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On a new theory of general integration problems

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INTRODUCTION

1. – Any system (S) of first-order ordinary differential equations corresponds to a linear partial differential equation (E) whose solutions are the first integrals of (S). The correspondence is reciprocal, and the integration of the system (S) and that of the equation (E) constitute equivalent problems. There then exists a sort of duality between (S) and (E), and one can say that the system and that equation are *correlative*. Those terms are all the more justified because the origin of the equivalence between the two integration problems of (S) and (E) is found in the bilinear relation:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0 \tag{1}$$

that exists between the partial derivatives of any solution of (E) and differentials of the coordinates of the current point of any integral of (S).

A similar duality exists between the completely-integrable systems of Pfaff equations and complete systems of homogeneous linear first-order partial differential equations. Cartan gave it the following form, in which a generalization of formula (1) appears: Let:

$$\omega_i = \sum_{\alpha=1}^{n} a_{\alpha,i}(x_1, \ldots, x_n)\, dx_\alpha = 0 \qquad (i = 1, 2, \ldots, s) \tag{S}$$

be a completely-integrable system of s Pfaff equations in n variables, and let:

$$X_j F = \sum_{\alpha=1}^{n} \xi_{\alpha,i}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_\alpha} = 0 \qquad (j = 1, 2, \ldots, m) \tag{E}$$

be a complete system of $m = n - s$ linear partial differential equations. The two systems (S) and (E) are correlative, and the integration of each of them will imply that of the other, if one has an identity of the form:

(2) $$df = \varpi_1 X_1 + \ldots + \varpi_m X_m + \omega_1 Z_1 + \ldots + \omega_s Z_s ,$$

in which $\varpi_1, \ldots, \varpi_m$ are new linear forms in $dx_1, \ldots, dx_n$, and $Z_1, \ldots, Z_s$ are new linear forms in $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$.

I propose to extend that *notion of duality* to the case in which the system (S) is an arbitrary Pfaff system, and to establish the principles of the correlative theory that must result from that extension, while taking my inspiration from Cartan's beautiful theory of Pfaff systems. In the following pages, I shall give a first glimpse into the new theory ([1]).

2. – If one is given $n = m + s$ Pfaff expressions $\omega_1, \ldots, \omega_s, \varpi_1, \ldots, \varpi_m$ then the identity (2) will define the linear operators $X_1, \ldots, X_m, Z_1, \ldots, Z_s$, under the single condition that $\omega_1, \ldots, \omega_s, \varpi_1, \ldots, \varpi_m$ must be linearly-independent forms in the $dx_1, \ldots, dx_n$.

If one is given only the Pfaff system (S) then $\omega_1, \ldots, \omega_s$ will be defined only up to a linear substitution:

$$\omega_i' = \sum_{\alpha=1}^{s} p_{i,\alpha}(x_1,\ldots,x_n)\,\omega_\alpha \qquad (i = 1, 2, \ldots, s) .$$

On the other hand, one can choose $\varpi_1, \ldots, \varpi_m$ arbitrarily in such a way that if one makes an initial choice then their most general values will have the form:

$$\omega_j' = \sum_{\alpha=1}^{s} q_{j,\alpha}(x_1,\ldots,x_n)\,\omega_\alpha + \sum_{\beta=1}^{m} r_{j,\beta}(x_1,\ldots,x_n)\,\varpi_\beta \qquad (j = 1, 2, \ldots, m) .$$

One immediately recognizes that such modifications to the choices of ω_i and ω_j will have the effect of replacing $X_1, \ldots, X_s$ with homogeneous linear combinations of the form:

(3) $$U f = u_1 (x_1, \ldots, x_n) X_1 f + \ldots + u_m (x_1, \ldots, x_n) X_m f ,$$

and can give forms for $Z_1, \ldots, Z_s$ that are entirely arbitrary in $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$.

By interpreting expressions such as $X_j f$ as the symbols of *infinitesimal transformations* (which conforms to the ideas of Sophus Lie), I conclude that under the type of duality that is defined by means of the identity (2), any system (S) of Pfaff equations will correspond to *a sheaf of infinitesimal transformations:*

([1]) An analogous duality manifests itself between Monge equations and non-linear partial differential equations, and it is also susceptible to extension.

$$(\mathcal{F}) \qquad \{X_1, \ldots, X_m\} \qquad (m = n - s)\ ,$$

i.e., the set of infinitesimal transformations that are given by (3), in which $X_1, \ldots, X_m$ are well-defined divergent ([1]) infinitesimal transformations, while the coefficients $u_1, \ldots, u_m$ remain arbitrary.

Conversely, any sheaf $\{X_1, \ldots, X_m\}$ will correspond to a Pfaff system $\omega_1 = \ldots = \omega_s = 0$ in the same way.

An integral multiplicity of the Pfaff system (S) will be an integral multiplicity of the correlative sheaf $(\mathcal{F})$, and conversely, if one adopts the following definition: A p-dimensional multiplicity is called an *integral* of a sheaf of infinitesimal transformation if it is invariant for p divergent transformations of the sheaf.

Here, one is led to consider complete integrals instead of the isolated integral multiplicities, and that is one difference between the two correlative theories. I say *complete integral* to mean a family of integral multiplicities such that one and only one multiplicity of that family passes through each point of space.

Any p-dimensional complete integral is provided by a complete system of p equations $U_1 f = \ldots = U_p f = 0$ whose left-hand sides are transformations of the sheaf. Those transformations $U_1, \ldots, U_p$ then define what one calls a *complete subsheaf* of the given sheaf.

The problem of integrating a Pfaff system then has the following correlate: *Determine all of the complete subsheaves that are contained in a given sheaf of infinitesimal transformations*. That is what one can call *integrating the sheaf.*

3. – Cartan's theory is based upon considering the bilinear covariants:

$$\omega_i' = \delta\omega_i\,(d) - d\omega_i\,(\delta)\ .$$

It gives rise to the identities-congruences:

$$(4) \qquad \omega_i' \equiv \sum_{(\alpha,\beta)} c_{\alpha,\beta,i}(x_1,\ldots,x_n)\cdot\varpi_\beta\,\varpi_\alpha \qquad (\text{mod } \omega_1, \ldots, \omega_s) \qquad (i = 1, 2, \ldots, s)\ ,$$

in which $\varpi_\beta\,\varpi_\alpha$ symbolically denotes the determinant:

$$\varpi_\beta\,(\delta)\ \varpi_\alpha\,(d) - \varpi_\beta\,(d)\ \varpi_\alpha\,(\delta)\ .$$

([1]) I intend the word *divergent*, which justifies the geometric interpretation of infinitesimal transformations, to mean that there exists no identity of the form:

$$u_1\,(x_1, \ldots, x_m)\,X_1 + \ldots + u_m\,(x_1, \ldots, x_m)\,X_m = 0$$

between the $X_1, \ldots, X_m$.

Those identities define the *structure* of the Pfaff system, which is a structure that depends upon the nature of the problem of integrating that system.

In our theory, the Jacobi brackets:

$$(X_j f, X_h f) = X_j (X_h f) - X_h (X_j f)$$

replace the bilinear covariants, and it is the identities-congruences:

$$(5) \qquad (X_j, X_h) \equiv \sum_{\gamma} c_{j,h,\gamma}(x_1, \ldots, x_n) Z_\gamma \qquad (\text{mod } X_1, \ldots, X_m) \qquad (j, h = 1, 2, \ldots, m)$$

that define the *structure of the sheaf* $\{X_1, \ldots, X_m\}$.

The equivalence of the two viewpoints results from a calculation whose principle is once more due to Cartan. One deduces the following identity from the identity (2):

$$0 = \delta(df) - d(\delta f)$$

$$= \sum_{\alpha} \varpi'_\alpha X_\alpha + \sum_{\gamma} \varpi'_\gamma Z_\gamma + \sum_{(\alpha,\beta)} \varpi_\beta \varpi_\alpha (X_\alpha, X_\beta) + \sum_{(\alpha,\gamma)} \varpi_\beta \varpi_\gamma (Z_\gamma, X_\alpha) + \sum_{\varepsilon,\gamma} \omega_\varepsilon \omega_\gamma (Z_\gamma, Z_\varepsilon),$$

and one immediately concludes from this that the functions $c_{j,h,i}$ are the same in formulas (4) and (5).

One attaches the notion of *derived sheaf* to the *structure formulas* (5), which is the correlate of the notion of derived system that was introduced by Cartan into the theory of Pfaff equations. The derived sheaf of a given sheaf is obtained by adding to it the brackets (X_j, X_h) that are formed from its basis transformations $X_1, \ldots, X_m$: In order to do that, it will suffice to add to them the linear forms $Z_1, \ldots, Z_s$ that appear in the right-hand sides of formulas (5), which are mutually independent. One notes that the word "derived" is used here in a sense that is entirely analogous to the one that S. Lie gave it in the theory of transformation groups ([1]).

