface (S). Any curve that is traced on that surface will provide an integral band that is defined by contact elements that are common to the curve and the surface. Among them, we shall seek the ones that have integral curves for their support. The elementary cone (K) at each point A of the surface (S) touches the tangent plane (P) to the surface along the integral linear element of the integral contact element [A, (P)]. We will then be reduced to finding the curves on (S) that have the integral linear element thus-defined for the linear element at each of their points A. From equations (3) of the preceding paragraph, that will amount to integrating the differential equation:

(4) $$\frac{dx}{\dfrac{\partial F}{\partial p}} = \frac{dy}{\dfrac{\partial F}{\partial q}},$$

in which one must suppose that z, p, q are replaced as functions of x and y by means of the equation:

(5) $$z = \Phi(x, y)$$

of the surface (S). Equation (4) will then be an ordinary differential equation, and one (and only one, in general) integral curve that is situated on (S) will pass through each point of (S). The surface (S) will then be generated by those curves (C).

Now consider the integral band that is circumscribed on the surface along one of those curves (C). The elements already satisfy equations (3) of the preceding paragraph, which we write as:

(6) $$F(x, y, z, p, q) = 0, \quad dx = \frac{\partial F}{\partial p}\,d\theta, \quad dy = \frac{\partial F}{\partial q}\,d\theta, \quad dz = \left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\right)d\theta,$$

upon introducing an auxiliary variable θ.

Moreover, it will satisfy the equations:

(7) $$dp = r\,dx + s\,dy, \qquad dq = s\,dx + t\,dy,$$

in which r, s, t are the second derivatives of the function (5). Now, that function satisfies equation (1) identically, so one will deduce by differentiation that:

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + r\frac{\partial F}{\partial p} + s\frac{\partial F}{\partial q} = 0, \qquad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} + s\frac{\partial F}{\partial p} + t\frac{\partial F}{\partial q} = 0,$$

and upon taking equations (6) and (7) into account, those equations will give:

(8) $$dp = -\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)d\theta, \qquad dq = -\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)d\theta.$$

It results from this that: *The contact elements on any integral surface subdivide into* ∞^1 *bands that belong to the* ∞^3 *bands that are defined by equations* (6) *and* (8). *Those* ∞^3 *bands are called the characteristic bands of the partial differential equation* (1). *The curves that serve as their supports are the characteristic curves, or more simply, the characteristics.*

The characteristic bands do, in fact, depend upon three arbitrary constants. Indeed, the differential equations:

(9) $$dx = \frac{\partial F}{\partial p}d\theta, \qquad dy = \frac{\partial F}{\partial q}d\theta, \qquad dz = \left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\right)d\theta,$$

$$dp = -\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)d\theta, \qquad dq = -\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)d\theta$$

will reduce to four equations if one eliminates $d\theta$. Moreover, they will imply the combination:

(10) $$0 = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial z}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial p}dp + \frac{\partial F}{\partial q}dq\,,$$

and conversely, if that combination $dF = 0$ is verified then those equations (9) will reduce to three. Therefore, if one takes into account the equation:

(1) $$F(x, y, z, p, q) = 0,$$

while solving it for q (for example) and substituting it into equations (9), what will remain is a system of three first-order differential equations in x, y, z, p whose general integral indeed depends upon three arbitrary constants.

On the contrary, suppose that we have integrated the system (9) in that way. We will get functions of θ:

(11) $$x = \xi(\theta; x_0, y_0, z_0, p_0, q_0), \qquad y = \eta(\theta; x_0, y_0, z_0, p_0, q_0),$$
$$z = \zeta(\theta; x_0, y_0, z_0, p_0, q_0),$$
(12) $$p = \varpi(\theta; x_0, y_0, z_0, p_0, q_0), \qquad q = \chi(\theta; x_0, y_0, z_0, p_0, q_0)$$

that will reduce to the initial values x_0, y_0, z_0, p_0, q_0 for $\theta = 0$, for example. They will have equation (10) for a consequence; i.e.:

(13) $$F(x, y, z, p, q) = F(x_0, y_0, z_0, p_0, q_0),$$

in such a way that they will define a characteristic band, provided that the initial contact element $(x_0, y_0, z_0, p_0, q_0)$ that figures in it is an integral contact element.

Therefore: *One and only one characteristic band will pass through any integral contact element,* and as a result *an integral surface that contains one integral contact element will contain the entire characteristic band that has that element for its initial element.*

We are in a position to construct all integral surfaces then, because *if one is given an arbitrary integral band on an arbitrary integral surface then that surface will be generated by the characteristic bands that have the various elements of that band for their initial elements.* That will result from the foregoing.

Conversely: *The characteristic bands that have the elements of an arbitrary integral band for their initial elements will generate an integral surface.*

Indeed, suppose that we replace the constants x_0, y_0, z_0, p_0, q_0 in equations (11) and (12) with the functions:

(14) $$x = x_0(u), \qquad y = y_0(u), \qquad z = z_0(u), \qquad p = p_0(u), \qquad q = q_0(u),$$

which define the given integral band by means of the parameter u. Due to the identity (13), all of the contact elements obtained will be integral, and equations (11) will define a surface as a function of the parameters θ and u. In order to prove that it is, in fact, the stated surface, it is sufficient to verify that it has the elements (11) and (12) for contact elements; i.e., that if one denotes the differentiations with respect to θ and u by d and δ, respectively, then the functions (11), (12) of θ and u will satisfy the two identities:

(15) $$D \equiv dz - p\,dx - q\,dy = 0, \qquad \Delta \equiv \delta z - p\,\delta x - q\,\delta y = 0.$$

As far as the first one is concerned, it results from equations (9). The second one is a consequence of the identity:

$$d\Delta - \delta D = -\,dp \cdot \delta x - dq\,\delta y + dx \cdot \delta p + dy\,\delta q.$$

Upon taking equations (9) into account, the left-hand side will, in fact, become:

$$\delta F - \frac{\partial F}{\partial z}(\delta z - p\,\delta x - q\,\delta y)\,d\theta \equiv \delta F - \Delta\frac{\partial F}{\partial z}\,d\theta.$$

The elements (11), (12) are both integral, so δF will be zero. Therefore, since the first condition (15) is realized, what will remain is:

(16) $$\frac{d\Delta}{d\theta} = -\frac{\partial F}{\partial z}\Delta.$$

One must suppose that the variables are replaced by the functions (11) and (13) in the factor $\partial F / \partial z$. One will then have an equation for Δ that has the form:

(17) $$\frac{d\Delta}{d\theta} = M(\theta, u) \cdot \Delta.$$

Now, Δ is annulled for $\theta = 0$, since the initial elements (14) form a band of elements, and one such equation (17) will not admit any solution besides the solution $\Delta \equiv 0$, which is annulled for $\theta = 0$. Hence, the second condition (15) is indeed verified for any θ and u.

In summary, *one and only one integral surface will pass through any integral band.*

The bands that are an exception to that are the characteristic bands. An infinitude of integral surfaces will pass through a characteristic band that coincide all along the characteristic that serves as the support of the band.

If we now return to the particular case in which equation (1) is the one that defines the surfaces of a complex then we will see, upon comparing the preceding analysis with that of § 2, that since the integral curves are curves of the complex, the characteristics that are situated on an integral surface will constitute the family of ∞^1 curves of the complex that are the asymptotic lines of that surface. The condition for that to be true is that equations (6) and (8) must have the consequence that:

$$dp\, dx + dq\, dy = 0;$$

i.e., that equation (1) must have the consequence that:

$$\frac{\partial F}{\partial p}\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right) + \frac{\partial F}{\partial q}\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right) = 0.$$

That is equation (8) of § 3. From the results of § 3, we can then conclude that *the first-order partial differential equations for which the characteristics are the asymptotic lines of the integral surfaces are (if one ignores linear equations) the equations whose elementary cones are the cones of the complexes of lines.*

Remark 1. – If *equation* (1) *is linear in p*, *q* then the elementary cone will reduce to a line. The characteristic curves will be defined independently of the characteristic bands by the equations *in x*, *y*, *z*:

$$\frac{dx}{\dfrac{\partial F}{\partial p}} = \frac{dy}{\dfrac{\partial F}{\partial q}} = \frac{dz}{p\dfrac{\partial F}{\partial p} + q\dfrac{\partial F}{\partial q} - F}.$$

There are more than ∞^2 characteristic curves, even though there are always ∞^3 characteristic bands, each of which is defined by a characteristic and a neighboring characteristic.

The integral surfaces are the ones that are generated by ∞^1 characteristics. The characteristics are asymptotes for all integral surfaces in the case where they are lines, and only in that case.

Remark 2. – If the cone of the complex reduces to a plane then the complex will be called a *linear complex.* The cone will not have a tangential equation then, and the preceding theory will no longer apply to it.

The case of linear complexes will be studied in the following chapter.

Geometric properties of characteristics

5. – In what follows, we will discard the linear equations. Consider a contact element (x, y, z, p, q) of a characteristic band and the infinitely-close element. The intersection of the planes of the two elements is defined by the two equations:

$$Z - z - p\,(X - x) - q\,(Y - y) = 0, \qquad (X - x)\,dp + (Y - y)\,dq = 0.$$

Indeed, the second one will result from differentiating the first one, while taking into account that:

$$dz - p\,dx - q\,dy = 0.$$

If one compares this with equations (7) of § 3, while taking equations (8), § 4 into account, then one will see that *the intersection of the plane of a contact element of a characteristic band with that of the infinitely-close element will be the characteristic linear element.* That will explain the name that we have given to that linear element [cf., Chap. VII, § 4, pp. 171].

That property will suffice to define the characteristic bands among the ones that have an integral curve for their support, except in the case where the partial differential equation is that of the surfaces of a complex of lines. Because of the equations:

$$dx = \frac{\partial F}{\partial p}\,d\theta, \quad dy = \frac{\partial F}{\partial q}\,d\theta, \quad dz = \left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\right)d\theta,$$

$$dp = -\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)d\tau, \qquad dq = -\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)d\tau,$$

one will conclude upon substituting them into $dF = 0$ that:

$$\left[\frac{\partial F}{\partial p}\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)+\frac{\partial F}{\partial q}\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\right](d\theta - d\tau) = 0;$$

i.e., $d\theta = d\tau$, if one excludes the reserved case. Moreover, the preceding equations are the ones that define the characteristic bands of the equation:

(1) $$F(x, y, z, p, q) = 0.$$

One sees that in all cases the integral linear element and the characteristic linear element of an (integral) contact element of an integral surface have conjugate directions on that surface. Those directions coincide in the case of the surfaces of a complex, which will indeed correspond to the fact that the characteristics are then asymptotes of the integral surfaces.

As for the *characteristic curves* of an integral surface, their fundamental property is that if one excludes the singular solutions then *in order for* ∞^1 *characteristics to generate an integral surface, it is necessary and sufficient that each of them must meet the infinitely-close characteristic.*

The results that were obtained in the preceding paragraph in the generation of integral surfaces by the characteristics (11) can, in fact, be stated thus: In order for a family of ∞^1 curves (11) to generate an integral surface, it is necessary and sufficient that one must take x_0, y_0, z_0, p_0, q_0 to be functions of one parameter u such that one will have both:

(2) $$F(x_0, y_0, z_0, p_0, q_0) = 0,$$

(3) $$\delta z_0 - p_0\, \delta x_0 - q_0\, \delta y_0 = 0.$$

The first of them is assumed to be realized if equation (11) represents the ∞^3 characteristics of equation (1). We shall see that the second one express the idea that two infinitely-close characteristics will meet.

Indeed, we seek to express the idea that this is the case. We continue to let d and δ denote the differentiations that relate to θ and u. We must express the idea that equations (11) are compatible with the equations that one will deduce from them by differentiation under the hypothesis that x, y, z are constant; they are:

(4) $$\frac{d\xi}{d\theta}\delta\theta + \delta\xi = 0, \qquad \frac{d\eta}{d\theta}\delta\theta + \delta\eta = 0, \qquad \frac{d\zeta}{d\theta}\delta\theta + \delta\zeta = 0.$$

Since those equations do not contain x, y, z, it will suffice to eliminate θ and $\delta\theta$ from them. Now, upon remarking that one has:

$$\frac{d\zeta}{d\theta} - \varpi\frac{d\xi}{d\theta} - \chi\frac{d\eta}{d\theta} = 0$$

identically, one will conclude the combination:

(5) $$\delta\zeta - \varpi\,\delta\xi - \chi\,\delta\eta = 0$$

from equations (4).

For $\theta = 0$, that will reduce to (3), which will then be a consequence of it. Moreover, we have seen in the preceding paragraph that if (3) is true then (5) will be verified for any θ. Hence, upon excluding the possible singular solutions that are due to the presence of the factor $\partial F / \partial z$ in the fundamental formula (16), we conclude that the combination (5) of equations (4) is equivalent to equation (3), which contains neither θ nor $\delta\theta$. That will then result from the elimination of θ and $\delta\theta$ from equations (4). They will then indeed express the intersection condition for two infinitely-close characteristics.

Moreover, we see that *this conditional equation* (3) *is linear and homogeneous with respect to the differentiations with arbitrary constants* that figure in the general equations of the characteristics. Without altering that character, one can suppose that the equations of the characteristics have been put into the form:

(6) $$P(x, y, z;\ \alpha, \beta, \gamma) = 0, \qquad Q(x, y, z;\ \alpha, \beta, \gamma) = 0,$$

because x_0, y_0, z_0, p_0, q_0 are expressed in terms of α, β, γ by means of the equations:

$$P(x_0, y_0, z_0; \alpha, \beta, \gamma) = 0, \quad Q(x_0, y_0, z_0; \alpha, \beta, \gamma) = 0, \quad F(x_0, y_0, z_0; \alpha, \beta, \gamma) = 0, \quad \begin{vmatrix} \frac{\partial P}{\partial x_0} & \frac{\partial P}{\partial y_0} & \frac{\partial P}{\partial z_0} \\ \frac{\partial Q}{\partial x_0} & \frac{\partial Q}{\partial y_0} & \frac{\partial Q}{\partial z_0} \\ p_0 & q_0 & -1 \end{vmatrix} = 0.$$

δx_0, δy_0, δz_0 will be homogeneous linear forms in $\delta\alpha$, $\delta\beta$, $\delta\gamma$, whose coefficients will be functions of α, β, γ, and the condition (3) will become a *Pfaff equation* in α, β, γ:

(7) $$A(\alpha, \beta, \gamma)\,\delta\alpha + B(\alpha, \beta, \gamma)\,\delta\beta + C(\alpha, \beta, \gamma)\,\delta\gamma = 0.$$

Complete integrals

One can recover this result, as well as its converse, by considering complete integrals. One will call any family of ∞^2 integral surfaces:

(8) $$H(x, y, z;\ \alpha, \beta) = 0$$

a *complete integral* of equation (1), with the reservation that any integral contact element must belong to one of the surfaces of the family. The method of generating integral surfaces that was obtained before proves the existence of an infinitude of complete integrals for any nonlinear equation (1).

Let (S) be an arbitrary integral that is not included in the complete integral (8), and take an integral band of that surface. Each contact element (E) of that band belongs to one and only one of the surfaces (8). One thus defines ∞^1 surfaces (8), each of which has a characteristic band in common with (*S*) that is defined by the initial element (*E*), because that characteristic band is defined entirely on (*S*) and on the surface (8) considered. Therefore: *Any integral surface is the envelope of* ∞^1 *surfaces that belong to the complete integral.*

Conversely, any envelope of ∞^1 surfaces (8) has elements of those surfaces for contact elements – i.e., integral contact elements. It will then be an integral surface.

Moreover, since one will then obtain all integral surfaces, *the characteristics are the intersection curves of the various surfaces of the complete integral with an arbitrary infinitely-close surface.*

An arbitrary integral surface is then defined by two equations of the form:

$$H(x, y, z;\ \alpha, \beta) = 0, \qquad 0 = \delta H \equiv \frac{\partial H}{\partial \alpha}\delta\alpha + \frac{\partial H}{\partial \beta}\delta\beta, \tag{9}$$

in which α and β are coupled by an arbitrary relation $\beta = \varphi(\alpha)$.

The characteristics that are situated on that surface are defined by the same equations for the various values of α.

The set of characteristics is represented by the equations:

$$H(x, y, z;\ \alpha, \beta) = 0, \qquad \frac{\partial H}{\partial \alpha} + \gamma\frac{\partial H}{\partial \beta} = 0, \tag{10}$$

with three arbitrary constants α, β, γ.

The intersection condition of a characteristic (10) and an infinitely-close characteristic are obtained by eliminating x, y, z between equations (10) and the equations:

$$\frac{\partial H}{\partial \alpha}\delta\alpha + \frac{\partial H}{\partial \beta}\delta\beta = 0, \quad \delta\left(\frac{\partial H}{\partial \alpha} + \gamma\frac{\partial H}{\partial \beta}\right) = 0,$$

which gives:

$$\delta\beta - \gamma\,\delta\alpha = 0. \tag{11}$$

That is indeed a Pfaff equation, and it expresses the idea that:

$$\beta = \varphi(\alpha), \qquad \gamma = \varphi'(\alpha).$$

One then recovers the condition that one must replace α, β, γ in order for the characteristics (10) to be the ones that generate an integral surface.

The preceding results are then indeed proved once again.

Furthermore, let us study the converse. We first remark that *any Pfaff equation:*

$$A\,\delta\alpha + B\,\delta\beta + C\,\delta\gamma = 0 \tag{12}$$

can be reduced to the integrable form $\delta\alpha = 0$ *or the form* (11) $\delta\beta - \gamma\,\delta\alpha = 0$ *by a change of variables.*

Indeed, set:

(13) $$\beta = \psi(\alpha, \gamma; \alpha_0)$$

in (12), in which α_0 is an arbitrary constant, and ψ is chosen arbitrarily. We will then get a differential equation α and γ whose general integral will have the form:

(14) $$\beta_0 = \chi(\alpha, \gamma; \alpha_0),$$

in which β_0 denotes a new arbitrary constant. We then determine ∞^2 integral curves of the Pfaff equation by equations (13), (14).

Having said that, make the changes of variables in (12) that is defined by formulas (13) and (14), while considering α_0, β_0 to be new variables and inferring α and β. Since the function ψ is arbitrary, one can then suppose that this solution is possible. It will give a Pfaff equation in α_0, β_0, γ that must be verified for arbitrary constant values of α_0 and β_0; viz., for $\delta\alpha_0 = \delta\beta_0 = 0$, it will reduce to the form:

$$A_0\ \delta\alpha_0 + B_0\ \delta\beta_0 = 0$$

or:

$$\delta\beta_0 - \gamma_0\ (\alpha_0, \beta_0, \gamma)\ \delta\alpha_0 = 0.$$

If γ_0 does not depend upon γ then what will remain is an equation of degree one in only α_0 and β_0 that can be written $\delta\alpha_1 = 0$, if its general integral is:

(15) $$\alpha_1 = M(\alpha_0, \beta_0) \equiv N(\alpha, \beta, \gamma) \qquad (\alpha_1 = \text{const.}).$$

On the contrary, if the function γ_0 does depend upon γ then one can take it to be a new variable, in place of γ, and the Pfaff equation will be reduced to the form:

(16) $$\delta\beta_0 - \gamma_0\ \delta\alpha_0 = 0.$$

In this case, the general solution of (12) is:

$$\beta_0 = \varphi(\alpha_0), \qquad \gamma_0 = \varphi'(\alpha_0);$$

there is no surface that satisfies the equation then.

On the contrary, in the preceding case, equation (12) is equivalent to:

$$N(\alpha, \beta, \gamma) = \text{const.},$$

which defines a family of surfaces that satisfy the equation, as well as any curve that is traced on one of its surfaces. In that case, one says that the Pfaff equation is *integrable*.

Having said that, suppose that one has a complex of curves (6) such that the intersection condition of the infinitely-close curves has the Pfaff form (7), and suppose that this equation is not integrable. One can suppose that one has made a preliminary change of parameters, such that this relation reduces to the canonical form (11):

(11) $$\delta\beta - \gamma\,\delta\alpha = 0.$$

Moreover, we can suppose that the equations of the complex of curves have been solved in the form:

(17) $$z = K\,(x, y\,;\,\alpha, \beta), \qquad \gamma = L\,(x, y\,;\,\alpha, \beta),$$

since otherwise, upon inferring γ from one of the equations (6) and substituting it into the other one, what will remain is a relation that is independent of the coordinates x, y, z.

We express the idea that the curve (17) meets the infinitely-close curve. One must eliminate x and y from:

$$\gamma = L\,(x, y;\,\alpha, \beta), \qquad \frac{\partial K}{\partial \alpha}\delta\alpha + \frac{\partial K}{\partial \beta}\delta\beta = 0, \qquad \delta\gamma = \frac{\partial L}{\partial \alpha}\delta\alpha + \frac{\partial L}{\partial \beta}\delta\beta\,.$$

In order for that to reproduce equation (11), it is necessary and sufficient that one must have:

$$\frac{\partial K}{\partial \alpha} + L\frac{\partial K}{\partial \beta} = 0,$$

in such a way that equations (17) can be written:

(18) $$z = K\,(x, y;\,\alpha, \beta), \quad \frac{\partial K}{\partial \alpha} + \gamma\frac{\partial K}{\partial \beta} = 0.$$

In order to prove that they represent a family of characteristics, it will suffice, moreover, to prove that there exists one and only one partial differential equation that has the complete integral:

(19) $$z = K\,(x, y;\,\alpha, \beta),$$

since equations (10) will become equations (18) if one replaces H with $(z - K)$.

Now, the functions (19) of x and y satisfy the equations:

(20) $$p = \frac{\partial K}{\partial x}, \qquad q = \frac{\partial K}{\partial y},$$

and one can eliminate α and β from (19) and (20), which will indeed give an equation of the form (1):

(1) $$F\,(x, y, z, p, q) = 0.$$

Nonetheless, one must verify that this elimination will give only one equation; i.e., that K, $\partial K / \partial x$, $\partial K / \partial y$ are coupled by only one relation when considered to be functions of α, β. If things were otherwise then the functional determinants:

$$\frac{\partial^2 K}{\partial x\,\partial\alpha}\frac{\partial K}{\partial\beta}-\frac{\partial^2 K}{\partial x\,\partial\beta}\frac{\partial K}{\partial\alpha}, \qquad \frac{\partial^2 K}{\partial y\,\partial\alpha}\frac{\partial K}{\partial\beta}-\frac{\partial^2 K}{\partial y\,\partial\beta}\frac{\partial K}{\partial\alpha}$$

would both be identically zero. One would then have the simultaneous identities:

$$\frac{\partial K}{\partial\alpha}+L\frac{\partial K}{\partial\beta}=0, \qquad \frac{\partial^2 K}{\partial x\,\partial\alpha}+L\frac{\partial^2 K}{\partial x\,\partial\beta}=0, \qquad \frac{\partial^2 K}{\partial y\,\partial\alpha}+L\frac{\partial^2 K}{\partial y\,\partial\beta}=0.$$

Upon differentiating the first one in x and y and comparing it to the other two, one will conclude that $\partial L / \partial x \equiv \partial L / \partial y \equiv 0$. However, the second equation (18), which is $L = \gamma$, will not contain x and y then, which is impossible.

We then conclude that *in order for a complex of curves to be composed of* ∞^3 *characteristics of the same first-order partial differential equation, it is necessary and sufficient that the intersection condition for two infinitely-close curves of the complex is expressed by a non-integrable Pfaff equation for the three parameters that those* ∞^3 *curves depend upon.*

Determination of the integral curves

6. – It remains for us to show how to integrate of the Monge equation:

(2) $$G(x, y, z, dx, dy, dz) = 0,$$

which is (as we saw in § 3) associated with the partial differential equation:

(1) $$F(x, y, z, p, q) = 0;$$

i.e., to determine the integral curves of that equation, which will result from the preceding considerations.

Now, any integral curve is the envelope of the characteristics that are defined by the initial contact elements that one obtains by associating the tangent plane each point M of the integral curve that is drawn to the elementary cone (K) with summit M with the generator of that cone that is tangent to the curve at M. Since each of those characteristics has an envelope, they will generate an integral surface, since each of them meets the infinitely-close characteristic.

Conversely, any family of characteristics that generates an integral surface has an envelope, since each of them will meet the infinitely-close characteristic, and that enveloping curve will be an integral surface, since every linear element of a characteristic is an integral linear element.

One will then obtain all integral curves by looking for the most general integral surface, and the envelope of the characteristics on it that it generates.

The result is presented in an explicit form if one is given a complete integral:

(3) $$H(x, y, z;\ \alpha, \beta) = 0.$$

An arbitrary integral surface is defined by the characteristics:

$$(4) \qquad H = 0, \qquad \frac{\partial H}{\partial \alpha} + \varphi'(\alpha)\frac{\partial H}{\partial \beta} = 0 \qquad [\beta = \varphi(\alpha)],$$

and the envelope of those characteristics is defined by the three equations:

$$(5) \qquad H = 0, \;\; \frac{\partial H}{\partial \alpha} + \varphi'(\alpha)\frac{\partial H}{\partial \beta} = 0, \;\; \frac{\partial^2 H}{\partial \alpha^2} + 2\varphi'(\alpha)\frac{\partial^2 H}{\partial \alpha\,\partial \beta} + \varphi'^2(\alpha)\frac{\partial^2 H}{\partial \beta^2} + \varphi''(\alpha)\frac{\partial H}{\partial \beta} = 0,$$

in which β must be replaced with the arbitrary function $\varphi(\alpha)$.

Remark. – There is only one integral curve on an integral surface that is not a characteristic then, and it is the envelope of the characteristics. The integral surfaces of the same partial differential equation then exhibit a remarkable analogy with the developable surfaces. Characteristics will then play the role of generators, and the non-characteristic integral curve will play the role of the edge of regression. That analogy will become an identity in the particular case that defines the object of the following paragraph.

Special complexes

7. – We say that a complex is *special* when the homography that exists between the points and planes of a line of the complex is special. Any element of a system will always correspond to the same element in the associated system, except for one element of the first system whose correspondent is indeterminate. The equation of the homography that relates to the complex:

$$(1) \qquad \varphi(a, b, f, g) = 0$$

is [§ 1, eq. (10)]:

$$\lambda\left(\frac{\partial \varphi}{\partial a} - Z\frac{\partial \varphi}{\partial f}\right) + \frac{\partial \varphi}{\partial b} - Z\frac{\partial \varphi}{\partial g} = 0,$$

so the condition for that homography to be special is that:

$$(2) \qquad \frac{\partial \varphi}{\partial a}\cdot\frac{\partial \varphi}{\partial g} - \frac{\partial \varphi}{\partial b}\cdot\frac{\partial \varphi}{\partial f} = 0.$$

The complex (1) will then be special if that equation (2) is a consequence of equation (1).

The *complex of the lines that are tangent to a surface* gives an example of a special complex. Indeed, consider a congruence of that complex. The developables of the congruence are circumscribed by the surface, so one of the focal planes will then be independent of the congruence that one considers. One will get the same result if one considers the *complex of lines that meet a given curve*. One will then obtain some special complexes. We shall show that there are no other ones.

Indeed, take the equation of a complex in the form:

$$\varphi = g - \Psi\,(a, b, f) = 0;$$

the condition (2) is written:

(3) $$\frac{\partial \Psi}{\partial a} + \frac{\partial \Psi}{\partial b} \cdot \frac{\partial \Psi}{\partial f} = 0.$$

That relation no longer contains g; it must then be an identity with respect to a, b, f.

Consider a line (D) of the complex then, and the infinitely-close lines that it meets. We have obtained the intersection condition [§ 1, eq. (5)], which is written:

$$da \cdot d\Psi - db \cdot df = 0$$

here, or:

$$db \cdot df - da\left(\frac{\partial \Psi}{\partial a}da + \frac{\partial \Psi}{\partial b}db + \frac{\partial \Psi}{\partial f}df\right) = 0.$$

Replace $\partial\Psi / \partial a$ with its value that is inferred from (3), which will give:

$$\frac{\partial \Psi}{\partial b} \cdot \frac{\partial \Psi}{\partial f} da^2 - \frac{\partial \Psi}{\partial b} da \cdot db - \frac{\partial \Psi}{\partial f} da \cdot df + db \cdot df = 0,$$

or

(4) $$\left(\frac{\partial \Psi}{\partial b}da - df\right)\left(\frac{\partial \Psi}{\partial f}da - db\right) = 0.$$

For example, suppose that it is the first factor that is annulled. The point at which the line (D) meets the corresponding infinitely-close line is given [§ 1, eq. (6)] by:

(5) $$z = -\frac{df}{da} = -\frac{\partial \Psi}{\partial b},$$

in such a way that all of the lines considered will cut (D) at the same point F:

(6) $$x = az + f, \qquad y = bz + \Psi, \qquad z = -\frac{\partial \Psi}{\partial b}.$$

Differentiate those formulas:

$$dx = a\,dz + z\,da + df, \qquad dy = b\,dz + z\,db + d\Psi,$$

so that upon replacing z with its value, one will get:

$$dx - a\,dz = -\frac{\partial \Psi}{\partial b}\,da + df, \qquad dy - b\,dz = \frac{\partial \Psi}{\partial a}da + \frac{\partial \Psi}{\partial f}df\,.$$

Upon eliminating *df* and taking the relation (3) into account, one will conclude that:

$$(7) \qquad -\frac{\partial \Psi}{\partial f}(dx - a\,dz) + dy - b\,dz = 0.$$

The differentials *dx*, *dy*, *dz* are then coupled by a homogeneous, linear relation. As a result, the functions *x*, *y*, *z* will be linked by at least one relation.

If there is only one relation then the locus of points *F* will be a surface, and equation (7), which defines the infinitely-small tangent displacements, will show that the line (*D*) is tangent to that surface. If there are two relations then the locus of points *F* will be a curve, and any line (*D*) will meet that curve, since each point *F* is on one of the lines (*D*). The only two cases that are possible for special complexes will then indeed be the indicated cases.

Remark 1. – Up to now, we have considered only the factor $\left(\frac{\partial \Psi}{\partial b}da - df\right)$ in equation (4). If one annuls the other factor then:

$$\frac{db}{da} = \frac{\partial \Psi}{\partial b},$$

so we will have the lines of the complex that, from equation (7) of § 1, will all be situated on the same plane with (*D*). That plane:

$$(X - a\,Z - f)\frac{\partial \Psi}{\partial f} - (Y - b\,Z - y) = 0$$

will be the singular plane of the homography, and from equation (7), it will be tangent to the locus of points *F*. One will then see that upon taking one or the other factor, one will define the same locus by the points and tangent planes.

Remark 2. – If the equation of the complex contains neither *f* nor *g* then it will define a relation between the direction coefficients of the line (*D*). One will then have the complex of lines that meet the same curve at infinity.

Remark 3. – The preceding calculation can be interpreted in the case of an arbitrary complex. Equation (2), which is no longer a consequence of the equation of the complex then, will define the congruence of lines of the complex on which the homography is special when it is joined with that equation of the complex. They are the *singular lines* of the complex. Hence: *All of the ruled surfaces of the complex that pass through a singular line will have the same tangent plane at the point F of the line that was defined previously*, since that tangent plane is parallel to the plane:

$$-\frac{\partial\Psi}{\partial f}(x - az) + y - bz = 0.$$

If the locus of singular points is a surface then equation (7) will show that this surface is also the envelope of the singular plane and the singular lines to which they are tangent. *The surface of singularities is one of the sheets of the focal surface of the congruence of the singular lines. The singular points and singular planes are the focal elements of that congruence that are not associated with each other. If the locus of singular points is a curve then,* from (7), *the singular planes will be tangent to that curve, which is a focal curve of the congruence of singular lines.*

Remark 4. – In particular, consider the case of the *second-degree complexes.* The plane that is associated with an arbitrary point is tangent to the cone of the complex; it is unique and well-defined. It will be indeterminate only if the cone of the complex that is associated with that point decomposes. *The surface of singularities is then the locus of points where the cone of the complex decomposes. It is also the envelope of planes for which the curve of the complex decomposes,* as one will verify by an analogous argument by assuming the correlative viewpoint.

