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On the mechanical interpretation of infinitesimal contact transformations

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Lie has briefly pointed out, in various passages in his *Géometrie des transformations de contact,* that the propagation of a wave motion, conforming to the law of Huyghens, is the image of a one-parameter group of contact transformations; i.e., of an infinitesimal contact transformation. It seemed to be of interest to us to discuss this idea along with the implications that it entails. We have dealt with three-dimensional space and indicated how the results thus obtained can be extended to n-dimensional spaces. Here are the main results:

The mode of propagation is assumed to be defined by a system of ∞^3 characteristic wave surfaces that are associated with the ∞^3 points (x, y, z) of space. Each of them represents the limiting form that is approached by the surface that consists of all points that are reached after a time interval dt by a disturbance that is produced at the corresponding point (x, y, z). Suppose that the origin of the coordinates has been transported to that point, and that the equation of a tangent plane to that surface is taken in the form:

(1) $$pX + qY - Z - w = 0,$$

in such a way that (p, q, w) are the current tangential coordinates for that surface. The system of characteristic surfaces is then defined by an equation of the form:

(2) $$w = W(x, y, z, p, q),$$

and one may say that this equation defines the mode of propagation.

If one imagines an arbitrary initial wave then the successive forms of the variable wave that result from it are the transforms of the original wave by the various transformations of a group of contact transformations in the parameter t, which represents time. This group is defined by the classical equations:

(3) $$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial W}{\partial p}, \quad \dfrac{dy}{dt} = \dfrac{\partial W}{\partial q}, \quad \dfrac{dz}{dt} = p\dfrac{\partial W}{\partial p} q\dfrac{\partial W}{\partial q} - W, \\ \dfrac{dp}{dt} = -\dfrac{\partial W}{\partial x}, \qquad \dfrac{dq}{dt} = -\dfrac{\partial W}{\partial y} - q\dfrac{\partial W}{\partial z}, \end{cases}$$

where W is the function that appears in the tangential equation of the characteristic surface.

Upon integrating the system (3) one completely determines the mode of propagation in question. However, one may also determine it by seeking the families of waves, i.e., the family of successive waves that issue from an arbitrary initial wave. One such family will be given by an equation:

$$t = f(x, y, z), \tag{4}$$

in such a way that one must find a function t of the independent variables (x, y, z). It is defined by the partial differential equation:

$$\frac{\partial t}{\partial z} W\left(x, y, z, -\frac{\partial t/\partial x}{\partial t/\partial z}, -\frac{\partial t/\partial y}{\partial t/\partial z}\right) + 1 = 0, \tag{5}$$

whose integration is therefore equivalent to the integration of the system (3).

From this, one obtains an extremely simple geometrical theory of first-order partial differential equations that comprises not only the theory of complete integrals and the theorem of Jacobi, but also the theory of characteristics. The only peculiarity of equation (5) is that it does not explicitly contain the unknown function t.

One arrives at these results in a more symmetric form by supposing that the equation of the tangent plane of the characteristic surface is taken in the form:

$$\alpha X + \beta Y + \gamma X - 1 = 0. \tag{6}$$

The tangential equation is then of the form:

$$\Pi(x, y, z, \alpha, \beta, \gamma) = 1, \tag{7}$$

where one may always assume that Π is homogeneous of degree *one* in α, β, γ. Equation (7), when one sets:

$$\frac{\partial t}{\partial x} = \alpha, \qquad \frac{\partial t}{\partial y} = \beta, \qquad \frac{\partial t}{\partial z} = \gamma, \tag{8}$$

is also the partial differential equation of the family of waves. The equations of the group of contact transformations take the known homogeneous form, which is analogous to that of the canonical Hamilton equations:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \Pi}{\partial \alpha}, & \dfrac{dy}{dt} = \dfrac{\partial \Pi}{\partial \beta}, & \dfrac{dz}{dt} = \dfrac{\partial \Pi}{\partial \gamma}, \\ \dfrac{d\alpha}{dt} = -\dfrac{\partial \Pi}{\partial x}, & \dfrac{d\beta}{dt} = -\dfrac{\partial \Pi}{\partial y}, & \dfrac{d\gamma}{dt} = -\dfrac{\partial \Pi}{\partial z}. \end{cases} \tag{9}$$

The intervention of the group of contact transformations in the wave theory leads to the consideration of the trajectories of propagation, which will be defined by the system (3), for example, upon considering p, q to be the auxiliary unknowns. One defines these trajectories directly by a differential system that is analogous to the Lagrange equations in mechanics by defining the characteristic wave surfaces by their general pointwise equation:

$$\Omega(x, y, z, X, Y, Z) = 1, \tag{10}$$

where one may suppose that Ω is homogeneous of degree *one* in X, Y, Z. The differential system of the trajectory is then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial x'}\right) - \dfrac{\partial \Omega}{\partial x} = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial y'}\right) - \dfrac{\partial \Omega}{\partial y} = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial z'}\right) - \dfrac{\partial \Omega}{\partial z} = 0, \end{cases} \tag{11}$$

with the condition equation:

$$\Omega(x, y, z, x', y', z') = 1. \tag{12}$$

It expresses the notion that the variation of the integral:

$$\int \Omega(x, y, z, dx, dy, dz) \tag{13}$$

is null, and that the time t is precisely the corresponding value:

$$t = \int \Omega(x, y, z, dx, dy, dz) \tag{14}$$

of that integral. One may, moreover, interpret the integral (13), when taken along an arc of an arbitrary curve, as the time that a disturbance takes to propagate from one extremity of that arc to the other when one follows the curve.

We thus have a geometric interpretation of a very general problem in the calculus of variations. The classical condition for the minimum may then be stated in the following form: Let M be an arbitrary point of a trajectory. This point is the origin of an infinitesimally small elementary wave:

$$\Omega(x, y, z, dx, dy, dz) = dt. \tag{15}$$

The trajectory, when one starts at M, must pierce that elementary wave at a point where its two curvatures have the same sign and it is concave towards M. The arcs of the corresponding trajectories therefore always have a minimum for the duration of the propagation when the characteristic wave surfaces always have the same sign for their curvatures and are always concave towards their respective origins. Of course, this is with the reservation that the arc considered does not contain pairs of conjugate points (in the sense of Weierstrass).

The preceding theory thus sheds light on a most interesting new day, not only on the diverse aspects of the theory of partial differential equations, but also its relationship with the calculus of variations.

It also makes a useful contribution to all of the questions in which the integral (13) is involved: brachistochrones, the equilibrium of strings, geodesic lines, the general problem of dynamics, etc. In these diverse cases, it intuitively restores the analogous theorems of Thomson and Tait, because they express only that the arcs of the trajectories connect two waves of the same family that correspond to equal times and that at each point of one wave the direction of the trajectory is linked with that of the tangent plane to the wave by the geometric relation that results from the first three formulas (3) or (9).

Lie has pointed out ([1]) the fact that the integration of the partial differential equation of mechanics:

$$\left(\frac{\partial S}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial S}{\partial x_n}\right)^2 - 2U(x_1, \ldots, x_n) - 2h = 0, \tag{16}$$

comes down to the determination of the group of contact transformations that has the characteristic function:

$$\frac{\sqrt{p_1^2 + \cdots + p_n^2}}{\sqrt{2(U+h)}}. \tag{17}$$

We recover this result in a more general form, and with its true origin, which is in the principle of least action.

If the *vis viva* $\dfrac{2T(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)}{dt^2}$ depends only upon the coordinates of the system, and not on time, and the same is true for the force function $U(x_1, \ldots, x_n)$, then the trajectories are the same for the mode of wave propagation in which the characteristic wave surface has the general pointwise equation:

$$2\sqrt{U(x_1, \cdots, x_n)T(x_1, \cdots, x_n \mid X_1, \cdots, X_n)} = 1. \tag{18}$$

However, this corresponds to the time t of this mode of propagation, and it is not generally the time t of the dynamical motion, but the action:

$$\tau = 2\int \sqrt{U(x_1, \cdots, x_n)T(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)}, \tag{19}$$

in such a way that one has:

([1]) *Leipziger Berichte*, v. XLI, pp. 145.

(20) $$dt = \frac{d\tau}{2U(x_1, \cdots, x_n)}.$$

In this statement, one assumes that one has given a fixed value to the constant of *vis viva*, and that one has entered it into the function U, in such a way that the equation for the *vis viva* will be:

(21) $$T(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = U(x_1, \ldots, x_n)\, dt^2.$$

This formal identity between the trajectories of the problems of dynamics and certain problems of wave propagation is certainly curious, although the difference between the laws that govern the trajectories robs it of some of its interest. The classical parallel between the theory of emission and the theory of undulations may be regarded as a special case if one supposes that in the theory of emission the luminous corpuscles obey laws that are analogous to the ones in our dynamics.

In another work, we shall study the consequences of pursuing this general parallelism for the integration of the equations of mechanics. There will likewise be good reason to study the problem of refraction and the propagation of a disturbance in a medium whose nature varies with time. In this note we would like to limit ourselves to the simplest facts.

I. – WAVES AND CONTACT TRANSFROMATIONS.

1. Consider a well-defined elastic medium and suppose that a disturbance of a specified nature propagates in this medium. We first assume that if this disturbance is produced with its origin at an isolated point A of the medium then after each time interval t it has reached all of the points of a surface $\Phi_{A,t}$ whose form is well-defined for each point A and each duration t.

We then assume that if the disturbance is produced with its origin at any point of a curve C or surface S then after each time interval t it reaches all of the points of the multiplicity M that envelops the surfaces $\Phi_{A,t}$ that correspond to the various points of C (S, resp.).

If we say *wave* when we mean the set of points that are reached by the disturbance at the same instant then C or S is the initial wave and M is what this initial wave becomes after a time interval t. The principle that we have assumed may be called the *principle of the enveloping wave.*

This has an immediate interpretation in the theory of contact transformations. Indeed, for each value of t the system of surfaces $\Phi_{A,t}$ defines a contact transformation T_t, and M is the transform of the initial wave (C or S) by this transformation.