In certain cases, there is good reason to complete the analysis of the structure of the sheaf by considering its *successive derivatives*.

4. – Any infinitesimal transformation of a sheaf ($\mathcal{F}$) defines an infinitesimal displacement at each point that is, in fact, what Cartan called an *integral element* of the correlative Pfaff system (S), and one then obtains all of the integral elements.

I say that two transformations of the sheaf are *in involution* when their bracket belongs to the sheaf. That is the case when two integral elements that are in involution at each point, in the sense

([1]) Indeed, the derived sheaf of a sheaf $\{X_1, \ldots, X_m\}$ is composed of the brackets of the *various infinitesimal transformations of the sheaf* when they are taken pair-wise in all possible ways, and the derived group of a finite group of transformation $\{X_1, \ldots, X_m\}$ is also composed of the brackets of the various infinitesimal transformations of the group taken pair-wise in all possible ways. However, there is not, in general, an identity between the bases of the derived group and the derived sheaf that is deduced from the same infinitesimal transformations $X_1, \ldots, X_m$.

that Cartan gave to that term. The search for complete subsheaves is then subordinate to the algebraic problem that consists of finding *involutions of the sheaf* with an arbitrary *degree p*, i.e., the subsheaves $\{U_1, .., U_p\}$ that satisfy the congruences ([1]):

$$(U_i, U_j) \equiv 0 \qquad (\text{mod } X_1, \ldots, X_m) \qquad (i, j = 1, 2, \ldots, p) .$$

Those involutions provide the integral elements of various orders that were considered by Cartan.

A sheaf will be called *involutive of order p* if the general transformation of the sheaf belongs to an involution of degree at least 2, the general involution of degree 2 belongs to an involution of degree at least 3, etc., and finally, if the general involution of degree $p - 1$ belongs to an involution of degree at p.

That definition, which is copied from Cartan's definition of Pfaff systems in involution, leads one to introduce the notion of *genus* and *character* of the sheaf, which are the same integers that Cartan introduced under the same names in his study of Pfaff systems. (One considers a sheaf and a correlated Pfaff system.)

5. – The preceding generalities are the subject of the first two sections of this work. In the second section, I will also establish the existence of complete subsheaves of degree p (i.e., p-dimensional complete *integrals*) for any involutive sheaf or order p, and I will specify the nature of the indeterminacy in the general complete subsheaf of degree p.

I will then indicate how one can *prolong* a sheaf in such a manner as to obtain a new sheaf whose p-dimensional integrals are themselves the multiplicities that are obtained by prolonging – in the sense that Lie gave to that word – the p-dimensional integral multiplicities of the initial sheaf by adding their contact elements.

In order to not overextend that sketch of the new theory, I shall confine myself to stating two fundamental theorems without proof, which are, moreover, consequences of some analogous theorems in Cartan's theory: Upon prolonging an involutive sheaf, one will obtain an involutive sheaf, and the indefinite prolongation of an arbitrary sheaf will lead to an involutive sheaf after a finite number of operations.

In the last two sections, I have preferred to indicate how the theory of characteristics flows out of the study of the structure of the sheaf of infinitesimal transformations, and I will show, by some examples, that the new theory lends itself to the applications with great ease. To that end, I have addressed only some applications of a classical character that relate to partial differential equations in two independent variables and one unknown function. The passage from partial differential equations to the sheaf that they correspond to is achieved immediately without appealing to Pfaff systems, moreover.

The *Cauchy characteristics* are provided (in all of the cases where they exist) by transformations of the sheaf that leave a sheaf invariant. Those *distinguished transformations* form a complete subsheaf whose integrals are the characteristics, and the use of those integrals for

([1]) For a complete subsheaf, one has the defining congruences:

$$(U_i, U_j) \equiv 0 \qquad (\text{mod } U_1, \ldots, U_p) \qquad (i, j = 1, 2, \ldots, p) .$$

integrating the given sheaf is immediate. As examples, I have taken the first-order partial differential equation and systems of two second-order partial differential equations that are in involution.

In an analogous manner, the *Monge characteristics* provide some transformations of the sheaf that, without being distinguished transformations, are nonetheless *singular transformations* from the standpoint of the study of involutions of degree 2 ([1]).

Those transformations (when they exist) form one or more subsheaves (which are not generally complete) whose integrals are, if applicable, the characteristics in question. In that way, I have studied the characteristics of the second-order partial differential equations in the case where the two systems of characteristics are distinct. Their various properties are then obtained quite rapidly.

It is worthy of note that the *invariants* of one or the other of those systems of characteristics, as Goursat defined them in his studies of second-order partial differential equations (which are now classical), are precisely the invariants that are common to the infinitesimal transformations of the *characteristic subsheaves* in question: Of course, those subsheaves must be prolonged up to the order of the invariants that wishes to consider.

Moreover, the use of those invariants of all order for the integration of the second-order equations will flow out of our method for finding complete integrals with remarkable simplicity.

Finally, the case of first-order invariants will lead quite naturally to the notion of first-order characteristics: When they exist, they will be provided by *families of infinitesimal transformations* that will become sheaves only in the case of the Monge-Ampère equation.

I have considered partial differential equations that are solved for their derivatives of maximum order everywhere. On another occasion, I shall indicate how the method of sheaves of transformations can be adapted to the study of equations that are not solved. In a later work, I will also show how one can develop the theory of continuous transformation groups (whether finite or infinite) by the method that Cartan has based upon his theory of Pfaff systems in involution.

I. –INFINITESIMAL TRANSFORMATIONS AND THE GENERAL INTEGRATION PROBLEM.

1. *Sheaves of infinitesimal transformations*. – Let m infinitesimal transformations in n variables be given:

$$X_k f = \sum_{\alpha=1}^{n} \xi_{k\alpha}(x_1,\ldots,x_n)\frac{\partial f}{\partial x_\alpha} \quad (k = 1, 2, \ldots, m) . \tag{1}$$

When one starts from an arbitrary point $(x_1, \ldots, x_n)$ in n-dimensional space, each of them will define an infinitesimal displacement:

$$dx_1 = \xi_{k1}(x_1, \ldots, x_n)\, dt , \qquad \ldots, \qquad dx_n = \xi_{kn}(x_1, \ldots, x_n)\, dt . \tag{2}$$

([1]) More generally, one can similarly consider involutions that are singular relative to the study of involutions of higher degree.

If there exists no identity of the form:

(3) $$\sum_{\alpha=1}^{m} \lambda_\alpha (x_1, \ldots, x_n)\, X_\alpha f = 0$$

with coefficients λ_α that are not all zero then those displacements will define an m-dimensional planar element, and we will say that the transformations $X_1, \ldots, X_m$ are *divergent* ([1]). That demands that $m \leq n$.

The set of all infinitesimal transformations:

(4) $$U f = \sum_{\alpha=1}^{m} u_\alpha (x_1, \ldots, x_n)\, X_\alpha f\,,$$

in which the u_α are arbitrary functions of $x_1, \ldots, x_n$ will then be called a *sheaf of infinitesimal transformations of degree m*.

The transformations $X_1, \ldots, X_m$ constitute the *basis* for that sheaf. However, one can take the basis that defines the sheaf to be m other arbitrary, but divergent, transformations of the sheaf. That amounts to performing a homogeneous linear substitution on $X_1, \ldots, X_m$ whose coefficients are arbitrary functions of $x_1, \ldots, x_n$.

In particular, one can exhibit the sheaf in a *form that is solved* for m of the derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$, i.e., suppose, for example, that $X_1, \ldots, X_m$ have the form:

(5) $$X_k f = \frac{\partial f}{\partial x_k} + \sum_{\alpha=m+1}^{n} \xi_{k,\alpha}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_\alpha} \qquad (k = 1, 2, \ldots, m)\,.$$

2. *Complete sheaves. Derived sheaves.* – The Jacobi brackets:

$$(X_h f, X_k f) = X_h X_k f - X_k X_h f \qquad (h, k = 1, 2, \ldots, m)$$

are infinitesimal transformations that are covariants to the transformations $X_1, \ldots, X_m$. The properties of the sheaf (4) or the sheaf $\{X_1, \ldots, X_m\}$ depends essentially upon the nature of those transformations. If they all belong to the sheaf then we say that the sheaf is *complete*, and we write:

$$(X_h, X_k) \equiv 0 \qquad (\text{mod } X_1, \ldots, X_m) \qquad (n, k = 1, 2, \ldots, m)$$

([1]) Since $X_1, \ldots, X_m$ are linear forms in $\partial f / \partial x_1, \ldots, \partial f / \partial x_n$, it would be natural to employ the term *independent* transformations. However, in the terminology of Sophus Lie, that word would express the idea that there exists no identity of the form (3) between the $X_1, \ldots, X_m$ that has *constant* coefficients λ_α.

to indicate that those brackets are expressed a homogeneous linear functions of $X_1, \ldots, X_m$. In a more abbreviated manner, if the letter $\mathcal{F}$ denotes the sheaf then we also write:

$$(X_h, X_k) \equiv 0 \qquad (\text{mod } \mathcal{F}) \qquad (n, k = 1, 2, \ldots, m)$$

in this case.