Surfaces and curves of special complexes

Let us return to the special complex: First, consider the case of the complex of tangents to a surface (Φ). The cones of the complex are the cones that are circumscribed on that surface. An arbitrary surface integral will then be the envelope of ∞^1 tangent planes to (Φ); i.e., an arbitrary developable that is circumscribed by (Φ). The characteristics, which are, in general the contact curves of the surface integral with the surfaces that belong to the complete integral that it envelops, are the rectilinear generators of those developables; i.e., lines of the complex. Finally, one will obtain the integral curves by taking the envelope of the characteristics on the integral surfaces. They will then be the edges of regression of the developables that are circumscribed by (Φ), which are curves of the complex.

Now consider the complex of lines that meet a curve. One will likewise see that the surfaces of the complex are the developables that pass through the curve, so the characteristics will be the lines of the complex, and the curves of the complex will be the edges of regression.

Hence: *In the special complexes, the first-order partial differential equation upon which the search for the surfaces of the complex will depend will have the lines of the complex for its characteristics. Conversely, any first-order partial differential equation whose characteristics are lines will be associated with a special complex.*

Indeed, let:

$$F(x, y, z, p, q) = 0$$

be a partial differential equation whose characteristics are lines. One will get its integral surfaces by taking an integral curve and drawing the tangent characteristics. Hence, the integral surfaces will be developables, and the tangent plane will be the same along each characteristic; i.e., $dp = 0$, $dq = 0$ must be consequences of the characteristic equations. That amounts to saying that $F = 0$ must imply the equations:

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \qquad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0$$

as a consequence. Suppose that z figures in the partial differential equation and set:

$$F \equiv z - \theta(x, y, p, q).$$

The preceding conditions will be written:

$$\frac{\partial \theta}{\partial x} - p = 0, \qquad \frac{\partial \theta}{\partial y} - q = 0,$$

so θ will have the form:

$$\theta = px + qy + \Psi(p, q),$$

and the partial differential equation will be:

$$z - px - qy = \Psi(p, q).$$

The tangent plane to any of the integral surfaces will then be:

$$pX + qY - Z + \Psi(p, q) = 0.$$

The set of all those planes will then have an envelope that is a surface or a curve. The elementary cone that is associated with any point is the cone that is circumscribed by that surface or curve, and the partial differential equations will indeed be associated with a special complex.

Remark. – We have supposed that z figures in the partial differential equation. If that were not true then, as one could predict by changing the role of the coordinates, that equation would contain neither x nor y, because one could write, for example:

$$F \equiv x - \theta(y, p, q) = 0,$$

so the condition:

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0$$

would not be verified. Hence, the partial differential equation will then take the form:

$$\Phi(p, q) = 0,$$

which will give the complex of lines that meet a curve at infinity.

For example, consider the equation:

$$1 + p^2 + q^2 = 0;$$

it defines the *complex of isotropic lines*. The curves of the complex are the minimal curves, and one will get them without integration as the edges of regression of the isotropic developables. That is how we determined the minimal curves in Chap. III, § 4.

Surfaces normal to the lines of a complex

8. – We now propose to look for the *surfaces whose normals belong to the complex* that is defined by the equation:

(1) $$\varphi(a, b, f, g) = 0.$$

A normal to a surface of the complex is defined by the equations:

$$\frac{X-x}{p} = \frac{Y-y}{q} = -(Z-z),$$

or

$$X = -pZ + x + pz, \qquad Y = -qZ + y + qz,$$

in such a way that the desired surfaces will be defined by the partial differential equation:

(2) $$\varphi(-p, -q, x + pz, y + qz) = 0.$$

If a surface meets that requirement then all of the surfaces that are parallel to it will also meet that requirement.

If the complex is special then the problem will amount to the search for a congruence of normals when one knows one of the focal multiplicities. If the focal multiplicity is a curve (φ) then the desired surfaces will be the envelopes of the spheres that have their centers on (φ), from what we said in Chap. VII, § 2, pp. 162. Moreover, those spheres will constitute an obvious complete integral of the equation of the problem.

If the focal multiplicity is a surface (Φ) then the problem will amount to the determination of the geodesics lines of that surface [Chap. VII, § 2, pp. 161].

In the case of an arbitrary complex, we shall look for the normal congruences that belong to the complex. One will then find the surfaces by means of a quadrature. In order for the ∞^2 lines:

$$\frac{x-f}{a} = \frac{y-g}{b} = \frac{z-0}{1}$$

to be the normals to the same surface, upon setting:

$$\alpha = \frac{a}{\sqrt{a^2+b^2+1}}, \qquad \beta = \frac{b}{\sqrt{a^2+b^2+1}}, \qquad \gamma = \frac{1}{\sqrt{a^2+b^2+1}},$$

it is necessary and sufficient that $\alpha\, df + \beta\, dg$ must be an exact total differential [Chap. VII, § 1, pp. 159]. Now, when the equation of the complex is solved for β, it can be written:

(3) $$\beta = \Psi\,(\alpha, f, g),$$

and $\alpha\, df + \Psi\,(\alpha, f, g)\, dg$ must be a total differential with respect to the two independent variables. For example, determine α as a function of f, g, which will give the condition:

(4) $$\frac{\partial\alpha}{\partial g} = \frac{\partial\Psi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial f} + \frac{\partial\Psi}{\partial f}.$$

We seek a solution of the form:

$$\theta\,(\alpha, f, g) = \text{const.}$$

Upon differentiating this with respect to f, g, we will get:

$$\frac{\partial\theta}{\partial f} + \frac{\partial\alpha}{\partial f}\cdot\frac{\partial\theta}{\partial\alpha} = 0, \qquad \frac{\partial\theta}{\partial g} + \frac{\partial\alpha}{\partial g}\cdot\frac{\partial\theta}{\partial\alpha} = 0,$$

and the condition (4) will become:

$$\frac{\partial\theta}{\partial g} - \frac{\partial\Psi}{\partial\alpha}\cdot\frac{\partial\theta}{\partial f} + \frac{\partial\Psi}{\partial f}\cdot\frac{\partial\theta}{\partial\alpha} = 0.$$

That is a partial differential equation whose integration reduces to the system of ordinary differential equations:

$$dg = \frac{df}{-\dfrac{\partial\Psi}{\partial\alpha}} = \frac{d\alpha}{-\dfrac{\partial\Psi}{\partial f}},$$

which determine the characteristics.

Having thus calculated α as a function of f and g, one will deduce β by using equation (3), and one will have $\gamma = \sqrt{1-\alpha^2-\beta^2}$. One performs the quadrature of the total differential:

$$u = -\int \alpha\, df + \Psi\, dq,$$

and the desired surfaces will be defined [Chap. VII, 1, pp. 159] by the formulas:

$$x = f + \alpha u, \qquad y = g + \beta u, \qquad z = \gamma u.$$

REMARK. – *The developables of the desired surfaces are the surfaces for which* ∞^1 *geodesics are the curves of the complex.* They are the focal surfaces of the congruences considered.

CHAPTER X

LINEAR COMPLEXES

Generalities on algebraic complexes

1. – Let:

(1) $$x = az + f, \qquad y = bz + g$$

be a line. An *algebraic complex* will be defined by an algebraic relation between *a*, *b*, *f*, *g*:

$$\varphi(a, b, f, g) = 0.$$

If one considers the lines of the complex that pass through a point *A* and are situated on a plane (*P*) that passes through that point then they will be the generators of the intersection of the plane (*P*) with the cone of the complex that is associated with the point *A* or the tangents to the curve of the complex that is situated on the plane (*P*) and issues from *A* [Chap. IX, § 1]. If the complex is algebraic then the cone and the curve will be algebraic, and one will say that *the order of the cone of the complex is equal to the class of the planar curve of the complex.* Their common value is called the *degree of the complex*; it is the number of lines of the complex that are situated in a plane and pass through a point of that plane.

If that number is equal to 1 then the complex will be called a *linear complex.* The cone of the complex that is associated with the point *A* is a plane that one calls the *focal plane* or *polar plane* of a point *A*. The curve of the complex that is situated in a plane (*P*) will reduce to a point that one calls the *focus* of *pole* of the plane (*P*). If the plane (*P*) is the polar plane to the point *A* then the point *A* will be the pole of the plane (*P*). *There is reciprocity between a pole and its polar plane* from viewpoint of the duality principle; the duality transformations will not alter the degree of an arbitrary algebraic complex.

Homogeneous coordinates

2. – For the study of algebraic complexes, it is advantageous to replace *a*, *b*, *f*, *g* with the homogeneous coordinates of lines.

Plücker coordinates. – Consider the equations of a line in Cartesian coordinates:

(2) $$\frac{X-f}{a} = \frac{Y-g}{b} = \frac{Z-h}{c},$$

which are equations that contain equations (1) as a particular case. We take the six quantities:

(3) $$a, \qquad b, \qquad c, \qquad p = gc - hb, \quad q = ha - fc, \quad r = fb - ga$$

to be the *Plücker coordinates* of the line. As one sees immediately, those coordinates are coupled by the homogeneous relation:

$$pa + qb + rc = 0. \tag{4}$$

Those six parameters, which are defined only up to the same factor, and which are coupled by one homogeneous relation, reduce to four in reality. a, b, c are the projections onto the axes of a certain vector that is carried by the line. p, q, r are the moments of that vector with respect to the axes (in rectangular coordinates). One can also define them to be the coefficients of the equations of the three projection of the line onto the three coordinate planes, which are supposed to be put into the form:

$$cY - bZ - p = 0, \qquad aZ - cX - q = 0, \qquad bX - aY - r = 0. \tag{5}$$

Let us see what the equation of the complex will become. One infers from (2) that:

$$X = \frac{a}{c}Z - \frac{q}{c}, \qquad Y = \frac{b}{c}Z + \frac{p}{c},$$

and the equation:

$$\varphi(a, b, f, g) = 0$$

will become

$$\varphi\left(\frac{a}{c}, \frac{b}{c}, -\frac{q}{c}, \frac{p}{c}\right) = 0.$$

When that equation is rendered homogeneous, it will take the form:

$$\Psi(a, b, c, p, q) = 0.$$

One can introduce r by virtue of equation (4), and one will finally obtain a homogeneous equation of degree equal to the degree of the complex:

$$\chi(a, b, c, p, q, r) = 0 \tag{6}$$

that will define the complex in Plückerian coordinates. *Conversely,* due to its homogeneity, if one sets $c = 1$, $h = 0$ in formulas (3) then any equation of the preceding form can be reduced to the original form for the equation of a complex:

$$\chi(a, b, 1, g, -f, fb - ga) = 0. \tag{7}$$

We seek the *cone of the complex* whose summit is (x, y, z). Let X, Y, Z denote the current coordinates; it will result from the definition of the Plückerian coordinates that:

$$\begin{cases} a = X - x, & b = Y - y, & c = Z - z, \\ p = cY - bZ, & q = aZ - cX, & r = bX - aY. \end{cases}$$

The equation of the cone of the complex is obtained by replacing *a*, *b*, *c*, *p*, *q*, *r* with the preceding values in the equation of the complex. It will then be:

$$\chi(X - x, Y - y, Z - z, yZ - zY, zX - xZ, xY - yY) = 0.$$

If one transports the origin of the coordinates by translation to the summit of the cone then that equation will be simply:

$$\chi(X, Y, Z, yZ - zY, zX - xZ, xY - yY) = 0.$$

If one seeks a *curve of the complex* then one will take:

$$\begin{cases} a = dx, & b = dy, & c = dz, \\ p = y\,dx - z\,dy, & q = z\,dx - x\,dz, & r = x\,dy - y\,dx, \end{cases}$$

by which the differential equation of the curves of the complex will be:

$$\chi(dx, dy, dz, y\,dz - z\,dy, z\,dx - x\,dz, x\,dy - y\,dx) = 0.$$

The condition for a complex to be special is that [Chap. IX, § 7]:

$$\frac{\partial\varphi}{\partial a}\cdot\frac{\partial\varphi}{\partial g} - \frac{\partial\varphi}{\partial b}\cdot\frac{\partial\varphi}{\partial f} = 0;$$

here, it will become:

(8)
$$\frac{\partial\chi}{\partial a}\cdot\frac{\partial\chi}{\partial p} + \frac{\partial\chi}{\partial b}\cdot\frac{\partial\chi}{\partial q} + \frac{\partial\chi}{\partial c}\cdot\frac{\partial\chi}{\partial r} = 0.$$

Indeed, upon taking the equation of the complex in the form (7) and taking the corresponding formulas:

$$c = r, \qquad p = q, \qquad q = -f, \qquad r = fb - ga$$

into account, it can be written:

$$\frac{\partial\chi}{\partial a}\cdot\frac{\partial\chi}{\partial p} + \frac{\partial\chi}{\partial b}\cdot\frac{\partial\chi}{\partial q} - \frac{\partial\chi}{\partial r}\left(a\frac{\partial\chi}{\partial a} + b\frac{\partial\chi}{\partial b} + p\frac{\partial\chi}{\partial p} + q\frac{\partial\chi}{\partial q} + r\frac{\partial\chi}{\partial r}\right) = 0.$$

If one wishes to obtain equation (8) then it will suffice to take the equation:

$$a\frac{\partial\chi}{\partial a} + b\frac{\partial\chi}{\partial b} + p\frac{\partial\chi}{\partial p} + q\frac{\partial\chi}{\partial q} + r\frac{\partial\chi}{\partial r} = 0$$

into account, which is deduced from (6) by means of the Euler identity on homogeneous functions, or, due to its homogeneity, one can give an arbitrary value to *c*, while the other coordinates take on values that correspond to that value of *c*.

In the case of an arbitrary algebraic complex, equation (8), when combined with that of complex, will define *the congruence of the singular lines.*

Recall the homography between lines and planes of lines of the complex. The coefficients of that homography are $\frac{\partial\varphi}{\partial a}, \frac{\partial\varphi}{\partial b}, \frac{\partial\varphi}{\partial f}, \frac{\partial\varphi}{\partial g}$, and as a result, in homogeneous coordinates, they will be homogeneous, linear combinations of the derivatives $\frac{\partial\chi}{\partial a}$, … $\frac{\partial\chi}{\partial r}$. Consider the line of the complex (a_0, b_0, c_0, p_0, q_0, r_0).

The equation:

$$\sum a \frac{\partial\chi}{\partial a_0} + \sum p \frac{\partial\chi}{\partial p_0} = 0$$

defines a linear complex that contains the line considered, and on that line, the homography for that linear complex will be precisely the same as it was for the original complex. That linear complex is called *tangent* to the given complex.

Remark. – If we define a line by two points (x, y, z) and (x', y', z') then we will see that:

$$\begin{cases} a = x' - x, & b = y' - y, & c = z' - z, \\ p = yz' - zy', & q = zx' - xz', & r = xy' - yx'. \end{cases}$$

Hence, as above, the equation of the cone of the complex will be:

$$\chi(x' - x, y' - y, z' - z, yz' - zy', zx' - xz', xy' - yx') = 0. \tag{9}$$

Correlatively, we define the line by two planes (u, v, w, s), (u', v', w', s'). Upon deducing the equations of the projections of the line from the equations of those planes:

$$uX + vY + wZ + s = 0, \qquad u'X + v'Y + w'Z + s' = 0,$$

and upon reducing the latter to the form (5), one will deduce:

$$\begin{cases} a = vw' - uw', & b = wu' - uw', & c = uv' - vu', \\ p = su' - us', & q = sv' - vs', & r = sw' - ws'. \end{cases}$$

One will then obtain the tangent equation of a planar curve of the complex:

$$\chi(vw' - wv', wu' - uw', uv' - vu', su' - us', sv' - vs', sw' - ws') = 0, \tag{10}$$

and one will then see that the class of that curve, like the order of the cone of the complex, is equal to the degree of the equation of the complex.

General coordinates of Grassmann and Klein. – More generally, take an arbitrary reference tetrahedron, and let x_1, x_2, x_3, x_4 be the coordinates of a point, while u_1, u_2, u_3, u_4 are the coordinates of a plane. Consider the line to be defined by two points (x), (y). We take the quantities:

$$p_{ik} = \rho \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \qquad (i, k = 1, 2, 3, 4) \tag{11}$$

to be the coordinates of that line, in which ρ is an arbitrary homogeneity factor.

We remark that $p_{ii} = 0$ and $p_{ki} = - p_{ik}$, in such a way that one will then obtain only six distinct coordinates; for example, p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}. They are the relative moments, with respect to the vector of the two points (x), (y), of the vectors that are equal to 1 when taken on the six edges of the tetrahedron, or at least, quantities that are proportional to those moments.

Let (p_{ik}) and (p'_{ik}) be two lines. The relative moment M of the two corresponding vectors is given by the formula:

$$\mu M = p_{12}\, p'_{34} + p_{34}\, p'_{12} + p_{13}\, p'_{42} + p_{42}\, p'_{13} + p_{14}\, p'_{23} + p_{23}\, p'_{14},$$

in which μ is a constant factor.

If that moment is zero then the two lines will meet. Now consider the determinant:

$$\Theta = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix},$$

which is identically zero.

Develop it using Laplace's rule:

$$\Theta = 2\,(p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23}).$$

Upon introducing the function:

$$\Phi\,(p_{ik}) = p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23}, \tag{12}$$

the coordinates of an arbitrary lines will satisfy the condition:

$$\Phi\,(p_{ik}) = 0, \tag{13}$$

and the condition for two lines to meet can be written:

$$\sum p'_{ik} \frac{\partial \Phi}{\partial p_{ik}} = 0, \tag{14}$$

in which the summation extends over the six coordinates.

If we define the line by two planes (u), (v) then we can take the coordinates to be:

$$q_{ik} = \sigma \begin{vmatrix} u_i & u_k \\ v_i & v_k \end{vmatrix}, \tag{15}$$

in which σ is an arbitrary homogeneity factor. We seek the relations between the coordinates p_{ik} and the coordinates q_{ik}. Since the line is the intersection of the planes (u), (v), a point (x) of that line will be the intersection of the three planes (u), (v), (w). Hence:

$$u_1\ x_1 + u_2\ x_2 + u_3\ x_3 + u_4\ x_4 = 0,$$
$$v_1\ x_1 + v_2\ x_2 + v_3\ x_3 + v_4\ x_4 = 0,$$
$$w_1\ x_1 + w_2\ x_2 + w_3\ x_3 + w_4\ x_4 = 0.$$

Consider the determinant:

$$\Omega = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix};$$

one can take the coordinate x_i to be the coefficient $S_i = \partial\Omega / \partial s_i$ of s_i. In order to get another point (y) on the line, we define it by three planes (u), (v), (s), and then $y_i = W_i = \partial\Omega / \partial w_i$. Consider the adjoint of Ω:

$$\begin{vmatrix} U_1 & U_2 & U_3 & U_4 \\ V_1 & V_2 & V_3 & V_4 \\ W_1 & W_2 & W_3 & W_4 \\ S_1 & S_2 & S_3 & S_4 \end{vmatrix}.$$

We have the classical relationship between each second-order minor of Ω that is defined by the last two lines and the complementary minor of the adjoint, which can be written:

$$\frac{1}{\rho} p_{ik} = \Omega \frac{1}{\sigma} \cdot \frac{\partial \Phi(q_{ik})}{\partial q_{ik}},$$

with the notation that is defined by formula (12).

Upon disposing of the proportionality factors, one can write this more simply as:

$$p_{ik} = \frac{\partial \Phi(q_{ik})}{\partial q_{ik}}, \tag{16}$$

and similarly:

(17) $$q_{ik} = \frac{\partial \Phi(p_{ik})}{\partial p_{ik}}.$$

The equation of then complex will then be $F(p_{ik}) = 0$, or $F(q_{hl}) = 0$, in which the indices i, k; h, l will correspond in such a manner that $p_{hl} = \partial\Phi / \partial p_{ik}$; one will then have the equations of the cone or the curve of the complex. The condition for the complex to be special is that:

(18) $$\frac{\partial F}{\partial p_{12}} \cdot \frac{\partial F}{\partial p_{34}} + \frac{\partial F}{\partial p_{13}} \cdot \frac{\partial F}{\partial p_{24}} + \frac{\partial F}{\partial p_{14}} \cdot \frac{\partial F}{\partial p_{32}} = 0.$$

Remark. – One can define the coordinates p_{ik} by the remark that the line considered is found in the planes:

$$p_{ik}\, x_l + p_{kl}\, x_i + p_{li}\, x_k = 0,$$

and one can deduce the relations between the p_{ik} and the q_{hl} the from this. The condition $\Phi(p_{ik}) = 0$ expresses the necessary and sufficient condition for those four planes to pass through the same line if one supposes that $p_{ik} = -p_{ki}$. It is then necessary and sufficient that the p_{ik} should be the coordinates of a line.

Linear complexes

3. – Let us study the linear complexes in more detail. The equation of such a complex is:

(1) $$\sum A_{hl}\, p_{ik} = 0,$$

with the notations that were adopted.

The complex will be special if it satisfies the relation:

(2) $$A_{12}\, A_{34} + A_{13}\, A_{42} + A_{14}\, A_{23} = 0,$$

and that equation expresses the idea that the A_{ik} are the coordinates of a line. The equation of the complex expresses the idea that any line of the complex meets that line. *A special linear complex is then composed of the lines that meet a fixed line,* which one calls the *directrix of the complex.*

Let (D) be a line of an arbitrary linear complex, let M be a point of that line, and let (P) be its polar plane. The cone of the complex reduces to the plane (P) here, so the homography of the complex is that of the planes (P) of the line (D) that are associated with their poles M.

Pencils of complexes

4. – Let:

(1) $$\sum A_{hl}\, p_{ik} = 0,$$

(2) $$\sum B_{hl}\, p_{ik} = 0$$

be two linear complex; the equation:

$$\sum (A_{hl} + \lambda\, B_{hl})\, p_{ik} = 0$$

will represent a *pencil of complexes.* Let us look for the special complexes in that pencil. They are defined by the equation:

$$(3)\qquad (A_{12} + \lambda\, B_{12})\,(A_{34} + \lambda\, B_{34}) + (A_{13} + \lambda\, B_{13})\,(A_{12} + \lambda\, B_{12}) + (A_{14} + \lambda\, B_{14})\,(A_{23} + \lambda\, B_{23}) = 0,$$

which is an equation of degree two. *There are then two special complexes in any pencil of linear complexes.* Let look for the conditions under which those two special complexes might coincide.

To that effect, suppose that $\lambda = 0$ is a root of equation (3). The necessary and sufficient condition for that to be true is:

$$\sum A_{12}\, A_{34} = 0,$$

and the preceding equation will reduce to:

$$(4)\qquad \lambda\,(A_{12}\, B_{34} + A_{34}\, B_{12} + \ldots) + \lambda^2\,(B_{12}\, B_{34} + \ldots) = 0.$$

We call the expression:

$$(5)\qquad \Delta_A = A_{12}\, A_{34} + A_{13}\, A_{24} + A_{14}\, A_{23}$$

the *invariant of the complex* (1), and the expression:

$$(6)\qquad \Delta_{AB} = \sum B_{ik} \frac{\partial \Delta_A}{\partial A_{ik}}$$

is the *simultaneous invariant* of the two complexes (1) and (2). With those notations, equation (4) can be written:

$$(7)\qquad \lambda\, \Delta_{AB} + \lambda^2\, \Delta_B = 0.$$

In order for $\lambda = 0$ to be a double root, it is necessary that $\Delta_{AB} = 0$, addition. Now, $\Delta_A = 0$ expresses the idea that the A_{ik} are the coordinates of a line, so $\Delta_{AB} = 0$ will express the idea that the line belongs to the second complex that defines the pencil. Obviously, it belongs to the first one, so it will belong to all complexes of the pencil. One then concludes that: *In order for one of the special complexes to be double, it is necessary and sufficient that it must belong to all complexes of the pencil.*

In order for the equation to reduce to an identity – i.e., in order for all of the complexes of the pencil to be special – it is further necessary that one must have $\Delta_B = 0$. It is therefore necessary that the complexes must be special and their directrices must meet.

We call the set of lines that are common to two linear complexes a *linear congruence.* In general, one and only one line of that congruence will pass through every point of space: It is the intersection of the polar planes to the point in the two complexes. One likewise sees that there is generally one and only one line of the congruence in any plane, and it joins the foci of that plane in the two complexes. Consider the pencil that is determined by the two complexes that define the congruence. If that pencil contains two distinct special complexes then all of the lines of the congruence will belong to those special complexes, and as a result they will meet two fixed directrices; conversely, *a linear congruence is generally composed of the lines that meet two fixed directrices.*

If the special complexes coincide then let (A) be their common directrix. Consider an arbitrary complex (C) of the pencil. (A) is a line of the complex (C). Homographically, each point M of (A) corresponds to its polar plane (P) with respect to the complex (C). The lines of the congruence that pass through M and belong to the complex (C) will be in that polar plane (P). Now, the points of (A) have the same polar plane with respect to all of the complexes of the pencil. The lines of the congruence meet the line (A), and for each point of that line, they will be situated in the corresponding polar plane.

Conversely, if one is given a homography arbitrarily and makes each point M of a fixed line (A) correspond to a plane (P) that passes through that line then the set of ∞^2 lines that each pass through a point M and are located in the plane (P) that is associated with that point M will be a linear congruence, and the special complexes of the corresponding pencil will coincide.

Indeed, take (A) to be the z-axis. A point M of (A) will be defined by its parameter z, and a plane (P) that passes through (A) will be defined by its equation $y - mx = 0$. The equation of the given homography will then be written:

$$P + Bz + Qm - Amz = 0. \tag{8}$$

The Plückerian coordinates a, b, c, p, q, r of a ray of the congruence in question will first satisfy:

$$r = 0, \tag{9}$$

which expresses the idea that the ray meets Oz. If a and b are not both zero then suppose, for example, that $a \neq 0$. The ray meets Oz at the point whose parameter is $z = q / a$, and it will be found in the plane $bx - ay = 0$. Upon taking into account that $ap + bq + cr = 0$, along with equation (9), the relation (8) will then give:

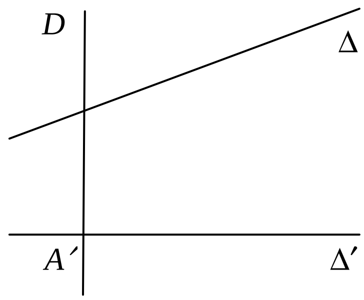
$$Ap + Bq + Pa + Qb = 0. \tag{10}$$

If $a = b = 0$, and if p, q are not both zero, then the ray will meet Oz at infinity, and its equations will be $cy = p$, $cx = -q$. The relation (8) then gives $Ap + Bq = 0$, and equation (10) will be once more verified. It will still be true for $a = b = p = q = r = 0$, which corresponds to the singular ray (A).

In summary, the congruence will be defined by equations (9), (10). Now, they define two linear complexes: The invariant of the first one is zero, as well as their simultaneous invariant. One has then returned to the indicated case.

Complexes in involution

5. – Recall the preceding pencil of complexes. The two basic complexes are said to be *in involution* of $\Delta_{AB} = 0$. Consider a line (*D*) that is common to two complexes, in the general case. A point *M* of that line will correspond homographically to its polar plane in each of the complexes, so let (*P*), (*Q*) be those planes. A homographic correspondence (*H*) between the planes (*P*), (*Q*) of the line will then result. Similarly, upon starting with a plane of the line, one will see that there exists a homography (*H*′) between the points of the line.



We seek the double planes of the homography (*H*). To that effect, consider one of the directrices (Δ) of the linear congruence that is defined by the two complexes and the plane (*D*)(Δ). The pole of that plane with respect to each of the two complexes is the intersection *A*′ of (*D*) with the second directrix (Δ), because all of the lines that pass through *A*′ and meet (Δ) will belong to the congruence, and as a result, to the two complexes. Therefore, in each of the two complexes, *A*′ will be the focus of the plane (*D*)(Δ), and similarly, *A*, which is the intersection of (*D*) and (Δ), will be the focus of the plane (*D*)(Δ′). It will then result that those planes correspond to themselves under the homography (*H*), and consequently that those two planes will be the desired double planes.

One likewise sees that the points *A* and *A*′ are the double points of the homography (*H*′). Having said that, we shall show that the condition $\Delta_{AB} = 0$ expresses the idea that each of the two homographies (*H*) and (*H*′) is an involution.

Indeed, in order for the homography (*H*) between the planes (*P*) and (*Q*) to be an involution, it is necessary and sufficient that the planes (*P*), (*Q*) must be conjugate with respect to its double planes. The equation of the polar plane of a point with respect to an arbitrary complex of the pencil is:

$$(A_{hl} + \lambda B_{hl}) \begin{vmatrix} X_i & X_k \\ x_i & x_k \end{vmatrix} = 0,$$

which is an equation of the form:

$$P + \lambda Q = 0.$$

We point out that it will then result that all of the polar planes of a point with respect to the complexes of a pencil will form a pencil of planes. The axis of that pencil of planes is the line of the linear congruence that is common to the two complexes that pass through the point considered. Consider four arbitrary complexes of the pencil then. The anharmonic ratio of the four polar planes of the same point in those four complexes will be equal to the anharmonic ratio of the four corresponding quantities λ. In particular, take the two basic complexes and the special complexes. The values of λ are 0, ∞, and the roots of the equation:

$$\sum (A_{14} + \lambda B_{14}) (A_{23} + \lambda B_{23}) = 0,$$

and the condition for the first two to be harmonically conjugate with respect to the other two is that:

$$\lambda_1 + \lambda_2 = 0,$$

or $\Delta_{AB} = 0$. Now, if the point considered is found on the line (D) then its polar planes with respect to the two special complexes will be precisely the planes $(D)(\Delta)$ and $(D)(\Delta')$. Hence*: If two complexes are in involution then the polar planes of a point in those two complexes will be harmonic conjugates with respect to the planes that pass through that point and through the directrices of the congruence that is common to the two complexes, and conversely.*

That is equivalent to saying that the homography (H) in an involution. The analogous property that relates to the homography (H') will be established similarly by utilizing the tangential coordinates q_{hl} instead of the point-wise ones p_{ik} . The property of two complexes being in involution will then correspond to itself under duality, and one can further say: *The poles of an arbitrary plane with respect to the complexes of a pencil are on a line that meets the directrices of the congruence that is common to those complexes. If two complexes are in involution then the poles of any plane with respect to those complexes are harmonic conjugates with respect to the points of the intersection of the line that joins them with the two directrices of the congruence that is common to those complexes, and conversely.*

Symmetric coordinates of a line. – One can further generalize the coordinates of lines. Recall the fundamental relation:

(1) $$ap + bq + cr = 0;$$

it is homogeneous and has degree two. Now, there exists a remarkable type of second-degree equation in which only the squares appear. In order to reduce the preceding relation to that form, it will suffice to set, for example:

(2) $$\begin{cases} a+p=t_1, & b+q=t_3, \quad c+r=t_5, \\ a-p=it_2, & b-q=it_4, \quad c-r=it_6. \end{cases}$$

The condition will then become:

(3) $$t_1^2+t_2^2+t_3^2+t_4^2+t_5^2+t_6^2 = 0.$$

One introduces the t_k as homogeneous coordinates, which are homogeneous linear functions of the Plückerian coordinates. Upon equating those six coordinates to 0, one

one will obtain the equations of the six complexes that are pair-wise in involution, because will easily see that the condition for the two complexes:

$$\sum A_k\, t_k = 0, \qquad \sum B_k\, t_k = 0$$

to be an involution is that

(4) $$\sum A_k\, B_k = 0.$$

Those results will persist if one replaces a, b, c, p, q, r with the general coordinates p_{ik} in the definition of the coordinates t_k , and if, even more generally, one replaces the t_k with the coordinates that one deduces from them by an orthogonal, homogeneous linear transformation in six variables.