Therefore, *to each elastic medium and each type of disturbance that may be produced in this medium there corresponds a family of contact transformations T_t, in such a way that after a time interval t any initial wave becomes a new wave that is the transform of the initial wave by the transformation T_t.*

Let x_0, y_0, z_0 be the coordinates of an arbitrary point A and let x, y, z be the current coordinates. The surface $\Phi_{A,t}$ has the equation:

(1) $$\Phi(x, y, z \mid x_0, y_0, z_0 \mid t) = 0,$$

and, as one knows, this equation defines the contact transformation T_t upon adding the equations:

$$
\left\{
\begin{aligned}
&\frac{\partial\Phi}{\partial x} + p\frac{\partial\Phi}{\partial z} = 0, \qquad \frac{\partial\Phi}{\partial y} + q\frac{\partial\Phi}{\partial z} = 0, \\
&\frac{\partial\Phi}{\partial x_0} + p_0\frac{\partial\Phi}{\partial z_0} = 0, \quad \frac{\partial\Phi}{\partial y_0} + q_0\frac{\partial\Phi}{\partial z_0} = 0.
\end{aligned}
\right. \tag{2}
$$

2. Moreover, assume that the disturbances in question propagate in conformity to *Huyghens's principle:* i.e., after an arbitrary time interval t the initial wave Σ_0 becomes a certain wave Σ and continues to propagate as if this wave Σ were the initial wave when starting with this instant.

This amounts to saying that $T_{t+t'}$ is identical to the product $T_tT_{t'}$, i.e., that *the contact transformations T_t form a one-parameter group that has t for its canonical parameter.*

Moreover, under the hypotheses that were made: *To each elastic medium and each type of disturbance that might propagate in this medium there corresponds one infinitesimal contact transformation $\mathcal{T}$, and any wave that exists at an arbitrary instant is successively modified according to the contact transformations T_t of the one-parameter group that $\mathcal{T}$ generates.*

According to the well-known results of Lie, if $W(x, y, z, p, q)$ is the characteristic function of T then M_t is obtained by integrating the following equations from 0 to t with the initial values x_0, y_0, z_0, p_0, q_0 then:

$$
\left\{
\begin{aligned}
&\frac{dx}{dt} = \frac{\partial W}{\partial p}, \qquad \frac{dy}{dt} = \frac{\partial W}{\partial q}, \qquad \frac{dz}{dt} = \frac{\partial W}{\partial r}, \\
&\frac{dp}{dt} = -\frac{\partial W}{\partial x} - p\frac{\partial W}{\partial z}, \qquad \frac{dq}{dt} = -\frac{\partial W}{\partial y} - q\frac{\partial W}{\partial z}.
\end{aligned}
\right. \tag{3}
$$

The function W is therefore characteristic for the medium and the nature of the disturbances in question.

It is coupled with the following geometric considerations: Recall the surface $\Phi_{A,t}$ and construct its homothetic image with the respect to the homothety center A and homothety ratio $1/t$. When t goes to zero this homothety will go to a limiting form Ψ_A that we call the *characteristic surface of the medium with A for its origin* because the system of these surfaces Ψ_A defines, as we shall see, the mode of propagation of the disturbances under consideration.

Indeed, upon setting:

$$W_0 = W(x_0, y_0, z_0, p_0, q_0),$$

and considering p_0, q_0 to be parameters, $\Phi_{A,t}$ has the equations:

$$x = x_0 + t\frac{\partial W_0}{\partial p_0} + \ldots,$$

$$y = y_0 + t\frac{\partial W_0}{\partial q_0} + \ldots,$$

$$z = z_0 + t\left(p_0\frac{\partial W_0}{\partial p_0} + q_0\frac{\partial W_0}{\partial q_0} - W_0\right) + \ldots,$$

the unwritten terms being of higher order in t. The homothety considered will have the equations:

$$x = x_0 + \frac{\partial W_0}{\partial p_0} + \ldots,$$

$$y = y_0 + \frac{\partial W_0}{\partial q_0} + \ldots,$$

$$z = z_0 + p_0\frac{\partial W_0}{\partial p_0} + q_0\frac{\partial W_0}{\partial q_0} - W_0 + \ldots,$$

and the unwritten terms contain the fact t. The surface Ψ_A is therefore represented, *upon transporting the axes to the point A*, by the equations:

$$(4) \qquad X = \frac{\partial W_0}{\partial p_0}, \qquad Y = \frac{\partial W_0}{\partial q_0}, \qquad Z = p_0\frac{\partial W_0}{\partial p_0} + q_0\frac{\partial W_0}{\partial q_0} - W_0 .$$

If one differentiates with respect to p_0, q_0 then one concludes:

$$p_0\, dX + q_0\, dY - dZ = 0,$$

and, as result, p_0, q_0, -1 are the direction coefficients of the tangent plane to Ψ_A at the point X, Y, Z, which has the curvilinear coordinates p_0, q_0 , and the equation of the tangent plane is:

$$(5) \qquad p_0\, X + q_0\, Y - Z - W_0 = 0,$$

The characteristic function W thus corresponds to the tangential equation of the characteristic surface Ψ_A *of the medium*, in the sense that if the equation of the tangent plane to the surface at the origin A is, upon transporting the axes to A:

$$(6) \qquad w_0 = W(x_0, y_0, z_0, p_0, q_0),$$

where x_0, y_0, z_0 are the coordinates of the point A. The surfaces Ψ_A thus define the function W entirely, and, as a consequence, the mode of propagation of the disturbances considered in the medium considered.

One may further substitute for the surface Ψ_A , its homothetic image when A is taken to be the center of the homothety and dt is the homothety ratio; this is what we call the *elementary wave* that has A for its origin. These equations, which always have A for the origin, are:

$$X = \frac{\partial W_0}{\partial p_0} dt, \qquad Y = \frac{\partial W_0}{\partial q_0} dt, \qquad Z = \left(p_0 \frac{\partial W_0}{\partial p_0} + q_0 \frac{\partial W_0}{\partial q_0} - W_0 \right) dt .$$

If A' is the point of this elementary wave for which the tangent plane has the direction coefficients p_0, q_0, -1 then the components of the vector AA' are the differentials dx_0, dy_0, dz_0 for the contact element $E(x_0, y_0, z_0, p_0, q_0)$.

To obtain the differential dp_0, dq_0 it suffices to find the contact element that is common to all of the elementary waves that have their origins at the points of the contact element E that are infinitely close to A. Since the elementary wave of origin A has the equations:

$$x = x_0 + \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial p} dt,$$

$$y = y_0 + \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial q} dt,$$

$$z = z_0 + \left[p \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial p} + q \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial q} - W(x_0, y_0, z_0, p, q) \right] dt,$$

we may use the following equations to find that characteristic contact element:

$$0 = \delta x_0 + \delta \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial p} dt,$$

$$0 = \delta y_0 + \delta \frac{\partial W(x_0, y_0, z_0, p, q)}{\partial q} dt,$$

$$0 = \delta z_0 + \left[p \frac{\partial W(x_0, \cdots, p, q)}{\partial p} + q \frac{\partial W(x_0, \cdots, p, q)}{\partial q} - W(x_0, \cdots, p, q) \right] dt,$$

$$0 = p_0 \delta x_0 + q_0 \delta y_0 - \delta z_0,$$

from which one concludes, δx_0, δy_0 being arbitrary:

$$p_0 - p - \left[\frac{\partial W(x_0, \cdots, p, q)}{\partial x_0} + p_0 \frac{\partial W(x_0, \cdots, p, q)}{\partial z_0} \right] dt = 0,$$

$$q_0 - q - \left[\frac{\partial W(x_0, \cdots, p, q)}{\partial y_0} + q_0 \frac{\partial W(x_0, \cdots, p, q)}{\partial z_0} \right] dt = 0.$$

One thus sees that p, q tend to p_0, q_0 when dt tends to zero and that the principal parts of $p - p_0$, $q - q_0$ are given precisely by the same equations that define the differentials dp, dq in equations (3).

The geometrical interpretation of equations (3) by means of the elementary wave is therefore complete.

From the principle of successive approximations, one may further say that *the propagation of the disturbance comes about by successive elementary waves,* since dt is then infinitely small.

3. One may replace equations (3) for the propagation of waves with more symmetric equations that are analogous to the *canonical equations of Hamilton*. It suffices to put the tangential equation for the characteristic surface Ψ_A into a symmetric form.

In the sequel, we will write x, y, z for the coordinates of A and p, q in place of p_0, q_0. If the current tangent plane of $\Psi_{A'}$ is:

$$pX + qY - Z - w = 0, \tag{8}$$

then its tangential equation will be:

$$w = W(x, y, z, p, q). \tag{9}$$

Take the tangent plane in the form:

$$\alpha X + \beta Y + \gamma Z - \varpi = 0, \tag{10}$$

and its tangential equation may be written:

$$\varpi = \Pi(x, y, z, \alpha, \beta, \gamma), \tag{11}$$

where Π will be homogeneous of first degree in α, β, γ. One will have, moreover, the identities:

$$p = -\frac{\alpha}{\gamma}, \qquad q = -\frac{\beta}{\gamma}, \qquad w = -\frac{\varpi}{\gamma}, \tag{12}$$

$$\pi(x, y, z, \alpha, \beta, \gamma) = -\gamma W\left(x, y, z, -\frac{\alpha}{\gamma}, -\frac{\beta}{\gamma}\right), \tag{13}$$

$$W(x, y, z, p, q) = \Pi(x, y, z, p, q, -1). \tag{14}$$

One immediately concludes, in view of the homogeneity of Π and its derivatives:

$$\frac{dx}{dt} = \frac{\partial \Pi}{\partial \alpha}, \qquad \frac{dy}{dt} = \frac{\partial \Pi}{\partial \beta}, \qquad \frac{dz}{dt} = \frac{\partial \Pi}{\partial \gamma}; \tag{15}$$

and, by a simple calculation:

$$\frac{\dfrac{d\alpha}{dt} + \dfrac{\partial \Pi}{\partial x}}{\alpha} = \frac{\dfrac{d\beta}{dt} + \dfrac{\partial \Pi}{\partial y}}{\beta} = \frac{\dfrac{d\gamma}{dt} + \dfrac{\partial \Pi}{\partial z}}{\gamma} = \frac{\dfrac{d\varpi}{dt}}{\varpi}. \tag{16}$$

The last ratio is obtained by combining the other ones upon taking into account the homogeneity of Π, equation (11), and equations (15).