If the sheaf is not complete then the brackets (X_h, X_k) will be expressed as homogeneous linear functions of $X_1, \ldots, X_m$ and some other infinitesimal transformations $X_{m+1}, \ldots, X_{m'}$ that one can choose in such a manner that $X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m'}$ will be divergent, and the sheaf $\{X_1, \ldots, X_{m'}\}$ is called the *derived sheaf* of the sheaf $\{X_1, \ldots, X_m\}$.

One sees that a sheaf is always contained in its derived sheaf, which one can express by saying that it is a *subsheaf*, and a complete sheaf is a sheaf that is identical to its derived sheaf.

If the derived sheaf of a sheaf is not complete then one might be led to consider the derived sheaf of that derived sheaf, or *second derived sheaf* of the proposed one, and more generally, the *successive derived sheaves* of the given sheaf.

Since the number m of transformations in a basis for a sheaf cannot exceed the number n of variables, one will necessarily arrive at a *final derived sheaf*, which will be complete.

If that final derived sheaf has degree n', which is less than n, then one will get a complete system upon equating the infinitesimal transformations of its basis to zero, and upon introducing $n - n'$ (independent) integrals of that complete system as variables in place of the variables $x_{n'}$, $\ldots, x_n$, the proposed sheaf, along with its successive derived sheaves, will, in fact, no longer depend upon the variables $x_1, \ldots, x_{n'}$, since the derivatives $\dfrac{\partial f}{\partial x'_{n'+1}}, \ldots, \dfrac{\partial f}{\partial x'_{n'}}$ will no longer enter in. That result is effortlessly completed in such a way that one arrives at the following one:

The degree of the final derived sheaf of a sheaf of infinitesimal transformations is equal to the minimum number of effective variables to which one can reduce that sheaf by a change of variables.

The other (ineffective) variables will appear only as arbitrary parameters.

3. *Integrals of a sheaf of transformations*. – We say that a multiplicity M_p of the space $(x_1, \ldots, x_n)$ whose dimension is $p \leq m$:

$$F_h(x_1, \ldots, x_n) = 0 \qquad (h = 1, 2, \ldots, n - p)$$

is an *integral multiplicity* of the sheaf $\{X_1, \ldots, X_m\}$ if it remains invariant under p divergent transformations of the sheaf. In that definition, it is implicit that none of the transformations in question leave every point of the multiplicity invariant. It will then be generated by the trajectories of each of those transformations $U_1, \ldots, U_p$, and also by the trajectories of any transformation of the sheaf $\{U_1, \ldots, U_p\}$. Conversely, any family of ∞^{p-1} curves that is not exceptional and generates

the multiplicity M_p is composed of the trajectories of one of the transformations of that proposed subsheaf $\{U_1, \ldots, U_p\}$.

In particular, the one-dimensional integral multiplicities of a sheaf are the trajectories of the various transformations of that sheaf.

If the sheaf $\{X_1, \ldots, X_m\}$, where $m < n$, is complete then there will be m-dimensional integral multiplicities whose general equations are obtained in the form:

$$F_h(x_1, \ldots, x_n) = c_h \qquad (h = 1, 2, \ldots, n - m) \tag{6}$$

upon equating $n - m$ independent arbitrary solution to the complete system $X_1 = X_2 = \ldots = X_m = 0$ to constants. There are also integral multiplicities with an arbitrary number $p < m$ of dimensions that are easily deduced from the preceding ones.

In general, to *integrate* a sheaf of infinitesimal transformations is to determine all of its integral multiplicities. We shall see that this problem is no different from the one that consists of integrating the most general differential systems.

4. *Sheaves of transformations and Pfaff systems.* – The theory of sheaves of infinitesimal transformations and the theory of Pfaff equations are two equivalent theories that correspond by a sort of *duality*. One can show that by a method that is due to Cartan. Associate the transformations of the basis for the given sheaf $\{X_1, \ldots, X_m\}$ with $n - m = s$ arbitrary infinitesimal transformations $Z_1, \ldots, Z_s$ in such a manner that $X_1, \ldots, X_m, Z_1, \ldots, Z_s$ will be collectively divergent, and we will have an identity of the form:

$$df = \varpi_1 X_1 + \ldots + \varpi_m X_m + \omega_1 Z_1 + \ldots + \omega_s Z_s\,, \tag{7}$$

in which $\varpi_1, \ldots, \varpi_m, \omega_1, \ldots, \omega_s$ are n independent Pfaff expressions. It then results that the infinitesimal displacements that satisfy the Pfaff system:

$$\omega_1 = \omega_2 = \ldots = \omega_s = 0 \tag{8}$$

are precisely the ones that correspond to the various infinitesimal transformations of the sheaf $\{X_1, \ldots, X_m\}$.

Any integral multiplicity of the sheaf $\{X_1, \ldots, X_m\}$ is then an integral multiplicity of the system (8) and conversely. The integral multiplicities are defined in the two cases by only two different procedures.

Conversely, the same method will permit one to make any system of Pfaff equations correspond to an equivalent sheaf of transformations. Cartan appealed to it in order to exhibit the correspondence between the *completely-integrable* Pfaff systems and the *complete systems* of homogeneous linear partial differential equations.

We say that sheaf $\{X_1, \ldots, X_m\}$ and a system $\omega_1 = \ldots = \omega_s = 0$ that it corresponds to are *correlative* or *dual* to each other. It results from what we just recalled that if the sheaf is complete then the Pfaff system will be completely integrable, and conversely.

5. *Sheaves of transformations and partial differential equations.* – The foregoing will suffice to show that the integration of any system of partial differential equations depends upon the integration of a sheaf of infinitesimal transformations. However, it is useful to give the direct proof.

Imagine an arbitrary differential system (Σ). One can suppose that they have order one by taking, if necessary, a certain number of derivatives of the unknown functions to be auxiliary variables. Therefore, let $x_1, \ldots, x_p$ be independent variables, let $y_1, \ldots, y_q$ be unknown functions, and let $y_{j,i} = \partial v_j / \partial x_i$ be their derivatives.

On an arbitrary p-dimensional multiplicity, $y_1, \ldots, y_q$ and their derivatives $y_{j,i}$ will be well-defined functions $\bar{y}_1, \ldots, \bar{y}_q, \ldots, \bar{y}_{j,i}, \ldots$ of $x_1, \ldots, x_p$, and that multiplicity will admit the p infinitesimal transformations:

$$\frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{q} \bar{y}_{\alpha,i} \frac{\partial f}{\partial x_\alpha} \qquad (i = 1, 2, \ldots, p) . \tag{9}$$

As a result, it will admit any system of p infinitesimal transformations:

$$X_i f = \frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{q} \eta_{\alpha,i}(x_1, \ldots, x_p, y_1, \ldots, y_q) \frac{\partial f}{\partial x_\alpha} \qquad (i = 1, 2, \ldots, p) , \tag{10}$$

such that the $\eta_{\alpha,i}$ will become identical to the $\bar{y}_{\alpha,i}$ with the same indices when one replaces the variables y_j with the corresponding functions $\bar{y}_j$.

Hence, if the system (Σ) is solved for all of the derivatives $y_{j,i}$, i.e., it has the form:

$$y_{j,i} = \eta_{j,i}(x_1, \ldots, x_p, y_1, \ldots, y_q) \qquad (i = 1, 2, \ldots, p ; j = 1, 2, \ldots, q) , \tag{11}$$

then any integral multiplicity of (Σ) will admit the transformations (10) as soon as one replaces the $\eta_{j,i}$ in them with the right-hand sides of equations (11), and conversely any p-dimensional integral multiplicity of the sheaf (10) thus-defined will be an integral multiplicity of the system (Σ). To integrate (Σ) will then be to find the p-dimensional integral multiplicities of the sheaf (10), which are deduced from it immediately.