Conjugate lines

6. – Consider a complex (C) that is not special and a line (Δ) that does not belong to that complex. Consider the congruence that is common to (C) and the special complex whose directrix is (Δ). That congruence has a second directrix (Δ') that is called the *line that is conjugate* to (Δ). There is obviously reciprocity between those two lines. *All of the lines of the complex* (C) *that meet the line* (Δ) *will meet its conjugate* (Δ'), since they are lines of the congruence, and conversely, *any line that meets both of two conjugate lines* (Δ), (Δ') *will belong to the congruence, and in turn, to the complex.* If one considers a point A of (Δ) then its polar plane will pass through (Δ'), since all of the lines that pass through A and meet (Δ') will belong to the complex. Therefore, (Δ') *is the envelope of the planes polar to the points of its conjugate* (Δ). One likewise sees that (Δ') *is the locus of polar of the planes that pass through its conjugate* (Δ). If the line (Δ) belongs to the complex (C) then the two directions of the preceding congruence will coincide. *The lines of the complex are their own conjugates.*

Let the equation of the complex be:

$$F(a, b, c, p, q, r) = Pa + Qb + Rc + Ap + Bq + Cr = 0.$$

Let us look for the coordinates ($a_2, b_2, c_2, p_2, q_2, r_2$) of the conjugate to a line ($a_1, b_1, c_1, p_1, q_1, r_1$). It suffices to express the idea that the given complex and the special complexes that have the lines ($a_1, b_1, c_1, p_1, q_1, r_1$), ($a_2, b_2, c_2, p_2, q_2, r_2$) for their directrices will belong to the same pencil, which gives:

$$P + \lambda_1\, p_1 + \lambda_2\, p_2 = 0 \qquad \text{and its analogues.}$$

Multiply this by $a_1, b_1, c_1, p_1, q_1, r_1$, respectively, and add corresponding sides, so the coefficient of λ_1 will disappear, and we will get:

$$F(a_1, b_1, c_1, p_1, q_1, r_1) + \lambda_2 \sum (a_1\, p_1 + a_2\, p_2) = 0.$$

Set:

$$\sum (a_1\, p_1 + a_2\, p_2) = \sigma,$$

to abbreviate, which will give:

(1) $$F(a_1, b_1, c_1, p_1, q_1, r_1) + \lambda_2\,\sigma = 0.$$

If we multiply this equation by $a_2, b_2, c_2, p_2, q_2, r_2$, resp., and add the resulting equations then the coefficient of λ_2 will disappear, and we will have:

(2) $$F(a_1, b_1, c_1, p_1, q_1, r_1) + \lambda_1\,\sigma = 0.$$

Finally, if we multiply by A, B, C, P, Q, R, resp., then upon setting:

$$\Delta = AP + BQ + CR,$$

we will get:

$$2\Delta + \lambda_1 F(a_1, b_1, c_1, p_1, q_1, r_1) + \lambda_2 F(a_2, b_2, c_2, p_2, q_2, r_2) = 0,$$

which will be written:

$$\Delta = \lambda_1 \lambda_2' \sigma,$$

upon taking (1) and (2) into account, so:

$$\lambda_1 = \frac{\Delta}{\lambda_2' \sigma} = -\frac{\Delta}{F(a_1, b_1, c_1, p_1, q_1, r_1)}.$$

We can then take the coordinates of the conjugate line to be:

$$a_2 = A - \frac{\Delta a_1}{F(a_1, b_1, c_1, p_1, q_1, r_1)}, \qquad \text{and its analogues,}$$

or

(3) $$a_2 = A - AF(a_1, b_1, c_1, p_1, q_1, r_1) - \Delta a_1, \qquad \text{and its analogues.}$$

Suppose that one takes two conjugate lines to be the opposite edges of the reference tetrahedron. If we call the tetrahedral coordinates x, y, z, t then we have seen that:

$$\begin{cases} a = xt' - tx', & b = yt' - ty', & c = zt' - tz', \\ p = yz' - zy', & q = zx' - xz', & r = xy' - yx'. \end{cases}$$

Suppose that one takes the lines ($x = 0$, $y = 0$) and ($z = 0$, $t = 0$) to be conjugate lines. Their coordinates are:

$$\begin{array}{llllll} a_1 = 0, & b_1 = 0, & c_1, & p_1 = 0, & q_1 = 0, & r_1 = 0, \\ a_2 = 0, & b_2 = 0, & c_2 = 0, & p_2 = 0, & q_2 = 0, & r_2\,. \end{array}$$

We express the idea that these lines are conjugate. The conditions that were found before give us:

$$0 = AF(a_1, \ldots), \quad 0 = BF(a_1, \ldots), \quad 0 = CF - \Delta c_1, \quad 0 = PF, \quad 0 = QF, \quad r_2 = RF.$$

Now:

$$F(a_1, b_1, c_1, p_1, q_1, r_1) = F(0, 0, c_1, 0, 0, 0) = R\, c_1 .$$

Since Δ is non-zero, by hypothesis, it will result that:

$$A = 0,\ \ B = 0,\ \ P = 0,\ \ Q = 0,\ \ R \neq 0,\ \ C \neq 0.$$

Hence:

$$\Delta = RC,$$

and the equation of the complex will take the reduced form:

$$Cr + Rc = 0,$$

or

(4) $$r = kc.$$

In particular, we seek to perform that reduction in Cartesian axes. We take the conjugate lines to be the axis *Oz* and the line at infinity in the *xy*-plane. One must first show that there are lines whose conjugates can be pushed out to infinity. In order for a line (a_1, b_1, c_1, p_1, q_1, r_1) to be at infinity, it is necessary and sufficient that $a_1 = 0$, $b_1 = 0$, $c_1 = 0$, and from the formulas that were found before, the conjugates to those lines will be such that:

$$\frac{a_2}{A} = \frac{b_2}{B} = \frac{c_2}{C} = \frac{F(0,0,0,p_1,q_1,r_1)}{\Delta}.$$

a_2, b_2, c_2 are then proportional to fixed quantities. *The conjugates of the lines at infinity are parallel to the same direction. Those lines are the loci of the poles of the planes parallel to a fixed plane.* One calls them *diameters*, and the parallel planes whose poles are on a diameter are said to be *conjugate to that diameter*. Upon referring a complex to a diameter and a conjugate plane, the equation of the complex will then take the form:

$$r = kc.$$

One can obtain that reduction in rectangular axes. Indeed, there exists an infinitude of lines perpendicular to their conjugates. They are defined by the relation:

$$a_1\, a_2 + b_1\, b_2 + c_1\, c_2 = 0,$$

or

$$(Aa_1 + Bb_1 + Cc_1)\, F(a_1, b_1, c_1, p_1, q_1, r_1) - \Delta(a_1^2 + b_1^2 + c_1^2) = 0.$$

Those lines then constitute a second-degree complex.

Take an arbitrary diameter (a_1, b_1, c_1, p_1, q_1, r_1) of the linear complex. The conjugate plane that passes through the origin and the line at infinity that is conjugate to the diameter (0, 0, 0, p_2, q_2, r_2) will have the equation:

$$p_2X + q_2Y + r_2Z = 0.$$

The condition for it to be perpendicular to the diameter is:

$$\frac{a_1}{p_2} = \frac{b_1}{q_2} = \frac{c_1}{r_2},$$

or

$$\frac{a_1}{PF_1 - \Delta p_1} = \frac{b_1}{QF_1 - \Delta q_1} = \frac{c_1}{RF_1 - \Delta r_1},$$

in which we have set:

$$F_1 = F(a_1, b_1, c_1, p_1, q_1, r_1),$$

to abbreviate.

Since the conjugate line to the diameter is at infinity, $a_2 = b_2 = c_2 = 0$. Hence, from formulas (3), a_1, b_1, c_1 will be proportional to A, B, C, which will give:

$$\frac{A}{PF_1 - \Delta p_1} = \frac{B}{QF_1 - \Delta q_1} = \frac{C}{RF_1 - \Delta r_1}.$$

Now:

$$a_1\, p_1 + b_1\, q_1 + c_1\, r_1 = 0,$$

which gives:

$$A\, p_1 + B\, q_1 + C\, r_1 = 0,$$

here, so:

$$F_1 = F(a_1, b_1, c_1, p_1, q_1, r_1) = P\, a_1 + Q\, b_1 + R\, c_1 .$$

If we multiply the two terms in the preceding ratios by A, B, C, respectively, and add them then we will get a ratio that equals $\sum A^2 / \Delta F_1$. We can then take $a_1 = A$, $b_1 = B$, $c_1 = C$, so $F_1 = \Delta$, and finally:

$$\frac{A}{PF - p_1\Delta} = \frac{\sum A^2}{\Delta^2}, \quad \text{and analogous equations.}$$

We will then have the defining formulas:

$$(5) \qquad a_1 = A, \quad b_1 = B, \quad c_1 = C, \quad p_1 = P - \frac{A\Delta}{\sum A^2}, \quad q_1 = Q - \frac{B\Delta}{\sum B^2}, \quad r_1 = R - \frac{C\Delta}{\sum C^2}.$$

We then obtain one and only one diameter that is perpendicular to the conjugate plane: It is the *axis of the complex*. Upon taking it to be the z-axis, we will get the reduced equation in rectangular coordinates:

$$r - mc = 0.$$

The form of the complex depends upon only one parameter m, which is its invariant with respect to the group of motion.

If $r = 0$, $c = 0$ then the equation will be satisfied. Now, $r = 0$, $c = 0$ are the coordinates of the lines that meet Oz and are perpendicular to it. *The complex contains all lines that meet the axis and are perpendicular to it; c, r* are coordinates that do not change when one turns the line around Oz; similar statements will be true if one displaces it parallel to Oz. In other words: *A helicoidal motion with axis Oz will leave the complex unaltered. It will then result that if one has* ∞^1 *lines that belong to the complex and are derived from each other by a helicoidal motion then one will obtain all lines of the complex by subjecting that system of lines to the preceding rotations and translations.* Consider the lines whose coordinates a, p are zero, and look for those lines among them that belong to the complex. One will find the lines:

$$bx = mc, \qquad cy - bz = 0,$$

which constitute a family of generators of the paraboloid:

$$xy - mz = 0.$$

Consequently, *in order to obtain the lines of a complex, it will suffice to take a system of generators of an equilateral paraboloid and to subject it to all of the helicoidal displacements that have the axis of the paraboloid for their axis.*

Nets of complexes

7. – If $\Phi = 0$, $\Phi' = 0$, $\Phi'' = 0$ are the equations of three linear complexes then a *net of complexes* will be defined by the equation:

$$\lambda\,\Phi + \lambda'\,\Phi' + \lambda''\,\Phi'' = 0.$$

Consider the lines that are common to all of the complexes of the net – i.e., common to the three complexes $\Phi = 0$, $\Phi' = 0$, $\Phi'' = 0$; there are ∞^1 of them. They belong to the special complexes of the net, so one can define them, *in general*, by means of three of those special complexes. Now, a special complex is composed of all the lines that meet its directrix. Since the preceding lines will then meet three arbitrary fixed lines, they will constitute a system of generators of a quadric, and the second system of generators will consist of the directrices of the special complexes of the net.

Application. – One can define a complex by five lines that do not belong to the same linear congruence. Indeed, let the lines be 1, 2, 3, 4, 5; we pick a point P and seek its polar plane. Consider the lines 1, 2, 3, 4; there exist two lines (Δ), (Δ') that meet those four lines. Those lines are conjugate with respect to the complex, so the line that passes through P and is supported by (Δ), (Δ') will belong to the complex. Similarly, upon considering the lines 2, 3, 4, 5, we will get a second line that passes through P and belongs to the complex; the polar plane to P will be then determined by those two lines.

Remark. – In order to find the lines that are common to four complexes:

$$\Phi = 0, \quad \Phi' = 0, \quad \Phi'' = 0, \quad \Phi''' = 0,$$

one can likewise, *in general*, replace these complexes with four of the special complexes that are contained in the family of ∞^3 complexes:

$$\lambda\,\Phi + \lambda'\,\Phi' + \lambda''\,\Phi'' + \lambda'''\,\Phi''' = 0.$$

The problem then amounts to finding the lines that meet four arbitrary fixed lines, and as one knows, one will have two solutions.

Curves of a linear complex

8. – We propose to determine the curves of the complex:

$$r = kc.$$

Consider a line that passes through a point (x, y, z) and the direction coefficients a, b, c. In order for them to belong to the complex, in it necessary and sufficient that:

$$bx - ay = kc.$$

The differential equation of the curves of the complex is then:

$$x\,dy - y\,dx = k \cdot dz. \tag{1}$$

That equation can be written:

$$x^2\,d\left(\frac{y}{x}\right) = d\,(kz).$$

Set:

$$kz = Y, \qquad \frac{y}{x} = X, \qquad x^2 = P. \tag{2}$$

The preceding equation will become:

$$dY - P\,dX = 0;$$

it shows that P is the derivative of Y with respect to X. One will then obtain the general integral to (1) by setting:

$$X = \varphi(t), \qquad X = \varphi(t), \qquad P = \frac{d\psi}{d\varphi} \tag{3}$$

in equation (2). One will then obtain x, y, z, expressed as functions of one arbitrary variable t, by means of two arbitrary functions. If one takes the independent variable to be X then it will suffice to set:

$$Y = f(X), \qquad P = f(X);$$

hence, the equations of the curve will be:

$$(4) \qquad kz = f\left(\frac{y}{x}\right), \quad x^2 = f'\left(\frac{y}{x}\right).$$

Upon finally setting:

$$\frac{y}{x} = u,$$

one will obtain the expressions for *x*, *y*, *z* as functions of *u*:

$$(5) \qquad x = \sqrt{f'(u)}\,, \quad y = u\sqrt{f'(u)}\,,\ z = \frac{1}{k}\,f(u)\,.$$

It is easy to obtain some remarkable curves of the complex by specializing the form of the function *f*.

1. One will get all of the algebraic curves of the complex by taking *f* to be an algebraic function of *u*. In particular, set:

$$f(u) = \frac{u^3}{3},$$

so

$$f'(u) = u^2,$$

and therefore:

$$(6) \qquad x = u, \qquad y = u^2, \qquad z = \frac{u^3}{3k}.$$

Those are the equations of a twisted cubic that osculates the plane at infinity in the direction $x = 0$, $y = 0$. Conversely, one can reduce the equations of any twisted cubic to the preceding form by a projective transformation, so it will result that *the tangent to any twisted cubic belongs to a linear complex.*

2. The general formulas (5) will contain a radical, provided that one has set $x^2 = P$. One can make the radical disappear by choosing the parameter in such a fashion that *P* is a perfect square. In order to do that, consider the plane curve $X = \varphi(t)$, $Y = \psi(t)$, which is the envelope of the line:

$$Y - u^2 X + 2\,\theta(u) = 0.$$

X, *Y* are such that:

$$\frac{dY}{dX} = u^2,$$

and the envelope is defined by the equation of the line and by:

$$-\,u\,X + \theta'(u) = 0.$$

Hence, one infers that:

$$X = \frac{\theta'(u)}{u}, \qquad Y = u\,\theta'(u) - 2\theta(u);$$

hence:

(7) $$x = u, \qquad y = \theta'(u), \qquad z = \frac{1}{k}[u\,\theta'(u) - 2\theta(u)].$$

These formulas permit one to find all of the unicursal curves of the complex. One only has to take u to be a rational function of an arbitrary parameter and to take θ to be a rational function of u.

3. The differential equation (1) is then written:

$$(x^2 + y^2)\; d\left(\arctan\frac{y}{x}\right) = k\,dz.$$

Set:

$$kz = Y, \qquad \arctan\frac{y}{x} = X, \qquad x^2 + y^2 = P = \frac{dY}{dX}.$$

Upon taking X to be an independent variable, one will obtain the general integral in the form:

$$\arctan\frac{y}{x} = \omega, \qquad kz = f(\omega), \qquad x^2 + y^2 = f'(\omega),$$

which is written:

(8) $$x = \sqrt{f'(\omega)}\cdot\cos\omega, \qquad y = \sqrt{f'(\omega)}\cdot\sin\omega, \qquad z = \frac{1}{k^2}f(\omega).$$

One will get some particular curves by setting:

$$f(\omega) = R^2\omega + C;$$

hence:

(9) $$x = R\cos\omega, \quad y = R\sin\omega, \quad z = \frac{R^2}{k}\omega + a.$$

Those are helices that are traced on cylinders of revolution around the axis of the complex. The pitch of those helices $2\pi R^2 / k$ is uniquely a function of R; therefore, *all of the helices of the complex that are traced on the same cylinder that has the axis of the complex for its axis will have the same pitch.*

General properties of the curves of a complex

It results immediately from the definition of the curves of a complex that *in a linear complex, the polar plane to a point of a curve of the complex is the osculating plane to* *the curve at that point* [Chap. IX, § 1]. Consider the osculating planes to a curve of the complex that issues from a point *P*. Let *A* be one of the contact points. The osculating plane at *A* is the polar plane to *A*, so the line *PA* will belong to the complex, and in turn, it will be in the polar plane of *P*. It will then result that *the contact points of the osculating planes that issue from a point on a curve of a linear complex are in the same plane that passes through that point.* In particular, *the contact points of the osculating planes that issue from a point of a twisted cubic are in the same plane that passes through that point.*

Take formulas (7). We find that:

$$A = y'z'' - z'y'' = \frac{1}{k}\theta'\theta''' = \frac{y}{k}\theta''',$$

$$B = z'x'' - x'z'' = -\frac{u}{k}\theta''' = -\frac{x}{k}\theta''',$$

$$C = x'y'' - y'z'' = \theta''',$$

and

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = \frac{1}{k}\theta'''^2.$$

One then sees that the torsion at the point (*x*, *y*, *z*) is given by:

$$T = -\frac{x^2 + y^2 + k^2}{k}.$$

It depends upon only the point, and not on the curve. Therefore, *all curves of the linear complex that pass through a point will have the same torsion at that point (Sophus Lie).*

Surfaces normal to the rays of a complex

9. – There is no reason to search for the surfaces of a linear complex. Indeed, let:

$$ay - bx + kc = 0$$

be a linear complex. The polar plane to the point (*x*, *y*, *z*) is parallel to the plane:

$$Xy - Yx + kZ = 0,$$

and in order for a surface $z = f(x, y)$ to be tangent to that plane, it is necessary that:

$$\frac{p}{x} = \frac{q}{-x} = \frac{-1}{k},$$

or

$$p = -\frac{y}{k}, \qquad\qquad q = \frac{x}{k}.$$

Now, the integrability condition:

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$

is not realized. The problem is therefore insoluble.

We then seek the surfaces whose normals are lines of the complex. We will have to integrate the partial differential equation:

$$py = qx = k = 0,$$

which amounts to the integration of the system:

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{k} = -dt,$$

which is precisely the system to which one arrives when one looks for the normal curves to the polar planes of their points. That system is written:

$$dx = -y \cdot dt, \quad dy = x \cdot dt, \qquad dz = -k \cdot dt,$$

and is integrated immediately. Since t defined only up to an additive constant, the general integral will be written:

$$x = R \cos t, \qquad y = R \sin t, \qquad z = -kt + h.$$

The orthogonal trajectories depend upon two arbitrary constants. They are circular helices that all have the same pitch, and thus the trajectories of a uniform helicoidal motion of pitch $-2k\pi$.

One then has the *kinematical interpretation of the linear complex:* Consider a uniform helicoidal motion. Each point M corresponds to the velocity at that point, and the polar plane to the point M in the complex is the plane perpendicular to that velocity. *The linear complex is composed of the normals to the velocities of instantaneous motion of a solid body.*

The surfaces that are normal to the complex are defined by the equations:

$$x = v \cos u, \qquad y = v \sin u, \qquad z = -ku + \varphi(v),$$

because they are generated by the preceding helices. They are the helices that are generated by an arbitrary profile in the preceding motion. The preceding equations represent the most general helicoid, moreover. It will then result that *the normals that issue from a point of a helicoid are in the same plane* (viz., the polar plane to that point).

Remark. – The helices that are orthogonal trajectories to the polar planes are obtained by setting v = const., and their orthogonal trajectories are the curves of the complex that are situated on the preceding surfaces. Let us look for them. We form the linear element on those surfaces:

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= (\cos u \cdot dv - v \sin u \cdot du)^2 + (\sin u \cdot dv + v \cos u \cdot du)^2 + (-k\, du + \varphi' \cdot dv)^2$$

or:

$$ds^2 = (v^2 + k^2)\, du^2 - 2k\, \varphi' \cdot du\, dv + (1 + \varphi'^2) \cdot dv^2.$$

The orthogonal trajectories of the helices v = const., $dv = 0$ are defined by the equation:

$$(v^2 + k^2)\, du - k\, \varphi' \cdot dv = 0;$$

hence:

$$u = \int \frac{k\,\varphi'}{k^2 + v^2}\, dv.$$

Their determination depends upon one quadrature.

Ruled surfaces of a complex

10. – Consider a ruled surface whose generators belong to the complex. Let (G) be one of its generators. It belongs to the complex, and therefore each of its points M will correspond to a plane (P) that is the focal plane. On the other hand, the point M also corresponds homographically to the tangent plane to the surface at that point. It will then result that *there is a homographic correspondence between the polar plane to a point of the generator and the tangent plane to the surface at that point.* There are two double elements to that homography, and therefore *there will exist two points A, B on each generator of the surface such that polar planes to those points are tangent to the surface.* Consider the locus of points A on the surface. The tangent plane to the surface at each of those points is the polar plane to A. The tangent to the curve, which is in the tangent plane to the surface, will then be in the polar plane. Hence: *The locus of points A, and also the locus of points B (which can coincide algebraically, moreover) will be curves of the complex.* The osculating plane at each point is the polar plane, so it will be tangent to the surface. *Those curves are asymptotes to the ruled surface then.* Moreover, the asymptotes are determined by means of only one quadrature [Chap. V, § 10].

It can happen that the generators of the surface belong to a linear congruence. They will then belong to an infinitude of linear complexes, and for each complex, one will have asymptotic lines that are curves of the complex. One will then obtain all of the asymptotes without any integration. *The generators of the preceding surface are then supported by two fixed directrices.* That is the case for conoids with a director plane and the general third-order ruled surfaces [Chap. V, § 10, pp. 115]. Conversely, one will easily see that an arbitrary curve of the complex is asymptotic to an infinitude of ruled surfaces of the complex. One can then find an arbitrary curve of the complex by means of those ruled surfaces.

If the generators of the surface belong to a special linear complex then the curves of the complex are plane curves whose planes contain the directrix of the complex: *The normal surfaces of the complex are the surfaces of revolution around the directrix; the ruled surfaces of the complex are surfaces whose generators meet a fixed line.* That directrix is an asymptote of the surface, and the other asymptotes will be determined by two quadratures.

CHAPTER XI

CONTACT TRANSFORMATIONS. – DUALITY TRANSFORMATIONS. – SOPHUS LIE'S TRANSFORMATION THAT CHANGES LINES INTO SPHERES

1. – First recall the notions of the geometry of contact elements that was introduced in Chapter IV, § 4, and was used frequently in the chapters that followed, while completing those notions:

A *contact element* is the set that consists of a point M and a plane (P) that passes through that point. Such an element is defined by its five *coordinates:* viz., the coordinates (x, y, z) of the point and the coefficients $(p, q, -1)$ of the normal to the plane.

Consider a point A. The contact elements at that point are composed of that point and all of the planes that pass through the point. The coordinates x, y, z are then fixed, while p, q are arbitrary. A point then possesses ∞^2 contact elements.

Consider a curve. One of its contact elements is composed of a point of the curve and a plane that is tangent to the curve at that point. The coordinates are x, y, z, which are functions of one arbitrary parameter u, and p, q are coupled by the relation:

$$p\frac{dx}{du}+q\frac{dy}{du}-\frac{dz}{du}=0.$$

There are then two arbitrary parameters. A curve possesses ∞^2 contact elements.

Now consider a surface. One of its contact elements is composed of a point and the tangent plane at that point. Its coordinates are $x, y, z = f(x, y)$, $p = \partial f / \partial x$, $q = \partial f / \partial y$. There are two arbitrary parameters then, and a surface therefore possesses ∞^2 contact elements. We remark that p, q might depend upon just one parameter. That is the case for developable surfaces, which then possess ∞^2 points and ∞^1 tangent planes, and correspond, by duality, to curves, which possess ∞^1 points and ∞^2 tangent planes.

The points, curves, and surfaces that are generated by ∞^2 contact elements are called *multiplicities* M_2. More generally, one calls any family of contact elements whose coordinates verify the relation:

(1) $$dz - p\,dx - q\,dy = 0$$

a *multiplicity.* If those coordinates depend upon only one arbitrary parameter then one will have a *multiplicity* M_1. If they depend upon two arbitrary parameters then one will have a *multiplicity* M_2.

We seek to determine all multiplicities M_2. The coordinates x, y, z, p, q are functions of two arbitrary parameters:

$$x = f(u, v),\qquad y = g(u, v),\qquad z = h(u, v),\qquad p = k(u, v),\qquad q = l(u, v).$$

Consider the first three relations. One can eliminate u, v from them, and one can obtain one, two, or three relations as a result of that elimination.

First suppose that obtains a relation:

$$F(x, y, z) = 0.$$

z, for example will then be a function of x, y, and if one writes that the relation (1) is satisfied for any x, y then one will get:

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}.$$

That will give the contact elements of a surface.

Suppose that one has two relations:

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0.$$

Two of the coordinates will then be functions of the third one; for example, x, y might be functions of z:

$$x = \varphi(z), \qquad y = \psi(z).$$

Those equations define a curve, and equation (1) will become:

$$dz - p\,\varphi'(z)\,dz - q\,\psi'(z)\,dz = 0,$$

or

$$p\,\varphi'(z) + q\,\psi'(z) - 1 = 0.$$

The plane of the contact element is then tangent to the curve, and is subject to only that condition: One then obtains the contact elements of a curve.

Finally, if one obtains three relations then x, y, z will be constants. Equation (1) will be verified for any p, q, which are then arbitrary parameters, and one will have the contact elements of a point.

We now look for the multiplicities M_1. x, y, z, p, q are functions of just one parameter:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad p = k(t), \qquad q = l(t).$$

Consider the first three equations, and eliminate t from them. We will then obtain two or three relations.

If there are two relations then the locus of points of the multiplicity, which one also calls the *support of the multiplicity*, is a curve, and the planes depend upon only one parameter, so each point of the curve will correspond to a well-defined tangent plane. One then has a *strip of contact elements.*

If there are three relations then x, y, z will be constant, and the support will be a point. One will then have a family of planes that depend upon one parameter and pass through a fixed point. That is what one calls an *elementary cone.*

Consider two multiplicities M_2 . They can have zero or one contact element in common, or even an infinitude of them.

Consider the case of *one common contact element.* If the multiplicities are two points A, A' then they will have a common contact element only if the two points coincide, and there will be ∞^2 common contact elements.

If the multiplicities are a point and a curve then the point will be on the curve, and all of the tangent planes to the curve at that point will belong to common contact elements, which will then be ∞^1 in number.

If the multiplicities are a point and a surface then the point will be on the surface, and the common contact element will be unique and composed of the point and the tangent plane to the surface at that point.

Consider two curves. If they have a common contact element then they will meet at a point, and if they are not tangent then there will be only one common contact element.

Consider a curve and a surface. They will have a common contact element if the curve is tangent to the surface.

Finally, two surfaces will have a common contact element if they are tangent at a point.

There will be ∞^1 *common contact elements* for a point on a curve, two curves that are tangent at a point, a curve that is situated on a surface, and two surfaces that are circumscribed along a curve.

Consider a *point that describes a curve.* We have a family of ∞^1 points, each of which will give ∞^1 contact elements to the curve.

Consider a *surface that is generated by a curve.* We have ∞^1 curves, each of which has a strip in common with the surface, and will in turn give ∞^1 contact elements to the surface.

Consider the *enveloping surface of* ∞^1 *surfaces.* Each envelope has a strip of ∞^1 contact elements in common with the envelope. In the three cases, we have ∞^1 multiplicities M_2 of generators.

Consider the case in which each generating element gives, on the contrary, only one contact element to the generated multiplicity: ∞^2 points generate a surface. ∞^2 curves define a congruence of curves. (In this case, as in that of congruences of lines, there will generally be a focal surface that is tangent to each of those curves and has one common contact element with each of them.) Finally, if one considers ∞^2 surfaces then they will have an envelope that has one contact element in common with each of them.

Remarks:

1. In the three preceding cases, when we said that each generator element gave one contact element to the multiplicity, we necessarily meant that the multiplicity could be decomposed into sheets, and that the statement then applied to each of the sheets separately.

2. There is an exceptional case, namely, that of ∞^1 curves that have a curve for their envelope. One will then have ∞^1 curves that each give ∞^1 contact elements to that envelope.

CONTACT TRANSFORMATIONS

2. – One calls any transformation of the contact elements that changes any multiplicity M_2 into a multiplicity M_2 a *contact transformation*. Such a transformation is defined by five equations:

$$x' = f(x, y, z, p, q), \quad y' = g(x, y, z, p, q), \quad z' = f(x, y, z, p, q), \quad p' = k(x, y, z, p, q), \quad q' = l(x, y, z, p, q). \tag{1}$$

If the variable contact element (x, y, z, p, q) belongs to a multiplicity then its coordinates will verify the condition:

$$dz - p\,dx - q\,dy = 0, \tag{2}$$

and for the transformed element (x', y', z', p', q') to also belong to a multiplicity, it is necessary and sufficient that one must have:

$$dz' - p'\,dx' - q'\,dy' = 0. \tag{2'}$$

A contact transformation is then defined by equations (1), such that each of the Pfaff equations (2), (2′) transforms into the other one when one makes the change of variables that is defined by those equations. That is what one expresses by saying that contact transformations are the transformations of x, y, z, p, q that leave the Pfaff equation (2) invariant.

Such a transformation changes two multiplicities that have a common contact element into two multiplicities that have a common contact element. Similarly, it will transform two multiplicities that have ∞^1 common contact elements into two multiplicities that have ∞^1 common contact elements. A contact transformation changes points, curves, and surfaces into points, curves, or surfaces, indistinctly.

Recall the equations of the transformation, and eliminate p, q, p', q' from them. We will then get one, two, or three relations between $x, y, z; x', y', z'$.