To obtain simple formulas, one imposes the condition:

(17) $$\varpi = 1,$$

which will give the equations:

(18) $$\frac{d\alpha}{dt} = -\frac{\partial \Pi}{\partial x}, \quad \frac{d\beta}{dt} = -\frac{\partial \Pi}{\partial y}, \quad \frac{d\gamma}{dt} = -\frac{\partial \Pi}{\partial z},$$

with the condition:

(19) $$\Pi(x, y, z, \alpha, \beta, \gamma) = 1.$$

The latter equation remains the tangential equation of Ψ_A, but, from now on, the equation of the tangent plane is written:

(20) $$\alpha X + \beta Y + \gamma Z - 1 = 0.$$

The stated equations are equations (15) and (18), where one must observe that Π is homogeneous of degree 1 in α, β, γ. It must be integrated while taking (19) into account. However, one must remark that the first integral Π = const. results from (15) and (18), and that one may replace the hypothesis $\varpi = 1$ by the more general hypothesis ϖ = const. without altering the preceding calculations. The only peculiarity that actually remains thus relates to the homogeneity of Π.

We finally remark that, due to this homogeneity, the condition (19) may be replaced by:

(21) $$\alpha\, dx + \beta\, dy + \gamma\, dz - dt = 0,$$

which also results from the geometric interpretation of dx/dt, dy/dt, dz/dt because it is then just the equation (20) of the tangent plane to Ψ_A.

4. One may further rid oneself of the hypothesis that relates to the homogeneity of Π. Indeed, suppose that the equation of the current tangent plane to the characteristic surface Ψ_A is always supposed to be written in the form:

(20) $$\alpha X + \beta Y + \gamma Z - 1 = 0.$$

so the tangential equation of that surface will be given in an arbitrary form:

(22) $$\Psi(x, y, z, \alpha, \beta, \gamma) = 0.$$

This equation is equivalent to (19), which is deduced by solving equation (22), made homogeneous, for ϖ, i.e.:

(23)
$$\bar{\Psi} = \Psi\left(x, y, z, \frac{\alpha}{\varpi}, \frac{\beta}{\varpi}, \frac{\gamma}{\varpi}\right) = 0,$$

and upon making $\varpi = 1$ in equation (11) that one obtains.

From this, one concludes that by means of (23) one has:

$$d\Pi = \frac{-\frac{\partial\bar{\Psi}}{\partial x}dx - \frac{\partial\bar{\Psi}}{\partial y}dy - \frac{\partial\bar{\Psi}}{\partial z}dz - \frac{1}{\varpi}\left[\frac{\partial\bar{\Psi}}{\partial\left(\frac{\alpha}{\varpi}\right)}d\alpha + \frac{\partial\bar{\Psi}}{\partial\left(\frac{\beta}{\varpi}\right)}d\beta + \frac{\partial\bar{\Psi}}{\partial\left(\frac{\gamma}{\varpi}\right)}d\gamma\right]}{-\frac{1}{\varpi^2}\left[\alpha\frac{\partial\bar{\Psi}}{\partial\left(\frac{\alpha}{\varpi}\right)} + \beta\frac{\partial\bar{\Psi}}{\partial\left(\frac{\beta}{\varpi}\right)} + \gamma\frac{\partial\bar{\Psi}}{\partial\left(\frac{\gamma}{\varpi}\right)}\right]}$$

identically, and consequently:

$$\frac{\partial\Pi}{\partial x} = \bar{M}\frac{\partial\bar{\Psi}}{\partial x}, \qquad \frac{\partial\Pi}{\partial y} = \bar{M}\frac{\partial\bar{\Psi}}{\partial y}, \qquad \frac{\partial\Pi}{\partial z} = \bar{M}\frac{\partial\bar{\Psi}}{\partial z},$$

$$\frac{\partial\Pi}{\partial\alpha} = \frac{\bar{M}}{\varpi}\frac{\partial\bar{\Psi}}{\partial\left(\frac{\alpha}{\varpi}\right)}, \qquad \frac{\partial\Pi}{\partial\beta} = \frac{\bar{M}}{\varpi}\frac{\partial\bar{\Psi}}{\partial\left(\frac{\beta}{\varpi}\right)}, \qquad \frac{\partial\Pi}{\partial\gamma} = \frac{\bar{M}}{\varpi}\frac{\partial\bar{\Psi}}{\partial\left(\frac{\gamma}{\varpi}\right)},$$

$$\frac{1}{\bar{M}} = \frac{1}{\varpi^2}\left[\alpha\frac{\partial\bar{\Psi}}{\partial\left(\frac{\alpha}{\varpi}\right)} + \beta\frac{\partial\bar{\Psi}}{\partial\left(\frac{\beta}{\varpi}\right)} + \gamma\frac{\partial\bar{\Psi}}{\partial\left(\frac{\gamma}{\varpi}\right)}\right].$$

Moreover, under the hypothesis (17), i.e., (19), i.e., (22), one thus has:

$$\frac{\partial\Pi}{\partial x} = M\frac{\partial\Psi}{\partial x}, \qquad \frac{\partial\Pi}{\partial y} = M\frac{\partial\Psi}{\partial y}, \qquad \frac{\partial\Pi}{\partial z} = M\frac{\partial\Psi}{\partial z},$$

$$\frac{\partial\Pi}{\partial\alpha} = M\frac{\partial\Psi}{\partial\alpha}, \qquad \frac{\partial\Pi}{\partial\beta} = M\frac{\partial\Psi}{\partial\beta}, \qquad \frac{\partial\Pi}{\partial\gamma} = M\frac{\partial\Psi}{\partial\gamma},$$

$$\frac{1}{M} = \alpha\frac{\partial\Psi}{\partial\alpha} + \beta\frac{\partial\Psi}{\partial\beta} + \gamma\frac{\partial\Psi}{\partial\gamma}.$$

Similarly, the system (15), (18), (19) may be replaced by the system:

(24) $$\frac{d\alpha}{-\frac{\partial\Psi}{\partial x}} = \frac{d\beta}{-\frac{\partial\Psi}{\partial y}} = \frac{d\gamma}{-\frac{\partial\Psi}{\partial z}} = \frac{dx}{\frac{\partial\Psi}{\partial\alpha}} = \frac{dy}{\frac{\partial\Psi}{\partial\beta}} = \frac{dz}{\frac{\partial\Psi}{\partial\gamma}} = d\tau,$$

(22) $$\Psi(x, y, z, \alpha, \beta, \gamma) = 0,$$

(25) $$dt = \left(\alpha\frac{\partial\Psi}{\partial\alpha} + \beta\frac{\partial\Psi}{\partial\beta} + \gamma\frac{\partial\Psi}{\partial\gamma}\right)d\tau\,,$$

whose integration amounts to that of the system (24), (22) in a form that is entirely analogous to that of the system (15), (18), (19), and one quadrature. Furthermore, in these equations the function Ψ is absolutely arbitrary.

However, if one replaces equation (22) by another one of the form:

$$\Psi(x, y, z, \alpha, \beta, \gamma) = \text{const.}$$

then one no longer obtains a representation of the same group of contact transformations. By comparison, in the preceding section replacing (19) by an equation Π = const. amounted to only replacing t with kt, where k is a constant, which did not alter the group of contact transformations considered.

From the geometric viewpoint, the surfaces of characteristic surfaces $\Psi = k$ (where k is constant) is essentially different from that of the surfaces (22), while the system of surfaces $\Pi = k$ has the same form as that of the surfaces $\Pi = 1$.

We further remark that equation (25) may be replaced with the equation:

(21) $$\alpha\,dx + \beta\,dy + \gamma\,dz - dt = 0.$$

This also results from the geometric equation of the quantities dx/dt, dy/dt, dz/dt, as in the preceding section.

One may suppose, in particular, that the equation (22) is of the form:

$$\Psi = G(x, y, z, \alpha, \beta, \gamma) - 1 = 0,$$

G being homogeneous of degree m in α, β, γ. One then has the canonical system:

(24 cont.) $$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{\partial G}{\partial\alpha}, & \dfrac{dy}{d\tau} = \dfrac{\partial G}{\partial\beta}, & \dfrac{dz}{d\tau} = \dfrac{\partial G}{\partial\gamma}, \\[2mm] \dfrac{d\alpha}{d\tau} = -\dfrac{\partial G}{\partial x}, & \dfrac{d\beta}{d\tau} = -\dfrac{\partial G}{\partial y}, & \dfrac{d\gamma}{d\tau} = -\dfrac{\partial G}{\partial z}, \end{cases}$$

with the condition:

(22 cont.) $$G(x, y, z, \alpha, \beta, \gamma) = 1,$$

and to determine the time, one has the simple formula:

$$dt = m\, d\tau. \tag{25 cont.}$$

One may thus replace τ with t without altering the group of contact transformations, and, for the reasons that were described above, one may also suppress condition (22 cont.). All of this amounts to changing just the unit of time.

II. – INTEGRATION PROBLEMS.

5. The problem of integration in the preceding theory consists of determining the propagation of an arbitrary wave knowing the system of characteristic surfaces, i.e., the system of elementary waves that corresponds to each point of the medium.

This problem will be solved if one determines the finite equations of the group of contact transformations that correspond to the mode of propagation considered, i.e., if one integrates the system (3). Indeed, let:

$$\begin{cases} x = \mathcal{X}(x_0, y_0, z_0, p_0, q_0 \mid t), \\ y = \mathcal{Y}(x_0, y_0, z_0, p_0, q_0 \mid t), \\ z = \mathcal{Z}(x_0, y_0, z_0, p_0, q_0 \mid t), \\ p = \mathcal{P}(x_0, y_0, z_0, p_0, q_0 \mid t), \\ q = \mathcal{Q}(x_0, y_0, z_0, p_0, q_0 \mid t), \end{cases} \tag{26}$$

be the equations thus obtained, where $t = 0$ corresponds to the identity transformation. If the initial wave is given then one may suppose that its contact elements (x_0, y_0, z_0, p_0, q_0) are given as functions of two parameters, and upon substituting these expressions in formulas (26) one will have the wave that results after a time interval t.