In the general case, the equations of the system (Σ) can be put into the form:

$$y_{j,i} = P_{j,i}(x_1, \ldots, x_p, y_1, \ldots, y_q, w_1, \ldots, w_r) \qquad (i = 1, 2, \ldots, p ; j = 1, 2, \ldots, q), \tag{120}$$

in which $w_1, \ldots, w_r$ are conveniently-chosen indeterminates ([1]). They can be some of the derivatives $y_{j,i}$, and more generally, well-defined functions of the coordinates of a point of the multiplicity and the derivatives $y_{j,i}$. Be that as it may, if one takes into account the equations (12)

([1]) If the derivatives of some of the unknown functions do not appear in the equations of the system (Σ) then one can delete those functions from the list $y_1, \ldots, y_q$ and introduce them among the $w_1, \ldots, w_r$. In the considerations that follow, that will imply some formal modifications that I shall omit, to abbreviate.

of the system then the set of values of $x_1, \ldots, x_p, y_1, \ldots, y_q, w_1, \ldots, w_r$ will collectively define a *contact element on any integral multiplicity*.

One will have then realized a *prolongation* (in the sense of Sophus Lie) of the integral multiplicity, and it is those *prolonged integral multiplicities* that we propose to find. On each of them, not only will the y_j be well-defined functions $\bar{y}_j$ of the variables $x_1, \ldots, x_p$, but the w_k will also be well-defined functions $\bar{w}_k$ of those variables that are, moreover, defined once one is given the initial multiplicity, due to equations (12).

Each (initial) integral multiplicity will then admit the transformations:

$$X_i f = \frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{q} P_{\alpha,i}(x_1,\ldots,x_p,y_1,\ldots,y_q,w_1,\ldots,w_r)\frac{\partial f}{\partial y_\alpha} \qquad (i = 1, 2, \ldots, p) \tag{13}$$

as soon as one replaces the y_j and the w_k in them with the functions $\bar{y}_j$ and $\bar{w}_k$ that correspond to that multiplicity. However, if one would like to pass to the prolonged multiplicity then one must introduce the derivatives:

$$w_{k,i} = \frac{\partial w_k}{\partial x_i}$$

and replace the transformations (9) with the *prolonged transformations:*

$$\frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{q} \bar{y}_{\alpha,i}\frac{\partial f}{\partial y_\alpha} + \sum_{\beta=1}^{r} \bar{w}_{\beta,i}\frac{\partial f}{\partial w_\beta} \qquad (i = 1, 2, \ldots, p)\ . \tag{14}$$

Now the equations of the system (Σ) give no indication in regard to those functions $\bar{w}_{\beta,i}$ (which are the derivatives of the functions $\bar{w}_\beta$). One can then confirm only that any prolonged integral multiplicity will admit p infinitesimal transformations of the form:

$$X_i f + \sum_{\beta=1}^{r} w_{\beta,i}\frac{\partial f}{\partial w_\beta} \qquad (i = 1, 2, \ldots, p), \tag{15}$$

in which the X_i are the transformations (13) that are known, and the $w_{\beta,i}$ are functions of the $x_1, \ldots, x_p, y_1, \ldots, y_q, w_1, \ldots, w_r$ that remain to be chosen conveniently.

The converse is immediate, and one can conclude that integrating the system (Σ) is equivalent to determining the p-dimensional integral multiplicities of the sheaf, which are known from equations (12) and formulas (13):

$$\left\{ X_1,\ldots,X_p, \frac{\partial f}{\partial w_1},\ldots,\frac{\partial f}{\partial w_r} \right\}\ . \tag{16}$$

6. *Examples.* –

1. The sheaf of infinitesimal transformations that corresponds to the partial differential equations:

$$\frac{\partial y}{\partial x_p} = \Phi\left(x_1,\ldots,x_p,y,\frac{\partial y}{\partial x_1},\ldots,\frac{\partial y}{\partial x_{p-1}}\right)$$

is defined by the transformations:

$$X_i = \frac{\partial f}{\partial x_i} + y_i\frac{\partial f}{\partial y} \quad (i = 1, 2, \ldots, p-1),$$

$$X_p = \frac{\partial f}{\partial x_p} + \Phi(x_1,\ldots,x_p,y,y_1,\ldots,y_{p-1})\frac{\partial f}{\partial y} \ ,$$

and the transformations $\frac{\partial f}{\partial y_i}, \ldots, \frac{\partial f}{\partial y_{p-1}}$.

2. The sheaf that corresponds to the second-order partial differential equation:

$$t = \Phi\,(x, y, z, p, q, r, s)$$

will have the four transformations:

$$\frac{\partial f}{\partial x} + r\frac{\partial f}{\partial z} + x\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q}, \qquad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + \Phi\frac{\partial f}{\partial q}, \qquad \frac{\partial f}{\partial r}, \qquad \frac{\partial f}{\partial s}$$

for a basis.

3. The sheaf that corresponds to the system of second-order partial differential equations:

$$s = \Phi\,(x, y, z, p, q, r), \qquad t = \Psi\,(x, y, z, p, q, r)$$

is defined by the three transformations:

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + \Phi\frac{\partial f}{\partial q}, \qquad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + \Phi\frac{\partial f}{\partial p} + \Psi\frac{\partial f}{\partial q}, \qquad \frac{\partial f}{\partial r}.$$

7. *Complete integrals*. – I say, to abbreviate, that a family of multiplicities is *regular* if one and only one multiplicity of the family passes through each point in space ([1]), and I call any regular family of integrals of a sheaf with an arbitrary number of dimensions a *complete integral* of that sheaf.

Let $\{X_1, \ldots, X_m\}$ be a sheaf, and let $\omega_1 = \omega_2 = \ldots = \omega_s = 0$ be the dual Pfaff system [no. **4**]. If the sheaf admits a complete integral then one can reduce it to the form $x_{p+1} = c_1, \ldots, x_n = c_{n-p}$. The Pfaff system must then be verified under the hypothesis that $dx_{p+1} = \ldots = dx_n = 0$ for any $x_1, \ldots, x_p$, $dx_1, \ldots, dx_p$ when one replaces $x_{p+1}, \ldots, x_n$ with arbitrary constants, i.e., it will be verified identically as soon as one sets $dx_{p+1} = \ldots = dx_n = 0$. Hence, $\omega_1, \ldots, \omega_s$ are linear forms in the $dx_{p+1}, \ldots, dx_n$.

Furthermore, one can satisfy the identity (7) [no. **4**] by taking $\varpi_1, \ldots, \varpi_p$ to be the differentials $dx_1, \ldots, dx_p$, and taking $\varpi_{p+1}, \ldots, \varpi_m$ to be forms in $dx_{p+1}, \ldots, dx_n$. It will then result that one can suppose that $X_1 = \partial f / \partial x_1, \ldots, X_p = \partial f / \partial x_p$, i.e., that there exists a complete subsheaf in the given sheaf that has the supposed complete integral for its general integral.

The search for the complete integrals of a sheaf is then equivalent to the search for complete subsheaves of that sheaf. It is implicit that once those complete subsheaves have been found, it will still remain for one to integrate them. However, one does that by integrating ordinary differential equations, and the goal of the general theory of general differential systems is to carry out or simplify their integration by integrating ordinary differential equations.

The first part of our theory of integration will then have the goal of discussing the existence of complete subsheaves of a sheaf of infinitesimal transformations that are supposed to be given. The following section will be devoted to that question.

As for the isolated integral multiplicities, some of them will belong to complete integrals, and their determination will depend upon the theory in question: One can say that they are *particular integrals*.

From the same standpoint, the other ones will be *singular integrals*. Indeed, part of the essential character of our theory must be that it is independent of any change of variables. Now, it is easy to show, by examples, that conveniently-chosen changes of variables can make integrals of the type that we call singular appear or disappear in a sheaf. Hence, the sheaf $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial z}$ has no two-dimensional integral multiplicity. Meanwhile, it will take on a (singular) integral $x' = 0$ under the change of variables $x' = e^x$, $y' = y$, $z' = z$. Similarly, it will take on the (singular) integral $y' = 0$ under the change of variables $x, y = y\,x', z = z'$. Indeed, those singular integrals are introduced in favor of a singularity in the changes of variables themselves.

Moreover, we shall return to the determination of *all* integral multiplicities (whether singular or not) at the end of the following section.

([1]) In that general study, we ignore all singularities. Here, it is then implicit that we shall conveniently limit the space and multiplicities as appropriate. Restrictions of the same type are made implicitly in all of the analogous cases.