Prolonged point-like transformations. – If one obtains three relations:

$$x' = f(x, y, z), \qquad y' = g(x, y, z), \qquad z' = h(x, y, z) \tag{3}$$

in the contact transformation then it will contain a point-like transformation. Such a transformation changes a point into a point, a curve into a curve, and a surface into a surface. Two curves that meet will transform into two curves that meet, and two tangent surfaces will transform into two tangent surfaces. A contact element that is common to two multiplicities will correspond to a contact element that is common to two transformed multiplicities. One will obtain p', q' as functions of p, q by considering z' to be a function of x', y'. Hence:

$$dx' = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}\,(p\,dx + q\,dy), \qquad dy' = \ldots, \qquad dz' = \ldots$$

Eliminating dx, dy from these three relations, one will obtain an equation of the form:

$$dz' = k(x, y, z, p, q)\,dx' + l(x, y, z, p, q)\,dy',$$

so

$$p' = k(x, y, z, p, q), \qquad q' = l(x, y, z, p, q).$$

In that case, one says that the contact transformation is a *prolonged point-like transformation.*

Case of just one directrix equation

3. – Now suppose that one obtains a relation by elimination:

$$\Omega(x, y, z; x', y', z') = 0. \tag{4}$$

Consider a point A (x, y, z) in the first space. Look for the multiplicity that it corresponds to in the second space. It is generated by the contact elements whose points are linked to the point A by equation (4), which represents a surface S'_A. The multiplicity that corresponds to a point is a surface. If one has a curve that is the locus of points A then it will correspond to a family of ∞^1 surfaces, and the multiplicity that is generated by those surfaces – i.e., their envelope – will be the transform of the curve. Finally, if one has a surface that is a locus of ∞^2 points A then it will correspond to ∞^2 surfaces, whose envelope will correspond to the given surface.

Equation (4) is called the *directrix equation* of the transformation. It defines surfaces in the second space that are homologous to surfaces in the first space, and conversely.

Duality transformations

In particular, suppose that the relation (4) is bilinear in x, y, z; x', y', z'. Each point of the first space corresponds to a plane in the second space, and conversely. ∞^3 points in the first space will correspond to ∞^3 distinct planes. Let:

$$\Omega = Ax' + By' + Cz' + D,$$

in which:

$$A = u\,x + v\,y + w\,z + h, \qquad B = u'\,x + \ldots, \;\; C = u''\,x + \ldots, \;\; D = u'''\,x + \ldots$$

In order to have the transform of a surface:

$$f(x', y', z') = 0,$$

one must take the envelope of planes $\Omega = 0$, x', y', z' that are coupled by the preceding relation, which gives:

$$\frac{A}{\dfrac{\partial f}{\partial x'}} = \frac{B}{\dfrac{\partial f}{\partial y'}} = \frac{C}{\dfrac{\partial f}{\partial z'}} = \frac{D}{\dfrac{\partial f}{\partial t'}}.$$

Those are the equations of the transformation. It is necessary that one must be able to infer $x, y, z,$ so the forms A, B, C, D must be independent, and then the set of planes $\Omega = 0$ will indeed constitute the set of all planes in space. The preceding transformation is a *duality transformation.*

We remark that the set of contact transformations obviously forms a *group* [cf., Chap. VIII, § 8, pp. 227]. As a result, a contact transformation can often decompose into simpler contact transformations. We shall see that this is the case for duality transformations.

Take the new variables to be:

$$X = \frac{A}{D}, \qquad Y = \frac{B}{D}, \qquad Z = \frac{C}{D},$$

so

$$\Omega = X\,x' + Y\,y' + Z\,z' + 1 = 0,$$

and the transformation will be a transformation by polar reciprocals with respect to the sphere:

$$x^2 + y^2 + z^2 + 1 = 0.$$

Hence, any duality transformation will reduce to the preceding transformation followed by a projective transformation, and conversely.

Remark. – One sees, in an analogous manner, that any duality transformation can also be reduced to the same transformation by polar reciprocals, *preceded* by a projective transformation. Therefore, if one performs two duality transformations in succession then the final result that is obtained (viz., the *product* of the two operations) will be a projective transformation.

Involutive duality transformations. – Look for all the duality transformation that are *symmetric* – or *involutive*; i.e., such that the plane that is homologous to a point is the same, regardless of whether one considers the point to belong to one or the other space. The equations:

$$\Omega\,(x, y, z; x', y', z') = 0, \qquad \Omega\,(x', y', z'; x, y, z) = 0$$

must be equivalent. There will then exist a constant factor k such that:

$$\Omega\,(x, y, z; x', y', z') \equiv k\,\Omega\,(x', y', z'; x, y, z).$$

Set $x' = x, y' = y, z' = z$:

$$\Omega\,(x, y, z; x, y, z) \equiv k\,\Omega\,(x, y, z; x, y, z).$$

One will then have either $\Omega\,(x, y, z; x, y, z) = 0$ or $k = 1$.

If $\Omega = 0$ then the corresponding plane will have a point that passes through that point. For any x, y, z, one will have:

$$x\,(ux + vy + wz + h) + y\,(u'x + v'y + w'z + h') + z\,(u''x + \ldots) + u'''\,x + \ldots \equiv 0,$$

which amounts to writing that the determinant:

$$\begin{vmatrix} u & v & w & h \\ u' & v' & w' & h' \\ u'' & v'' & w'' & h'' \\ u''' & v''' & w''' & h''' \end{vmatrix}$$

is a skew-symmetric determinant, so it will have the form:

$$\begin{vmatrix} 0 & C & -B & P \\ -C & 0 & A & Q \\ B & -A & 0 & R \\ -P & -Q & -R & 0 \end{vmatrix}.$$

The directrix equation will then be written:

$$\Omega = x'\,(Cy - Bz + P) + y'\,(-Cy + Az + Q) + z'\,(Bx - Ay + R) - Px - Qy - Rz = 0,$$

or:

$$A\,(yz' - zy') + B\,(zx' - xz') + C\,(xy' - yx') + P\,(x - x') + Q\,(y - y') + R\,(z - z') = 0.$$

This is the equation of a linear complex, and the locus of points (x', y', z') that are associated with the point (x, y, z) is the polar plane to the point (x, y, z) by the relationship that the complex defines. The polar plane to a point is the multiplicity that is the transform of that point, and conversely. As a result, the transform of a line is its conjugate, and a line of the complex is its own homologue. Two homologous multiplicities M_2 are the two focal multiplicities of a congruence of lines of the complex, and conversely. Since a multiplicity M_2 can always be considered to be a focal multiplicity of the congruence of the ∞^2 lines of the complex that have at least one contact element in common with that multiplicity, and since those lines are homologous to themselves, the transformed multiplicity of M_2 must have at least one contact element in common with each of those lines.

A curve will generally correspond to a developable. A curve of the complex will correspond to the developable of its tangents.

If we now take the solution $k = 1$ then we will have:

$$x'\,(ux + vy + wz + h) + \ldots = x\,(ux' + vy' + wz' + h) + \ldots$$

The form Ω will then be symmetric in x, y, z; x', y', z', and will be written:

$$\Omega = Axx' + Byy' + Czz' + M\,(yz' + zy') + N\,(zx' + xz') + P\,(xy' + yx')$$
$$+ Q\,(x + x') + R\,(y + y') + S\,(z + z') + T.$$

The two points (x, y, z), (x', y', z') are conjugate with respect to the quadric:

$$Ax^2 + By^2 + Cz^2 + 2Myz + 2Nzx + 2Pxy + 2Qx + 2Ry + 2Sz + T = 0.$$

We then obtain the most general transformation by polar reciprocals.

The *Legendre transformation* is given by the quadric $x^2 + y^2 - 2\,z = 0$. The directrix equation is $xx' + yy' - z - z' = 0$, and the equations of the transformation are $x = p$, $y = q$, $p = x$, $q = y$, $z = px + qy - z$.

Remark. – In order to get the equations of a contact transformation that is defined by just one directrix equation $\Omega = 0$, one must write down that the equation:

(2′) $$dz' - p'\,dx' - q'\,dy' = 0$$

is a consequence of the equations:

(2) $$dz - p\,dx - q\,dy = 0,$$

(5) $$d\Omega = 0,$$

which is equivalent to posing an identity of the form:

(6) $$dz' - p'\,dx' - q'\,dy' \equiv \lambda\,(dz - p\,dx - q\,dy) + \mu\,d\Omega.$$

Indeed, let $\Omega = 0$, $\Omega_1 = 0$, …, $\Omega_4 = 0$ be five distinct equations in x, y, z, p, q; x', y', z', p', q' that define the transformation. The invariance of equation (2) is expressed by an identity of the form:

$$dz' - p'\,dx' - q'\,dy' \equiv \lambda\,(dz - p\,dx - q\,dy) + \mu\,d\Omega + \mu_1\,d\Omega_1 + \ldots + \mu_4\,d\Omega_4\,.$$

If not all four $\mu_1, \ldots, \mu_4$ are zero then one will conclude from this that $(\mu_1\,d\Omega_1 + \ldots + \mu_4\,d\Omega_4)$ contains only the differentials dx, dy, dz; dx', dy', dz' without being identically zero. The equations of the transformation then imply two linear relations that are homogeneous in dx, dy, dz; dx', dy', dz', namely:

$$d\Omega = 0, \qquad \mu_1\,d\Omega_1 + \ldots + \mu_4\,d\Omega_4 = 0.$$

They then imply two relations between the variables x, y, z; x', y', z', which is contrary to hypothesis.

One will then identify the two sides of the equation (6), which will give six equations. If one eliminates λ, μ from them then one will have four equations that will give x', y', z', p', q' as functions of x, y, z, p, q, or conversely, when they are combined with $\Omega = 0$.

Case of two directrix equations

4. – We now pass on to the case in which one obtains two relations:

$$\Omega\,(x, y, z; x', y', z') = 0, \qquad \Theta\,(x, y, z; x', y', z') = 0 \tag{7}$$

by eliminating p, q; p', q' from the equations (1) of the contact transformation considered. A point M (x, y, z) in the first space will correspond to a curve (C') in the second space that is defined by those equations (7) in x', y', z'. A curve that is the locus of ∞^1 points M will correspond to a surface that is generated by the homologous ∞^1 curves (C') while a surface (S) that is a locus of ∞^2 points will correspond to the congruence of curves (C') that are homologous to those points. In general, such a congruence has a focal surface that is tangent to all of those curves, and which will be the transform of the surface (S).

In order to get the equations of such a transformation, one must write down that the relation:

$$dz' - p'\,dx' - q'\,dy' = 0$$

is a consequence of the relations:

$$dz' - p'\,dx' - q'\,dy' = 0, \qquad d\Omega = 0, \qquad d\Theta = 0,$$

which will give an identity of the form:

$$dz' - p'\,dx' - q'\,dy' \equiv \lambda\,(dz - p\,dx - q\,dy) + \mu\,d\Omega + \nu\,d\Theta. \tag{8}$$

One proves the effective existence of such an identity as above. Upon identifying coefficients, one will have six equations. If one eliminates λ, μ, ν from them then one will have three equations that will give the formulas for the transformation when they are combined with $\Omega = 0$, $\Theta = 0$.

Sophus Lie's transformation that changes lines into spheres

In particular, suppose that equations (7) are bilinear. A point M (x, y, z) corresponds to a line (D'). The ∞^3 points M correspond to a complex of such lines (D'), namely, (K'). Similarly, every point in the second space will correspond to a complex (K) in the first space. We study the nature of those complexes. To that effect, consider just one of equations (7). It defines a duality transformation in which each point M has a plane (P') for its homologue. The other equation likewise defines a duality transformation that makes the same point M correspond to a plane (Q'), and the line (D') is the intersection of the planes (P'), (Q'), which then corresponds to the point M under those two duality transformations. Now, we have see that the product of two duality transformations is a projective transformation. Hence, the complex (K') is the complex of lines along which planes that correspond under a projective transformation intersect. Such a complex is called a *Reye complex*, or *tetrahedral complex*. We recall the properties in the general case. The lines of the complex are cut by the tetrahedron that is defined by the four invariant planes of the homography at four points whose anharmonic ratio is constant. The anharmonic ratio of the four planes that are drawn through a line of the complex and through the four summits of the same tetrahedron is constant (Von Staudt). The complex (K') has degree two, and the surface of singularities is composed of the four faces of the tetrahedron.

Having said that, we return to our contact transformation. A curve (C) corresponds to a ruled surface of the complex (K'). A surface (S) corresponds to a congruence of lines that belong to the complex (K'); that congruence will admit two focal multiplicities. Therefore, a contact element in the first space will correspond to two contact elements in the other one.

We seek the equations of the two complexes (K) and (K'). Let:

$$\Omega = Ax' + By' + Cz' + D, \qquad \Theta = Lx' + My' + Nz' + P,$$

in which $A, B, \ldots, P$ are linear functions of x, y, z.

Let $M'(x', y', z')$ be a point of the second space; let (D) be the corresponding line. If (x, y, z) and (x_0, y_0, z_0) are two points on that line then one will have:

$$\Omega\,(x, y, z; x', y', z') = 0, \qquad \Theta\,(x, y, z; x', y', z') = 0,$$
$$\Omega\,(x_0, y_0, z_0; x', y', z') = 0, \qquad \Theta\,(x_0, y_0, z_0; x', y', z') = 0.$$

If we eliminate x', y', z' from those four equations then, upon letting $A_0, B_0, \ldots, P_0$ denote what the linear functions $A, B, \ldots, P$ will become when one replaces x, y, z with x_0, y_0, z_0 in them, we will get:

$$\begin{vmatrix} A & B & C & D \\ A_0 & B_0 & C_0 & D_0 \\ L & M & N & P \\ L_0 & M_0 & N_0 & P_0 \end{vmatrix} = 0.$$

That is the equation of the complex. Upon developing it by Laplace's rule, one will get a second-degree equation in the coordinates of the line that is defined by means of the two points (x, y, z), (x_0, y_0, z_0). The complex (K), and likewise the complex (K'), will then indeed be of second degree, in general.

A curve (C) corresponds to a ruled surface that is generated by the line (D'). Let us see whether that ruled surface can be developable. The lines (D') have the equations:

$$Ax' + By' + Cz' + D = 0, \qquad Lx' + My' + Nz' + P = 0,$$

in which x, y, z, and in turn, A, B, C, D are functions of one parameter u. We express the idea that this line meets the infinitely-close line. We combine its equations with the equations:

$$x'\,dA + y'\,dB + z'\,dC + dD = 0, \qquad x'\,dL + y'\,dM + z'\,dN + dP = 0.$$

Hence, one has the condition that defines the curve (C):

$$\begin{vmatrix} A & B & C & D \\ L & M & N & P \\ dA & dB & dC & dD \\ dL & dM & dN & dP \end{vmatrix} = 0.$$

However, upon setting:

$$A_0 - A = \Delta A, \quad B_0 - B = \Delta B, \quad \ldots, \qquad P_0 - P = \Delta P,$$

the equation of the complex (K) can be written in the form:

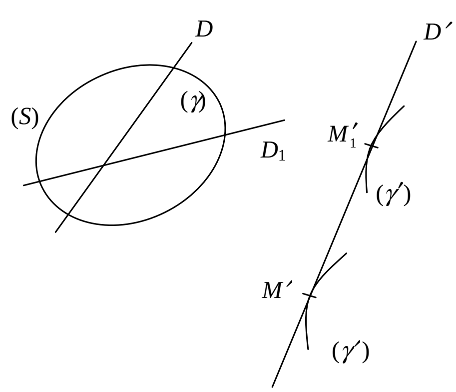
$$\begin{vmatrix} A & B & C & D \\ L & M & N & P \\ \Delta A & \Delta B & \Delta C & \Delta D \\ \Delta L & \Delta M & \Delta N & \Delta P \end{vmatrix} = 0.$$

Now $A, B, \ldots, P$ are linear functions, so the increments $\Delta A, \ldots, \Delta P$ are formed from:

$$\Delta x = x_0 - x, \qquad \Delta y = y_0 - y, \qquad \Delta z = z_0 - z,$$

in the same way that the differentials $dA, \ldots, dP$ are formed from dx, dy, dz. The equation of the curve (C) is then deduced from the equation of the complex by replacing $x_0 - x$, $y_0 - y$, $z_0 - z$ with dx, dy, dz. It is then such that its tangent belongs to the complex (K).

The curves of the first complex then correspond to developables whose generators are lines of the second complex, and whose edges of regression are, in turn, curves of the second complex. Each point M of a curve (C) of the first complex corresponds to a generator (T') of a developable. Let M' be its point of contact with the edge of regression. If one considers the linear element that is composed of a point M and the line (T) in the first complex that passes through point and is tangent to (C) then it will correspond to the well-defined linear element of the second complex that is composed of M' and (T'). The curves of the two complexes will then correspond by points and tangents.



Let (S) be a surface, and suppose that the complex (K) is effectively of second degree. Consider a point M on the surface and the tangent plane (P). The cone of the complex (K) whose summit is M is cut by the plane (P) along two lines (D), (D_1) that belong to the complex (K). Two lines of the complex (K) then pass through each point of (S) that are tangent to the surface. Two curves (γ), (γ_1) of the complex (K) pass through any point of the surface (S) and belong to that surface. The point M corresponds to a line (D') of the complex (K'). The line (D) of the complex (K) corresponds to a point M' of (D'), and similarly the line (D_1) corresponds to a point M_1' of (D'). The curves (γ), (γ') of the complex (K) correspond to two curves (γ_1), (γ_1'), resp., of the complex (K) that are tangent to the line (D') at M', M_1', resp. If the point M describes the curve (γ) then the corresponding lines (D') have the curve (γ') for their envelope, and if M describes (γ_1) then (D') will envelop (γ_1').

If one considers the congruence of lines (D') that correspond to the points M of the surface (S) then the curves (γ') will be edges of regression of a family of developables in that congruence, and the curves (γ_1') will be the edges of regression of the other family. The curves (γ') generate one of the sheets of the focal surfaces, while the curves (γ_1') generate the other sheet. The tangent plane at M to the focal multiplicity is the osculating plane to (γ_1'), and in turn, the tangent plane to the cone of the complex (K') whose summit is M_1'.

A contact element corresponds to the element (M, P) that is composed of a point M' and the tangent plane to the cone of the complex (K') that has M_1' for its summit.

If the surface (S) is a surface of the complex (K) that is tangent at each of its points to the cone of the complex then the lines (D), (D_1) will coincide. The two contact elements that correspond to the element (M, P) will coincide, and the surface (S') that is defined by those elements will be a surface of the complex (K').

Remarks. – The only possible cases are the following ones:

1. The complexes (K), (K') are effectively of second degree. As we have said before, one will then prove that they are both tetrahedral.

2. Just one of the complexes is linear. One proves that the other is composed of lines that meet a conic. That case will give us *Sophus Lie's transformation* that changes lines into spheres.

3. Both complexes are linear; one will then prove that they are both special. That case will give us *Ampère's transformation*, in particular, which is defined by the directrix equations:

$$xx' + z + z' = 0, \qquad y + y' = 0,$$

and whose equations are:

$$x' = p, \quad y' = -y, \quad z' = -z - px, \quad p' = x, \quad q' = -q.$$

Transformation of lines into spheres. – Suppose, in particular, that:

$$\Omega = x - iy + x'z - z' = 0, \qquad \Theta = x'(x + iy) - z - y' = 0.$$

The equation of the first complex is:

$$\begin{vmatrix} z - z_0 & 0 & 0 & x - iy - (x_0 - iy_0) \\ z_0 & 0 & -1 & x_0 - iy_0 \\ x + iy & -1 & 0 & -z \\ x_0 + iy_0 - (x + iy) & 0 & 0 & z - z_0 \end{vmatrix} = 0,$$

which becomes:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 0;$$

i.e.:

$$(K) \qquad a^2 + b^2 + c^2 = 0.$$

The complex (K) is the complex of minimal lines.

We seek the second complex. It suffices to consider two points (x', y', z'), (x'_0, y'_0, z'_0) that correspond to the same point (x, y, z). That will give:

$$\begin{vmatrix} 0 & 0 & x' - x' & -z' + z'_0 \\ 1 & -i & x'_0 & -iz'_0 \\ x' - x'_0 & i(x' - x'_0) & 0 & -y' + y'_0 \\ x'_0 & ix'_0 & -1 & -y'_0 \end{vmatrix} = 0,$$

which will become:

$$(x' - x'_0)(x'y'_0 - y'x'_0) + (z' - z'_0)(x' - x'_0) = 0;$$

i.e., with the classical notations for the Plückerian coordinates:

$$a\,(r - c) = 0.$$

The solution $a = 0$ is singular, and one gets:

$$(K') \qquad r - c = 0$$

for the complex (K'). We then have a *correspondence between a second-degree special complex and a linear complex.* The cones of the complex (K) are isotropic cones. Each contact element in the first space corresponds to two contact elements of the second space that are conjugate with respect to the complex (K'), because, in a general fashion, the points M', M'_1 are on a line (D') of (K'), and the plane that is associated with M' is the polar plane to M'_1 here, and conversely.

Start with a sphere: Take two generators of a system; they will be the minimal lines (D), (D_1). The second system of generators is entirely well-defined, because each of them must meet (D), (D_1), and the imaginary circle at infinity. Two lines (D), (D_1) correspond to M', M_1', resp. Consider an isotropic generator (Δ) that meets (D), (D_1); it corresponds to a point μ'. (Δ) meets the line (D), the line $M'\mu'$ is a line of the linear complex, and similarly, so is $M_1'\mu'$. Therefore, μ' is the pole of a plane that passes through $M'M_1'$. When (Δ) describes the sphere, the plane $\mu_1' M' M_1'$ will turn around $M'M_1'$. The sphere then corresponds to a line. Upon starting with the second system of generators, one will likewise obtain a line. (D) and (D_1) give the points M', and that line will then be the line $M'M_1'$ that is conjugate to the preceding one. Therefore: *A sphere corresponds to two lines that are conjugate with respect to the linear complex* (K').

One can see this by calculation. Take the line (Δ'), whose Plückerian coordinates are $a_0, b_0, c_0, p_0, q_0, r_0$:

$$(\Delta') \qquad c_0\, x' = a_0\, z' - q_0\,, \qquad c_0\, y' = b_0\, z' + p_0\,.$$

The corresponding ruled surface is generated by the lines:

$$c_0\,(x - iy) + z\,(a_0\, z' - q_0) - c_0\, z' = 0,$$
$$(a_0\, z' - q_0)\,(x + iy) - c_0\, z - b_0\, z' - p_0 = 0$$

that are obtained by substituting the values for x' and y' that are inferred from equations (Δ') into $\Omega = 0$, $\Theta = 0$. Order them in terms of z', and get:

$$c_0\,(x - iy) - q_0\, z + z'\,(a_0\, z - c_0) = 0,$$
$$[q_0\,(x + iy) + c_0\, z + p_0] - z'\,[a_0\,(x + iy) - b_0] = 0.$$

Upon eliminating z', one will get the desired surface:

$$[c_0\,(x - iy) - q_0\, z]\,[a_0\,(x + iy) - b_0] + (a_0\, z - c_0)\,[q_0\,(x + iy) + c_0\, z + p_0] = 0,$$

or, upon taking into account that $a_0\, p_0 + b_0\, q_0 + r_0\, c_0 = 0$:

$$(\Sigma) \qquad a_0\,(x^2 + y^2 + z^2) - b_0\,(x - iy) - q_0\,(x + iy) - (c_0 + r_0)\, z - p_0 = 0.$$

That is the equation of a sphere, and it is easy to see that it can be an arbitrary sphere by choosing (Δ') conveniently.

We seek the conjugate (Δ_1') to (Δ') with respect to (K'); let its coordinates be a_0', b_0', c_0', p_0', q_0', r_0'. We must express the idea that the complex (K') and the special complexes (Δ'), (Δ_1') belong to the same sheaf. If λ, λ', and μ are unknown auxiliary variables then that will give:

$$\lambda a_0 + \lambda' a_0' = 0, \qquad \lambda b_0 + \lambda' b_0' = 0, \qquad \lambda p_0 + \lambda' p_0' = 0,$$
$$\lambda q_0 + \lambda' q_0' = 0, \qquad \lambda c_0 + \lambda' c_0' + \mu = 0, \qquad \lambda r_0 + \lambda' r_0' - \mu = 0.$$

Since the coordinates are defined only up to a factor, one can replace a_0, b_0, … with λa_0, λb_0, …, and a_0', b_0', … with $-\lambda' a_0$, $-\lambda' b_0$, … That amounts to setting $\lambda = 1$, $\lambda' = -1$, and will give the simplified equations:

$$a_0' = a_0, \qquad b_0' = b_0, \qquad p_0' = p_0, \qquad q_0' = q_0\,, \qquad c_0' = c_0 + \mu, \qquad r_0' = r_0 - \mu.$$

The condition:

$$a_0'\, p_0' + b_0'\, q_0' + c_0'\, r_0' = 0$$

then gives:

$$\mu\,[\mu + c_0 - r_0] = 0,$$

and upon dropping the trivial solution $\mu = 0$, what will remain is:

$$\mu + c_0 - r_0 = 0.$$

One will then find that $c_0' = r_0$ and $r_0' = c_0$, and one sees that one will recover the same sphere (Σ) upon starting with (B_1'), instead of (B').

Equations of the transformation. – The formulas of the transformation are obtained by the general method. One finds that:

$$\begin{cases} x = \dfrac{z'}{2} - \dfrac{1}{2}\dfrac{x'(px' + qy') - y' - p'}{x' - q'}, \\ y = \dfrac{iz'}{2} + \dfrac{i}{2}\dfrac{x'(px' + qy') + y' + p'}{x' - q'}, \\ z = \dfrac{p'x' + q'y'}{x' - q'}, \\ p = -\dfrac{q'x' - 1}{q' + x'}, \\ q = i\dfrac{q'x' + 1}{q' + x'}. \end{cases}$$

This transformation of Sophus Lie, which changes lines that meet into tangent spheres – i.e., lines that have a common contact element into spheres that have a common contact element – realizes the correspondence between lines and spheres that was announced in the preceding chapters.

For example, it transforms a ruled surface into a canal surface, a quadric into a Dupin cyclide, a developable surface into an isotropic canal surface, and an asymptotic strip on a surface into a curvature strip on its transform, and in such a way that one can say that *it transforms the asymptotic lines into curvature lines.*

One easily verifies that it transforms a linear complex of lines into a family of ∞^2 contact spheres that cut a fixed sphere at a constant angle, and that the constant angle will be a right angle when the linear complex is in involution with the complex (K').

Lie's transformation in penta-spherical coordinates. – The last results will become immediate when one remarks that from the equation that was found above, the sphere (Σ), which is the homologue of the line (Δ') (whose Plückerian coordinates are a_0, b_0, c_0, p_0, q_0, r_0), will have the homogeneous penta-spherical coordinates [Ch. VIII, § 6, pp. 219]:

$$c_1 = a_0 + p_0 , \qquad c_2 = -i\,(a_0 - p_0), \qquad c_3 = b_0 + q_0 ,$$
$$c_4 = -i\,(b_0 - q_0), \qquad c_5 = c_0 + r_0, \qquad c_6 = -i\,(c_0 - r_0).$$

Now, from the formulas of [Chap. X, § 5, pp. 268], these are precisely the *symmetric coordinates* $t_1, t_2, \ldots, t_6$ of the line (Δ').

Therefore, *the Lie transformation translates into the interpretation of the penta-spherical coordinates of spheres as the symmetric coordinates of lines,* in absolutely the same way that the duality transformation translates into the interpretation of point coordinates as line coordinates.

The equation $\sum_{k=1}^{6} C_k\, t_k = 0$ of a linear complex then becomes, in particular, the equation $\sum_{k=1}^{6} C_k\, c_k = 0$, which expresses [Chap. VIII, § 6, pp. 219] the idea that a sphere cuts a sphere at a constant angle; that angle will be a right angle if C_6 is zero. Now, the equation of the complex (K') is $t_6 = 0$ in symmetric coordinates, in such a way that the condition $C_6 = 0$ indeed expresses [Chap. X, § 6] the idea that the complex is in involution with the complex $\sum_{k=1}^{6} C_k\, t_k = 0$.

Transformation of asymptotic lines

5. – We propose to find all of the contact transformations that change the asymptotic lines of an arbitrary surface into asymptotic lines of the transform of that surface; i.e., they change every asymptotic strip into an asymptotic strip. To that effect, we remark that such a transformation will change any multiplicity M_2 for which the asymptotic strips do not depend solely upon arbitrary constants, but upon arbitrary functions, into a multiplicity M_2 of the same nature. Now, the asymptotic strips (or strips of regression) are defined by the equations:

$$dz - p\,dx - q\,dy = 0, \qquad dp\,dx + dq\,dy = 0,$$

so one must also consider that in the present question, ∞^1 contact elements that have the same point in common – i.e., an elementary cone – will form an asymptotic strip, because the coordinates of those elements satisfy the preceding equations, since they are such that $dx = dy = dz = 0$.

Moreover, the particular M_2 in question are planes, lines, and points. The desired transformation then exchanges the figures that consist of lines, points, and planes amongst themselves. There are several cases of that to examine:

1. If the transformation is point-like then it will change points into points, planes into planes, and lines into lines. As a result, it is a *projective transformation* (or *homography*).

2. If the transformation is a contact transformation of the first kind – i.e., it makes every point of the first space (E) correspond to a surface in the second space (E') – then it will change the points of (E) into planes of (E'), and since it will also make each point of (E') correspond to a surface in (E) then, it will change the points of (E') into planes in (E). Therefore, it will change points into planes, planes into points, and lines into lines. Therefore, if one composes it with a transformation by polar reciprocals then one will obtain a homographic transformation, and in turn, it will be obtained when one composes a homographic transformation with a transformation by polar reciprocal. It will then be a *duality transformation.*

3. If the transformation is a contact transformation of the second kind – i.e., any point of one of the spaces corresponds to a curve in the other one – then any point in one of the spaces will correspond to a line in the other one. Now, take four points P_1, P_2, P_3, P_4 in the space (E) that are not situated in the same plane, and let (D_1), (D_2), (D_3), (D_4), resp. be the lines that correspond to them in the space (E'). There exists at least one line (Δ) that has a common contact element with each of the four lines (D_1), (D_2), (D_3), (D_4), and (Δ) must correspond to a point, plane, or line in (E) that has a common contact element with each of the four points P_1, P_2, P_3, P_4 . But, it does not exist. Hence, *the third case is impossible.*

The only transformations that can answer the question are then homographies or duality transformations. However, every contact transformation that changes lines into lines will answer the question, because it will change the family of generators of a developable, each of which will have a common contact element with the infinitely-close generator, into the family of generators of another developable. As a result, the strip of regression of the first developable will change into the strip of regression of the second one.

One then deduces that:

1. *The homographic transformations and the duality transformations change asymptotic lines into asymptotic lines, and they are the only contact transformations that possess that property.*

2. *Those transformations are also the only contact transformations that change every line into a line.*

Remark. – The transformations thus-obtained form two distinct families (projective transformations and duality transformations) of ∞^{15} transformations, but the product of two duality transformations will be a projective transformation, as we saw above, and the set of all transformations that are obtained will form a group, as was obvious *a priori*.

Transformation of lines of curvature

6. – Lie's contact transformation of lines into spheres permits one to immediately deduce all of the contact transformations that change lines of curvature on an arbitrary surface into lines of curvature on its transform from the preceding results.

One sees, moreover, that they are also the ones that change any sphere into a sphere. One can then say that they constitute the *sphere group.* From the preceding, there will be two families that each have ∞^{15} transformations.

In order to obtain the preceding result, one can repeat an argument that is directly analogous to the one in § 5, while starting from multiplicities M_2 for which the curvature strips depend upon arbitrary functions.

More especially, look for the transformations in question that are point-like transformations. Under Lie's transformation, the points in a space (E) will correspond to lines in a linear complex (K'). The desired transformations then provide projective or duality transformations that leave that complex invariant. Upon composing them with the transformation by polar reciprocals that the complex (K') defines, one will then obtain any of the projective transformations that leave the complex invariant.

Hence, one finds that a correspondence between the *projective group of a linear complex* and the group of point-like transformations that change every sphere into a sphere has been established. The latter is, as one saw in Ch. VIII, § 8, the *conformal group.* One knows that its transformations are obtained by combining inversions, homotheties, and displacements.