However, one may approach the question in another way by looking for the *family of waves* directly, i.e., the set of waves that successively issue from the same initial wave. The general equation of such a family may be assumed to be presented in the form:

$$f(x, y, z) = t, \tag{27}$$

and everything comes down to finding the corresponding function f.

If we associate equation (27) with the system:

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0, \tag{28}$$

in such a manner as to obtain the system that defines the set of contact elements of the initial wave, then the necessary and sufficient condition that determines f will be obtained by writing that this system (27), (28) is invariant under the infinitesimal transformation:

$$\frac{\partial F}{\partial t}+\frac{\partial W}{\partial p}\frac{\partial F}{\partial x}+\frac{\partial W}{\partial q}\frac{\partial F}{\partial y}+\left(p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}-W\right)\frac{\partial F}{\partial x}$$
$$-\left(\frac{\partial W}{\partial x}+p\frac{\partial W}{\partial z}\right)\frac{\partial F}{\partial p}-\left(\frac{\partial W}{\partial y}+q\frac{\partial W}{\partial z}\right)\frac{\partial F}{\partial q}.$$

Since this transformation changes any multiplicity into one multiplicity, it will suffice to operate on equation (27). One verifies, moreover, by direct calculation that the conditions that one obtains by operating on equations (28) are consequences of the ones that we shall obtain.

We thus write that:

$$\frac{\partial W}{\partial p}\frac{\partial f}{\partial x}+\frac{\partial W}{\partial q}\frac{\partial f}{\partial y}+\left(p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}-W\right)\frac{\partial f}{\partial z}-1=0$$

is a consequence of equations (27), (28), which reduces to the equation:

$$\frac{\partial f}{\partial z}W\left(x,y,z,-\frac{\partial f/\partial x}{\partial f/\partial y},-\frac{\partial f/\partial y}{\partial f/\partial z}\right)+1=0. \tag{29}$$

The desired necessary and sufficient condition is therefore that f must be an integral of that partial differential equation (29).

The preceding considerations then furnish us with all of the essential facts relating to the integration of that equation.

First, it admits one and only one solution such that equation (27) reduces to the equation of the given surface for $t = 0$, which characterizes the degree of generality of the general integral of (29).

As a result, the integration of (29) results from that of system (3), since, if one replaces x_0, y_0, z_0, p_0, q_0 in equations (26) with functions of the two parameters u, v that correspond to an arbitrary initial wave then one only has to solve the first three of equations (26) for t, upon eliminating u and v, to obtain the desired general solution (12). One may, for example, take the initial wave to be:

$$\begin{cases} x_0=\dfrac{\partial\Theta}{\partial u}, \quad y_0=\dfrac{\partial\Theta}{\partial v}, \quad z_0=u\dfrac{\partial\Theta}{\partial u}+v\dfrac{\partial\Theta}{\partial v}, \\ \qquad p_0=u, \qquad q_0=v, \end{cases} \tag{30}$$

Θ being an arbitrary function of only u and v.

6. One also sees that conversely the integration of the partial differential equation (29) implies that of the system (3) to which it is associated. This is because in order to obtain the motion of an arbitrary contact element it will suffice to take two initial waves that have only that contact element in common. The waves that result from it will constantly have a contact element in common that will be the position of the initial contact element considered for each value of t.

In order to apply this method it likewise suffices to know ∞^3 families of waves, because among the ∞^3 initial waves there will be one that passes through an arbitrarily given contact element E_0, which one may consider to be common with the initial wave and two other conveniently chosen initial waves that are infinitely close.

To that effect, suppose that one knows an integral of (29) that depends *essentially* on two arbitrary non-additive constants. Let $f(x, y, z, a, b)$ be this integral. The ∞^3 initial waves that one must consider will be defined by the equation:

$$f(x, y, z, a, b) = c. \tag{31}$$

The point of contact of one of them with two arbitrary infinitely close waves (of the same system) is obtained by adjoining to equation (31) the two equations:

$$\frac{\partial f}{\partial a} = a', \qquad \frac{\partial f}{\partial b} = b', \tag{32}$$

and the common contact element is defined by (31), (32), and:

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0. \tag{33}$$

In order to pass to the position of that element after a time interval t, one need only replace equation (31) with the equation:

$$f(x, y, z, a, b) = c + t, \tag{34}$$

which gives what then become the ∞^3 waves (31).

In summation, the general integral of system (3) is given by:

$$\begin{cases} f(x, y, z, a, b) = c + t, \\ \dfrac{\partial f}{\partial a} = a', \qquad \dfrac{\partial f}{\partial b} = b', \\ \dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z} = 0, \quad \dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z} = 0, \end{cases} \tag{35}$$

where the five constants a, b, c, a', b' must be determined by the initial conditions.

Finally, since an arbitrary initial wave may be regarded as the envelope of ∞^3 conveniently chosen initial waves (31):

$$f(x, y, z, a, b) = \chi(a, b),$$

which becomes, after a time interval t:

(36) $$f(x, y, z, a, b) = \chi(a, b) + t.$$

The general solution of equation (29) will be obtained deducing a and b from the equations:

(37) $$\frac{\partial f}{\partial a} - \frac{\partial \chi}{\partial a} = 0, \qquad \frac{\partial f}{\partial b} - \frac{\partial \chi}{\partial b} = 0,$$

and substituting the values that one finds in (36); the function χ is then an arbitrary function.

7. Upon recalling the notations of nos. **3** and **4**, one may replace the partial differential equation (29) with an arbitrary equation that does not contain the unknown function.

The identity (14) gives:

$$W\left(x, y, z, -\frac{\partial f/\partial x}{\partial f/\partial z}, -\frac{\partial f/\partial y}{\partial f/\partial z}\right) = \Pi\left(x, y, z, -\frac{\partial f/\partial x}{\partial f/\partial z}, -\frac{\partial f/\partial y}{\partial f/\partial z}, -1\right),$$

and due to the homogeneity of Π equation (29) reduces to:

(38) $$\Pi\left(x, y, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = 1.$$

To simplify the notations, we remark that finding an equation of the form (29) amounts to calculating t as a function of x, y, z, i.e., one may replace the letter f with the letter t in the preceding. Moreover, if one sets:

(39) $$\alpha = \frac{\partial t}{\partial x}, \qquad \beta = \frac{\partial t}{\partial y}, \qquad \gamma = \frac{\partial t}{\partial z}$$

then equation (19):

(19) $$\Pi(x, y, z, \alpha, \beta, \gamma) = 1$$

is the partial differential equation (38) to which we already arrived.

All of the theories in nos. **5**, **6** (theory of characteristics and the theory of complete integrals) apply immediately to that equation upon replacing system (3) with the system that one deduces in no. **3** by a change of variables; i.e.:

(40) $$\frac{dx}{\frac{\partial \Pi}{\partial \alpha}} = \frac{dy}{\frac{\partial \Pi}{\partial \beta}} = \frac{dz}{\frac{\partial \Pi}{\partial \gamma}} = -\frac{d\alpha}{\frac{\partial \Pi}{\partial x}} = -\frac{d\beta}{\frac{\partial \Pi}{\partial y}} = -\frac{d\gamma}{\frac{\partial \Pi}{\partial z}} = dt,$$

to which one must adjoin the equation of condition (19). Equation (33) must also be replaced by the equivalent ones:

$$\frac{\frac{\partial f}{\partial x}}{\alpha} = \frac{\frac{\partial f}{\partial y}}{\beta} = \frac{\frac{\partial f}{\partial z}}{\gamma}.$$

8. Finally, as in no. **4**, one may replace equation (19) with an equivalent equation of the arbitrary form:

(22) $$\Psi(x, y, z, \alpha, \beta, \gamma) = 0.$$

One only needs to replace the system (40) with the system (24), (25):

(41) $$\frac{dx}{\frac{\partial \Psi}{\partial \alpha}} = \frac{dy}{\frac{\partial \Psi}{\partial \beta}} = \frac{dz}{\frac{\partial \Psi}{\partial \gamma}} = \frac{dt}{\alpha\frac{\partial \Psi}{\partial \alpha} + \beta\frac{\partial \Psi}{\partial \beta} + \gamma\frac{\partial \Psi}{\partial \gamma}} = -\frac{d\alpha}{\frac{\partial \Psi}{\partial x}} = -\frac{\partial \beta}{\frac{\partial \Psi}{\partial y}} = -\frac{\partial \gamma}{\frac{\partial \Psi}{\partial z}}.$$

Finally, equation (21), which is implicitly contained in this, may be written:

(21) $$dt = \alpha\, dx + \beta\, dy + \gamma\, dz$$

and agrees completely with the notations (39) for the partial derivatives.

9. One thus sees how the general equation of characteristic surfaces Ψ_A may be interpreted as the partial differential equation of a family of waves. One may explain this fact geometrically, without appealing to the theory of groups of transformations that was invoked in no. **5**.

Indeed, write that the surface:

(42) $$f(x, y, z) = t + \delta t$$

is tangent to each of the elementary waves that issue from the various points (x_0, y_0, z_0) belonging to the wave at the instant t; i.e., ones such that one has:

(43) $$f(x_0, y_0, z_0) = t.$$

Equation (42), when one transports the origin to such a point, becomes:

$$f(x_0 + \mathrm{X}, y_0 + \mathrm{Y}, z_0 + \mathrm{Z}) = t + \delta t,$$

and the tangent plane to one of its points $\mathrm{X} = \delta x$, $\mathrm{Y} = \delta y$, $\mathrm{Z} = \delta z$ is:

$$(\mathrm{X} - \delta x)\frac{\partial f}{\partial(x_0 + \delta x)} + (\mathrm{Y} - \delta y)\frac{\partial f}{\partial(y_0 + \delta y)} + (\mathrm{Z} - \delta z)\frac{\partial f}{\partial(z_0 + \delta z)} = 0.$$

On the other hand, the tangential equation of the elementary wave being:

$$\delta t\, \Pi(x_0, y_0, z_0, \alpha, \beta, \gamma) - 1 = 0,$$

one has the condition:

$$\delta t\left[x_0, y_0, z_0, \frac{\partial f}{\partial(x_0+\delta x)}, \frac{\partial f}{\partial(y_0+\delta y)}, \frac{\partial f}{\partial(z_0+\delta z)}\right]$$
$$= \delta x\frac{\partial f}{\partial(x_0+\delta x)} + \delta y\frac{\partial f}{\partial(y_0+\delta y)} + \delta z\frac{\partial f}{\partial(z_0+\delta z)}\,.$$

Neglecting the infinitely small quantities of order greater than the first, this relation becomes:

$$\delta t\,\Pi\left(x_0, y_0, z_0, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial y_0}, \frac{\partial f}{\partial z_0}\right) = \delta x\frac{\partial f}{\partial x_0} + \delta y\frac{\partial f}{\partial y_0} + \delta z\frac{\partial f}{\partial z_0}\,.$$

Furthermore, since one also has:

$$f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) = t + \delta t,$$

which, upon again neglecting the terms of order higher than the first and taking (43) into account, reduces to:

$$\frac{\partial f}{\partial x_0}\delta x + \frac{\partial f}{\partial y_0}\delta y + \frac{\partial f}{\partial z_0}\delta z = dt,$$

what ultimately remains is:

$$\Pi\left(x_0, y_0, z_0, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial y_0}, \frac{\partial f}{\partial z_0}\right) = 1;$$

moreover, since the point (x_0, y_0, z_0) is arbitrary this is precisely the partial differential equation that we were looking for.