II. – INVOLUTIVE SHEAVES. EXISTENCE OF COMPLETE INTEGRALS. PROLONGATION OF A SHEAF.

8. *Involutions of a sheaf.* – Conforming to the conclusion of the preceding section, we shall move on to the search for complete subsheaves of an arbitrary given sheaf $\{X_1, \ldots, X_m\}$. If that given sheaf $\mathcal{F}$ is complete then it will admit a unique m-dimensional complete integral:

$$F_h(x_1, \ldots, x_n) = c_h \qquad (h = 1, 2, \ldots, n - m)\,, \tag{1}$$

and in order to obtain the most general p-dimensional complete integral of the sheaf $\mathcal{F}$, it will suffice to intersect it with an arbitrary family of regular multiplicities:

$$G_k(x_1, \ldots, x_n) = a_k \qquad (k = 1, 2, \ldots, m - p)\,. \tag{2}$$

We then suppose that the given sheaf $\mathcal{F}$ is not complete, and we introduce the derived sheaf $\{X_1, \ldots, X_m\,;\, Z_1, \ldots, Z_s\}$. Here, the sum $s + m$ is equal to at most the number n of variables. From the definition of a derived sheaf [no. **2**], we will have some identities of the form:

$$(X_i, X_j) = \sum_{\alpha=1}^{s} c_{i,j,\alpha}(x_1,\ldots,x_n)\, Z_\alpha + \sum_{\beta=1}^{m} g_{i,j,\beta}(x_1,\ldots,x_n)\, X_\beta \qquad (i, j = 1, 2, \ldots, m), \tag{3}$$

in which the functions $c_{i,j,h}$ define what we can call the *structure* of the sheaf ([1]).

The conditions that express the idea that the subsheaf:

$$U_h = \sum_{\alpha=1}^{m} u_{h,\alpha}(x_1,\ldots,x_n)\, X_\alpha \qquad (h = 1, 2, \ldots, p) \tag{4}$$

is complete are written, in turn, as:

$$0 = (U_h, U_k) = \sum_{\alpha=1}^{s}\left(\sum_{\gamma=1}^{m}\sum_{\delta=1}^{m} c_{\gamma,\delta,\alpha}\, u_{h,\gamma}\, u_{k,\delta}\right) Z_\alpha + \sum_{\beta=1}^{m}\left(U_h\, u_{k,\beta} - U_k\, u_{h,\beta} + \sum_{\gamma=1}^{m}\sum_{\delta=1}^{m} g_{\gamma,\delta,\beta}\, u_{h,\gamma}\, u_{k,\delta}\right) X_\beta\,, \tag{5}$$

which gives the conditions:

$$\sum_{\gamma=1}^{m}\sum_{\delta=1}^{m} c_{\gamma,\delta,\alpha}\, u_{h,\gamma}\, u_{k,\delta} = 0 \qquad (j = 1, 2, \ldots, s) \tag{6}$$

and

([1]) It will suffice to take the sheaf in the solved form [no. **1**] in order to make the functions $g_{i,j,k}$ disappear from formulas (3).

(7) $$U_h\, u_{k,\beta} - U_k\, u_{h,\beta} + \sum_{\gamma=1}^{m}\sum_{\delta=1}^{m} g_{\gamma,\delta,\beta}\, u_{h,\gamma}\, u_{k,\delta} = 0 \qquad (i = 1, 2, \ldots, m).$$

It is convenient to first consider the conditions (6), which have a purely-algebraic character. They are equivalent to the identities-congruences:

(8) $$(U_h, U_k) \equiv 0 \pmod{\mathcal{F}} \qquad (h, k = 1, 2, \ldots, p),$$

as one sees immediately from the expressions (5) for the brackets (U_h, U_k).

We say that two infinitesimal transformations of the sheaf $\mathcal{F}$ are *in involution* if their bracket belongs to the sheaf, i.e., if that bracket is congruent to zero (mod $\mathcal{F}$), and that a subsheaf $\{U_1, \ldots, U_p\}$ of $\mathcal{F}$ is *an involution of degree p* of that sheaf if its transformations are pair-wise in involution ([1]). It is among the involutions of degree p in the sheaf $\mathcal{F}$ that we will find the complete subsheaves of degree p.

9. *Involutive sheaves of order p*. – In order to determine the *general involution of degree p*, one can write the congruences (8) in the form:

$$\begin{array}{llllll}
(8_1) & (U_1, U_2) \equiv 0\,, & & & & \\
(8_2) & (U_1, U_3) \equiv 0\,, & (U_2, U_3) \equiv 0\,, & & & \\
\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & & & \\
(8_{p-1}) & (U_1, U_p) \equiv 0\,, & (U_2, U_p) \equiv 0\,, & \ldots, & (U_p, U_{p-1}) \equiv 0\,. &
\end{array}$$

If U_1 is given as a general transformation of the sheaf $\mathcal{F}$ then one takes U_2 to be the general solution of the identity-congruence:

$$(9_1) \qquad (U_1, U) \equiv 0 \qquad \Big[U = \sum_{\alpha=1}^{m} u_\alpha\,(x_1, \ldots, x_n)\, X_\alpha\,\Big]\,,$$

and then takes U_3 to be the general solution of the identities-congruences:

$$(9_2) \qquad (U_1, U) \equiv 0, \quad (U_2, U) \equiv 0,$$

and so on, until one ultimately takes U_p to be the most general solution of the identities-congruence:

$$(9_{p-1}) \qquad (U_1, U) \equiv 0\,, \quad (U_2, U) \equiv 0\,, \quad \ldots, \quad (U_{p-1}, U) \equiv 0\,,$$

([1]) The elementary displacements that the transformations of an involution of degree p determines at each point of space constitute an *integral element* of order p, in Cartan's terminology.

if no obstruction presents itself.

Of course, it is necessary that (9_1) must admit other solutions than the transformations of the sheaf $\{U_1\}$, that (9_2) must admit other solutions than the transformations of the sheaf $\{U_1, U_2\}$, and so on, and finally, that (9_{p-1}) must admit other solutions than the transformations of the sheaf $\{U_1, \ldots, U_{p-1}\}$.

If that is true then the sheaf $\mathcal{F}$ will be called an *involutive sheaf of order (at least) p. The definition of an involutive sheaf of order p is then the following one: The general transformation of the sheaf belongs to an involution of degree at least* 2. *The general involution of degree* 2 *belongs to an involution of degree at least* 3, *and so on. Finally, the general involution of degree* $p - 1$ *belongs to an involution of degree at least p*.

One notes that all systems (9_1), (9_2), …, (9_{p-1}) are equivalent to systems of equations that are homogeneous and linear in $u_1, \ldots, u_m$. The degree of indeterminacy of each of them is then fixed by the *rank* of the linear system that it then provided (viz., the degree of the principal determinant), and it cannot be raised if one successively specializes $U_1, U_2, \ldots, U_{p-i}$. We let $q_1, q_2, \ldots, q_{p-1}$ denote the ranks of those linear systems, which are calculated by keeping all of the indeterminacy in $U_1, U_2, \ldots, U_{p-1}$ that is found in each of those transformations, in succession, and is susceptible to the application of the preceding calculations.

In other words, q_1 *is the number of independent linear equations* (in $u_1, u_2, \ldots, u_m$) *that express the idea that the transformation* $U = u_1 X_1 + u_2 X_2 + \ldots + u_m X_m$ *of the sheaf is in involution with the general transformation (of the sheaf).* q_2 *is the number of independent linear equations that express the idea that U is in involution with each of the transformations of the general involution of degree* 2 *(of the sheaf), and so on.*

If $\mathcal{F}$ is an involutive sheaf of order p then, from the foregoing, one will have:

$$q_1 + 1 < m\,, \quad q_2 + 2 < m\,, \quad \ldots, \quad q_{p-1} + (p-1) < m\,, \tag{10}$$

since $m - q_1, m - q_2, \ldots, m - q_{p-1}$ are the numbers of independent solutions of the successive systems (9_1), (9_2), …, (9_{p-1}). We remark that, on the other hand:

$$q_1 \leq q_2 \leq \ldots \leq q_{p-1}\,, \tag{11}$$

since the equations of each of the linear systems considered belong to the following system. Finally, if one sets $q = m - p$ then the last of the inequalities (10) is written $q_{p-1} \leq q$.

10. *Genus, indices, and characters of a sheaf.* – The *genus g* of a sheaf $\mathcal{F}$ is the maximum order of involutivity of the sheaf. By definition, a sheaf of genus g is then involutive of order p if $p \leq g$ and it is not if $p > g$.

Consider the identities-congruences for such a sheaf:

$$(U_1, U) \equiv 0, \quad (U_2, U) \equiv 0, \quad \ldots, \quad (U_g, U) \equiv 0, \tag{12}$$

in which $\{U_1, \ldots, U_g\}$ is the general involution of degree g of sheaf. The rank of the corresponding linear system in $u_1, \ldots, u_m$ is then:

(13) $$q_g = m - g\,,$$

since the congruences (12) are satisfied only if U belongs to the sheaf $\{U_1, \ldots, U_g\}$, from the definition of the genus g.