That correspondence will be found effortlessly, moreover, by the use of symmetric coordinates for lines and penta-spherical homogeneous coordinates for spheres, as was indicated above.

Among the contact transformation that change lines of curvature into lines of curvature, one finds the *dilatations*, under which any contact element is subjected to a translation perpendicular to its plane that has a given amplitude; i.e., each surface will be replaced by a parallel surface. They are defined by the directrix equation $(x' - x)^2 + (y' - z)^2 + (z' - z)^2 = h^2$, in which h is an arbitrary constant.

Another class of contact transformations that changes any sphere into a sphere is defined by the directrix equations of the form:

$$(x' - x)^2 + (y' - z)^2 + z'^2 - 2mz'z + z^2 = 0,$$

in which m is an arbitrary constant. Each point (x, y, z) has its homologue in a sphere that cuts the xy-plane at a constant angle V ($\cot V = mi$). The circle of intersection is the one along which the isotropic cone whose summit is (x, y, z) cuts the same plane.

Those transformations are called *transformations by reciprocal semi-planes* (Ribaucour, Laguerre, Darboux), because they change a plane into a pair of planes that pass through the line of intersection of the first one with the xy-plane. Since they are involutive, the equation that defines them will be symmetric with respect to the two coordinate systems (x, y, z) and (x', y', z').

Among the transformations considered, one also finds the Ribaucour transformations, which will be defined in Chapter XIII.

Remark. – One proves that when one defines a sphere by its six homogeneous penta-spherical coordinates, the two families of ∞^{15} transformations of the sphere group will be defined by orthogonal, homogeneous, linear transformations that act upon those six variables. The two families are distinguished by the value (+ 1 or – 1) of the determinant of that substitution.

Apsidal transformations. Fresnel's wave surface.

7. – Finally, we point out an important class of contact transformations that are defined by two directrix equations. Each of them corresponds to a point in space, or *pole* of the transformation. If one takes the pole to be the coordinate origin then the directrix equations of the transformation will be:

$$(1) \qquad \begin{cases} x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2, \\ xx' + yy' + zz' = 0. \end{cases}$$

That transformation, which is called *apsidal*, will then be involutive, and transform each point M into a circle: It is the circle of radius OM that has O for its center and the line OM for its axis.

As a result, one can obtain the transform of a surface (S) by cutting it with the various planes (II) that pass through O and measuring out lengths OM along the normal to each of those planes at O that are equal to the radii of circles that are centered at O, situated in the plane in question, and tangent to the surface (S). Those radii are, moreover, lengths of the normals that are drawn through O to the section of (S) of by the plane (Π).

The apsidal surface of a sphere is a torus. – Indeed, let C be the center of the sphere (S), and let (Π_0) be a plane that passes through OC; let (γ) be the circle that is the section of the sphere (S) by that plane. Any plane (Π) that is perpendicular to (Π_0) and drawn through O will cut (γ) along a chord AB, and OA, OB will be the normals through O to the second of the sphere by (Π). The perpendicular that is drawn through O to (Π) is, moreover, situated in (Π_0). One then obtains the points P that are situated in the plane (Π) by making the chord AB turn in the plane (Π_0) through a right angle around O in one sense or the other. That operation, when repeated with all the chords of (γ) that pass through O, will give two circles (γ_1) and (γ_2) that are symmetric with respect to OC, and are obtained by subjecting (γ) to the same two rotations. Those circles constitute the meridians of the transform of (S), which must be, like (S), a surface of revolution around OC. The theorem is then proved.

Wave surface. – By definition, the wave surface is the apsidal transform of an ellipsoid with respect to its center. From the preceding, one will then obtain it by measuring out lengths along each diameter of the ellipsoid, starting from the center and in one direction or the other, that are equal to the semi-axes of the central section that is perpendicular to that diameter.

We calculate the wave surface directly by completing the equations of the transformation. To that effect, from the general theory of contact transformations, we must write down the identity:

$$dz' - p'\,dx' - q'\,dy' - \lambda\,(dz - p\,dx - q\,dy)$$

$$= \rho\,(x\,dx + y\,dy + z\,dz - x'\,dx' - y'\,dy' - z'\,dz') + \sigma\,(x\,dx' + y\,dy' + z\,dz' + x'\,dx + y'\,dy + z'\,dz),$$

which will give, by identification:

$$1 = -\rho z' + \sigma z, \qquad -p' = -\rho x' + \sigma x, \qquad -q' = -\rho y' + \sigma y,$$
$$-\lambda = \rho z + \sigma z', \qquad \lambda p = \rho x + \sigma x', \qquad \lambda q = \rho y + \sigma y'.$$

Upon eliminating λ, ρ, and σ:

$$(2) \qquad \begin{cases} p\,(yz' - zy') + q\,(zx' - xz') = (xy' - yx'), \\ p'\,(yz' - zy') + q'\,(zx' - xz') = (xy' - yx'), \\ pp' + qq' + 1 = 0. \end{cases}$$

The interpretation is immediate. Let M be the point of the contact element (x, y, z, p, q) and let (P) be its plane; let M', (P') be the point and plane of the element (x', y', z', p', q'). The radius OM', which is already perpendicular and equal to OM, is in the plane normal to (P) that is drawn through OM. The normal to (P') at M' is in the same plane MOM', and it is perpendicular to the normal to (P).

One then has the complete definition of the transformation of the contact elements.

Having said that, if one starts with the ellipsoid:

$$(3) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

then

$$\begin{vmatrix} \dfrac{x}{a^2} & \dfrac{y}{b^2} & \dfrac{z}{c^2} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = 0,$$

and that will be equivalent to some relations of the form:

$$\text{(4)} \qquad \mu' x' = x\left(\frac{1}{a^2} - \mu\right), \quad \mu' y' = y\left(\frac{1}{b^2} - \mu\right), \quad \mu' z' = z\left(\frac{1}{c^2} - \mu\right).$$

Upon taking into account the first one and (3), the second of equations (1) will then give:

$$0 = 1 - \mu\,(x^2 + y^2 + z^2), \qquad \text{or} \qquad \mu = \frac{1}{x'^2 + y'^2 + z'^2}.$$

All that remains is to substitute the values of x, y, z that are inferred from equations (4) into the homogeneous combination of the first equation (1) and (3):

$$x^2\left(\frac{1}{a^2} - \mu\right) + y^2\left(\frac{1}{b^2} - \mu\right) + z^2\left(\frac{1}{c^2} - \mu\right) = 0.$$

Upon suppressing the primes, one will get:

$$\frac{x^2}{\frac{1}{a^2} - \mu} + \frac{y^2}{\frac{1}{b^2} - \mu} + \frac{z^2}{\frac{1}{c^2} - \mu} = 0, \qquad \mu = \frac{1}{x^2 + y^2 + z^2},$$

or, after reductions:

$$\sum a^2\, x^2 \cdot c^2 - \sum (b^2 + c^2)\, a^2\, x^2 + a^2\, b^2\, c^2 = 0.$$

CHAPTER XII

TRIPLY-ORTHOGONAL SYSTEMS

Dupin's theorem

1. – The use of rectangular coordinates amounts to defining each point to be the intersection of three planes that are parallel to the three faces of the coordinate trihedron, respectively, and consequently they will be mutually orthogonal. It is then based upon the consideration of the *triply-orthogonal system* that is composed of three families of planes, such that each plane of one of the families is orthogonal to every plane of one of the other two families.

One can generalize that and employ a triple system as *coordinate surfaces*, which is a system that is composed of three families of surfaces:

$$\varphi(x, y, z) = u, \qquad \psi(x, y, z) = v, \qquad \chi(x, y, z) = w. \tag{1}$$

Each point P (x, y, z) will then found to be defined by the parameters u, v, w of three coordinate surfaces that cut at that point, and those values of u, v, w will be its *curvilinear coordinates* in the coordinate system thus-defined.

Formulas (1) transform the coordinates x, y, z into coordinates u, v, w. If we solve the preceding equations for x, y, z (which we assume to be possible) then we will have the equivalent formulas:

$$x = f(u, v, w), \qquad y = g(u, v, w), \qquad z = h(u, v, w). \tag{2}$$

In general, one employs a *triply-orthogonal system*. We then seek to express the idea that the equations (1) or (2) define a triply-orthogonal system. The pair-wise intersections of the surfaces must be orthogonal. The surfaces of the three families are obtained by setting u = const., v = const., w = const. in (2) in succession.

The pair-wise intersections of the surfaces are respectively:

$$(v = \text{const.}, w = \text{const.}), \qquad (w = \text{const.}, u = \text{const.}), \qquad (u = \text{const.}, v = \text{const.}),$$

and the directions of the tangents will be:

$$\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}; \qquad \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}; \qquad \frac{\partial f}{\partial w}, \frac{\partial g}{\partial w}, \frac{\partial h}{\partial w},$$

respectively.

The orthogonality conditions will then be:

$$\sum \frac{\partial f}{\partial v} \cdot \frac{\partial f}{\partial w} = 0, \qquad \sum \frac{\partial f}{\partial w} \cdot \frac{\partial f}{\partial u} = 0, \qquad \sum \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} = 0. \tag{3}$$

Let us interpret those conditions. Take the surface w = const. The third condition expresses the idea that the lines u = const., v = const. are orthogonal on that surface, and the first two express the idea that $\frac{\partial f}{\partial w}, \frac{\partial g}{\partial w}, \frac{\partial h}{\partial w}$ is a direction that is perpendicular to the tangents to those curves, and as a result, it is the normal direction. Let l, m, n be three coefficients of the direction of that normal. Differentiate the third relation with respect to w; we will get:

$$\sum \frac{\partial f}{\partial u}\frac{\partial^2 f}{\partial v\,\partial w} + \sum \frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial u\,\partial w} = 0,$$

or

$$\sum \frac{\partial f}{\partial u}\frac{\partial l}{\partial v} + \sum \frac{\partial f}{\partial v}\frac{\partial l}{\partial u} = 0.$$

Now:

$$\sum l \frac{\partial f}{\partial u} = 0, \qquad \sum l \frac{\partial f}{\partial v} = 0;$$

hence:

$$\sum l \frac{\partial^2 f}{\partial u\,\partial v} = -\sum \frac{\partial l}{\partial v}\frac{\partial f}{\partial u}, \qquad \sum l \frac{\partial^2 f}{\partial u\,\partial v} = -\sum \frac{\partial l}{\partial u}\frac{\partial f}{\partial v}.$$

The preceding condition can then be written:

$$\sum l \frac{\partial^2 f}{\partial u\,\partial v} = 0,$$

which expresses the idea (Chap. II, § 3, pp. 27) that the lines u = const., v = const. (i.e., the intersections of the surface w = const. with the surfaces u = const. and v = const.) are conjugate on that surface. Since those curves are already orthogonal, by hypothesis, they will be lines of curvature. Hence:

Dupin's theorem: *The intersections of each surface of a triply-orthogonal system with the other surfaces of that system are lines of curvature.*

Darboux's partial differential equation

2. – We propose to look for the triply-orthogonal systems. We take a family of surfaces:

(1) $$\varphi(x, y, z) = u$$

and seek to determine two other families that constitute a triply-orthogonal system along with that family. Take a point M in space. Pass a surface u through that point M. Take the tangents MT, MT' to its lines of curvature at M; those lines are perfectly well-defined. If p, q, – 1 are the coefficients of the direction of MT then they will be known functions of x, y, z, and similarly for MT'. One will then have a surface of another family at each point M; for example, let:

(2) $$\psi(x, y, z) = v$$

be normal to MT. One must then have that p, q are the partial derivatives of z with respect to x, y (z being defined by the preceding equation), hence that ψ is a solution of the equation:

(3) $$\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z} = 0, \qquad \frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z} = 0.$$

Those equations are not compatible, in general. In order for that to be true, from the theory of *complete systems* of homogeneous linear partial differential equations, it is necessary and sufficient that p and q must satisfy the condition:

(4) $$\frac{\partial q}{\partial x} + p\frac{\partial q}{\partial z} = \frac{\partial p}{\partial y} + q\frac{\partial p}{\partial z},$$

which is obtained by eliminating ψ from the preceding two equations by differentiation. It is a third-order partial differential equation, since p, q are expressed as functions of the first and second derivatives of φ with respect to x, y, z. Hence, *a family of given surfaces cannot, in general, belong to a triply-orthogonal system.* If the condition (4) is realized then the general solution to equations (3) will be an arbitrary function of a well-defined function of x, y, z, and we will have a second family of surfaces that are entirely well-defined, each of which cuts each of the surfaces (S) of the family $\varphi(x, y, z)$ = const. at right angles along a line of curvature of that surface (S). From Joachimsthal's theorem, the intersection of each surface (S_1) of that family with each surface (S) of the first will also be a line of curvature on (S_1).

In summary, we have two families of surfaces:

(S) $$\varphi(x, y, z) = \text{const.},$$
(S_1) $$\psi(x, y, z) = \text{const.},$$

which intersect orthogonally along curves that are each lines of curvature for both of the two corresponding surfaces. It remains to study whether one can determine a third family of surfaces:

(S_2) $$\chi(x, y, z) = \text{const.}$$

that constitutes a triply-orthogonal system with the first two; i.e., to study the system of linear partial differential equations that the unknown function χ depends upon:

(5) $$\begin{cases} \dfrac{\partial \varphi}{\partial x}\cdot\dfrac{\partial \chi}{\partial x} + \dfrac{\partial \varphi}{\partial y}\cdot\dfrac{\partial \chi}{\partial y} + \dfrac{\partial \varphi}{\partial z}\cdot\dfrac{\partial \chi}{\partial z} = 0, \\ \dfrac{\partial \psi}{\partial x}\cdot\dfrac{\partial \chi}{\partial x} + \dfrac{\partial \psi}{\partial y}\cdot\dfrac{\partial \chi}{\partial y} + \dfrac{\partial \psi}{\partial z}\cdot\dfrac{\partial \chi}{\partial z} = 0. \end{cases}$$

To abbreviate, introduce the differential operators:

$$Af = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial\chi}{\partial x}+\frac{\partial\varphi}{\partial y}\cdot\frac{\partial\chi}{\partial y}+\frac{\partial\varphi}{\partial z}\cdot\frac{\partial\chi}{\partial z},$$

$$Bf = \frac{\partial\psi}{\partial x}\cdot\frac{\partial\chi}{\partial x}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial\chi}{\partial y}+\frac{\partial\psi}{\partial z}\cdot\frac{\partial\chi}{\partial z}.$$

From the theory of complete systems of linear equations, the necessary and sufficient condition for the system (5) to be integrable is that the equation:

$$\left(A\frac{\partial\psi}{\partial x}-B\frac{\partial\varphi}{\partial x}\right)\cdot\frac{\partial\chi}{\partial x}+\left(A\frac{\partial\psi}{\partial y}-B\frac{\partial\varphi}{\partial y}\right)\cdot\frac{\partial\chi}{\partial y}+\left(A\frac{\partial\psi}{\partial z}-B\frac{\partial\varphi}{\partial z}\right)\cdot\frac{\partial\chi}{\partial z}=0$$

should be an algebraic consequence of equations (5); i.e., that φ and ψ should satisfy the condition:

$$\text{(6)}\qquad \begin{vmatrix} A\frac{\partial\psi}{\partial x}-B\frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial x} & \frac{\partial\psi}{\partial x} \\ A\frac{\partial\psi}{\partial y}-B\frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial y} & \frac{\partial\psi}{\partial y} \\ A\frac{\partial\psi}{\partial z}-B\frac{\partial\varphi}{\partial z} & \frac{\partial\varphi}{\partial z} & \frac{\partial\psi}{\partial z} \end{vmatrix} = 0.$$

That condition simplifies. Indeed, we remark that:

$$A\frac{\partial\psi}{\partial x}+B\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial x}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\varphi}{\partial z}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\varphi}{\partial y\,\partial x}+\frac{\partial\psi}{\partial z}\cdot\frac{\partial^2\varphi}{\partial z\,\partial x}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}\right\},$$

so, due to the orthogonality of the surfaces (S) and (S_1), one will have the identity:

$$A\frac{\partial\psi}{\partial x}+B\frac{\partial\varphi}{\partial x}=0,$$

and similarly, the analogous identities:

$$A\frac{\partial\psi}{\partial y}+B\frac{\partial\varphi}{\partial y}=0, \qquad A\frac{\partial\psi}{\partial z}+B\frac{\partial\varphi}{\partial z}=0.$$

As a result, the condition (6) will become:

$$\begin{vmatrix} A\dfrac{\partial\psi}{\partial x} & \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\psi}{\partial x} \\ A\dfrac{\partial\psi}{\partial y} & \dfrac{\partial\varphi}{\partial y} & \dfrac{\partial\psi}{\partial y} \\ A\dfrac{\partial\psi}{\partial z} & \dfrac{\partial\varphi}{\partial z} & \dfrac{\partial\psi}{\partial z} \end{vmatrix} = 0.$$

Now, for an arbitrary value of *x*, *y*, *z*, the derivatives $\frac{\partial\psi}{\partial x}$, $\frac{\partial\psi}{\partial y}$, $\frac{\partial\psi}{\partial z}$ will be the direction coefficients *l*, *m*, *n* of the normal to those of the surfaces (S_1) that pass through the point with coordinates *x*, *y*, *z*, and $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$, $\frac{\partial\varphi}{\partial z}$ will be the direction coefficients of the normal to those of the surfaces (*S*) that pass through the same point; i.e., of the tangent to a line of curvature (S_1). Upon denoting a displacement that is performed along the direction of that tangent by *dx*, *dy*, *dz*, one will have:

$$\frac{\partial\varphi}{\partial x} = \lambda\cdot dx, \qquad \frac{\partial\varphi}{\partial y} = \lambda\cdot dy, \qquad \frac{\partial\varphi}{\partial z} = \lambda\cdot dz,$$

and as a result:

$$A\frac{\partial\psi}{\partial x} = A\,l = \lambda\left(\frac{\partial l}{\partial x}dx + \frac{\partial l}{\partial y}dy + \frac{\partial l}{\partial z}dz\right) = \lambda\cdot dl,$$

and similarly:

$$A\frac{\partial\psi}{\partial y} = \lambda\cdot dm, \qquad A\frac{\partial\psi}{\partial z} = \lambda\cdot dn.$$

The condition (7) will then become:

$$\begin{vmatrix} dl & dx & l \\ dm & dy & m \\ dn & dz & n \end{vmatrix} = 0.$$

It is satisfied, since the displacement *dx*, *dy*, *dz* takes place along a line of curvature.

The integrability condition of the system (5) is then satisfied, and the third family (S_2) always exists and is entirely well-defined. One then has the following results:

1. *There exists a third-order partial differential equation* [viz., equation (4)] *that expresses the necessary and sufficient condition for a function* φ (*x*, *y*, *z*) *to provide a family of surfaces* (*S*) *that belong to a triply-orthogonal system. If the family* (*S*) *is given then the other families* (S_1) *and* (S_2) *will be entirely well-defined.*

2. *In order for two families of surfaces* (S) *and* (S_1) *to belong to a triply-orthogonal system, it is necessary and sufficient that they should intersect at a right angle, and that the intersections should be lines of curvature on the surfaces* (S) *or on the surfaces* (S_1).

Finally, one should note that if one knows the lines of curvature (C_1) of the surfaces (S_1), for example, that are not intersections of the surfaces (S_1) and the surfaces (S), and the lines of curvature (C) of just one surface (S), then each surface (S_2) will be generated by the curves (C_1) that rest upon the same curve (C).

Triply-orthogonal systems that contain a given surface

3. – One easily recognizes that any given surface can belong to a triply-orthogonal system. Indeed, trace out the lines of curvature on that surface (S), and draw the normals to the surface at all points of those lines. They will generate two families of developables that are orthogonal to the given surface. One will get a triply-orthogonal system upon adjoining the parallel surfaces to (S).

Remark I. – The surfaces that are parallel to a surface (S) are derived by the contact transformation that is defined by the equation:

$$(X-x)^2 + (Y-y)^2 + (Z-z)^2 - r^2 = 0,$$

in which r is an arbitrary constant. Indeed, the parallel surface is the envelope of a family of spheres of constant radius that have their centers on the surface (S). As we have seen, that contact transformation is called a *dilatation* [Cf., Chap. XI, § 6].

Remark II. – When one knows that a family of surfaces (S) belongs to a triply-orthogonal system, the determination of the other two families of that triple system can be accomplished as follows: One determines the lines of curvature of one of the surfaces (S), and on the other hand, looks for the curves (T) that are orthogonal trajectories of the surfaces (S). The other families of the system are generated by the orthogonal trajectories (T) that rest upon the lines of curvature that were found. In the particular case of a family of parallel surfaces, the orthogonal trajectories will be the normals to those surfaces, and one will recover the mode of construction that was indicated above.

Triply-orthogonal systems that contain a family of planes

4. – Consider a family of planes (P). As we saw in the context of milling surfaces, the orthogonal trajectories are obtained [Chap. VII, § 6] by rolling a moving plane around the developable that is the envelope of the planes (P). Take two systems of orthogonal curves in the plane, which is always possible, because if we give one of the systems:

$$\varphi(x, y) = a$$

then the other one is determined by the integration of the equation:

$$\frac{dx}{\frac{\partial \varphi}{\partial x}} = \frac{dy}{\frac{\partial \varphi}{\partial y}}.$$

One will generate the other families of the triply-orthogonal system by means of the curves of the planes (P) that are subject to meeting the orthogonal trajectories. Those families are then composed of the milling surfaces. One can then recover their lines of curvature by means of Dupin's theorem.

Triply-orthogonal systems that contain a family of spheres

5. – The fact that any family of planes belongs to a triply-orthogonal system is based upon the fact that any curve in a plane is a line of curvature of the plane, in such a way that a family of surfaces that are orthogonal to the given planes will satisfy the necessary and sufficient condition for the existence of a third family that completes the triply-orthogonal system.

The same fact will then also be true for a family of spheres, and in order to determine an arbitrary triply-orthogonal systems that contains the given family of spheres (S), it will suffice:

1. To take two families of orthogonal curves (C), (C_1) on one of the spheres, and
2. To determine the orthogonal trajectories (T) to the spheres (S),

because then the curves (T) that rest upon the curves (C) and the curves (T) that rest upon the curves (C_1) will generate the surfaces of the two families (S_1) and (S_2) that form the desired triply-orthogonal system with the spheres (S).

Everything then comes down to solving the following *two problems:*

1. *Determine an arbitrary orthogonal system on a sphere.*
2. *Determine the orthogonal trajectories to a family of spheres.*

The first problem immediately comes down to the analogous problem in the plane by means of a stereographic projection.

Let us study the second one then:

If we consider two spheres of the family then the orthogonal trajectories will establish a point-wise correspondence between them, and from the preceding, that correspondence will be such that an orthogonal system on one of the spheres will correspond to an orthogonal system on the other. Now, two rectangular directions are harmonically conjugate with respect to the isotropic direction. On the other hand, for an arbitrary point-wise correspondence between two surfaces, the anharmonic ratio of four tangents is an invariant, because one can suppose that the correspondence expresses the manner by which the coordinate curves u = const., v = const. are homologous, in such a way that the homologous points will have the same curvilinear coordinates (u, v), and then the anharmonic ratio of four tangents and that of the four homologous tangents will be equal to the same anharmonic ratio of the same four values of the ratio *dv* / *du*. Hence, under the correspondence in question, the isotropic directions on one of the spheres will transform into isotropic directions on the other one. The rectilinear generators of one of the spheres will then transform into rectilinear generators on the other one, and since the anharmonic ratio of the two arbitrary directions with the isotropic directions will remain constant, the angles will be preserved. *The transformation that is established between the spheres of one one-parameter family and their orthogonal trajectories is then a conformal transformation.*

Therefore, let:

(1) $$\sum (x - x_0)^2 - R^2 = 0$$

be the general equation of the spheres considered, which depend upon a parameter t. The preceding considerations lead us to introduce the rectilinear generators. Then set:

$$x - x_0 + i\,(y - y_0) = \lambda\,[(z - z_0) + R],$$
$$x - x_0 + i\,(y - y_0) = -\frac{1}{\lambda}\,[(z - z_0) - R],$$
$$x - x_0 + i\,(y - y_0) = \mu\,[(z - z_0) + R];$$

hence:

(2) $$\begin{cases} z - z_0 = R\dfrac{1-\lambda\mu}{1+\lambda\mu}, \\ x - x_0 + i(y - y_0) = R\dfrac{2\lambda}{1+\lambda\mu}, \\ x - x_0 - i(y - y_0) = R\dfrac{2\mu}{1+\lambda\mu}. \end{cases}$$

The differential equations of the orthogonal trajectories are:

$$\frac{dx}{x - x_0} = \frac{dy}{y - y_0} = \frac{dz}{z - z_0} = \frac{d(x+iy)}{x - x_0 + i\,(y - y_0)} = \frac{d(x-iy)}{x - x_0 - i\,(y - y_0)}.$$

Upon equating the third ratio to the other two in succession and setting:

$$dA = \frac{d(x_0 + iy_0)}{2R}, \qquad dB = \frac{d(x_0 - iy_0)}{2R}, \qquad dC = \frac{dz_0}{2R},$$

one will get the two Ricatti equations:

(3) $$\frac{d\lambda}{dt} = \lambda^2\frac{dB}{dt} + 2\lambda\frac{dC}{dt} - \frac{dA}{dt}, \quad \frac{d\mu}{dt} = \mu^2\frac{dA}{dt} + 2\mu\frac{dC}{dt} - \frac{dB}{dt}.$$

One can verify that since *A* and *B* will be conjugate-imaginary quantities in the case in which one works with real spheres, the solutions of the second of those Ricatti equations will be conjugate imaginaries of the solutions of the first one, in such a way that everything will come down to integrating one of them.

If one knows an orthogonal trajectory then one will know an integral of each equation, and the solution to the problem will reduce to two quadratures. If one knows two orthogonal trajectories then one will have only one quadrature to perform, and if one knows three orthogonal trajectories then the problem will have been solved without any quadratures. The general integral of the first equation is then provided by the formula:

$$\frac{\lambda-\lambda_1}{\lambda-\lambda_2}:\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2} = \frac{\lambda^0-\lambda_1^0}{\lambda^0-\lambda_2^0}:\frac{\lambda_3^0-\lambda_1^0}{\lambda_3^0-\lambda_2^0},$$

upon denoting the values that correspond to $t = t_0$ by the index zero. It will then be a relation of the form:

$$l = \frac{M\lambda^0+N}{P\lambda^0+Q}.$$

One will likewise have an integral of the form:

$$\mu = \frac{R\mu^0+S}{T\lambda^0+U}$$

for the second Ricatti equation, in which *R*, *S*, *T*, *U* are conjugate to *M*, *N*, *P*, *Q*, respectively, moreover.

Those two forms define the correspondence between the sphere that corresponds to the value t_0 of the parameter and the sphere that corresponds to the value *t* of the parameter that is established by the orthogonal trajectories.

One then sees that the transformation will change the circles on one of the spheres into circles on the other one, because the circles, which are plane sections of the sphere that is represented by equations (2) are defined by a relation that is homographic in λ, μ. By stereographic projection, it will become one of the planar transformations of the group of reciprocal radius vectors [Chap. VIII, pp. 228].

Particular triply-orthogonal systems

6. – As particular triply-orthogonal systems, recall the system of homofocal quadrics:

$$\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c-\lambda} - 1 = 0,$$

and the system of homofocal fourth-degree cyclides [Chap. VIII, § 7]:

$$\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c-\lambda}+\frac{(x^2+y^2+z^2-R^2)^2}{4R^2(d-\lambda)}-\frac{(x^2+y^2+z^2+R^2)^2}{4R^2(e-\lambda)}=0.$$

One verifies that one obtains another system that is composed of third-degree Dupin cyclides by considering the surfaces that are loci of contact points of tangent planes that are drawn from a point on one of the axes to a family of homofocal quadrics.

CHAPTER XIII

CONGRUENCES OF SPHERES AND CYCLIC SYSTEMS

Generalities

1. – We call a family of ∞^2 spheres (Σ):

(1) $$\sum (x - f)^2 - r^2 = 0$$

in which f, g, h, r are functions of the two parameters u, v, a *congruence of spheres*. The locus of the centers of those spheres is a surface (S):

(S) $$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

We seek the envelope of those spheres. We must append the two equations:

(2) $$\sum (x - f)\frac{\partial f}{\partial u} + r\frac{\partial r}{\partial u} = 0, \qquad \sum (x - f)\frac{\partial f}{\partial v} + r\frac{\partial r}{\partial v} = 0$$

to equation (1). Equations (2) represent a line, and thus the envelope of the spheres (Σ) touches each of the spheres at two points, which one calls *focal points*. The envelope (F), which one calls the *focal surface*, then decomposes into two sheets (F_1), (F_2).

Consider a family of ∞^1 spheres (Σ) in the congruence (1); it suffices to define u, v as functions of one parameter t. Those spheres admit an envelope that touches each of them along a characteristic circle whose plane has the equation:

(3) $$\sum (x - f)\, df + r\, dr = 0.$$

When the expressions for u, v as functions of t vary, all of the characteristic circles pass through two fixed points, which are the focal points of the sphere considered. The envelopes thus-obtained correspond to the ruled surfaces of congruences of lines; one calls them the *canal surfaces* of the congruence (1).

Among those canal surfaces, we seek the ones for which each sphere is tangent to the infinitely-close sphere. They are, in reality, ruled surfaces with isotropic generators [Chap. VII, § 3, pp. 168]. The circle that is defined by equations (1), (3) must reduce to a pair of isotropic lines. The plane (3) must then be tangent to the sphere (1), which will give the condition:

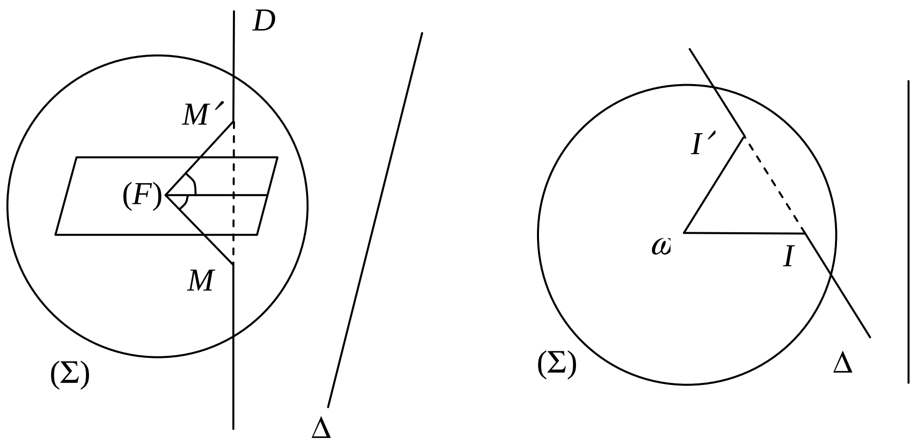
(4) $$\sum df^2 - dr^2 = 0,$$

which is a first-order, second-degree differential equation. There are then two special families of spheres in which each sphere touches the infinitely-close sphere. The point of contact is defined by the following equations, which one will get by writing down the equations of the normal to the plane (3) that is drawn through the center, and taking (1) and (4) into account:

$$\text{(5)} \qquad \frac{x-f}{df} = \frac{y-g}{dg} = \frac{z-h}{dh} = \frac{-r}{dr}.$$

One sees that *df*, *dg*, *dh* are direction coefficients of the radius of the point of contact; the direction cosines are:

$$-\frac{df}{dr}, \quad -\frac{dg}{dr}, \quad -\frac{dh}{dr}.$$

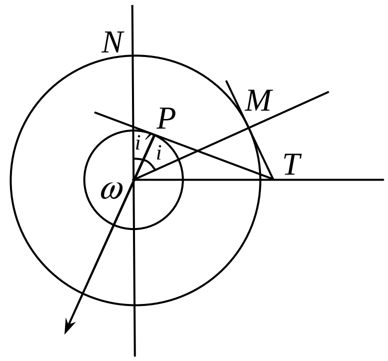


Let *I*, *I′* be the points of contact thus-found. Equation (4) defines two directions *ωI*, *ωI′* on the surface (*S*). Let *M*, *M′* be the points of contact of the corresponding sphere (Σ) with the focal surface (*F*). The line *MM′* is represented by the two equations (2), or furthermore, since the points *M*, *M′* are on all of the characteristic circles, by the two equations (3) that correspond to the special envelopes (isotropic canal surfaces). Now, in that case, equation (3) will represent the tangent plane to the sphere at one of the points *I*, *I′*. Hence, the lines *II′*, *MM′* will be polar reciprocals with respect to the sphere (Σ). Moreover, one sees that if one considers the ratio dv / du in equations (5) to be variable then the point that it defines will describe a line, which will contain the poles of the two planes (2) for $dv = 0$ and $du = 0$. That line, which is the line *II′*, will then be the conjugate to *MM′*.