III. – TRAJECTORIES.

10. We use the word *trajectory* for the mode of propagation considered to mean the set of successive positions of the point of contact of an arbitrary contact element under the motion of that contact element. These trajectories are thus obtained by the integration of the system (3) upon considering p, q to be the auxiliary unknowns; i.e., they are defined by the first three of equations (26). If one takes into account, since this has meaning in these equations, the manner in which they are described then they depend upon five arbitrary constants; however, they form only a system of ∞^4 curves in space.

One may define them by a differential system that is analogous to the *Lagrange equations* in dynamics.

To that effect, we introduce the pointwise equation for the characteristic surface of the medium; i.e., the surface Ψ_A . Let:

(44) $$\Omega(x, y, z, \mathrm{X}, \mathrm{Y}, \mathrm{Z}) = 1$$

be the equation. We may assume that Ω is homogeneous and of first degree in X, Y, Z; one arrives at it by starting with an arbitrary form for the equation of that surface, rendering it homogeneous, and solving it for the homogeneous variable, as one did under an analogous circumstance in no. **4**. One is, moreover, naturally led to introduce that particular form for the equation because it immediately gives the equation of the elementary wave, which will be:

(45) $$\Omega(x, y, z, \mathrm{X}, \mathrm{Y}, \mathrm{Z}) = dt.$$

Write that this surface (44) is the same as the one that is defined in tangential coordinates by equation (19):

(19) $$\Pi(x, y, z, \alpha, \beta, \gamma) = 1.$$

The tangent plane to (44) at an arbitrary point:

$$\mathrm{X} = x', \qquad \mathrm{Y} = y', \qquad \mathrm{Z} = z',$$

has the equation:

(46) $$\mathrm{X}\frac{\partial\Omega}{\partial x'} + \mathrm{Y}\frac{\partial\Omega}{\partial y'} + \mathrm{Z}\frac{\partial\Omega}{\partial z'} = x'\frac{\partial\Omega}{\partial x'} + y'\frac{\partial\Omega}{\partial y'} + z'\frac{\partial\Omega}{\partial z'},$$

upon setting, to abbreviate:

$$\Omega = \Omega(x, y, z, x', y', z'),$$

in such a way that equation (46) is simply:

$$\mathrm{X}\frac{\partial\Omega}{\partial x'} + \mathrm{Y}\frac{\partial\Omega}{\partial y'} + \mathrm{Z}\frac{\partial\Omega}{\partial z'} = 1.$$

One will thus obtain equation (19) upon setting:

(47) $$\alpha = \frac{\partial\Omega}{\partial x'}, \qquad \beta = \frac{\partial\Omega}{\partial y'}, \qquad \gamma = \frac{\partial\Omega}{\partial z'},$$

and eliminating x', y', z' between (47) and:

(48) $$\Omega(x, y, z, x', y', z') = 1.$$

As a result, the unique differential relation that results from (19), i.e.:

$$\frac{\partial \Pi}{\partial x}dx + \frac{\partial \Pi}{\partial y}dy + \frac{\partial \Pi}{\partial z}dz + \frac{\partial \Pi}{\partial \alpha}d\alpha + \frac{\partial \Pi}{\partial \beta}d\beta + \frac{\partial \Pi}{\partial \gamma}d\gamma = 0, \tag{49}$$

is a consequence of relations (47), (48), and the ones that one deduces by total differentiation.

Now, upon deducing the tangential equation (19) of the corresponding pointwise equation, one will be led to write the relations:

$$x' = \frac{\partial \Pi}{\partial \alpha}, \qquad y' = \frac{\partial \Pi}{\partial \beta}, \qquad z' = \frac{\partial \Pi}{\partial \gamma}, \tag{50}$$

which are, as a result, consequences of (47) and (48), in such a way that (49) becomes:

$$\frac{\partial \Pi}{\partial x}dx + \frac{\partial \Pi}{\partial y}dy + \frac{\partial \Pi}{\partial z}dz + x'd\alpha + y'd\beta + z'd\gamma = 0. \tag{51}$$

One then deduces from (47) that:

$$\begin{aligned} x'\,d\alpha + y'\,d\beta + z'\,d\gamma &= \left(x'\frac{\partial^2\Omega}{\partial x'\partial x} + y'\frac{\partial^2\Omega}{\partial y'\partial y} + z'\frac{\partial^2\Omega}{\partial z'\partial z}\right)dx \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \left(x'\frac{\partial^2\Omega}{\partial x'^2} + y'\frac{\partial^2\Omega}{\partial y'\partial x'} + z'\frac{\partial^2\Omega}{\partial z'\partial x'}\right)dx' \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

which reduces, due to the degree of homogeneity of Ω and its partial derivatives, to:

$$x'\,d\alpha + y'\,d\beta + z'\,d\gamma = \frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy + \frac{\partial \Omega}{\partial z}dz\,.$$

Furthermore, since this is the only differential relation in dx, dy, dz, $d\alpha$, $d\beta$, $d\gamma$ that one can deduce from (47) and (48) it must be identical to (51), if one takes into account the finite equations (47) and (48); i.e., one has, as a consequence of changing the tangential coordinates into pointwise coordinates, the identities:

$$\frac{\partial \Pi}{\partial x} + \frac{\partial \Omega}{\partial x} = 0, \qquad \frac{\partial \Pi}{\partial y} + \frac{\partial \Omega}{\partial y} = 0, \qquad \frac{\partial \Pi}{\partial z} + \frac{\partial \Omega}{\partial z} = 0. \tag{52}$$

Having said this, the change of variables in question is carried out immediately in the equations (15), (18), (19). Upon comparing (15) and (50), one sees that all one must do is to set:

(53) $$\frac{dx}{dt} = x', \qquad \frac{dy}{dt} = y', \qquad \frac{dz}{dt} = z'$$

in the preceding formulas. (This results, moreover, from the geometric considerations of no. **2**.) Furthermore, equations (18), when compared with (52) and (47), give the stated system:

(54) $$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial x'}\right) - \dfrac{\partial \Omega}{\partial x} = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial y'}\right) - \dfrac{\partial \Omega}{\partial y} = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial \Omega}{\partial z'}\right) - \dfrac{\partial \Omega}{\partial z} = 0, \end{cases}$$

to which one must adjoin equation (48):

(48) $$\Omega(x_0, y_0, z_0, x', y', z') = 1.$$

One thus finds the trajectories defined directly, and if one has integrated that system then one has deduced the motion of the contact themselves by means of equations (47). One thus has a new form of the equations for an infinitesimal contact transformation.

One may remark that one deduces from equations (54) upon multiplying by x', y', z' and adding:

$$\frac{d}{dt}\left(x'\frac{\partial \Omega}{\partial x'} + y'\frac{\partial \Omega}{\partial y'} + z'\frac{\partial \Omega}{\partial z'}\right) - \frac{1}{dt}d\Omega = 0,$$

which is an identity, in view of the homogeneity of Ω. In reality, these equations thus reduce to only two.

11. If one takes the equation for Ψ_A in an arbitrary form that is equivalent to:

(55) $$\Theta(x, y, z, x', y', z') = 0,$$

then one will have, by arguing as in no. **4**:

$$\frac{\partial \Omega}{\partial x} = \mathrm{L}\frac{\partial \Theta}{\partial x}, \qquad \frac{\partial \Omega}{\partial y} = \mathrm{L}\frac{\partial \Theta}{\partial y}, \qquad \frac{\partial \Omega}{\partial z} = \mathrm{L}\frac{\partial \Theta}{\partial z},$$
$$\frac{\partial \Omega}{\partial x'} = \mathrm{L}\frac{\partial \Theta}{\partial y'}, \qquad \frac{\partial \Omega}{\partial y'} = \mathrm{L}\frac{\partial \Theta}{\partial y'}, \qquad \frac{\partial \Omega}{\partial z'} = \mathrm{L}\frac{\partial \Theta}{\partial z'},$$
$$\frac{1}{\mathrm{L}} = x'\frac{\partial \Theta}{\partial x'} + y'\frac{\partial \Theta}{\partial y'} + z'\frac{\partial \Theta}{\partial z'}.$$

The first equation in (54) must therefore become:

$$\frac{d}{dt}\left(\mathrm{L}\frac{\partial\Theta}{\partial x'}\right)-\mathrm{L}\frac{\partial\Theta}{\partial x}=0,$$

i.e.:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial x'}-\frac{\partial\Theta}{\partial x}=-\frac{d\mathrm{L}/dt}{\mathrm{L}}\frac{\partial\Theta}{\partial x'}=\left(\frac{d}{dt}\log\frac{1}{\mathrm{L}}\right)\frac{\partial\Theta}{\partial x'}.$$

One will thus have to adjoin to equation (55), the system:

(56)
$$\frac{\dfrac{d}{dt}\dfrac{\partial\Theta}{\partial x'}-\dfrac{\partial\Theta}{\partial x}}{\dfrac{\partial\Theta}{\partial x'}}=\frac{\dfrac{d}{dt}\dfrac{\partial\Theta}{\partial y'}-\dfrac{\partial\Theta}{\partial y}}{\dfrac{\partial\Theta}{\partial y'}}=\frac{\dfrac{d}{dt}\dfrac{\partial\Theta}{\partial z'}-\dfrac{\partial\Theta}{\partial z}}{\dfrac{\partial\Theta}{\partial z'}}.$$

One sees, moreover, that the common value of the ratios in (56) is:

$$\frac{d}{dt}\log\left(x'\frac{\partial\Theta}{\partial x'}+y'\frac{\partial\Theta}{\partial y'}+z'\frac{\partial\Theta}{\partial z'}\right),$$

but this is a consequence of equation (55).