The *characters* ([1]) of the sheaf are, by definition, the positive or zero integers:

(14) $$s_1 = q_1\,, \qquad s_2 = q_2 - q_1\,, \quad s_3 = q_3 - q_2\,, \quad \ldots, \qquad s_g = q_g - q_{g-1}\,,$$

and conversely, one will have one calls the *indices of the sheaf* for the integers $q_1, q_2, \ldots, q_g$:

(15) $$q_1 = s_1\,, \qquad q_2 = s_1 + s_2\,, \quad q_3 = s_1 + s_2 + s_3\,, \qquad \ldots, \qquad q_g = s_1 + s_2 + \ldots + s_g\,.$$

We remark that if s_2 is the number of independent equations of the linear system that is provided by (91) and $s_1 + s_2$ is the number of (independent) equations of the linear system that is equivalent to (92) then s_2 is the number of (independent) equations that $(U_2, U) \equiv 0$ adds to the ones that are provided by $(U_1, U) \equiv 0$. Therefore, s_2 cannot be greater than the number of independent equations that are provided by $(U_2, U) \equiv 0$, when considered by itself, and the latter number cannot be greater than s_1 [no. **9**], but it can be less, since U_2 is in involution with U_1, so it will no longer be the most general transformation of the sheaf $\mathcal{F}$.

One then concludes that $s_2 \leq s_1$, and similarly, $s_3 \leq s_2$, and so on.

Hence, the characters of the sheaf are coupled by the inequalities:

(16) $$s_1 \geq s_2 \geq s_3 \geq \ldots \geq s_g\,.$$

As a result, if one of the characters is zero then all of the following ones will also be so.

11. *Solved form of the involutions.* – Let us return to the study of the involutions of a sheaf while preserving all of the preceding notations. In the general involution of degree p of the sheaf $\mathcal{F}$:

(17) $$U_h = u_{h,1}\, X_1 + \ldots + u_{h,m}\, X_m \qquad (h = 1, 2, \ldots, p;\ p \leq g),$$

which is supposed to be calculated by the method in no. **9**, the coefficients $u_{1,\alpha}$ ($\alpha = 1, 2, \ldots, m$) are arbitrary, the coefficients $u_{2,\alpha}$ are coupled by q_1 independent linear equations (whose coefficients depend upon the $u_{1,\alpha}$), the coefficients $u_{3,\alpha}$ are coupled by q_2 independent linear

([1]) The genus and characters of a sheaf are the same numbers that Cartan introduced under the same names for the dual Pfaff system of the sheaf. The inequalities that we establish in regard to them are therefore not new.

equations (whose coefficients depend upon the $u_{1,\alpha}$ and $m - q_1$ functions $u_{2,\alpha}$ that remain arbitrary), and so on. Taken together, the coefficients:

$$u_{\alpha\beta} \qquad (\alpha = 1, 2, \ldots, p\ ;\ \beta = 1, 2, \ldots, m)$$

are then coupled by $q_1 + q_2 + \ldots + q_{p-1}$ independent equations.

In order to have the number of arbitrary variables in any subsheaf (17), one must take it in its solved form, for example, in the form:

$$V_h = X_h + \sum_{\alpha=1}^{q} v_{\alpha,h}\, X_\alpha \qquad (h = 1, 2, \ldots, p\ ;\ q = m - p). \tag{18}$$

That number of arbitrary variables is therefore pq, and the number of *essential* arbitrary variables in the general involution of degree p of the sheaf $\mathcal{F}$ is equal to:

$$Q = pq - (q_1 + q_2 + \ldots + q_{p-1})\ . \tag{19}$$

In order to exhibit those Q arbitrary variables, recall the search for that general involution by putting into the form (18) from the outset and following the same route as in no. **9**. We must successively consider the systems of identities-congruences:

$$(20_1) \qquad (V_1, U) \equiv 0\ ,$$

$$\begin{array}{lllll}
(20_2) & (V_1, U) \equiv 0\ , & (V_2, U) \equiv 0\ , & & \\
\ldots & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, & & \\
(20_{p-1}) & (V_1, U) \equiv 0\ , & (V_2, U) \equiv 0\ , & \ldots, & (V_{p-1}, U) \equiv 0\ ,
\end{array}$$

in which we have further set:

$$U = u_1 X_1 + \ldots + u_m X_m\ .$$

If the transformation V_1 is non-singular [no. **9**] for arbitrary values of $v_{\alpha,1}$ then the general solution to (20_1) in terms of $u_1, \ldots, u_m$ will depend upon $m - q_1$ arbitrary ones. Furthermore, $u_1, \ldots, u_p$ cannot be coupled by any relation, because when the V_h are the ones that the first method gives, the system (20_1) will admit the solution:

$$u_1 V_1 + u_2 V_2 + \ldots + u_p V_p = u_1 X_1 + u_2 X_2 + \ldots + u_p X_p + \ldots,$$

in which the $u_1, \ldots, u_p$ are arbitrary. Hence, if $u_1, \ldots, u_p$ are arbitrary for a certain choice of the $v_{\alpha,1}$ then they will also be so *a fortiori* when the $v_{\alpha,1}$ are indeterminate. Hence, in the general solution of (20_1), the $u_1, \ldots, u_p$ are arbitrary, as well as the $m - p - q_1$ other coefficients of U, whereas the other coefficients of U are expressed in terms of them. It then results that there are solutions of the

form V_2 (which are obtained for $u_1 = 0$, $u_2 = 1$, $u_3 = 0$, …, $u_p = 0$), and that in the most general of them, q_1 of the $v_{\alpha,1}$ are expressed in terms of $q - q_1$ other ones, which will remain arbitrary.

If the general involution $\{V_1, V_2\}$ thus-obtained is non-singular [no. **9**] then one can argue similarly with the system (20_2), and so on.

Thus, the arbitrariness in the general involution (18) comes from: The q coefficients $v_{\alpha,1}$, $q - q_1$ of the $v_{\alpha,2}$, $q - q_2$ of the $v_{\alpha,3}$, …, $q - q_{p-1}$ of the $v_{\alpha,p}$. The $v_{\alpha,2}$ that are not arbitrary are expressed in terms of the $v_{\alpha,1}$, and the $v_{\alpha,2}$ remain arbitrary. The $v_{\alpha,3}$ that are not arbitrary are expressed in terms of the $v_{\alpha,1}$, and the $v_{\alpha,2}$ and $v_{\alpha,3}$ remain arbitrary, and so on.

One can remark, moreover, that if $q - q_{k-1}$ of the coefficients u_1, …, u_m are arbitrary in the general solution of (20_k) then they will be arbitrary *a fortiori* in the general solution to (20_{k-1}), which is contained in (20_k). One can then choose the notations in such a manner that the arbitrary coefficients are:

$$v_{1,1}, \ldots, v_{q,1}; \qquad v_{q_1+1,2}, \ldots, v_{q,2}; \qquad v_{q_2+1,3}, \ldots, v_{q,3}; \qquad \ldots, \qquad v_{q_{p-1}+1,p}, \ldots, v_{q,p}.$$

It remains to show that one can take X_1, …, X_p in such a manner that the successive involutions:

$$\{V_1\}, \quad \{V_1, V_2\}, \qquad \ldots, \qquad \{V_1, V_2, \ldots, V_{p-1}\}$$

to which an application of the method will lead are all non-singular.

To that effect, we remark that this will be the case when one takes X_1, …, X_p to be pair-wise in involution and such that the involutions:

$$\{X_1\}, \quad \{X_1, X_2\}, \qquad \ldots, \qquad \{X_1, X_2, \ldots, X_{p-1}\}$$

are all non-singular. That is true because, by hypothesis, that will be the case when one annuls all of the $v_{\alpha,h}$ ($h = 1, 2, \ldots, p$), and as a result, it will be true *a fortiori* when one subjects them to only the conditions $(V_h, V_j) \equiv 0$ ($h, j = 1, 2, \ldots, p$) that those conditions will be verified by hypothesis when one annuls all of the $v_{\alpha,h}$.

Having said that, start from an arbitrary basis $\{X_1^0, \ldots, X_m^0\}$ of the sheaf $\mathcal{F}$, and take:

$$X_h = \sum_{\alpha=1}^{m} u_{\alpha,h} X_\alpha^0 \qquad (i = 1, 2, \ldots, m)$$

in the foregoing, in which $u_{\alpha,i}$ are indeterminate (and functions of the x_1, …, x_n, like all of the arbitrary things that we introduce). V_1 will cease to be non-singular only when the coefficients $u_{\alpha,1}$ satisfy a certain system S_1 of algebraic equations. The involution $\{V_1, V_2\}$ that is then calculated by the preceding method will then cease to be non-singular only if the coefficients $u_{\alpha,1}$ and $u_{\alpha,2}$ satisfy a certain system S_2 of algebraic equations, and so on. None of those systems S_1, S_2, … is verified by all systems of values for the $u_{\alpha,\beta}$, since we have seen that one can choose the $u_{\alpha,\beta}$ in such a manner that none of the systems S_1, S_2, … is verified.