If we suppose that (Σ) is a real sphere then *I*, *I′* will be imaginary in the case where *M*, *M′* are real, and conversely. *ωI*, *ωI′* are in the tangent plane to the surface (*S*) at *ω*. *MM′* is perpendicular to that tangent plane. The points *M*, *M′*, and in turn, the lines *ωM*, *ωM′* are symmetric with respect to that tangent plane.

Now recall that *ωM* is normal to the first sheet of the focal surface, and *ωM′* is normal to the second one, and consider *ωM* to be an incident ray to the surface (*S*) and *ωM′* to be the reflected one. We then have a congruence of normals that reflects from the surface (*S*) into a congruence of normals. The surface (*S*) can be arbitrary, as well as the surface (F_1). Indeed, consider the spheres that have their centers on (*S*) and their tangents on (F_1). (F_1) will be one of the focal sheets of the congruence of spheres thus-obtained, and the congruence of normals to (F_1) reflect from (*S*) into the congruence of normals to (F_2), which is the second focal sheet. Hence, one has:

Malus's theorem: *The rays that are normal to an arbitrary surface reflect from an arbitrary surface along normals to a new surface.*



As one can see, this theorem extends to refracted rays. To that effect, recall the classical Huyghens construction. Consider a sphere of center ω. Let ωM be the incident ray, let ωN be the normal to the refringent surface, and let n be the index of refraction. Construct a second sphere with its center at ω, whose radius has the ratio of n with the radius of the first one. Consider the tangent plane to the refringent surface at ω. At the point M where the incident ray meets the first sphere, draw the plane that is tangent to that sphere, which will cut the plane ωT along a line (T). Through the line (T), draw the plane (T) P that is tangent to the second sphere. Upon letting i, i' denote the angles that ωM and ωP make with ωN, respectively, one will have immediately that:

$$\omega T = \frac{\omega M}{\sin i} = \frac{\omega P}{\sin i'},$$

so:

$$\frac{\sin i'}{\sin i} = \frac{\omega P}{\omega M} = n.$$

Hence, ωP will be the refracted ray. Start with a congruence of normals then. Let (F_1) be the surface normal, and let (Σ) be spheres tangent to (F_1) whose centers ω are on the refringent surface. In order to construct the refracted rays, one must consider the spheres (Σ') that are concentric to the spheres (Σ) and of radius nr. Now, the line (Δ) that relates to the spheres (Σ) is defined by equations (5), in which du, dv are variables, and those equations will not change when one replaces r with nr. The line (Δ) is then the same for a sphere (Σ) and for the concentric sphere (Σ'). On the other hand, since it is in the tangent plane to (S) at ω and in the tangent plane to M at (Σ), it will be the line (T) of Huyghens's construction, and since it keeps the same significance for (Σ'), it will belong to the tangent planes that are common to (Σ') and its envelope. Hence, P will be one of the contact points of (Σ') with its envelope, and the refracted rays ωP will be normal to one of the sheets of the focal surface of the congruence of spheres (Σ').

Special congruences

2. – We have associated four congruences of lines with the congruence of spheres considered: viz., the lines ωM that are normal to (F_1), the lines $\omega M'$ that are normal to (F_2), the lines (Δ), and the lines (D).

Suppose that M, M' coincide on each sphere (Σ); they will also coincide with I, I'. The two focal sheets will then coincide. The loci of coincident points I, I' that correspond to each family of spheres (Σ) that satisfies condition (4) will then be a line of curvature on the double focal surface (F), and the spheres (Σ) of that family will be corresponding curvature spheres. *The congruence of spheres is then composed of the curvature spheres of a surface* (F), *which corresponds to one of the families of the lines of curvature.*

Conversely, consider a surface (F) and its curvature spheres (Σ) of the same family. The surface (F) will be the double focal surface of the congruence of those curvature spheres, because one of the points I, I' that belongs to (F), which belongs to the focal surface, will coincide with one of the points M, M'. The two conjugate lines (Δ) and (D), which intersect, will be tangent to (Σ) at the same point, and the points I, I', M, M' will coincide at that point. One will then revert to the case in question.

All of the congruences of lines considered reduce to three here: viz., the normals to the surface (F), the lines (D) that are tangent to one family of lines of curvature of (F), and the lines (Δ) that are tangent to the other family. The surface (S) will then be one of the sheets of the development of the double focal surface. The lines of curvature, which are integrals of (4), will correspond to a family of geodesics [Chap. VII, § 2] on the surface (S).

Application to the search for geodesics

One will then be led to determine the geodesics of (S) upon writing down that equation (4) has a double root at du, dv. With the usual notations for the ds^2 of (S) [Chap. II], that equation can be written:

(6) $$E\,du^2 + 2F\,du\,dv + G\,dv^2 - \left(\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv\right)^2 = 0$$

or:

$$\left[E - \left(\frac{\partial r}{\partial u}\right)^2\right]du^2 + 2\left[F - \frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right]du\,dv + \left[G - \left(\frac{\partial r}{\partial v}\right)^2\right]dv^2 = 0.$$

In order for it to have a double root, it is necessary and sufficient that:

$$\left[E - \left(\frac{\partial r}{\partial u}\right)^2\right]\left[G - \left(\frac{\partial r}{\partial v}\right)^2\right] - \left[F - \frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right]^2 = 0$$

or

(7) $$H^2 - \left[E\left(\frac{\partial r}{\partial u}\right)^2 - 2F\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} + G\left(\frac{\partial r}{\partial v}\right)^2 \right] = 0,$$

which is a partial differential equation that determines r. Having calculated r, one will get the family of geodesics that corresponds, by integration, to the ordinary differential equation that one obtains by equating the square root in the left-hand side of (6) to zero. Indeed, the latter is the square of a linear form in du, dv, due to the condition (7).

The curves of (S) that are defined by the condition r = const. have the following significance, moreover: If that condition is realized then the center ω of (Σ) will describe a curve (γ) on (S), and the point of contact M of (Σ) with (F) will describe a curve (γ') on (F). Since ωM is normal to (F), (γ') will remain orthogonal to ωM, and since $\omega M = r$ is constant, (γ) will also be orthogonal to each of the lines ωM. Consequently, (γ) will cut each of the geodesics considered at a right angle, since ωM will be tangent to one of those geodesics at each point ω of (S).

Hence, the curves r = const. of (S) are the orthogonal trajectories of a family of geodesics [cf., Chap. III, § 9]. One verifies that immediately upon noting that equation (6), which has a perfect square for its left-hand side, has the consequence that for any δu and δv, one will have:

$$(E\,du + F\,dv)\,\delta u + (F\,du + G\,dv)\,\delta v = \left(\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv\right)\frac{\partial r}{\partial u}\delta u + \left(\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv\right)\frac{\partial r}{\partial}\delta v$$
$$\equiv dr\,\delta r.$$

The left-hand side will then be annulled if one supposes that $\delta r = 0$, which indeed expresses the orthogonality of the geodesics considered to the curves r = const.

Dupin's theorem

3. – Suppose that the focal surface (F) has two distinct sheets (F_1) and (F_2), and study their relationship to the surface (S) that is the locus of the centers of the spheres (Σ). If x, y, z denote the coordinates of M then the direction cosines of ωM, which is normal to one of the sheets, will be:

$$\lambda = \frac{x-f}{r}, \qquad \mu = \frac{y-g}{r}, \qquad \nu = \frac{z-h}{r}.$$

Hence, we will have:

(8) $$x = f + \lambda r, \qquad y = g + \mu r, \qquad z = h + \nu r$$

for the equations of the focal sheet considered. Substitute those values for x, y, z into equations (2). They will become:

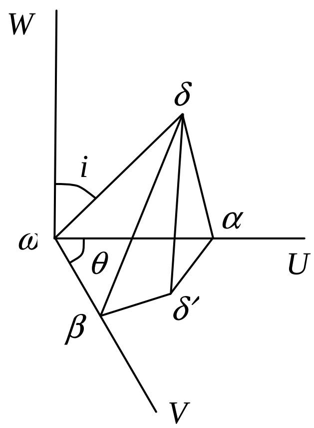
(9) $$\sum \lambda \frac{\partial f}{\partial u} + \frac{\partial r}{\partial u} = 0, \qquad \sum \lambda \frac{\partial f}{\partial v} + \frac{\partial r}{\partial v} = 0.$$

Those equations, when combined with $\lambda^2 = 1$, define the two systems of values of λ, μ, ν that correspond to the two sheets.

Let i, i' be the angles that ωM and $\omega M'$, resp., make with the normal ωN to the locus (S) of centers ω. Those angles are supplementary ($\cos i' = -\cos i$), and if l, m, n are the direction cosines of ωN then:

(10) $$\cos i = \sum \lambda l.$$

Calculate the angle i. It will suffice to infer λ, μ, ν from equations (9) and (10) and substitute the values obtained into $\sum \lambda^2$. In order to avoid the calculation, we employ another method. In the tangent plane to (S), let ωU, ωV be tangents to the curves $v = \text{const.}$, $u = \text{const.}$, resp., that are directed in the senses of increasing u and v, resp.



The direction cosines of ωU are:

$$\frac{1}{\sqrt{E}}\frac{\partial f}{\partial u}, \qquad \frac{1}{\sqrt{E}}\frac{\partial g}{\partial u}, \qquad \frac{1}{\sqrt{E}}\frac{\partial h}{\partial u}.$$

Those of ωV are:

$$\frac{1}{\sqrt{G}}\frac{\partial f}{\partial v}, \qquad \frac{1}{\sqrt{G}}\frac{\partial g}{\partial v}, \qquad \frac{1}{\sqrt{G}}\frac{\partial h}{\partial v}.$$

Let $\omega\delta$ be the vector of length 1 that is measured along the half-line ωM. From formulas (9), its orthogonal projections $\omega\alpha$, $\omega\beta$, on ωU, ωV, resp., will be:

$$\omega\alpha = A = -\frac{1}{\sqrt{E}}\frac{\partial r}{\partial u}, \qquad \omega\beta = B = -\frac{1}{\sqrt{G}}\frac{\partial r}{\partial v}.$$

The sine of i is the projection $\omega\delta$ of $\omega\delta'$ onto the plane UOV, and everything will come down to calculating $\omega\delta'$. Let θ be the angle between ωU and ωV:

$$\cos\theta = \frac{F}{\sqrt{EG}}, \qquad \sin\theta = \frac{\sqrt{EG - F^2}}{\sqrt{EG}} = \frac{H}{\sqrt{EG}}.$$

Since $\omega\delta'$ is the diameter of the circle that is circumscribed in the triangle $\omega\alpha\beta$ whose edge $\alpha\beta$ is $\sqrt{A^2 - 2AB\cos\theta + B^2}$, we will get immediately:

$$\sin^2 i = \omega\delta'^2 = \frac{A^2 - 2AB\cos\theta + B^2}{\sin^2\theta}.$$

Now:

$$\frac{A^2 - 2AB\cos\theta + B^2}{\sin^2\theta} = \frac{1}{H^2}\Phi\left(\frac{\partial r}{\partial v}, -\frac{\partial r}{\partial u}\right),$$

if we set:

$$\Phi(du, dv) = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

with our usual notations. We then obtain the desired formula:

$$\sin^2 i = \frac{1}{H^2}\Phi\left(\frac{\partial r}{\partial v}, -\frac{\partial r}{\partial u}\right). \tag{11}$$

We now return to equations (8), and we propose to determine the lines of curvature of the sheet of the focal surface that they represent. The lines of curvature are defined by the equation:

$$|\,dx \quad \lambda \quad d\lambda\,| = 0,$$

or:

$$|\,df + \lambda\,dr + r\,d\lambda \quad \lambda \quad d\lambda\,| = 0,$$

which reduces to:

$$|\,dx \quad \lambda \quad d\lambda\,| = 0.$$

Multiply that by the determinant:

$$\left|\lambda \quad \frac{\partial f}{\partial u} \quad \frac{\partial f}{\partial v}\right|,$$

which is not zero, since the normal ωM is not in the tangent plane to (S). The equation will become:

$$\begin{vmatrix} \sum \lambda\,df & \sum \lambda^2 & \sum \lambda\,d\lambda \\ \sum \frac{\partial f}{\partial u}df & \sum \lambda\frac{\partial f}{\partial u} & \sum \frac{\partial f}{\partial u}d\lambda \\ \sum \frac{\partial f}{\partial v}df & \sum \lambda\frac{\partial f}{\partial v} & \sum \frac{\partial f}{\partial v}d\lambda \end{vmatrix} = 0,$$

or, upon taking (9) into account:

$$\begin{vmatrix} -dr & 1 & 0 \\ \sum \frac{\partial f}{\partial u} df & -\frac{\partial r}{\partial u} & \sum \frac{\partial f}{\partial u} d\lambda \\ \sum \frac{\partial f}{\partial v} df & -\frac{\partial r}{\partial v} & \sum \frac{\partial f}{\partial v} d\lambda \end{vmatrix} = 0.$$

Multiply the first row by $\partial r / \partial u$ and add it to the second one, and multiply it by $\partial r / \partial u$ and add it to the third one. We then get the equation:

(12)
$$\begin{vmatrix} \sum \frac{\partial f}{\partial u} df - \frac{\partial r}{\partial u} dr & \sum \frac{\partial f}{\partial u} d\lambda \\ \sum \frac{\partial f}{\partial v} df - \frac{\partial r}{\partial v} dr & \sum \frac{\partial f}{\partial v} d\lambda \end{vmatrix} = 0.$$

The elements of the first column are one-half the partial derivatives with respect to *du*, *dv* of the quadratic form:

(13)
$$\sum df^2 - dr^2 = \Phi_1 (du, dv),$$

which defines the pair of directions ωI, $\omega I'$ on (*S*). Let us see if the elements of the second column are susceptible to an analogous interpretation. If we differentiate equations (9) then we will get:

$$\sum \frac{\partial f}{\partial u} d\lambda = -\sum \lambda\, d\left(\frac{\partial f}{\partial u}\right) - d\left(\frac{\partial f}{\partial u}\right).$$

Now, if one totally differentiates with respect to the independent variables *u*, *v*, while consequently supposing that $d^2u = d^2v = 0$, then:

$$d\left(\frac{\partial r}{\partial u}\right) = \frac{1}{2} \cdot \frac{\partial (d^2 r)}{\partial (du)}, \qquad d\left(\frac{\partial f}{\partial u}\right) = \frac{1}{2} \cdot \frac{\partial (d^2 f)}{\partial (du)},$$

and:

$$\sum \lambda \cdot d\left(\frac{\partial f}{\partial u}\right) = \frac{1}{2} \cdot \frac{\partial (\sum \lambda \cdot d^2 f)}{\partial (du)}.$$

Set:

(14)
$$\Theta (du, dv) = \sum \lambda\, d^2 f, \qquad \Omega (du, dv) = \Theta + d^2 r,$$

and the equation can be written:

(15)
$$\begin{vmatrix} \frac{\partial \Phi_1}{\partial du} & \frac{\partial \Omega}{\partial du} \\ \frac{\partial \Phi_1}{\partial dv} & \frac{\partial \Omega}{\partial dv} \end{vmatrix} = 0.$$

Hence, the principal directions of the sheet of (F) considered will be harmonic conjugates with respect to the two pairs $\Phi_1 = 0$ and $\Omega = 0$.

Calculate Θ. In order to do that, eliminate λ, μ, ν from equations (9), (10), and:

$$\sum \lambda\, d^2 f - \Theta = 0.$$

One will get:

(16)
$$\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & l & d^2 f \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & m & d^2 g \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & n & d^2 h \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} & -\cos i & -\Theta \end{vmatrix} = 0,$$

which will give:

$$\Theta H - H \cos i\, \Psi\,(du, dv) + H\,\chi(du, dv) = 0$$

when one develops it in the elements of the last row. As in [Chap. II, § 3], $\Psi\,(du, dv)$ denotes the form $\sum l\, d^2x$ in that formula; however, l, m, n are direction cosines here. The form $\chi(du, dv)$ is deduced from the left-hand side of (16) by replacing the elements $-\cos i$, $-\Theta$ with zeros and dividing by H. By combining the first two columns, one will get:

$$H\chi = \begin{vmatrix} \frac{\partial f}{\partial v}\frac{\partial r}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial r}{\partial v} & l & d^2 f \end{vmatrix},$$

as a third-degree determinant, and it will suffice to multiply the two sides by the determinant:

$$H = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & l \end{vmatrix}$$

in order to obtain:

(17)
$$H^2\chi = \begin{vmatrix} F\frac{\partial r}{\partial u} - E\frac{\partial r}{\partial v} & \sum \frac{\partial f}{\partial u} d^2 f \\ G\frac{\partial r}{\partial u} - F\frac{\partial r}{\partial v} & \sum \frac{\partial f}{\partial v} d^2 f \end{vmatrix},$$

which, from the calculations of [Chap. II, § 4, pp. 32], can be expressed in terms of E, F, G, and their derivatives. Moreover, we have:

$$\Omega = d^2r + \cos i\,\Omega\,(du, dv) - \chi(du, dv)$$

or:

(18)
$$\Omega = \Psi_1\,(du, dv) - \cos i \cdot \Psi\,(du, dv),$$

with

$$\Psi_1 = d^2r - \chi.$$

The lines of curvature of the second sheet are likewise tangents to the conjugate directions with respect to $\Phi_1 = 0$ and to the pair that one deduces from $\Omega = 0$ upon changing the sign of cos i; i.e.:

$$\Psi_1 (du, dv) - \cos i \cdot \Psi (du, dv) = 0.$$

Consider the points of contact of the same sphere (Σ) with its two sheets to be homologous on those two sheets. It then results from the preceding conclusions that in order for the lines of curvature to correspond on the two sheets – i.e., in order for them to be defined by the same quadratic equation (15) in du, dv – it is necessary and sufficient that there should exist a pair of variations du, dv that are conjugate with respect to the three pairs:

$$\Phi_1 = 0, \qquad \Psi_1 + \cos i \cdot \Psi = 0, \qquad \Psi_1 - \cos i \cdot \Psi = 0;$$

i.e., with respect to the pairs:

$$\Phi_1 = 0, \qquad \Psi_1 + \cos i \cdot \Psi = 0, \qquad \Psi = 0,$$

or furthermore:

$$\Phi_1 = 0, \qquad \Psi_1 = 0, \qquad \Psi = 0.$$

Equation (15) defines curves on the surface (S) along which the developables of the normals to one of the sheets of (F) will cut (S). The condition for those curves to also be intersections of (S) with the developables of the normals to the other sheet of (F) is then that at each point of (S) their directions must be harmonic conjugates with respect to the directions that are defined by $\Psi = 0$; i.e., that they must be conjugate directions on (S).

We then obtain:

Dupin's theorem: *If the lines of curvature correspond on the two focal sheets then the developables of the corresponding normals will cut the surface* (S) *along the same conjugate net, and conversely.* Moreover: *The necessary and sufficient condition for the developables of a congruence of normals to reflect from a surface into other developables is that it must determine conjugate nets on the surface.*

Congruence of lines (D)

4. – We seek the developables of the congruence of lines (D); they are defined by the equation:

$$\text{(19)} \qquad \begin{vmatrix} dx & dy & dz \\ l & m & n \\ dl & dm & dn \end{vmatrix} = 0.$$

x, y, z always denote the coordinates of M, and l, m, n are the direction cosines of the normal to (S) at ω, which is parallel to (D).

Now:

$$x = f + r\lambda, \qquad y = g + r\mu, \qquad z = h + r\nu,$$

from equation (8), and equation (19) will become:

$$| \, df + r\, d\lambda + \lambda\, dr \quad l \quad dl \, | = 0.$$

Multiply the left-hand side by the non-zero determinant:

$$H = \begin{vmatrix} l & \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{vmatrix};$$

we will get:

$$\begin{vmatrix} r\sum l\, d\lambda + dr\sum \lambda l & 1 & 0 \\ \sum\dfrac{\partial f}{\partial u} df + r\sum\dfrac{\partial f}{\partial u} d\lambda + dr\cdot\sum\lambda\dfrac{\partial f}{\partial u} & 0 & \sum\dfrac{\partial f}{\partial u} dl \\ \sum\dfrac{\partial f}{\partial v} df + r\sum\dfrac{\partial f}{\partial v} d\lambda + dr\cdot\sum\lambda\dfrac{\partial f}{\partial v} & 0 & \sum\dfrac{\partial f}{\partial v} dl \end{vmatrix} = 0,$$

or

$$\text{(20)} \qquad \begin{vmatrix} \sum\dfrac{\partial f}{\partial u} df + r\sum\dfrac{\partial f}{\partial u} d\lambda + dr\cdot\sum\lambda\dfrac{\partial f}{\partial u} & \sum\dfrac{\partial f}{\partial u} dl \\ \sum\dfrac{\partial f}{\partial v} df + r\sum\dfrac{\partial f}{\partial v} d\lambda + dr\cdot\sum\lambda\dfrac{\partial f}{\partial v} & \sum\dfrac{\partial f}{\partial v} dl \end{vmatrix} = 0.$$

The elements of the second column are one-half the partial derivatives of the form Ψ (du, dv) with respect to du, dv. As for the elements of the first one, we point out that from a calculation in the preceding paragraph:

$$\sum\frac{\partial f}{\partial u} d\lambda = -\frac{1}{2}\frac{\partial\Omega}{\partial(du)}, \qquad \sum\frac{\partial f}{\partial v} d\lambda = -\frac{1}{2}\frac{\partial\Omega}{\partial(dv)},$$

or, from (18):

$$\Omega = \Psi_1 + \cos i\,\Psi.$$

Finally, the points M, M' are defined by the relations (9):

$$\sum\lambda\frac{\partial f}{\partial u} + \frac{\partial r}{\partial u} = 0, \qquad \sum\lambda\frac{\partial f}{\partial v} + \frac{\partial r}{\partial v} = 0,$$

in such a way that with the notation that was introduced by formula (13):

$$\sum\frac{\partial f}{\partial u} df + dr\cdot\sum\lambda\frac{\partial f}{\partial u} = \sum\frac{\partial f}{\partial u} df - \frac{\partial r}{\partial u} dr = \frac{1}{2}\frac{\partial\Phi_1}{\partial(du)},$$

$$\sum \frac{\partial f}{\partial v} df + dr \cdot \sum \lambda \frac{\partial f}{\partial v} = \sum \frac{\partial f}{\partial v} df - \frac{\partial r}{\partial v} dr = \frac{1}{2} \frac{\partial \Phi_1}{\partial (dv)}.$$

The elements of the first columns will then be one-half the partial derivatives of the form $\Phi_1 - r [\Psi_1 + \Psi \cos i]$ with respect to du, dv.

Therefore: The developables of the congruence of lines (*D*) correspond to curves on the surface (*S*) whose tangents are conjugate at each point with respect to the pairs of directions that are defined by the equations:

$$\Psi = 0, \qquad \Phi_1 - r [\Psi_1 + \Psi \cos i] = 0,$$

or with respect to the pairs:

$$\Psi = 0, \qquad \Phi_1 - r \Psi_1 = 0. \tag{21}$$

As one should expect, that result will not change if one changes i into $\pi - i$, and *the developables of the congruence of lines* (*D*) *will correspond to a conjugate net on the surface* (*S*).

Consider the focal planes. One focal plane is parallel to the direction l, m, n, and to the direction dl, dm, dn, which corresponds to an infinitely-close line (*D*) on one of the developables that pass through (*D*). However:

$$l^2 + m^2 + n^2 = 1,$$

so:

$$l\, dl + m\, dm + n\, dn = 0.$$

dl, dm, dn then define the direction of the lines of the focal plane that that is parallel to the tangent plane to the surface. Now, the two directions will correspond to two focal planes, and thus to two developables, and since they are conjugate, if we define them by the characteristics d and δ then they will satisfy the equation:

$$\sum dl \cdot \delta f = 0,$$

which expresses the idea that the first focal plane is perpendicular to the direction δf, δg, δh, which corresponds to the other focal plane. *Each focal plane is perpendicular to the direction of the surface* (*S*) *that corresponds to the developable that is not tangent to the focal plane.*

Congruence of lines (Δ)

5. – The line (Δ) is the intersection of the tangent planes to the sphere at M and the surface (S) at ω, which have the equations:

$$\sum \lambda (X - f) - r = 0, \qquad \sum l (X - f) = 0,$$

respectively. We express the idea that the preceding line will meet the infinitely-close line. That gives:

$$\sum d\lambda \cdot (X - f) - \sum \lambda \, df - dr = 0, \qquad \sum dl \cdot (X - f) - \sum l \, df = 0,$$

which are conditions that will simplify upon remarking that:

$$\sum l \, df = 0 \qquad \text{and} \qquad \sum \lambda \, df + dr = 0$$

What remains will be:

$$\sum d\lambda \cdot (X - f) = 0, \qquad \sum dl \cdot (X - f) = 0. \tag{22}$$

If we express the idea that the equations that are obtained are compatible then we will get the equation that defines the developables:

$$| \, l \quad d\lambda \quad dl \, | = 0. \tag{23}$$

We further multiply this by the non-zero determinant:

$$\left| l \quad \frac{\partial f}{\partial u} \quad \frac{\partial f}{\partial v} \right|.$$

We get:

$$\begin{vmatrix} 1 & \sum l \, d\lambda & 0 \\ 0 & \sum d\lambda \cdot \dfrac{\partial f}{\partial u} & \sum dl \cdot \dfrac{\partial f}{\partial u} \\ 0 & \sum d\lambda \cdot \dfrac{\partial f}{\partial v} & \sum dl \cdot \dfrac{\partial f}{\partial v} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \sum \dfrac{\partial f}{\partial u} d\lambda & \sum \dfrac{\partial f}{\partial u} dl \\ \sum \dfrac{\partial f}{\partial v} d\lambda & \sum \dfrac{\partial f}{\partial v} dl \end{vmatrix} = 0. \tag{24}$$

The elements of the first column are, up to sign, one-half the partial derivatives of the form $\Omega = \Psi_1 + \Psi \cos i$ with respect to du, dv. Those of the second column are one-half the partial derivatives of Ψ with respect to du, dv. The developables of the congruence of lines (Δ) will then correspond to a net of curves on the surface (S) whose directions are harmonic conjugate at each point with respect to the pairs of directions that are defined by the equations:

$$(25) \qquad \Psi = 0, \qquad \Psi_1 = 0.$$

In particular, *the developables of the congruence of lines* (Δ) *correspond to a conjugate net on the surface* (*S*).

As for the focal points, they are defined by the equations of (Δ) and equations (22), which are compatible by virtue of the relation (23). One deduces from this that the directions that join ω to the focal points are defined by the relations:

$$\sum l \cdot \delta f = 0, \qquad \sum dl \cdot \delta f = 0, \qquad \sum d\lambda \cdot \delta f = 0.$$

The first one expresses the idea that those lines are in the tangent plane to (*S*), while the second one says that they are the tangents conjugate to the directions of (*S*) that correspond to the developables.

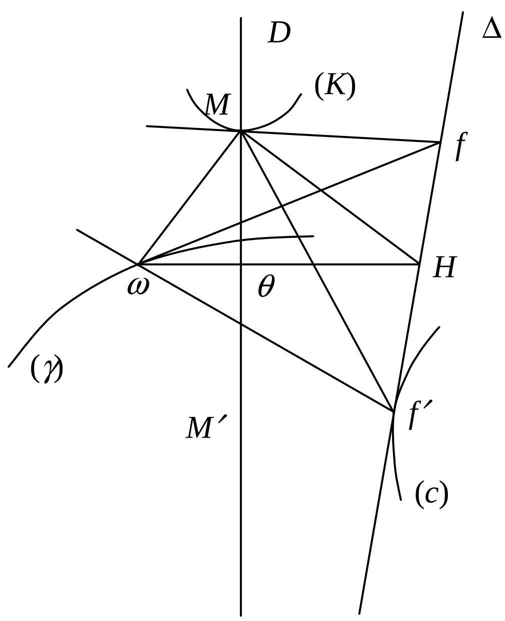
Special cases. – Suppose that the two preceding congruences correspond by developables. The two conjugate nets that we have determined on the surface (*S*) will then coincide. For that to be true, it is necessary and sufficient that the three pairs:

$$\Psi = 0, \qquad \Psi_1 = 0, \qquad \Phi_1 - r \cdot \Psi_1 = 0,$$

or:

$$\Psi = 0, \qquad \Psi_1 = 0, \qquad \Phi_1 = 0$$

must belong to the same involution, and then, from the results of § 2, the lines of curvature will correspond on the two sheets of the surface (*S*), and conversely.



In this case, we have a conjugate net (*R*) on the surface (*S*) that corresponds to the developables of the four congruences ωM, $\omega M'$, (*D*), (Δ). From what we just saw, the focal points f, f' of (Δ) are on the tangents to the two curves of the net that pass through

ω. The lines Mf, Mf' are the tangents at M to the lines of curvature of one of the sheets of the enveloping surface (F), because the tangent planes to the developables of the normals to that sheet are $M\omega f$, $M\omega f'$, since those developables cut (S) along the conjugate net (R) considered, and the plane M (Δ), which cuts the planes along Mf and Mf', is tangent to that same sheet of (F) at M. The line (D) is perpendicular to the plane $f\omega f'$ that is tangent to (S), and its focal places are perpendicular to ωf and $\omega f'$. The developables of the congruence of lines (D) cut the two sheets of the envelope (F) along their lines of curvature, moreover.

Ribacour's triple system

6. – We address the latter case. Let (γ) be one of the curves of the conjugate net (R) on the surface (S). When ω describes (γ), the point M will describe a line of curvature on the sheet of the surface (F) that is tangent to Mf, and the line (Δ) will envelope a curve (C) that is the locus of f'. Consider the sphere (σ) with its center at f that passes through M. That sphere will have a canal surface (E) for its envelope. Since the sphere (σ) has its radius Mf' perpendicular to Mf, it will be constantly tangent to the curve (K), so the point M will be a point of the characteristic circle (H). The plane of that circle is perpendicular to the line Δ that is tangent to (C), so its center will be the foot of the perpendicular that is based at M on Δ. That circle will then be orthogonal to the sphere (Σ) at the point M and the point M' that is symmetric with respect to the plane $f\omega f'$, and the surface (E) will be generated by the circles that are orthogonal to the sphere (Σ) at the points M, M'. That tangent circle to ωM at M will remain orthogonal to the line of curvature (K). Now, it is a line of curvature on the surface (E), so (K) will also be a line of curvature on the surface (E). If we vary (K) then we will get a family of surfaces (E) that will all be orthogonal to the two sheets (F_1), (F_2) of (F), and which will cut along the lines of curvature.