The general differential system of the trajectories is thus composed of only equations (56) and (55).

In particular, suppose that equation (55) is of the form:

(57)
$$\Theta = \mathrm{H}(x, y, z, x', y', z') - 1 = 0,$$

H being homogeneous of degree m in x', y', z'. One may always arrange for this to be the case by taking H to be a power of Ω. One then has:

$$x'\frac{\partial \mathrm{H}}{\partial x'}+y'\frac{\partial \mathrm{H}}{\partial y'}+z'\frac{\partial \mathrm{H}}{\partial z'}=m\mathrm{H},$$

and the common value of the ratios in (56) is zero. The equations of the trajectories then take the Lagrangian form:

(58)
$$\begin{cases}\dfrac{d}{dt}\dfrac{\partial \mathrm{H}}{\partial x'}-\dfrac{\partial \mathrm{H}}{\partial x}=0,\\[2mm] \dfrac{d}{dt}\dfrac{\partial \mathrm{H}}{\partial y'}-\dfrac{\partial \mathrm{H}}{\partial y}=0,\\[2mm] \dfrac{d}{dt}\dfrac{\partial \mathrm{H}}{\partial z'}-\dfrac{\partial \mathrm{H}}{\partial z}=0,\end{cases}$$

with the condition (57). However, if $m \neq 1$ then one may suppress condition (57), because, upon multiplying equations (58) by x', y', z' and adding them, one obtains:

$$\frac{d}{dt}\left(x'\frac{\partial \mathrm{H}}{\partial x'} + y'\frac{\partial \mathrm{H}}{\partial y'} + z'\frac{\partial \mathrm{H}}{\partial z'}\right) - \frac{d\mathrm{H}}{dt} = (m-1)\frac{d\mathrm{H}}{dt} = 0,$$

i.e.:

(59) $$\mathrm{H}(x, y, z, x', y', z') = h = \text{const.}$$

Now, replacing the term – 1 in equation (57) with the term – h amounts to replacing the characteristic surfaces Ψ_A with homothetic surfaces (with a constant homothety ratio) without essentially altering the trajectories. In a more precise manner, this amounts to replacing t with $h^{1/m}t$ in all of the equations.

Conversely, it is clear that if one starts with a system (58), where H is an arbitrary homogeneous function in x', y', z' then one may always interpret it as corresponding to an infinitesimal contact transformation and reduce it, in turn, to the canonical form.

12. Return to equations (54); they do not apparently contain time. Indeed, $\partial\Omega/\partial x'$, $\partial\Omega/\partial y'$, $\partial\Omega/\partial z'$ are of degree zero in x', y', z', in such a way that one has, for example:

$$\frac{\partial \Omega}{\partial x'} = \frac{\partial \Omega(x, y, z, dx, dy, dz)}{\partial(dx)}.$$

On the contrary, $\partial\Omega/\partial x$, $\partial\Omega/\partial y$, $\partial\Omega/\partial z$ are homogeneous of degree one, and one has, for example:

$$\frac{\partial \Omega}{\partial x}dt = \frac{\partial \Omega(x, y, z, dx, dy, dz)}{\partial x}.$$

Thus, upon setting, to abbreviate:

(57) $$\bar{\Omega} = \Omega(x, y, z, dx, dy, dz),$$

equations (54) may be written:

(58) $$\begin{cases} d\dfrac{\partial \bar{\Omega}}{\partial(dx)} - \dfrac{\partial \bar{\Omega}}{\partial x} = 0, \\ d\dfrac{\partial \bar{\Omega}}{\partial(dy)} - \dfrac{\partial \bar{\Omega}}{\partial y} = 0, \\ d\dfrac{\partial \bar{\Omega}}{\partial(dz)} - \dfrac{\partial \bar{\Omega}}{\partial z} = 0. \end{cases}$$

One then sees that they define the trajectories, independently of the law by which they are written. Of course, they reduce to only two distinct equations.

It is equation (48) that consequently determines the manner by which the trajectories are written, because it is written:

$$\Omega(x, y, z, dx, dy, dz) = dt, \tag{59}$$

i.e., t is given by the quadrature:

$$t = \int \Omega(x, y, z, dx, dy, dz) = \int \overline{\Omega}\,. \tag{60}$$

13. Equations (58) give a characteristic property of the trajectories. In effect, they express the fact that the variation of the integral:

$$\theta = \int \Omega(x, y, z, dx, dy, dz) \tag{61}$$

is null when one displaces along a trajectory and formula (60) shows that the time t is justifiably the corresponding value of that integral (61).

We seek to interpret then integral (61) when it is taken between two points A and B of an arbitrary curve. For this, it suffices to represent this curve as a canal of infinitely small diameter, and in whose interior the disturbance propagates without friction. We assume that if the disturbance arrives at an arbitrary instant at the point M of that curve whose coordinates are x, y, z then after a time dt it will arrive at the point M' on that curve that is found on the elementary wave with M as its origin; i.e., the time that it takes to go from M to M' is given, up to a higher-order infinitesimal, by the equation:

$$\Omega(x, y, z, dx, dy, dz) = dt.$$

Therefore, *the integral* (61) *represents the time that the disturbance takes to propagate from A to B when one follows the curve in question.*

Furthermore, *the trajectories are the curves for which the variation of this time (when one deforms it infinitesimally) is null.*

If we look for the condition that corresponds to a *minimum* time then, according to the classical theory of the second variation, we must demand that the quadratic form:

$$\sum \frac{\partial^2 \Omega}{\partial x'^2}\xi^2 + 2\sum \frac{\partial^2 \Omega}{\partial y' \partial z'}\eta\xi \tag{62}$$

be constantly positive, except for values of the form:

$$\xi = \lambda x', \qquad \eta = \lambda y', \qquad \zeta = \lambda z'.$$

We shall interpret this condition geometrically.

To that effect, consider the characteristic surface Ψ_A at an arbitrary point $A(x, y, z)$ of a trajectory. The tangent to A on that trajectory pierces Ψ_A at the point P, which has the coordinates x', y', z' (when the origin is transported to A), because one may suppose that one has:

(63) $$\Omega(x, y, z, x', y', z') - 1 = 0.$$

The tangent plane to Ψ_A at P has the equation (*see* no. **10**):

(64) $$\mathrm{X}\frac{\partial\Omega}{\partial x'} + \mathrm{Y}\frac{\partial\Omega}{\partial y'} + \mathrm{Z}\frac{\partial\Omega}{\partial z'} - 1 = 0.$$

We seek the position of an arbitrary point N of Ψ_A, which is assumed to be infinitely close to P with respect to that tangent plane. The coordinates of N being $x' + \xi$, $y' + \eta$, $z' + \zeta$, one must look for the sign of:

$$(x'+\xi)\frac{\partial\Omega}{\partial x'} + (y'+\eta)\frac{\partial\Omega}{\partial y'} + (z'+\zeta)\frac{\partial\Omega}{\partial z'} - 1,$$

which, on account of (63), reduces to:

(65) $$\xi\frac{\partial\Omega}{\partial x'} + \eta\frac{\partial\Omega}{\partial y'} + \zeta\frac{\partial\Omega}{\partial z'}.$$

Now, the point N being on Ψ_A, one has:

$$\Omega(x' + \xi, y' + \eta, z' + \zeta) - 1 = 0,$$

i.e., upon developing:

$$\Sigma\xi\frac{\partial\Omega}{\partial x'} + \frac{1}{1\cdot 2}\left(\Sigma\xi^2\frac{\partial^2\Omega}{\partial x'^2} + 2\eta\zeta\frac{\partial^2\Omega}{\partial y'\partial z'}\right) + \ldots = 0,$$

or again:

(66) $$\Sigma\xi\frac{\partial\Omega}{\partial x'} = -\frac{1}{1\cdot 2}\left(\Sigma\xi^2\frac{\partial^2\Omega}{\partial x'^2} + 2\eta\zeta\frac{\partial^2\Omega}{\partial y'\partial z'}\right) + \ldots$$

Thus, if the form (62) is positive then the result of the substitution is negative, i.e., of the same sign as at the point:

$$M(\mathrm{X} = 0, \mathrm{Y} = 0, \mathrm{Z} = 0).$$

The exceptional values $\xi = \lambda x'$, $\eta = \lambda y'$, $\zeta = \lambda z'$ do not correspond to any point, since the condition:

$$\Omega(x' + \lambda x', y' + \lambda y', z' + \lambda z') = 1$$

reduces, due to the homogeneity of Ω, to:

$$1 + \lambda = 1;$$

i.e.:

$$\lambda = 0,$$

which will give the point P itself.

Therefore, if the analytical condition for the minimum is satisfied then *both curvatures for the surface* Ψ_A *have the same sign at* P *and it is concave towards* M.

Conversely, suppose that this geometric condition is satisfied. The form (62) has null discriminant, due to the relations that give Euler's theorem when it is applied to the homogeneous functions of degree zero $\partial\Omega/\partial x'$, $\partial\Omega/\partial y'$,$\partial\Omega/\partial z'$ (*see* no. **10**). If this is a difference of two squares then it will be annulled for two relations of the form:

$$A\xi + B\eta + C\zeta = 0, \tag{67}$$

which are verified for $\xi = \lambda x'$, $\eta = \lambda y'$, $\zeta = \lambda z'$, since these values annul the partial derivatives of that form. Geometrically, these relations represent two planes passing through the line AP and which, under the hypotheses that were made on the form of Ψ_A, cut that surface along two curves that pass through P. Therefore, there will be some points on the surface that are infinitely close to P and for which, by virtue of equation (66), their distance to the tangent plane at P will be of order higher than the second. However, this contradicts the hypothesis that the curvatures both have the same sign.