The method can certainly be applied then without one being obligated to take the $X_1, \ldots, X_p$ in the way that we just did, viz., transformations that are pair-wise in involution.

I shall now say that *upon performing a change of the preliminary variables, if necessary, one can apply the method by taking* $X_1, \ldots, X_m$ *to be the basis for the sheaf when it has been solved for the derivatives* $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}$ [no. **1**].

Indeed, suppose that $\{X_0^1, \ldots, X_m^0\}$ is precisely that solved form:

$$X_k^0 = \frac{\partial f}{\partial x_1} + \sum_{\alpha=m+1}^{n} \xi_{k,\alpha}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_m} \qquad (k = 1, 2, \ldots, m),$$

and make a change of variables of the form:

$$x_i = \varphi_i(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) \qquad (i = 1, 2, \ldots, m). \tag{21}$$

Now solve the sheaf in the form:

$$Y_h = \frac{\partial f}{\partial y_h} + \sum_{\alpha=m+1}^{n} \alpha_{h,\alpha}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) \frac{\partial f}{\partial x_\alpha} \qquad (h = 1, 2, \ldots, m)\,.$$

Under a change of variables (21), Y_h will become, conversely, a transformation of the sheaf $\mathcal{F}$:

$$X_h = Y_h\,\varphi_1 \frac{\partial f}{\partial x_1} + Y_h\,\varphi_m \frac{\partial f}{\partial x_m} + \eta_{h,m+1} \frac{\partial f}{\partial x_{m+1}} + \cdots + \eta_{h,n} \frac{\partial f}{\partial x_n}\,,$$

so one will have the identity:

$$X_h = \sum_{\alpha=1}^{m} Y_h\,\varphi_\alpha\, X_\alpha^0 \qquad (h = 1, 2, \ldots, m)\,.$$

For an arbitrary choice of the functions φ_α, the $Y_h\,\varphi_\alpha = \frac{\partial \varphi_\alpha}{\partial Y_h} + \ldots$ cannot be coupled by any system of algebraic equations that are not of an identical nature. Hence, they will not generally satisfy any of the differential systems that one deduces from the systems $S_1, S_2, \ldots$ by setting $u_{\alpha,h} = Y_h\,\varphi_\alpha$, and that proves that our assertion is legitimate.

12. *Existence of complete subsheaves of degree* 2. – We can now establish the existence of complete subsheaves for all degrees p that do not exceed the genus g of the sheaf $\mathcal{F}$ considered,

i.e., complete integrals that have a given number p of dimensions that is equal to at most that genus g. We begin with the subsheaves of degree 2, and we pass on the general case by recurrence.

We take the transformations for the basis of the sheaf in the solved form [no. **1**, eq. (5)]:

$$X_k = \frac{\partial f}{\partial x_k} + \sum_{\alpha=m+1}^{n} \xi_{k,\alpha}(x_1,\ldots,x_m)\frac{\partial f}{\partial x_\alpha} \qquad (k = 1, 2, \ldots, m)\,, \tag{22}$$

in such a way that the identities (3) [no. **8**] will have the form:

$$(X_i\,, X_j) = \sum_{\alpha=1}^{s} c_{i,j,\alpha}(x_1,\ldots,x_m)\, Z_\alpha \qquad (i, j = 1, 2, \ldots, m)\,, \tag{23}$$

in which the Z_h are themselves transformations of the sheaf:

$$\left\{\frac{\partial f}{\partial x_{m+1}},\ldots,\frac{\partial f}{\partial x_n}\right\}.$$

Start from the general involution of degree 2 $\{V_1, V_2\}$ taken in the form:

$$V_1 = X_1 + \sum_{\alpha=1}^{q} v_{\alpha,1}\, X_{\alpha+2}\,, \qquad V_2 = X_2 + \sum_{\alpha=1}^{q} v_{\alpha,2}\, X_{\alpha+2} \qquad (q = m-2). \tag{24}$$

Since one has involution [no. **8**], one has identically:

$$(V_1, V_2) = \sum_{\alpha=1}^{q} (V_1\, v_{\alpha,2} - V_2\, v_{\alpha,1})\, X_{\alpha+2}\,,$$

in such a way that the sheaf $\{V_1, V_2\}$ will be complete under the conditions that:

$$V_1\, v_{\alpha,2} - V_2\, v_{\alpha,1} = 0 \qquad (\alpha = 1, 2, \ldots, q). \tag{25}$$

Since $\{V_1, V_2\}$ is the general involution of degree 2, one can consider [no. **11**] the $v_{\alpha,2}$ and $q - q_1$ of the $v_{\alpha,1}$ in formulas (24) to be arbitrary; the other $v_{\alpha,1}$ are functions of those arbitrary ones ([1]). If one replaces those $v_{\alpha,1}$ with their expressions in the condition equations (25) then one can consider the other $v_{\alpha,1}$ (for example, $v_{q_1+1,1}, \ldots, v_{q,1}$) to be arbitrarily-chosen functions of the $x_1, \ldots, x_n$, in such a way that they will be a system of partial differential equations that relate to only the $v_{1,1}, \ldots, v_{q,2}$.

([1]) One can take $X_1, \ldots, X_p$ in any order in the considerations of no. **11**.

It then results from the solved form for X_1 and X_2 that this system is a Kowalewski system. It will then admit a solution for which the functions $v_{\alpha,2}$ (x_1^0 , x_2, …, x_n), in which x_1^0 is an arbitrary numerical value of x_1, are arbitrarily-given functions, and that is the general solution of that system (25).

We then conclude that if a sheaf has genus ≥ 2 then it will contain complete subsheaves of degree 2, and the *general* complete sheaf of degree 2 will depend upon $n - 2 - q_1$ arbitrary functions of n arguments and $m - 2$ arbitrary functions of $n - 1$ arguments. Recall that n is the total number of variables and m is the degree of the sheaf.

13. *General existence theorem.* – We shall preserve the notations of the preceding sections and look for the complete subsheaves of arbitrary degree p, which is less than or equal to the genus g of the given sheaf $\mathcal{F}$. We suppose that such a subsheaf has been put into the solved form:

$$V_i = X_i + \sum_{\alpha=1}^{q} v_{\alpha,i}\, X_{\alpha+p} \qquad (i = 1, 2, \ldots, p\ ;\ q = m - p)\ . \tag{26}$$

From the result that was obtained for $p = 2$, it is natural to think that *such subsheaves exist and that for the most general of them, the arbitrary data are* $q - q_{p-1}$ *of the functions* $v_{\alpha,1}$, *the values* ([1]) *of* $q - q_{p-1}$ *of the functions* $v_{\alpha,2}$ *for* $x_1 = x_1^0$, *the values of* $q - q_{p-3}$ *of the functions* $v_{\alpha,3}$ *for* $x_1 = x_1^0$, $x_2 = x_2^0$, *etc., the values of* $q - q_1$ *of the functions* $v_{\alpha,p-1}$ *for* $x_1 = x_1^0$, …, $x_{p-2} = x_{p-2}^0$, *and finally the values of* q *of the functions* $v_{\alpha,p}$ *for* $x_1 = x_1^0$, …, $x_{p-1} = x_{p-1}^0$.

We shall suppose that this theorem was established for the subsheaves of degrees 2, 3, …, p – 1, and examine whether it persists for the subsheaves of degree p.

We will have some identities of the form:

$$(V_i, V_j) = \sum_{\alpha=1}^{q} A_{i,j,\alpha}\, X_{\alpha+\mu} + \sum_{\beta=1} C_{i,j,\beta}\, Z_\beta \qquad (i, j = 1, 2, \ldots, p) \tag{27}$$

upon setting:

$$A_{i,j,\alpha} = V_i\, v_{\alpha,j} - V_j\, v_{\alpha,i} \tag{28}$$

and upon letting $C_{i,j,k}$ denote the left-hand sides of the algebraic equations:

$$C_{i,j,k} = 0 \qquad (i, j = 1, 2, \ldots, p\ ;\ k = 1, 2, \ldots, s)\ , \tag{29}$$

([1]) We shall use the word *value*, to abbreviate, to refer to a function of all of the variables x_1, …, x_n other than the ones to which one gives the name of well-defined *numerical values* x_1^0, x_2^0, …

which expresses the idea that the subsheaf is an involution of $\mathcal{F}$. In order for the subsheaf to be complete (*cf.*, no. **8**), one must add the differential equations:

(30) $$A_{i,j,k} = 0 \qquad (i, j = 1, 2, \ldots, p\ ;\ k = 1, 2, \ldots, q)\ .$$

First consider the equations:

(31) $$C_{i,j,k} = 0 \qquad (j = 2, 3, \ldots, p\ ;\ k = 1, 2, \ldots, s)\ ,$$

which are linear in $v_{1,1}, \ldots, v_{q,1}$.