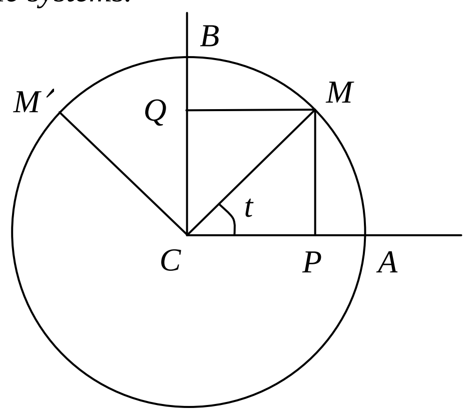
Now, if we look for the second system of lines of curvature on (F_1) and (F_2) then we must consider the spheres whose centers are at f and pass through M. The characteristic circle will again be the circle (H). Moreover, since fM and $f'M$ are perpendicular, the corresponding spheres (σ), (σ') are orthogonal, so their envelopes (E), (E') will also be orthogonal.

We will then have two families of canal surfaces that cut orthogonally along the lines of curvature, which are the circles (H). They will then belong to a triply-orthogonal system. In other words, the circles (H) are orthogonal to a family of surfaces to which the two sheets (F_1), (F_2) of (F) belong, and they will establish a correspondence between the points of any two of those surfaces that is like the one between the points M, M', such that there is a correspondence between the lines of curvature of those surfaces.

Conversely, if two surfaces (F_1), (F_2) are orthogonal to a family of ∞^2 circles (H), and if M and M' are the points where one of those circles cuts (F_1) and (F_2), respectively, then the sphere (Σ) that is orthogonal to the circle at those two points will be tangent to (F_1) and (F_2), which will then be the two sheets of the envelope of spheres (Σ) thus-defined. Moreover, if the circles (H) that have their feet on (F_1) along one line of curvature also cut (F_2) at various points of a line of curvature then the lines of curvature will correspond on the two sheets of the envelope of the spheres (Σ), and one will get back to the special case that was just studied.

Therefore:

If the circles (*H*) *of a congruence are orthogonal to two surfaces* (F_1), (F_2), *and if they establish a correspondence between the lines of curvature on those two surfaces then they will be orthogonal to an infinitude of surfaces on which the lines of curvature correspond. Those surfaces will belong to a triply-orthogonal system whose other two families will be composed of canal surfaces, each of which is generated by the circles* (*H*) *that rest upon one of the lines of curvature of* (F_1) *or* (F_2). Such congruences of circles are called (Ribaucour) *cyclic systems.*



Congruences of circles and cyclic systems

7. – We shall re-address the question of cyclic systems analytically. Consider a family of ∞^2 circles and then look for the existence of normal surfaces to all of those circles. Let (*K*) one of them, let *C* (x_0, y_0, z_0) be its center, and let ρ be its radius, in which x_0, y_0, z_0, ρ are functions of the two parameters *u*, *v*. We define the plane of that circle by way of the direction cosines of two rectangular directions *CA*(*a*, *b*, *c*) and *CP*(a', b', c') that pass through the center of the circles, and we fix the position of a point *M* on the circle by the angle (*CA*, *CM*) = *t*, which is measured positively from *CA* to *CB*. The coordinates of *M* with respect to the system *CAB* are $\rho\cos t$, $\rho\sin t$, and its coordinates *x*, *y*, *z* are:

$$(1) \qquad \begin{cases} x = x_0 + \rho(a\cos t + a'\sin t) = x_0 + \rho\alpha', \\ y = y_0 + \rho(b\cos t + b'\sin t) = b_0 + \rho\beta', \\ z = z_0 + \rho(c\cos t + c'\sin t) = z_0 + \rho\gamma'. \end{cases}$$

We seek to determine *t* as a function of *u*, *v*, in such a fashion that the surface that is the locus of the corresponding points will admit the tangent to the circle at the point *M* for its normal, and we denote its direction cosines by α, β, γ. To that effect, we have the condition:

$$(7) \qquad \sum \alpha\, dx = 0,$$

which is the total differential equation of the desired surfaces. We develop that equation, where α, β, γ are the projections of the directing segment of the direction CM' that corresponds to $t + \pi / 2$:

$$\alpha = -a \sin t + a' \cos t, \qquad \beta = -b \sin t + b' \cos t, \qquad \gamma = -c \sin t + c' \cos t.$$

On the other hand:

$$dx = dx_0 + \alpha' \cdot d\rho + \rho\alpha' dt + \rho (\cos t \cdot da + \sin t \cdot da'), \qquad dy = \ldots, \qquad dz = \ldots,$$

and upon taking into account that:

$$\sum \alpha^2 = 1, \qquad \sum \alpha\alpha' = 0,$$

we will conclude that:

$$\begin{aligned} \sum \alpha\, dx &= \sum \alpha\, dx_0 + \rho \cdot dt + \rho [\cos t \cdot \sum \alpha\, d\alpha + \sin t \cdot \sum \alpha\, d\alpha'] \\ &= -\sin t \cdot \sum \alpha\, dx_0 + \cos t \cdot \sum \alpha' dx_0 + \rho\, dt + \rho [\cos^2 t \cdot \sum \alpha' d\alpha - \sin^2 t \cdot \sum \alpha\, d\alpha'] = 0. \end{aligned}$$

However:

$$\sum \alpha\alpha' = 0,$$

so upon differentiating:

$$\sum \alpha\, d\alpha' + \sum \alpha' d\alpha = 0,$$

and equation (2) will be written simply:

$$dt = \sum \alpha' d\alpha + \frac{1}{\rho} \sum \alpha\, dx_0 \cdot \sin t - \frac{1}{\rho} \sum \alpha' dx_0 \cdot \cos t. \tag{3}$$

Set:

$$\tan \frac{t}{2} = w, \tag{4}$$

so

$$t = 2 \arctan w,$$

and we will get:

$$2\, dw = (1 + w^2) \sum \alpha' d\alpha + \frac{2w}{\rho} \sum \alpha\, dx_0 + \frac{w^2 - 1}{\rho} \sum \alpha' dx_0 . \tag{5}$$

That equation enjoys some properties that are analogous to those of the Riccati equation. In particular, one can verify that the anharmonic ratio of four solutions has a vanishing total differential, and consequently, it will be constant. It can be put into the form:

$$dw = A\, du + A' dv + w (B\, du + B' dv) + w^2 (C\, du + C' dv),$$

and will decompose into two partial differential equations:

(6) $$\frac{\partial w}{\partial u} = A + B\,w + C\,w^2, \qquad \frac{\partial w}{\partial v} = A' + B'w + C'w^2.$$

Upon writing down the idea that the $\frac{\partial^2 w}{\partial u\,\partial v}$ that is inferred from the first one is equal to the $\frac{\partial^2 w}{\partial v\,\partial u}$ that one infers from the second one, one will deduce that:

(7) $$\frac{\partial A}{\partial v} + w\frac{\partial B}{\partial v} + w^2\frac{\partial C}{\partial v} + (B + 2Cw)(A' + B'w + C'w^2)$$
$$- \left[\frac{\partial A'}{\partial v} + w\frac{\partial B'}{\partial v} + w^2\frac{\partial C'}{\partial v} + (B' + 2C'w)(A + Bw + Cw^2)\right] = 0.$$

Any integral of the system (6) will then satisfy that condition, which will take the form:

(8) $$L + M\,w + Nw^2 = 0.$$

If that condition is not satisfied identically then there will be no other solutions than those of equation (8), which will admit two of them. If one prefers that there should be an infinitude of them then that condition must be satisfied identically, and since it has degree two, it will be sufficient that it is satisfied by three functions. The conditions for that to be true will then be:

(9) $$\begin{cases} L = \dfrac{\partial A}{\partial v} - \dfrac{\partial A'}{\partial u} + BA' - AB' = 0, \\ M = \dfrac{\partial B}{\partial v} - \dfrac{\partial B'}{\partial u} + 2(CA' - AC') = 0, \\ N = \dfrac{\partial C}{\partial v} - \dfrac{\partial C'}{\partial u} + CB' - BC' = 0. \end{cases}$$

It results from the theory of partial differential equations that if these are identities then the system (6) will have effectively an infinitude of solutions.

Therefore: *If the circles of a congruence are normal to three surfaces then they will be normal to an infinitude of surfaces.*

It is easy to construct the circles that are normal to two arbitrary surfaces, because there exist ∞^2 spheres that are tangent to those two surfaces, and the circles that are orthogonal to the spheres at the contact points will be normal to the two surfaces. If the lines of curvature on the two surfaces correspond then, as we have seen, we will have a cyclic system that is composed of circles that are normal to the ∞^1 surfaces.

We remark that if the given family of ∞^2 circles is composed of circles that are normal to the two surfaces then we must expect that the integrability conditions (9) will reduce to just one. On the other hand, if we have an envelope of spheres then in order to express the idea that the lines of curvature on the two sheets correspond, we will also get just one condition. It remains for us to examine whether those conditions are identities.

First, suppose that there exists a surface (F_1) that is normal to all of the circles (1). We can do that in such a way that it corresponds to $t = 0$ or $w = 0$. Equation (5) will then admit the solution $w = 0$, so one will have the condition:

$$\sum a\, da' - \frac{1}{\rho}\sum a'dx_0 = 0,$$

and equation (5) will then become:

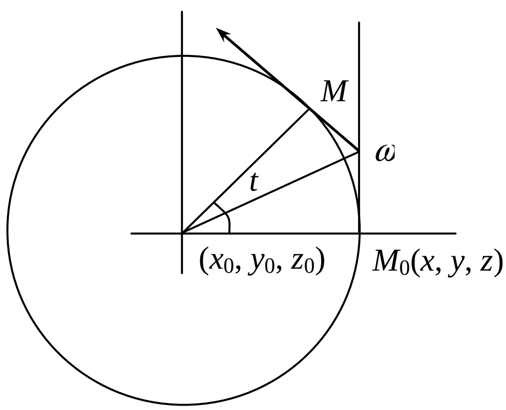
$$dw = w^2 \sum a\, da' + \frac{w}{\rho}\sum a\, dx_0 . \tag{10}$$

Let $M_0(x, y, z)$ be the point that corresponds to $t = 0$:

$$\begin{aligned} x &= x_0 + \rho\, a, & y &= y_0 + \rho\, b, & z &= z_0 + \rho\, c, \\ x_0 &= x - \rho\, a, & y_0 &= y - \rho\, b, & z_0 &= z - \rho\, c, \\ dx_0 &= dx - \rho\, da - a\, d\rho, & &\ldots, & &\ldots, \end{aligned}$$

so

$$\sum a\, dx_0 = \sum a\, dx - d\rho.$$



If we now consider the normal (l, m, n) to (F_1) at M_0 then it will be tangent to the circle, and (10) will become:

$$dw = w^2 \sum a\, dl + \frac{w}{\rho}\left(\sum a\, dx - d\rho\right)$$

or

$$\frac{dw}{w} + \frac{d\rho}{\rho} = w \cdot \sum a\, dl + \frac{1}{\rho}\sum a\, dx.$$

We then introduce the quantity:

$$\rho\, w = r, \tag{11}$$

and get:

$$\frac{dr}{r} = \frac{r}{\rho}\sum a\,dl + \frac{1}{\rho}\sum a\,dx$$

or

(12) $$dr = \frac{r^2}{\rho}\sum a\,dl + \frac{r}{\rho}\sum a\,dx .$$

Now, from (4):

$$r = \rho \tan \frac{t}{2},$$

which shows that r is the radius of the sphere (Σ) that is tangent to the surfaces that are the loci of M and M_0 . Its center is the point ω, which is the intersection of the tangents to the circle at M and M_0 .

Now suppose that there exists a second surface (F_2) that is normal to the circles. Set:

(13) $$\frac{1}{r} = S,$$

so

$$dr = -r^2 \cdot dS,$$

and equation (12) will become:

$$dS + \frac{S}{\rho}\sum a\,dx + \frac{1}{\rho}\sum a\,dl = 0.$$

Let S_1 be the known solution:

(14) $$dS_1 + \frac{S_1}{\rho}\sum a\,dx + \frac{1}{\rho}\sum a\,dl = 0,$$

so, upon subtracting, one will get:

$$d(S - S_1) + \frac{S - S_1}{\rho}\sum a\,dx = 0,$$

or

(15) $$d \ln (S - S_1) = -\frac{1}{\rho}\sum a\,dx .$$

In order for that equation to have other integrals, it is necessary and sufficient that $\frac{1}{\rho}\sum a\,dx$ must be an exact differential. Now, from (4), we have:

(16) $$\frac{\partial S_1}{\partial u} + \frac{S_1}{\rho}\sum a\frac{\partial x}{\partial u} + \frac{1}{\rho}\sum a\frac{\partial l}{\partial u} = 0, \qquad \frac{\partial S_1}{\partial v} + \frac{S_1}{\rho}\sum a\frac{\partial x}{\partial v} + \frac{1}{\rho}\sum a\frac{\partial l}{\partial v} = 0.$$

Suppose that the coordinate lines are lines of curvature on (F_1). Upon denoting the radii of principal curvature by R, R', the formulas of Olinde Rodrigues will give:

$$\frac{\partial l}{\partial u} = -\frac{1}{R}\frac{\partial x}{\partial u}, \qquad \frac{\partial m}{\partial u} = -\frac{1}{R}\frac{\partial y}{\partial u}, \qquad \frac{\partial n}{\partial u} = -\frac{1}{R}\frac{\partial z}{\partial u},$$

$$\frac{\partial l}{\partial v} = -\frac{1}{R'}\frac{\partial x}{\partial v}, \qquad \frac{\partial m}{\partial v} = -\frac{1}{R'}\frac{\partial y}{\partial v}, \qquad \frac{\partial n}{\partial v} = -\frac{1}{R'}\frac{\partial z}{\partial v}.$$

Set:

(17) $$-\frac{1}{R} = T, \qquad -\frac{1}{R'} = T',$$

and we will then have:

(18) $$\begin{cases} \dfrac{\partial l}{\partial u} = T\dfrac{\partial x}{\partial u}, & \dfrac{\partial m}{\partial u} = T\dfrac{\partial y}{\partial u}, & \dfrac{\partial n}{\partial u} = T\dfrac{\partial z}{\partial u}, \\ \dfrac{\partial l}{\partial v} = T\dfrac{\partial x}{\partial v}, & \dfrac{\partial m}{\partial v} = T\dfrac{\partial y}{\partial v} & \dfrac{\partial n}{\partial v} = T\dfrac{\partial z}{\partial v}. \end{cases}$$

Hence:

$$\sum a\frac{\partial l}{\partial u} = T \cdot \sum a\frac{\partial x}{\partial u}, \qquad \sum a\frac{\partial l}{\partial v} = T' \cdot \sum a\frac{\partial x}{\partial v},$$

and the integrability conditions (16) for S_1 will become:

$$\frac{\partial S_1}{\partial u} + (S_1 + T)\frac{\sum a\frac{\partial x}{\partial u}}{\rho} = 0, \quad \frac{\partial S_1}{\partial v} + (S_1 + T)\frac{\sum a\frac{\partial x}{\partial v}}{\rho} = 0,$$

so

$$-\frac{1}{\rho}\sum a\,dx = \frac{1}{S_1 + T}\frac{\partial S_1}{\partial u}du + \frac{1}{S_1 + T}\frac{\partial S_1}{\partial v}dv.$$

If we now express the idea that the right-hand side is an exact differential then, upon suppressing the index on S_1, we will get the partial differential equation:

(19) $$\Delta \equiv \frac{\partial}{\partial v}\left(\frac{1}{S+T}\frac{\partial S}{\partial u}\right) - \frac{\partial}{\partial u}\left(\frac{1}{S+T'}\frac{\partial S}{\partial v}\right) = 0$$

for the definition of the systems of circles that are normal to the ∞^1 surfaces. In that equation, T and T' are the principal curvatures of a surface, referred to its lines of curvature:

$$u = \text{const.}, \qquad v = \text{const.}$$

S is the inverse of the radius of a sphere (Σ) that is tangent to that surface at the point (u, v), and the system of ∞^2 circles that is defined by a solution of that equation is composed of the circles that are orthogonal to the corresponding spheres (Σ) at their contact points with their envelope. Moreover, the given surface will be one of the sheets of that envelope.

We shall see that *equation* (19) *expresses precisely the idea that the lines of curvature on the two sheets of the envelope will correspond*. From Dupin's theorem, in order for that to be true, it is necessary and sufficient that the lines of curvature of the given surface (F_1) must correspond to a conjugate net on the surface that is the locus of ω. Let X, Y, Z be the coordinates of ω:

(20) $$X = x + \frac{1}{S} l, \quad Y = y + \frac{1}{S} m, \quad Z = z + \frac{1}{S} n .$$

In order for the curves u = const., v = const. to form a conjugate net on the surface, it is necessary and sufficient that one must have:

(21) $$\begin{vmatrix} \frac{\partial^2 X}{\partial u\,\partial v} & \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \end{vmatrix} = 0.$$

However, upon taking the formulas of Olinde Rodrigues (18) into account:

$$\frac{\partial X}{\partial u} = \frac{\partial x}{\partial u} + \frac{T}{S}\frac{\partial x}{\partial u} + l\frac{\partial(1/S)}{\partial u} = \left(1 + \frac{T}{S}\right)\frac{\partial x}{\partial u} + l\frac{\partial(1/S)}{\partial u}, \;\ldots, \quad \ldots,$$

$$\frac{\partial X}{\partial v} = \left(1 + \frac{T'}{S}\right)\frac{\partial x}{\partial v} + l\frac{\partial(1/S)}{\partial v},$$

which are relations that one can further write:

(22) $$\begin{cases} \dfrac{\partial X}{\partial u} = \dfrac{S+T}{S^2}\left[S\dfrac{\partial x}{\partial u} - \dfrac{1}{S+T}\dfrac{\partial S}{\partial u} l\right], \quad \ldots, \quad \ldots, \\[2ex] \dfrac{\partial X}{\partial v} = \dfrac{S+T'}{S^2}\left[S\dfrac{\partial x}{\partial v} - \dfrac{1}{S+T'}\dfrac{\partial S}{\partial v} l\right], \quad \ldots, \quad \ldots \end{cases}$$

We can replace $\dfrac{\partial^2 X}{\partial u\,\partial v}$ and the other elements of the first column in the determinant (21) with:

$$\frac{\partial}{\partial v}\left(M\frac{\partial X}{\partial u}\right) - \frac{\partial}{\partial u}\left(N\frac{\partial X}{\partial v}\right),$$

and the analogous quantities, under the condition that (M – N) must not be identically zero. We take:

$$M = \frac{S^2}{S+T}, \quad \text{and} \quad N = \frac{S^2}{S+T'},$$

in such a way that, upon taking into account (18), (19), and (22):

$$\frac{\partial}{\partial v}\left(M\frac{\partial X}{\partial u}\right)-\frac{\partial}{\partial u}\left(N\frac{\partial X}{\partial v}\right) = \frac{\partial S}{\partial v}\frac{\partial x}{\partial u}-\frac{\partial S}{\partial u}\frac{\partial x}{\partial v}-\frac{1}{S+T}\frac{\partial S}{\partial u}\cdot T'\frac{\partial x}{\partial v}+\frac{1}{S+T'}\frac{\partial S}{\partial v}\cdot T\frac{\partial x}{\partial u}-\Delta l.$$

We must then express the idea that:

$$\begin{vmatrix} -\frac{\partial S}{\partial u}\frac{\partial x}{\partial v}\cdot\frac{S+T+T'}{S+T}+\frac{\partial S}{\partial v}\frac{\partial x}{\partial u}\cdot\frac{S+T+T'}{S+T}-\Delta l & \cdots & \cdots \\ S\frac{\partial x}{\partial u}+\frac{1}{S+T}\frac{\partial S}{\partial u}l & \cdots & \cdots \\ S\frac{\partial x}{\partial v}-\frac{1}{S+T}\frac{\partial S}{\partial v}l & \cdots & \cdots \end{vmatrix} = 0.$$

Multiply the second row by $-\frac{S+T+T'}{S(S+T')}\frac{\partial S}{\partial v}$ and the third one by $\frac{S+T+T'}{S(S+T')}\frac{\partial S}{\partial u}$ and add them to the first one. After simplification, we will get:

$$-\Delta\cdot S^2\begin{vmatrix} l & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\end{vmatrix} = 0.$$

Now, the determinant in this is not zero, and neither is S, so that condition will be equivalent to $\Delta = 0$, as we asserted.

One can then define a cyclic system to be a congruence of circles that are normal to ∞^1 *surfaces.*

Ribaucour's contact transformation

Consider a fixed sphere with center ω and the ∞^4 circles (*H*) that are orthogonal to that sphere. On the other hand, consider a surface (*S*), one of its points *M*, and the contact element at that point. There is one and only one circle (*H*) that passes through *M* and is normal to the surface (*S*) at that point. Hence, the surface (*S*) will correspond to a congruence of circles (*H*) that are orthogonal to it. Furthermore, those circles will be orthogonal to the sphere (ω) at two points, so they will be orthogonal to three surfaces; they will then constitute a cyclic system. Let *P*, *P′* be the points where the circle (*H*) meets the sphere. Determine the point *M* on that circle such that the anharmonic ratio (*M*, *M′*, *P*, *P′*) are equal to a given constant *C*. The locus of the point *M′* is a surface that is normal to (*H*), since equation (5) has the same properties as the Ricatti equation in just one variable. For each value of *C*, the contact element of the surface (*S*) at the point (*M*) will then correspond to a contact element on another surface. The lines of curvature will then correspond on the two surfaces, and we will then have a group of ∞^1 contact transformations that preserve the lines of curvature.

These results will obviously persist if one takes the circles (*H*) to be normal to a fixed plane.

Weingarten Surfaces

8. – We have considered some congruences of spheres such that the lines of curvature correspond on the two focal sheets. S. Lie's transformation makes spheres correspond to lines and lines of curvature correspond to asymptotic lines. It is then natural to also consider congruences of lines such that the asymptotes correspond on the two focal sheets. We confine ourselves to the case in which the congruence is a normal congruence, and the problem will then amount to looking for surfaces such that the asymptotes correspond on the two sheets of the developable.

Therefore, let (Σ) be a surface on which we take the lines of curvature to be the coordinate lines. Let l, m, n be the direction cosines of the normal, and let R, R' be the radii of principal curvature. The two sheets of the developable are defined by the equations:

$$(S) \qquad X = x + R\,l, \qquad Y = y + R\,m, \qquad Z = z + R\,n,$$
$$(S) \qquad X' = x + R'l, \qquad Y' = y' + R'm, \qquad Z' = z + R'n.$$

We seek the asymptotes of (*S*), (*S′*) and express the idea that the differential equations in *u*, *v* that define them are the same. Here, the coordinate lines form an orthogonal, conjugate net:

$$ds^2 = E\,du^2 + G\,dv^2,$$
$$\sum l\,d^2x = L\,du^2 + N\,dv^2,$$

and [Chap. III, § 10 and Chap. IV, § 2]:

$$\frac{1}{R} = \frac{L}{E}, \qquad \frac{1}{R'} = \frac{N}{G},$$

so:

$$\sum l\,d^2x = \frac{E}{R}du^2 + \frac{G}{R'}dv^2.$$

The formulas of O. Rodrigues give:

$$\frac{\partial l}{\partial u} = -\frac{1}{R}\frac{\partial x}{\partial u}, \qquad \frac{\partial m}{\partial u} = -\frac{1}{R}\frac{\partial y}{\partial u}, \qquad \frac{\partial n}{\partial u} = -\frac{1}{R}\frac{\partial z}{\partial u},$$

and

$$\frac{\partial l}{\partial v} = -\frac{1}{R'}\frac{\partial x}{\partial v}, \qquad \frac{\partial m}{\partial v} = -\frac{1}{R'}\frac{\partial y}{\partial v}, \qquad \frac{\partial n}{\partial v} = -\frac{1}{R'}\frac{\partial z}{\partial v},$$

and consequently:

$$dX = dx + R\,dl + l\,dR = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv - R\left(\frac{1}{R}\frac{\partial x}{\partial u}du + \frac{1}{R'}\frac{\partial x}{\partial v}dv\right) + l\,dR,$$

or

$$(1) \qquad dX = \left(1 - \frac{R}{R'}\right)\frac{\partial x}{\partial v}dv + l\,dR.$$

As one could have predicted, that formula and its analogues show that the normal to (S) will have the direction coefficients:

$$\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}.$$

Furthermore, one concludes that for that surface (S):

$$(2) \qquad ds^2 = \left(1 - \frac{R}{R'}\right)^2 G\,dv^2 + dR^2,$$

which exhibits a family of geodesics v = const. on the surface (S) and their orthogonal trajectories R = const. [Cf., Chap. III, § 9, Chap. VII, § 2, and Chap. XIII, § 2].

The differential equation of the asymptotes is:

$$\sum dl \cdot dX = 0,$$

or

$$\sum d\left(\frac{\partial x}{\partial u}\right) \cdot dX = 0.$$

We develop that equation upon appealing to the formulas (1). The coefficient of $\left(1 - \frac{R}{R'}\right) \cdot dv$ is:

$$\sum \frac{\partial x}{\partial v} d\left(\frac{\partial x}{\partial u}\right) = du \cdot \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2} + dv \cdot \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\,\partial v}.$$

Now:

$$\sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0,$$

so

$$\sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2} = -\sum \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\,\partial v} = -\frac{1}{2}\frac{\partial E}{\partial v},$$

and

$$\sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\,\partial v} = \frac{1}{2} \cdot \frac{\partial G}{\partial u}.$$

On the other hand, the coefficient of dR is:

$$\sum l\, d\left(\frac{\partial x}{\partial u}\right) = \sum l \frac{\partial^2 x}{\partial u^2}\cdot du + \sum l \frac{\partial^2 x}{\partial u\, \partial v} dv = \frac{E}{R} du,$$

so the equation of the asymptotes will be:

$$(3) \qquad \frac{1}{2}\left(1 - \frac{R}{R'}\right)\left[-\frac{\partial E}{\partial v} du\, dv + \frac{\partial G}{\partial u} dv^2\right] + \frac{E}{R} dR\, du = 0.$$

From the general properties of the developables of the congruences [Chap. VI, § 2], the curves u = const., v = const. correspond to conjugate curves on the surface (S). Hence, the coefficient of $du\, dv$ in the preceding equation will be zero:

$$(4) \qquad -\frac{1}{2}\left(1 - \frac{R}{R'}\right)\frac{\partial E}{\partial v} + \frac{E}{R}\frac{\partial R}{\partial v} = 0,$$

and equation (3) will become:

$$\frac{1}{2}\left(1 - \frac{R}{R'}\right)\frac{\partial G}{\partial u} dv^2 + \frac{E}{R}\frac{\partial R}{\partial u} du^2 = 0.$$

Similarly, on the surface (S'), one will get the condition:

$$(5) \qquad -\frac{1}{2}\left(1 - \frac{R'}{R}\right)\frac{\partial G}{\partial u} + \frac{E}{R'}\frac{\partial R'}{\partial u} = 0,$$

in such a way that the equation for the asymptotes on (S) can be written:

$$-\frac{G}{R'^2}\frac{\partial R'}{\partial u} dv^2 + \frac{E}{R^2}\frac{\partial R}{\partial u} du^2 = 0,$$

or

$$(6) \qquad G\frac{\partial(1/R')}{\partial u} dv^2 - E\frac{\partial(1/R)}{\partial u} du^2 = 0.$$

Similarly, the differential equation for the asymptotes of (S') is:

$$(7) \qquad E\frac{\partial(1/R)}{\partial v} du^2 - G\frac{\partial(1/R')}{\partial v} dv^2 = 0.$$

In order for those equations to be identical, it is necessary and sufficient that:

$$\begin{vmatrix} \frac{\partial(1/R)}{\partial v} & \frac{\partial(1/R')}{\partial v} \\ \frac{\partial(1/R)}{\partial v} & \frac{\partial(1/R')}{\partial v} \end{vmatrix} = 0;$$

i.e., that $1 / R$ is a function of $1 / R'$. *The radii of curvature are functions of each other (Ribaucour).* The surfaces that satisfy that condition are called *Weingarten surfaces or W-surfaces.* The minimal surfaces are a special case of them ($R + R' = 0$).

Suppose that we start with a surface (W) as the surface (Σ) in the preceding calculations. R' is a function of R, and the condition (5) will show that:

$$\frac{\partial \ln G}{\partial u} = \Psi\,(R)\,\frac{\partial R}{\partial u},$$

so

$$\ln G = \chi(R) + \theta(v),$$

and

$$G = e^{\chi(R)}\,e^{\theta(R)} = F(R)\,K(v).$$

Formula (2) gives the ds^2 of the developable, and it will then be written in the form:

$$ds^2 = \Theta^2(R)\,K(v)\,dv^2 + dR^2.$$

Set:

$$\sqrt{K(v)}\,dv = dV,$$

and it will become:

(8) $$dS^{\,2} = dR^2 + \Theta^2(R)\,dV^{\,2},$$

which is the characteristic form of the element of arc length for a surface of revolution with respect to the meridians and parallels. If we refer the meridian to its arc length σ then its equations will be:

$$x = \Theta(\sigma), \qquad y = 0, \qquad z = \Theta_1(\sigma),$$

and those of the surface of revolution will be:

$$x = \Theta(\sigma)\cos V, \qquad y = \Theta(\sigma)\sin V, \qquad z = \Theta_1(\sigma),$$

so due to the fact that $\Theta'^2 + \Theta_1'^2 = 1$, one will deduce that the ds^2 of the surface is:

$$ds^2 = d\sigma^2 + \Theta^2(\sigma)\,dV^{\,2}.$$

Upon setting $\sigma = R$, this will be formula (8).

One then sees that *the developables of any surface* (W) *can be mapped to a surface of revolution, such that the meridians will correspond to a family of geodesics and the parallels to their orthogonal trajectories.*

Application. – Suppose that the surface (*W*) has constant negative total curvature [Chap. IV, § 6]. Upon changing units, one can always suppose that this total curvature is equal to – 1. One will then have:

$$RR' = -1,$$

or

$$R' = -\frac{1}{R}.$$

The condition (5) will then be written:

$$\left(1+\frac{1}{R^2}\right)\frac{\partial G}{\partial u} = -\frac{2G}{R}\frac{\partial R}{\partial u},$$

or

$$\frac{\partial \ln G}{\partial u} = -\frac{2R}{R^2+1}\frac{\partial R}{\partial u} = -\frac{\partial \ln(R^2+1)}{\partial u}.$$

One concludes from this that:

$$G = \frac{1}{R^2+1}K(v),$$

and if one again sets $dV = \sqrt{K(v)}\,dv$ then one will infer from formula (2) that:

$$ds^2 = (R^2 + 1) \cdot dV^2 + dR^2.$$

Then set:

$$\Theta(R) = \sqrt{R^2+1},$$

and from the calculation above, the meridian of the surface of revolution will be such that one has:

$$x = \sqrt{\sigma^2+1},$$

so

$$\sigma = \sqrt{x^2-1}.$$

We look for *z*. It suffices to write:

$$dx^2 + dz^2 = d\sigma^2 = dx^2 \cdot \frac{x^2}{x^2-1},$$

and one concludes that:

$$dz^2 = \frac{dx^2}{x^2-1},$$

or:

$$dz = \frac{dx}{\sqrt{x^2-1}}.$$

Hence:

$$z = \ln (x + \sqrt{x^2 - 1}),$$

so

$$x + \sqrt{x^2 - 1} = e^z ;$$

thus:

$$x - \sqrt{x^2 - 1} = e^{-z}.$$

Therefore, we finally get the catenary (*chaînette*):

$$x = \tfrac{1}{2}(e^z + e^{-z}) = \cosh z$$

for the desired meridian. The catenary:

$$x = a \cosh \frac{z}{a}$$

will likewise correspond to a constant total curvature that is equal to $(-a^2)$. Therefore:

The two sheets of the development of a surface of negative constant total curvature can be mapped to an alysséide; i.e., the surface that is generated by a catenary that turns around its base.