The form (62) might no longer reduce to a perfect square, because it will now be annulled by all points of a plane passing through AP, and the same contradiction presents itself.

The geometrical condition that we found is equivalent to the classical analytical condition.

We conclude that *the trajectories are the curves along which the disturbances propagate the fastest whenever the elementary waves are the surfaces that have both curvatures with the same sense at each point and whose concavity always points towards their respective origins.*

The application of this result to the various cases that one envisions in optics is immediate. From the theory of the calculus of variation, it is clear that it is not true in general, i.e., besides the arcs of trajectories that do not contain pairs of foci (Weierstrass conjugate points).

14. The property of trajectories that corresponds to the vanishing of the variation of an integral persists for any differential system that one defines.

The system (15), (18), (19) is provided by the condition:

$$\delta\int \alpha\, dx + \beta\, dy + \gamma dz = 0, \tag{68}$$

when α, β, γ are coupled by the condition (19):

$$\Pi(x, y, z, \alpha, \beta, \gamma) = 1 \tag{19}$$

and t is given by (21):

$$dt = \alpha\, dx + \beta\, dy + \gamma dz. \tag{21}$$

The system (3) is given by the equivalent condition:

(69) $$\delta\int\frac{dz - p\,dx - q\,dy}{\mathrm{W}} = 0,$$

with the equation that defines time, i.e.:

(70) $$dt = \frac{p\,dx + q\,dy - dz}{\mathrm{W}}.$$

However, these new forms of the theorem, which lead back to the results that were presented by Yoshiye ([1]) and E.-R. Hedrick ([2]), from ideas of Hilbert, are less susceptible to simple geometrical interpretations.

15. For the sake of applications, we insist upon the property of trajectories, relative to families of waves, that they result from assimilating the propagation of waves into a group of contact transformations. Moreover, in order to make the statements more concise we introduce the following nomenclature:

Let A be an arbitrary point, and let E a contact element at that point; a well-defined trajectory corresponds to that element. We say that the direction of the tangent to the trajectory is conjugate to the element and also that the trajectory itself is conjugate to the element. If α, β, γ are the direction coefficients of the normal to the element and x', y', z' are those of the trajectory then the conditions that express the fact that the trajectory is conjugate to the element are equations (47), which we write, upon supposing that α, β, γ are defined only up to a factor here, along with x', y', z':

(71) $$\frac{\alpha}{\dfrac{\partial\Omega}{\partial x'}} = \frac{\beta}{\dfrac{\partial\Omega}{\partial y'}} = \frac{\gamma}{\dfrac{\partial\Omega}{\partial z'}}.$$

We likewise say that the trajectory is conjugate at A to any surface and any curve admitting the element E for one of its contact elements. In the case of a surface, this fact will always be expressed by equations (71), where α, β, γ will be the direction coefficients of the normal to the surface. In the case of a curve having ξ, η, ζ for the direction coefficients of its tangent, one will have the condition:

(72) $$\xi\frac{\partial\Omega}{\partial x'} + \eta\frac{\partial\Omega}{\partial y'} + \zeta\frac{\partial\Omega}{\partial z'} = 0.$$

It expresses the fact that the tangent to the curve is parallel to one of the lines that are tangent to the elementary wave at the point where that wave is pierced by the direction of the trajectory.

One may, moreover, state the following facts:

([1]) *Math. Annalen*, v. LVII, pp. 185.
([2]) *Ann. of Math.*, 2nd series, v. IV, pp. 141, 157.

If ∞^2 *trajectories are conjugate to a surface then they are conjugate to* ∞^1 *surfaces and the arcs of the trajectories that lie between two of these surfaces correspond to equal times.*

If ∞^1 *trajectories are conjugate to a curve then they are conjugate to* ∞^1 *curves and the arcs of the trajectories that lie between two of these curves will correspond to equal times.*

In effect, these two stated results result in the propagation of an initial wave, whether in the form of a surface or curve, by means of the group of contact transformations considered. Relative to the second statement, one remarks that the ∞^1 curves issuing from the given curve are traced on the superficial waves issuing from the curve that constitutes the initial wave. One also remarks that when one is given ∞^1 trajectories that form a continuous system there always exists family of ∞^1 curves to which they are conjugate because the coordinates x, y, z of a point of any one of these trajectories are functions of t and a parameter s:

$$x = f(t, s), \qquad y = g(t, s), \qquad z = h(t, s), \tag{74}$$

and the desired curves will be defined by the differential equation:

$$\frac{\partial \Omega}{\partial x'} dx + \frac{\partial \Omega}{\partial y'} dy + \frac{\partial \Omega}{\partial z'} dz = 0. \tag{75}$$

where x, y, z, dx, dy, dz must be replaced with the functions (74) and their differentials, and where x', y', z' must have the values:

$$x' = \frac{\partial f}{\partial t}, \; y' = \frac{\partial g}{\partial t}, \; z' = \frac{\partial h}{\partial t}. \tag{76}$$

Finally, one may append to the preceding two statements the following one:

The ∞^2 *trajectories issuing from a point A are conjugate to* ∞^1 *surfaces, and the arcs of the trajectories between A and one of these surfaces correspond to equal times.*

The surfaces in question are, in effect, the surfaces $\Phi_{A,t}$ that we started with in no. **1**.

IV. – GENERAL SUMMARY. APPLICATIONS.

16. The preceding considerations may be reformulated, with no essential modifications, upon passing from ordinary space to a space of n dimensions. We state the most important points.

I. That space being considered to represent a medium of a constant nature in which disturbances of a certain nature propagate according to a well-defined law, the mode of propagation is determined by the *system of elementary waves* that have their origins at the various points of the medium.

This propagation may be considered to be a displacement of contact elements in space that is defined by a one-parameter group of contact transformations. The elementary wave that has its origin at an arbitrary point $(x_1, x_2, \ldots, x_n)$ is the locus of extremities of elementary displacements $(dx_1, dx_2, \ldots, dx_n)$ that move that point, which is considered to be associated with all of its successive contact elements, when the time varies by dt.

Its general equation is therefore of the form:

$$\Omega(x_1, x_2, \ldots, x_n \mid dx_1, dx_2, \ldots, dx_n) = dt, \tag{77}$$

Ω being homogeneous of degree *one* in $dx_1, dx_2, \ldots, dx_n$.

One may likewise define the system of elementary waves by their general equation when written in tangential coordinates, which will be of the form:

$$\Pi(x_1, x_2, \ldots, x_n \mid p_1, p_2, \ldots, p_n) = \frac{1}{dt}, \tag{78}$$

where Π is homogeneous of degree *one* in $p_1, p_2, \ldots, p_n$. Here, one supposes that the general equation of a tangent plane, with the transported origin being at $(x_1, x_2, \ldots, x_n)$, is taken to have the form:

$$p_1 X_1 + p_2 X_2 + \ldots + p_n X_n - 1 = 0. \tag{79}$$

One may consider, instead of elementary waves, characteristic surfaces that are related to the extremities of velocity $(dx_1 / dt, \ldots, dx_n / dt)$. Upon setting:

$$\frac{dx_1}{dt} = x_1', \qquad \frac{dx_2}{dt} = x_2', \qquad \ldots, \qquad \frac{dx_n}{dt} = x_n', \tag{80}$$

they are defined by one or the other of two equations:

$$\Omega(x_1, x_2, \ldots, x_n \mid x_1', x_2', \ldots, x_n') = 1, \tag{81}$$

or:

$$\Pi(x_1, x_2, \ldots, x_n \mid p_1, p_2, \ldots, p_n) = 1, \tag{82}$$

according to whether one takes the pointwise viewpoint or the tangential one. Equation (82) is verified identically by the formulas:

$$p_1 = \frac{\partial \Omega}{\partial x_1'}, \qquad p_2 = \frac{\partial \Omega}{\partial x_2'}, \qquad \ldots, \qquad p_n = \frac{\partial \Omega}{\partial x_n'}, \tag{83}$$

and equation (81) is verified identically by the formulas:

$$x_1' = \frac{\partial \Pi}{\partial p_1}, \qquad x_2' = \frac{\partial \Pi}{\partial p_2}, \qquad \ldots, \qquad x_n' = \frac{\partial \Pi}{\partial p_n}. \tag{84}$$

One must also associate these formulas with the condition:

$$p_1 x_1' + p_2 x_2' + \cdots + p_n x_n' = 1. \tag{85}$$

II. An arbitrary contact element will have the coordinates $(x_1, x_2, \ldots, x_n \mid p_1, p_2, \ldots, p_n)$; i.e., $(x_1, x_2, \ldots, x_n)$ will be the coordinates of its point and $(p_1, p_2, \ldots, p_n)$ will be the direction coefficients of its normal. Moreover, the group of contact elements considered will be defined by the canonical equations ([1]):

$$\frac{dx_i}{dt} = \frac{\partial \Pi}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \Pi}{\partial x_i} \qquad (i = 1, 2, \ldots, n). \tag{86}$$

In order for them to define the propagation considered precisely – i.e., in order for the parameter t to actually represent time in them and not time multiplied by a constant – one must combine them with equation (82) or the condition (85), which may be written:

$$dt = p_1\,dx_1 + p_2\,dx_2 + \ldots + p_n\,dx_n. \tag{87}$$

III. The same mode of propagation may be defined by directly searching for the ∞^1 surfaces that are provided by an arbitrary initial wave. One such *family of waves* being represented by an equation of the form:

$$t = f(x_1, x_2, \ldots, x_n), \tag{88}$$

the problem amounts to seeking t as a function of $(x_1, x_2, \ldots, x_n)$. The formula (87) being then supposed to represent the total differential of t, the solution of the problem consists of integrating equation (82), which is considered to be a partial differential equation.

Furthermore, if one has an integral of that equation that refers to $(n - 1)$ essential arbitrary constants, none of which are additive:

$$t = f(x_1, x_2, \ldots, x_n \mid a_1, a_2, \ldots, a_{n-1}), \tag{89}$$

then the integration of the system (86), (87) is given by the formulas ([2]):

([1]) This form of equations for an infinitesimal contact transformation was given by Lie. *See*, for example, *Theorie der Transformations-Gruppen*, v. II, pp. 263.