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EXERCISES

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FIRST CHAPTER

1. – Find the instantaneous axis of rotation and sliding for the Serret trihedron. Confirm that it meets the principal normal at the central point of the ruled surface that is generated by that principal normal [Chap. V, § 8, pp. 106].

2. – Find the circular helices that osculate a skew curve at one of its points. Determine those of its helices that have the same torsion as the given curve.

3. – Determine the fundamental elements (arc length, curvature, torsion) of the locus of centers of the osculating sphere to a skew curve. Conclude from that study that in order for a curve to be a spherical curve, it is necessary and sufficient that the radius of its osculating sphere must be constant. [Cf., Chap. V, § 10, pp. 118].

5 [*sic*]. –

a. Show that in order for the principal normals of a curve (C) to also be the principal normals of a second curve (C'), it is necessary and sufficient that the radii of curvature and torsion of (C) must satisfy an identity of the form:

$$\frac{h}{R} + \frac{k}{T} = 1 \qquad (h = \text{const.}, k = \text{const.}). \tag{1}$$

Find the relation that results for (C'). Examine the case in which the osculating planes to (C) and (C') at the points that were situated on the common principal normal are rectangular.

b. Show that if one is given the relation (1) and the spherical curve (γ) that is described by the point with coordinates:

$$\xi = \alpha \cos \theta + \alpha'' \sin \theta, \qquad \eta = \beta \cos \theta + \beta'' \sin \theta, \qquad \zeta = \gamma \cos \theta + \gamma'' \sin \theta$$

$$(h = m \cos \theta, \quad k = m \sin \theta)$$

then the Serret formulas will yield:

$$\alpha, \beta, \gamma; \qquad \alpha', \beta', \gamma'; \qquad \alpha'', \beta'', \gamma''; \qquad \frac{ds}{d\sigma}$$

as functions of the arc length s of (γ), and for the curve (C), they will lead to the equations:

$$(2) \qquad x = h \int \xi \, d\sigma - k \int (\eta \, d\zeta - \zeta \, d\eta), \qquad y = \ldots, \qquad z = \ldots$$

c. Verify that for any spherical curve (γ), formulas (2) will give a curve (C) that satisfies equation (1). (Such curves are called *Bertrand curves.*) Examine the special case:

$$R = h, \quad T = k,$$

which will yield the *curves of constant curvature* and the *curves of constant torsion.*

6. – Determine a curve (C) when one knows the expressions for the radius of curvature R and the radius of torsion as functions of the arc length s. One will appeal to the Serret formulas:

$$dx = \alpha \, ds, \qquad d\alpha = \frac{\alpha'}{R} ds, \qquad d\alpha'' = \frac{\alpha''}{R} ds, \quad d\alpha' = -\left(\frac{\alpha}{R} + \frac{\alpha''}{T}\right) ds,$$

upon pursuing the following path:

a. Consider $\alpha, \alpha', \alpha''$ to be coordinates of a point of the sphere (Σ) whose center is at O and whose radius is 1. Take the unknowns to be the parameters of the rectilinear generators of (Σ) by setting [Cf., Chap. IV, § 6]:

$$1 + \alpha' = - u\,(\alpha + i\alpha''), \qquad \alpha + i\alpha'' = v\,(1 + \alpha'),$$

and one will find that u, v are two solutions to the *Ricatti equation* [Chap. V, § 10, pp. 112]:

$$dW = (MW^2 + M_0)\, ds \qquad \left[M = \frac{1}{2}\left(\frac{1}{R} + \frac{i}{T}\right), \quad M_0 = \frac{1}{2}\left(\frac{1}{R} - \frac{i}{T}\right)\right].$$

b. Let:

$$u = \frac{Au_0 + B}{Cu_0 + D}, \quad v = \frac{Av_0 + B}{Cv_0 + D} \qquad (u_0 = \text{const.}, v_0 = \text{const.})$$

be two arbitrary solutions to that Ricatti equation. Show that the points $\alpha, \alpha', \alpha''$; β, β', β''; $\gamma, \gamma', \gamma''$ that correspond to the values:

$$u_0 = 1,\ v_0 = 1, \qquad u_0 = i,\ v_0 = -i; \qquad u_0 = 0,\ v_0 = \infty$$

provide a solution to the problem, and show how one can deduce the most general solution from it. – Conclude from this that there is an infinitude of curves (C) that meet the requirements of this problem and that they are all curves that can be superimposed on each other.

c. What would make the ratio R / T const? Do the calculations while supposing that R and T are constant.

d. *Remark*. – Upon considering $\alpha, \alpha', \alpha''$ to be the direction cosines of a given direction with respect to three rectangular coordinate axes, any change of coordinates, or (what amounts to the same thing) any rotation around the origin, will translate into the same projective transformation that is performed on u and v. The point at infinity in the direction considered will then be subjected to the most general projective transformation in the plane at infinity that leaves the imaginary circle at infinity invariant.

CHAPTER II

7. – Consider the surface S that is the locus of diametral circular sections of a family of homofocal ellipsoids. Determine the orthogonal trajectories on S of the circular sections that generate it.

8. – Determine all of the conformal representations of a sphere on the plane. Find all of the ones that give the known systems of cartographic projections (e.g., stereographic projection, Mercator projection).

9. – Suppose that the coordinate curves of a surface S are rectangular. Let MU and MV be their tangents, and let φ_0 be the angle (MU, MT). Calculate the expressions r_1 and r_1 in formula (9) [page 35]:

$$\frac{\sin\theta}{R} - \frac{d\varphi}{ds} = r_1 \frac{du}{ds} + r_2 \frac{dv}{ds}.$$

Generalize that, while supposing that the coordinates u and v are arbitrary.

10. – Establish the fundamental formulas that give $\frac{\cos\theta}{R}$, $\frac{\sin\theta}{R}$ by deducing the first terms in the series developments [Chap. I, § 5, pp. 7] of the coordinates of a point of the curve when referred to the trihedron $M \cdot TPB$ [Chap. II, § 4, pp. 28], and the series developments [that are deduced from $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$] of the coordinates of a point on the curve when it is referred to the trihedron $M \cdot TN'N$ [Chap. II, § 4, pp. 29]. – It will suffice to calculate the terms up to degree two in ds.

11. – A surface (S) is assumed to be defined to be the envelope of a family of surfaces (Σ_{uv}) that are given by an equation of the form $F(x, y, z; u, v) = 0$, in such a way that u, v are the curvilinear coordinates of a running point M on (S). Any curve (C) that is traced on (S) will then correspond to a family of ∞^1 surfaces (Σ_{uv}), each of which cuts the infinitely-close surface along a characteristic. Let (K) be those of the characteristics that pass through the point M of (C). Show that there is reciprocity between the directions of the tangents to (C) and (K) at M. – Examine the case in which the surfaces (Σ_{uv}) are planes.

CHAPTER III

12. – Consider the surface:

$$x = \frac{c^2 - b^2}{bc} \cdot \frac{uv}{u - v}, \qquad y = \frac{\sqrt{c^2 - b^2}}{b} \cdot \frac{v\sqrt{b^2 - u^2}}{u + v}, \qquad z = \frac{\sqrt{c^2 - b^2}}{c} \cdot \frac{u\sqrt{v^2 - c^2}}{u + v}.$$

Determine its lines of curvature and calculate the radii of principal curvature.

13. – Show that the surfaces:

$$e^{m(z - z_0)} = \cos m\,(x - x_0) \cos m\,(y - y_0)$$

are surfaces of translation whose two families of generators are plane curves that are situated in rectangular planes (parallel to zOx and zOy), and are such that the planar generators that pass through an arbitrary point of the surface are tangents to the conjugate diameters that are equal to the indicatrix there. – Examine the lines of curvature of those surfaces.

14. – Consider the surface:

$$x = \frac{1}{2}\int (1 - u^2)\, f(u)\, du + \frac{1}{2}\int (1 - v^2)\varphi(v)\, dv,$$

$$y = \frac{i}{2}\int (1 + u^2)\, f(u)\, du - \frac{i}{2}\int (1 + v^2)\varphi(v)\, dv,$$

$$z = \int u\, f(u)\, du + \int v\,\varphi(v)\, dv.$$

Calculate the radii of principal curvature and the coordinates of the centers of principal curvatures. Construct the differential equation for the lines of curvature and the asymptotic lines. Study the lines of curvature by taking:

$$f(u) = \frac{2m^2}{(m^2 + u^2)^2}, \qquad \varphi(v) = \frac{2m^2}{(m^2 + v^2)^2},$$

and introducing new coordinates by means of the formulas:

$$u = m \tan \frac{\lambda + i\mu}{2}, \qquad v = m \tan \frac{\lambda - i\mu}{2}.$$

15. – In rectangular coordinates, suppose that one has the equations:

$$x = \tfrac{1}{2} e^u \cos(v - \alpha) + \tfrac{1}{2} e^{-u} \cos(v + \alpha),$$

$$y = \tfrac{1}{2}e^u \sin(v - \alpha) + \tfrac{1}{2}e^{-u} \sin(v + \alpha),$$
$$z = u \cos \alpha + v \sin \alpha.$$

a. For each value of α, those formulas will define a surface S_α. Indicate a way of generating that surface. What are S_0 and $S_{\pi/2}$, in particular?

b. Consider two of those surfaces S_α and S_β, and make them correspond point-by-point in such a manner that the tangent planes to the corresponding points will be parallel. Prove that the tangents to the two corresponding curves that are drawn at two homologous points will define a constant angle.

c. Find the lines of curvature and the asymptotic lines of S_α and find a geometric property of the curves to which they correspond on S_0 under the preceding transformation. What will happen for $\alpha = \pi / 2$?

16. – Study the surfaces whose lines of curvature of one system are situated on concentric spheres. What can one say about the lines of curvature of the other system?

17. –

a. If the coordinate curves u = const., v = const. on a surface (S) are asymptotic lines of that surface, and if λ, μ, υ are the direction cosines of the normal to (S) at an arbitrary point of (S) then show that there exists a function θ such that one will have:

$$dx = \theta\left[\mu\left(\frac{\partial\upsilon}{\partial u}du - \frac{\partial\upsilon}{\partial v}dv\right) - \upsilon\left(\frac{\partial\mu}{\partial u}du - \frac{\partial\mu}{\partial v}dv\right)\right],$$

$$dy = \theta\left[\nu\left(\frac{\partial\lambda}{\partial u}du - \frac{\partial\lambda}{\partial v}dv\right) - \lambda\left(\frac{\partial\upsilon}{\partial u}du - \frac{\partial\upsilon}{\partial v}dv\right)\right],$$

$$dz = \theta\left[\lambda\left(\frac{\partial\mu}{\partial u}du - \frac{\partial\mu}{\partial v}dv\right) - \mu\left(\frac{\partial\lambda}{\partial u}du - \frac{\partial\lambda}{\partial v}dv\right)\right].$$

b. Find the ds^2 of the surface, the equations of the lines of curvature, and the equation for the radii of principal curvature when one starts with those formulas. Calculate the torsion of the asymptotic lines and show that it is expressed by means of only the radii of principal curvature.

c. If one sets:

$$l = \lambda\sqrt{\theta}, \qquad m = \mu\sqrt{\theta}, \qquad n = \upsilon\sqrt{\theta}$$

then one will get the *Lelieuvre formulas.* Show that l, m, n are three particular solutions of the same partial differential equations of the form $\frac{\partial^2 \omega}{\partial u \, \partial v} = K\omega$.

CHAPTER IV

18. – Establish the integrability conditions that couple the fundamental invariants while supposing that the surface is referred to its lines of curvature.

19. – Same question, but while supposing that the surface is referred to a family of geodesics and their orthogonal trajectories. Express the total curvature and the differential form $ds / R_g - d\varphi_0$ [Chap. II, pp. 34; Chap. III, pp. 55] as functions of the quantity H, and then recover the formula of Ossian Bonnet [Chap. IV, pp. 75].

20. – Find the integrability conditions that give the expression for the total curvature while supposing that the coordinates are arbitrary.

21. – Discuss the form of the meridian of the surfaces of constant total curvature when that curvature is either positive or negative.

22. –

a. The equations of the pseudo-sphere are [Chap. IV, pp. 81]:

$$x = R \cos\theta \cos\varphi, \qquad y = R\cos\theta \sin\varphi, \qquad z = R\left[\ln\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \sin\theta\right] \quad (1 < \theta < \pi / 2).$$

One will get one conformal representation of that surface on a half-plane by setting:

$$X = m\varphi, \qquad Y = \frac{m}{\cos\theta} \qquad (m \text{ positive constant, thus } Y > 0).$$

On the other hand, upon setting:

$$u = X + iY, \qquad v = X - iY,$$

one will reduce ds^2 to a form of type:

$$ds^2 = -4l^2 \frac{du\,dv}{(u-v)^2}.$$

b. Upon appealing to the coordinates u, v, find all of the transformations of the points of the surface that preserve the arc length. If one interprets this on the plane (X, Y) then one will find that leave the X-axis invariant, and that they change every circle into a circle.

c. With that same conformal representation, the geodesic lines of the pseudo-sphere are represented by the semi-circles that have their centers on the X-axis and are situated in the half-plane that is bounded by that axis and extends in the direction of positive Y.

d. Up to the factor l, the distance between two points is ln (M_1, M_2, A_1, A_2), if M_1, M_2 denote the homologues of the points in the XY-plane, and A_1, A_2 denote the points where the X-axis is cut by the circle that is the image of the geodesic that joins the two points. – The points of the X-axis play the role of points at infinity. – Two pairs of points whose separation distance is the same can be made to coincide by a displacement of the surface onto itself that is defined by the transformations that were found.

CHAPTER V

23. – Find the contact points of the isotropic planes that are drawn through an arbitrary generator of a ruled surface. What relationship do they have to the central point and the distribution parameter?

24. – Find the ruled surfaces whose asymptotic lines intersect equal segments on the generators.

25. – Find the ruled surfaces whose lines of curvature intersect equal segments on the generators.

26. – Find the lines of curvature and the geodesic lines of the developable that is the helicoid.

27. – Show that the lines of an arbitrary surface (S) for which $ds - R_g\, d\varphi_0 = 0$ (with the same notations as in exercise 9) are characterized by the property that if one draws a tangent to the curve v = constant through each of the points of one of them then the ruled surface that one obtains will have the line in question for its line of striction.

28. – Given a surface (S) and a curve (C) on that surface, consider the ruled surface (G) that is generated by the normals MN that are drawn to (S) at the various points M of (C). The central point of MN is called the *metacenter* of (S), which corresponds to the point (M) and to the tangent MT of (C).

a. Determine that metacenter, the asymptotic plane, and the distribution parameter. Discuss the variation of the metacenter when the curve (C) varies while always passing through M.

b. Show that the metacenter is the center of curvature of the cross-section of the cylinder that is circumscribed by (*S*) and whose generators are perpendicular to the asymptotic plane of *G*.

c. Suppose that one has several surfaces (*S*), and that one has endowed each of them with a numerical coefficient *a*. Consider the points *M* on those various surfaces (which are taken on each surface) to be homologous when the tangent planes to those various surfaces are parallel. Let M_0 be the center of the proportional distances of one such system of homologous points *M*, and relative to the system of coefficients *a*. Let (S_0) be the surface that is the locus of point M_0 . Show that it corresponds to each of the surfaces (*S*) by parallel tangent planes, and that if I_0 is the metacenter of (S_0) that corresponds to the various metacenters *I* of the surfaces (*S*) that are found to be associated under the correspondence considered then one will have:

$$(\sum a) \cdot M_0 I_0 = \sum (a \cdot MI).$$

29. – Suppose that one is given a skew curve (*R*) that is an edge of regression of a developable (Δ). Each of the generators (*G*) of such a surface is perpendicular to a tangent plane (*P*) to (Δ), and the point at which (*G*) and (*P*) meet is the central point of (*G*). Therefore, let (Σ) be one of its ruled surfaces, so each of the isotropic planes that pass through one if its generators will envelop a developable. Show that the locus of the midpoints of the segments whose extremities describe the edges of regression of those two developables, independently of each other, is a minimal surface that is inscribed in (Δ).

30. –

a. Construct the equations for the radii of principal curvature of a skew ruled surface (*S*) with the expressions for ds^2 and the form Ψ that were employed in § 11 of Chapter V.

b. One then deduces the relation:

$$KM = [\varphi(v) - PT]\sqrt{KT} - K'T\sqrt{1-KT},$$

in which:

$$M = \frac{1}{R_1} + \frac{1}{R_2}, \qquad T = \frac{1}{\sqrt{-R_1R_2}}.$$

Conclude from this that if the radii of principal curvature R_1, R_2 are functions of each other [Cf., Chap. XIII, § 8] then *P*, *K*, and $\varphi(v)$ will be constants.

c. Show that if that were true then the surface (*S*) would be a ruled helicoid or a skew surface of revolution.

CHAPTER VI

31. – Consider the congruence of tangents that are common to the two surfaces:

$$x^2 + y^2 = 2az, \qquad x^2 + y^2 = -2az.$$

Determine the developables of that congruence. Study their edges of regression, their contact curves, and their traces on the plane $z = 0$.

32. – If the two focal multiplicities of a congruence are isotropic developables (viz., an isotropic congruence) then all of the ruled surfaces that pass through the same line of the congruence will have the same central point and the same distribution parameter. The plane that is perpendicular to each line its of the congruence that is drawn at an equal distance to the two focal points will envelope a minimal surface. One can then obtain the most general minimal surface.

33. – Suppose that the rays (D) and (D') of two congruences correspond in such a manner that two corresponding rays will be parallel. If the developables of the two congruences correspond then the focal planes of (D) will be parallel to those of (D'). The lines (Δ), (Δ') that join the corresponding focal points will cut at a point M. The locus of that point admits (Δ) and (Δ') for conjugate tangents, and the conjugate curves that are enveloped by those lines will correspond to the developables of the two congruences.

CHAPTER VII

34. – Study the congruences that are composed of lines that are tangent to a sphere and normals to the same surface. Study the surfaces that are normal to the lines of such a congruence and their lines of curvature.

35. – Study the congruence that is composed of lines that are normal to a surface, one of whose families of lines of curvature is situated on concentric spheres.

36. – Show that in the case where one of the sheets of the developable is a cylinder or a cone, milling surfaces can be defined by the motion of a profile plane of invariable form whose plane remains constantly normal to a cylinder or a cone. Specify the motion of that profile. Determine whether one can say something analogous for the general milling surfaces.

37. – Show that the lines tangent to two homofocal quadrics constitute a normal congruence. If one makes all of those lines (when considered to be light rays) reflect from another quadric that is homofocal to the first two then what will be the focal multiplicities of that second congruence?

38. – Suppose that one is given two homofocal surfaces of degree two and a plane (P). If one draws tangent planes to the two surfaces through the lines (d') of the plane (P) then the lines (d) that join the corresponding contact points will be normal to a family of parallel surfaces. Let (δ) be the line that contains the poles of the plane (P) with respect to the two homofocal quadrics, and let (d') be the line of the plane (P) that corresponds to a line (d) of the congruence of normals considered. The plane that is drawn through (δ) perpendicular to (d') will cut (d) at a point m. The locus of the point m will be one of the desired surfaces; it is a cyclide. The developables of the congruence will cut out conjugate nets on the homofocal nets.

39. – Consider the congruence of lines in space on which three planes define a tri-rectangular trihedron that determines invariable segments. Prove that it is a normal congruence and determine the normal surfaces to the lines of the congruence. Determine the focal points on any of those lines. Determine the director cones of the developable of the congruence.

40. – Prove that there exist (isogonal) congruences such that the focal planes define a constant dihedron. What is the property of the edges of regression of the developables of the congruence with respect to the sheets of the focal surface that contain them? Find the differential equation of those curves on the focal surface, which is assumed to be given. What can one say in the cases where one of the sheets of the focal multiplicity is a developable, a curve, or a sphere?

41. – Consider a family of spheres whose locus of centers ω is a plane curve (C) and whose radii are proportional to the distances from the centers ω to a fixed line (Δ) in the plane of the curve (C). Show that all of the lines of curvature in the envelope of those spheres will be planar. What can one say about the planes of those lines of curvature? – Conversely, how can one get all canal surfaces whose lines of curvature are all planar?

CHAPTER VIII

42. – Suppose that one is given two curves (C), (C_1). Find all of the surfaces (S) on which the contact curves of the cones that are circumscribed by (S) and have their summits on (C) and (C_1) will form a conjugate net. Upon defining (C) and (C_1) by the equations:

$$x = f(\lambda), \qquad y = g(\lambda), \qquad z = h(\lambda), \qquad t = k(\lambda),$$
$$x = \varphi(\mu), \qquad y = \psi(\mu), \qquad z = \chi(\mu), \qquad t = \theta(\mu),$$

the most general surface that meets the requirements will be defined by the equations:

$$x = \int A(\lambda) f(\lambda)\, d\lambda + \int B(\mu)\, \varphi(\mu)\, d\mu,$$
$$y = \int A(\lambda)\, g(\lambda)\, d\lambda + \int B(\mu)\, \psi(\mu)\, d\mu,$$
$$z = \int A(\lambda)\, h(\lambda)\, d\lambda + \int B(\mu)\, \chi(\mu)\, d\mu,$$

$$t = \int A(\lambda)\, k(\lambda)\, d\lambda \;\; + \int B(\mu)\; \theta(\mu)\; d\mu.$$

Geometrically interpret the formulas that were obtained in such a fashion that one can find a geometric definition of these surfaces. Transform the results obtained by duality.

43. – Let (Σ) be a sphere whose center is at O and whose radius is equal to unity. Let (S) be any sphere and let (S') be its polar reciprocal with respect to (Σ). Let M be any point of (S) and let (P) be the tangent plane at that point. Let M' and (P') be the point and the tangent plane to (S') that correspond to (P) and M, resp., as polar reciprocals. Now, consider the congruence (K) of lines MM' and the congruence (K') of the intersections of the planes (P) and (P'). Show that their developables correspond, and that the developables of (K) cut out conjugate nets on (S) and (S'). How do the developables of (K) cut (Σ)? – Determine (S) in such a manner that (K) is a normal congruence. What can one then say about the developables of (K) and the surface (S)?

44. – Suppose that (C) is a skew curve through a fixed point O and draw segments OM that are equipollent to the various chords of (C). The locus of points M is a surface (S_0). Through each point M of that surface, draw the parallel (Δ) to the intersection of the osculating planes to (C) that is drawn through the points P and P_1 of (C) such that PP_1 is equipollent to OM. Let (S_1) and (S_2) be two sheets of the focal surface of the congruence of lines (Δ).

a. Determine (S_1) and (S_2), their ds^2, and their $\sum l\, d^2 x$. Show that the asymptotes will correspond on (S_1) and (S_2). What are the curves of (S_0) that they correspond to?

b. Find a necessary and sufficient condition that (C) must satisfy in order for the congruence of lines (Δ) to be a normal congruence, and then find a normal surface. Show that the radii of curvature of (Σ) are functions of each other.

c. While remaining in that case, refer the ds^2 of (S_1) to the geodesics that are tangent to the lines (Δ) and to their orthogonal trajectories. Conclude from this that (S_1) can be mapped to a paraboloid of revolution.

N. B. – The last two parts of this exercise are attached to the end of Chapter XIII.

CHAPTER IX

45. – Consider two rectangular planes and all of the lines such that the segment that is intersected on each of them by the preceding planes has a constant length. Find the normal congruences of the complex of those lines.

46. – Consider three planes that define a tri-rectangular trihedron and the lines that are such that the ratio of the segments that are determined by those three planes on each of them are constant. Find the surfaces whose normals belong to the complex of those

lines. Among those surfaces, there is an infinitude of second-order surfaces that admit the three given planes as symmetry planes. The preceding complex is that of the normals to a family of homofocal quadrics, or to a family of quadrics that are homothetic with respect to their center (viz., *Chasles complex*).

CHAPTER X

47. – Study the asymptotes of third-order ruled surfaces. Show that in the general case, they will be fourth-order unicursals, and that each generator will meet an asymptote at two harmonic conjugate points with respect to the points where the generator is supported by the double line and the singular line.

Examine the case in which the surface is a Cayley surface with a unique direction.

N. B. – As one knows, the equation of a skew ruled surface can reduce to either the form:

$$x^2 z - y^2 t = 0 \qquad \text{(general ruled surface)}$$

or the form:

$$x^3 + 2xyz - y^2 t = 0 \qquad \text{(Cayley surface)}$$

by a convenient choice of reference tetrahedron.

48. – Determine the asymptotes of the *Steiner surface.* For which curves is it represented in the parametric representation of the surface?

N. B. – One knows that the equations of a Steiner surface have the form:

$$x = \frac{f(u,v)}{k(u,v)}, \quad y = \frac{g(u,v)}{k(u,v)}, \quad z = \frac{h(u,v)}{k(u,v)},$$

in which *f, g, h, k* are four arbitrary second-degree polynomials. Upon excluding the special cases, one can reduce it to the form:

$$x = \frac{2u}{u^2+v^2+2}, \qquad y = \frac{2v}{u^2+v^2+2}, \qquad z = \frac{u^2-v^2}{u^2+v^2+2}$$

by a projective transformation and a convenient choice of parameters. Any section of the surface by a tangent plane will decompose into two conics. Upon interpreting *u, v* as the rectangular coordinates in a plane, the preceding formulas will realize the representation of the surface on a plane.

49. – Determine the most general canal surface whose lines of curvature are all spherical. Show that those lines of curvature can be determined without integration.

50. – What can one say about the determination of the lines of curvature of a canal surface that is envelope of ∞^1 spheres that cut a fixed sphere at a constant angle?

51. – Determine the ruled surfaces of a linear complex that admits a given line for its asymptotic line. Show that all of their asymptotes can be determined without integration, and that they will be algebraic if the given curve is algebraic.

CHAPTER XI

52. – Study the congruence of lines that are defined by the equations:

$$A\lambda + B\mu + C = 0, \qquad A_1\lambda + B_1\mu + C_1 = 0,$$

in which A, B, C, A_1, B_1, C_1 are linear functions of the coordinates and λ, μ are arbitrary parameters. In particular, discuss the questions of the lines that pass through a point, the lines that meet a fixed line, the lines that are situated in a plane, and focal multiplicities.

53. – Prove the results that were stated at the end of § 3 of this Chapter.

54. – Prove, by calculation, the properties of the Lie transformation that were stated at the end of § 4 of this chapter.

CHAPTER XII

55. – Consider a family of ∞^1 paraboloids (P) that have the same principal planes. How must one choose those paraboloids in order for the congruence of rectilinear generators of the same system to be a normal congruence for all of those paraboloids? Show that the paraboloids (P) will then constitute one of the three families of a triply-orthogonal system and find the other two families. Show that one can choose the paraboloids (P), more especially, in such a manner that one of those other families is again composed of paraboloids and give the geometric significance of the two families of paraboloids in that case.

CHAPTER XIII

56. – Let (S) be an arbitrary surface, and let (Π) be an arbitrary plane. Consider all of the spheres (U) that have their centers on (S) and cut the plane (Π) at a constant angle φ such that one will have $\cos \varphi = 1 / k$. Let (S') be the surface that is deduced from (S) by reducing the ordinates of (S) perpendicular to (Π) by the ratio $\sqrt{1-k^2} / 1$. The spheres (U) envelope a surface with two sheets. Show that their lines of curvature correspond point-wise with the ones on (S'). Examine the case in which (S) has degree two.

57. – Describe a circle (K) in the tangent plane to a surface (S) at each point M whose radius is equal to a given constant.

a. Determine the families of ∞^1 circles (*K*) that generate a surface on which those circles are lines of curvature. Find the loci of the centers of the spheres that have such a surface for their envelope.

b. Find the necessary and sufficient condition for the circles (*K*) to define a cyclic system. If that condition is assumed to be satisfied then let (S_1) be one of the normal surfaces to the circles (*K*). Show that the lines of curvature of (S_1) will correspond to the ones on (*S*) when one makes each point *M* of (*S*) correspond to the point M_1 of the corresponding circle where (S_1) is normal to (*K*).

c. Show that (S_1) has constant total curvature, and that the line congruence that has (*S*), (S_1) for its focal surfaces is a normal congruence.

d. Let *C* be one of the centers of principal curvature of (*S*) at *M*, and let C_1 be the center of principal curvature of (S_1) at M_1, which corresponds to *C*. Study the congruence of lines CC_1 .

58. – Given a surface (*S*), let (*C*) denote any of the lines of curvature of one of the family, and let (C') denote any of the lines of curvature of the other family, in such a way that a curve (*C*) and a curve (C') cross at a point *M* of (*S*). Let ω, ω' be the centers of principal curvature that correspond to those two curves, and let *G*, G' be the centers of geodesic curvature of those two curves.

a. What can one say about the congruences that are defined by the four lines *MG*, MG', $G\omega$, $G'\omega'$, respectively?

b. Let (γ) be the osculating circle to (*C*) at *M*. Prove that (γ) generates a canal surface when *M* describes a curve (C'). Find the spheres whose envelope is that canal surface.

c. Show that if (*S*) belongs to one of the families of a triply-orthogonal system then the osculating circles to the orthogonal trajectories of the surfaces of that family that are constructed at the various points of (*S*) will define a cyclic system.

59. – Let *O* be a fixed point, and let (*S*) be an arbitrary surface. Draw the tangent plane (*P*) to (*S*) at an arbitrary point, and drop a perpendicular to (*P*) from *O*; let *H* be its foot.

a. Find the curves of (*S*) that admit *MH* for their normal at each of their points *M*.

b. Let *HI* be the midpoint of the triangle *OHM*. The congruence of lines *HI* is a normal congruence. Find the surfaces that are normal to all of those lines. Show that their lines of curvature will correspond to a net of conjugate curves that are described by *M* on (*S*).

c. Let K be the point at which the plane perpendicular to MO meets MH, and let (γ) be the circle with its center at K that passes through O and is situated in the plane MOK. Show that the circles (γ) form a cyclic system.

60. – Describe a sphere (Σ) that is tangent to the plane xOy at each point M of the paraboloid:

$$(P) \qquad xy - az = 0.$$

Let A be the contact point of (Σ) with that plane, and let B be the second contact point of (Σ) with its envelope.

a. What sort of curve on (P) must M describe in order for AB to generate a developable? Those curves will form a conjugate net on (P), and their tangents at each point M will be perpendicular to the focal planes of the congruence that is generated by AB.

b. Determine the lines of curvature of the envelope of (Σ). The normals that are drawn to the envelope along each line of curvature will cut out a conjugate net on (P).

c. Consider the circle (C) that is normal to (Σ) at A and B. Show that there is an infinitude of surfaces that are normal to all of the circles (C) and determine them.

d. Show that those surfaces form one of the families of a triply-orthogonal system and succeed in determining that system.

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