([2]) This is the *Jacobi theorem*, in one of its forms. If one would like to abstract from the condition (87) then it suffices to multiply t by a new arbitrary constant in the formulas (90); this results in all of the preceding.

(90) $$\begin{cases} p_i = \dfrac{\partial f}{\partial x_i}, \qquad \dfrac{\partial f}{\partial a_k} = b_k, \qquad t = f(x_1, x_2, \cdots, x_n \mid a_1, a_2, \cdots, a_{n-1}) + a_n \\ \qquad\qquad (i = 1, 2, \cdots, n) \qquad\qquad\qquad (k = 1, 2, \cdots, n-1). \end{cases}$$

IV. If, during the motion of contact elements, one considers only the motion of the points of these elements then one obtains what one may call the *trajectories* of the propagation. They are defined directly by the differential system:

(91) $$d\frac{\partial \Omega}{\partial (dx_i)} - \frac{\partial \Omega}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n),$$

and the manner by which they are described is given by the equation:

(92) $$dt = \Omega(x_1, x_2, \ldots, x_n \mid dx_1, dx_2, \ldots, dx_n).$$

This is equivalent to saying that they annul the variation of the integral:

(93) $$\theta = \int \Omega(x_1, x_2, \ldots, x_n \mid dx_1, dx_2, \ldots, dx_n).$$

The value of that integral, when taken along an arc of an arbitrary curve, gives the time that it takes for the disturbance to propagate along that arc.

Finally, if one agrees to say that a trajectory is conjugate to a multiplicity if it corresponds to the motion of a contact element of that multiplicity then one has the following theorem:

If ∞^p trajectories are conjugate to a multiplicity of dimension p then they are conjugate to ∞^p multiplicities of the same nature, and the arcs of these trajectories that lie between two of these multiplicities all correspond to the same time interval.

17. The applications are numerous. First consider the propagation of light in an isotropic, but not homogeneous, medium. The elementary waves are spheres; i.e.:

$$\Omega = \omega(x, y, z)\sqrt{x'^2 + y'^2 + z'^2}\ .$$

The trajectories are the luminous rays and conditions (71) and (72) become the orthogonality conditions.

From this, one deduces the theorem that the luminous rays that issue from a point and are normal to one surface are normal to an infinitude of surfaces. The families of waves are the families of orthogonal surfaces to the same congruence of rays. These rays are curvilinear, in general.

If the medium is homogeneous, but not necessarily isotropic, then Ω is a function of only x', y', z'; one then concludes from equations (58) and (59) that x', y', z' are constants. The luminous rays are rectilinear and the velocity of propagation is constant on each ray.

To each ray direction there is associated one plane direction. It is the tangent plane to the wave surface at the point where it is pierced by the ray direction ([1]).

18. Any problem that involves an integral of the form (93) whose variation is null constitutes an application of the preceding, and the notion of contact transformation will prove useful. Examples of such problems are the *brachistochrone problem*, the general problem of the *equilibrium of filaments*, the problem of *geodesic lines*, and ultimately, the *general problem of dynamics*. The general theorem on multiplicities and conjugate trajectories gives the key to the theorems of Thomson and Tait, and their generalizations.

Without insisting upon the details, we examine the case of the equations of dynamics. Start with the Lagrange equations, where we suppose that neither the *vis viva* $2\mathrm{T}(x_1, x_2, \ldots, x_n \mid x_1', x_2', \ldots, x_n')$ of the system nor the force function $\mathrm{U}(x_1, x_2, \ldots, x_n)$ depends upon time. These equations are:

$$(94) \qquad \frac{d}{dt}\frac{\partial \mathrm{T}}{\partial x_i'} - \frac{\partial \mathrm{T}}{\partial x_i} = \frac{\partial \mathrm{U}}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

and upon supposing that one has given a particular value to the constant *vis viva,* one may write the equation of *vis viva* that must be appended to (94):

$$(95) \qquad \mathrm{T} = \mathrm{U}.$$

By means of that equation, we shall eliminate the time in equations (94). Set:

$$\bar{\mathrm{T}} = \mathrm{T}(x_1, x_2, \ldots, x_n \mid dx_1, dx_2, \ldots, dx_n),$$

and we write (95) in the form:

$$\bar{\mathrm{T}} = \mathrm{U}\, dt^2,$$

i.e.:

$$(96) \qquad dt = \sqrt{\frac{\bar{\mathrm{T}}}{\mathrm{U}}} = \mathrm{S}(x_1, x_2, \ldots, x_n \mid dx_1, dx_2, \ldots, dx_n).$$

Therefore:

$$\mathrm{T} = \frac{\bar{\mathrm{T}}}{\mathrm{S}^2}, \qquad \frac{\partial \mathrm{T}}{\partial x_i'} = \frac{1}{\mathrm{S}}\frac{\partial \bar{\mathrm{T}}}{\partial\, dx_i'}, \qquad \frac{\partial \mathrm{T}}{\partial x_i} = \frac{1}{\mathrm{S}^2}\frac{\partial \bar{\mathrm{T}}}{\partial x_i'}.$$

As a consequence:

$$\frac{\partial \mathrm{T}}{\partial x_i'} = \frac{1}{\mathrm{S}}\frac{\partial(\mathrm{US}^2)}{\partial\, dx_i} = 2\mathrm{U}\frac{\partial \mathrm{S}}{\partial\, dx_i} = 2\frac{\partial(\mathrm{US})}{\partial\, dx_i},$$

and the equations (94) become:

([1]) Compare: LÉVISTAL, *Recherches d'Optiques géometriques (Annales de l'École Normale*, 7th series, v. IV, pp. 195).

$$2\frac{1}{\mathrm{S}}\frac{\partial(\mathrm{US})}{\partial x_i} - 2\mathrm{U}\frac{\partial \mathrm{S}}{\partial x_i} - \frac{\partial \mathrm{U}}{\partial x_i} = \frac{\partial \mathrm{U}}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

or finally, upon setting:

(97) $$\Omega = 2\mathrm{US} = 2\sqrt{\mathrm{U\bar{\bar{T}}}},$$

one has:

$$d\frac{\partial \Omega}{\partial(dx_i)} - \frac{\partial \mathrm{U}}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n).$$

Also, since Ω is homogeneous of degree *one* with respect to the differentials, it is a system of the form (91), where Ω has only the particular form:

(98) $$\Omega = 2\sqrt{\mathrm{U}(x_1, \cdots, x_n)\mathrm{T}(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)},$$

which is characterized by the fact that T is a positive-definite quadratic form in the differentials.

This function Ω is the *elementary action* of the system, and we thus arrive, by a classical calculation, at the *principle of least action.* However, Ω has another meaning for us: The equation $\Omega = dt$ defines the system of elementary waves of one mode of wave propagation in which the trajectories are the same as those of the dynamical motion considered, except that what corresponds to time τ in the wave motion is the *action* of the dynamical motion.

In other words, the trajectories of any problem of dynamics are identical with those of one-parameter group of contact transformations; however, the canonical parameter of this group is not the time t, but the action:

(99) $$t = 2\int\sqrt{\mathrm{U\bar{\bar{T}}}}$$

of the dynamical problem.

It results from the preceding that upon preserving the use that action for the independent variable one may reduce the integration of problem to that of a canonical system (86) or a partial differential equation (82). One must then determine the time by the quadrature:

(100) $$t = \frac{1}{2}\int\frac{d\tau}{\mathrm{U}},$$

which provides the two formulas:

$$dt = \sqrt{\frac{\bar{\bar{\mathrm{T}}}}{\mathrm{U}}}, \qquad dt = 2\sqrt{\mathrm{U\bar{\bar{T}}}}.$$

Compare these calculations with those of Hamilton. In order to arrive at equation (82), we have to eliminate $x_1', \ldots, x_n'$ from equations (83), which are homogeneous of degree *zero*; i.e.:

(101) $$p_i = \sqrt{\frac{\mathrm{U}}{\mathrm{T}}} \cdot \frac{\partial \mathrm{T}}{\partial x_i'} \qquad (i = 1, 2, \ldots, n).$$

In the calculations of Hamilton, in order to arrive at the partial differential equation of Jacobi:

(102) $$\mathrm{H}(x_1, x_2, \ldots, x_n \mid p_1, \ldots, p_n) = 0,$$

one must eliminate $x_1', \ldots, x_n'$ between the equations:

(103) $$p_i = \frac{\partial \mathrm{T}}{\partial x_i'}, \qquad \mathrm{T} - \mathrm{U} = 0 \qquad (i = 1, 2, \ldots, n).$$

Now, one deduces the homogeneous equations (101) from these equations.

Equation (82):

(104) $$\Pi(x_1, x_2, \ldots, x_n \mid p_1, \ldots, p_n) - 1 = 0$$

is therefore equivalent to (102) of Jacobi-Hamilton ([1]), and if one writes (102) in the form:

$$\sqrt{\frac{\mathrm{H} + \mathrm{U}}{\mathrm{U}}} - 1 = 0,$$

then, the radical that appears here being homogeneous of degree *one*, one has identically:

(105) $$\Pi \equiv \sqrt{\frac{\mathrm{H}}{\mathrm{U}} + 1}.$$

As for our canonical system:

(106) $$\frac{dx_i}{d\tau} = \frac{\partial \Pi}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{\partial \Pi}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

in order to deduce the system of Hamilton one must concern oneself with not only formula (100), but also the equation of *vis viva* H = 0. The calculation results immediately from formula (105).

Finally, we remark that formula (105), which gives the characteristic function of the infinitesimal contact transformation, may be written:

(107) $$\Pi = \sqrt{\frac{\mathcal{T}}{\mathrm{U}}},$$

([1]) This fact was confirmed by Lie in the particular case of the motion of a material point (*Leipziger Berichte*, v. XLI, pp. 145).

upon denoting the adjoint form to T by $\mathcal{T}$, or, more precisely, what T becomes under a change of variables:

(108) $$p_i = \frac{\partial \mathrm{T}}{\partial x_i'} \qquad (i = 1, 2, \ldots, n).$$