In the general involution $\{V_1, V_2, \ldots, V_p\}$, they will determine V_1 when one supposes that the general involution $\{V_2, \ldots, V_p\}$ is given. There will then be q_{p-1} of those equations that are independent, and the other ones will be consequences of them. Isolate those q_{p-1} equations and let the notation:

(32) $$\mathrm{r}_h\,(v_{1,1}, \ldots, v_{q,1}) = 0 \qquad (h = 1, 2, \ldots, q_{p-1})$$

denote those of equations (31) that they provide when one supposes that the $v_{\alpha,p}$ are entirely undetermined. Those equations (32) are *a fortiori* independent, and one will have some identities of the form:

(33) $$C_{i,j,k} = \sum_{\alpha=1}^{q_{p-1}} \varphi_{j,k,\alpha}\, F_\alpha + \sum_{\alpha=2}^{p}\sum_{\beta=2}^{p}\sum_{\gamma=1}^{s} \psi_{j,k,j,\alpha,\beta,\gamma}\, C_{\alpha,\beta,\gamma}$$

in order to express the idea that the system (31) will reduce to the system (32) when one introduces the relations between the $v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,p}$ that express the idea that $\{V_2, \ldots, V_p\}$ is an involution, i.e., the equations:

(34) $$C_{i,j,k} = 0 \qquad (i, j = 2, 3, \ldots, p\ ;\ k = 1, 2, \ldots, s)\ .$$

The φ are functions of the $v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,p}$, and the ψ depend upon those indeterminates, along with those of the quantities $v_{\alpha,1}$ that are left arbitrary in equations (32). To fix ideas, we suppose that the latter are $v_{q_{p-1}+1,1}, \ldots, v_{q,1}$ ([1]).

([1]) Indeed, one can first write some identities of the form:

(33, *cont.*) $$C_{i,j,k} = \sum_{\alpha=1}^{q_{p-1}} \varphi_{j,k,\alpha}\, F_\alpha + \sum_{\beta=1}^{q-q_{p-1}} \rho_{j,k,\beta}\, v_{q_{p-1}+\beta,1} + \rho_{j,k,0}\ .$$

The equations $\rho_{j,k,l} = 0$ are consequences of equations (34) because if one supposes that those equations (34) are verified by the $v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,\nu}$ then the $C_{i,j,k}$, which are linear functions of the $v_{1,1}, \ldots, v_{q,1}$, will then become homogeneous combinations of nothing but linear functions (of those variables) $F_1, \ldots, F_{q_{p-1}}$.

Having made that remark, we consider the mixed system:

(35) $$F_h\,(v_{1,1}, \ldots, v_{q,1}) = 0 \qquad (h = 1, 2, \ldots, q_{p-1}),$$

(36) $$A_{1,j,k} = 0 \qquad (j = 2, 3, \ldots, p\,;\ k = 1, 2, \ldots, q).$$

We infer the values of $v_1, \ldots, v_{1,q_{p-1}}$ in equations (35) and substitute them in equations (36). If we then consider $v_{q_{p-1}+1,1}, \ldots, v_{q,1}$ to be arbitrarily-chosen functions of $x_1, \ldots, x_n$ then they will be equations of Kowalewski type in the $v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,p}$, because they are found to be solved for the expressions $V_1\, v_{j,k}$ ($j = 2, 3, \ldots, p\,;\ k = 1, 2, \ldots, q$), and V_1 is the only one of those operators V_i in which the derivative $\partial f / \partial x_1$ appears.

Hence, the system (35), (36) will determine all of the $v_{i,j}$ when one is given (arbitrarily) the expressions for $v_{q_{p-1}+1,1}, \ldots, v_{q,1}$, and the functions of $x_2, \ldots, x_n$ to which the various unknowns $v_{\alpha,2}$, $v_{\alpha,3}, \ldots, v_{\alpha,p}$ ($\alpha = 1, 2, \ldots, q$) reduce for $x_1 = x_1^0$.

14. *Continuation and conclusion.* – We choose those initial data in such a manner that the sheaf $\{V_2^{(0)}, \ldots, V_p^{(0)}\}$ is complete. Here and in what follows, the index (0) signifies that x_1 has been replaced with x_1^0. That should create no difficulties when the derivatives $\partial f / \partial x_1$ does not appear, which is the case for $V_2, \ldots, V_p$.

For the same reason, the existence of complete sheaves $\{V_2^{(0)}, \ldots, V_p^{(0)}\}$ results from that of complete subsheaves $\{V_2, \ldots, V_p\}$, i.e., from the hypothesis that we are reasoning by recurrence itself.

We will thus have completely determined a general type of subsheaf $\{V_2, \ldots, V_p\}$ in which the number and nature of the arbitrary data that were enumerated in the statement of our theorem (no.

Now consider equations (34). One knows (no. **11**) that one solves them step-by-step: One first solves q_1 of them with respect to q_1 of the $v_{\alpha,p-1}$, then solves q_2 other ones with respect to q_2 of the $v_{\alpha,p-2}$, and so on. Finally, one solves q_{p-2} of them with respect to q_{p-2} of the $v_{\alpha,2}$. All of the systems that one must solve are linear and have non-zero determinants: Let $H_1 = H_2 = 0$ be the set of equations of all those systems. If one considers the equations $H_1 = w_1$, $H_2 = w_2, \ldots$, in which $w_1, w_2, \ldots$ will be auxiliary indeterminates that are left to solved.

Substitute the values thus-obtained for the $v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,p-1}$ that one has solved for, and the $\rho_{j,k,l}$ will become rational functions of the $v_{\alpha,2}, \ldots, v_{\alpha,p-1}$ that remain arbitrary and the $w_1, w_2, \ldots$ Each of those rational functions that are annulled for $w_1 = w_2 = \ldots = 0$ will be written in the form $M_1\, w_1 + M_2\, w_2 + \ldots$, where the coefficients $M_1, M_2, \ldots$ can depend upon the $v_{\alpha,2}, \ldots, v_{\alpha,p-1}$ that remain arbitrary and $w_1, w_2, \ldots$ One will get an identical expression for that function ρ upon replacing w_1 with H_1, w_2 with $H_2, \ldots$ in the expression $M_1\, w_1 + M_2\, w_2 + \ldots$ Thus, every function $\rho_{j,k,l}$ can be written in the form of a homogeneous polynomial of degree one in a certain number of the $C_{i,j,k}$, which are the left-hand sides of equations (34). In order to obtain the identities (33) in the text, all that remains then is to substitute the expressions thus-obtained for the $\rho_{j,k,l}$ in the formulas (33, *cont.*).

13) will appear. It remains for us to verify whether the subsheaf thus-obtained is indeed complete, because for the moment we know only that it satisfies the conditions (35), (36), and:

(37) $$A^{(0)}_{i,j,h} = 0, \qquad C^{(0)}_{i,j,h} = 0 \qquad (i, j = 2, 3, \ldots, p\ ;\ h = 1, 2, \ldots, q\ ;\ k = 1, 2, \ldots, s).$$

To that end, start from the Jacobi identity:

$$(V_1, (V_i, V_j)) = (V_i, (V_1, V_j)) - (V_j, (V_1, V_i))\ .$$

In the context of the identities (27) and (36), it will give:

(38) $$\sum_{\alpha=1}^{q} V_1 A_{i,j,\alpha} \cdot X_{p+\alpha} + \sum_{\beta=1}^{s} V_1 C_{i,j,\beta} \cdot Z_\beta + \sum_{\alpha=1}^{q} A_{i,j,\sigma}(V_1, X_{p+\alpha}) + \sum_{\beta=1}^{s} C_{i,j,\sigma}(V_1, Z_\beta)$$

$$= \sum_{\beta=1}^{s} (V_i C_{1,j,\beta} - V_j C_{1,i,\beta}) \cdot Z_\beta + \sum_{\beta=1}^{s} C_{1,j,\beta}(V_i, Z_\beta) - \sum_{\beta=1}^{s} C_{1,i,\sigma}(V_j, Z_\beta)\ .$$

The brackets (V_1, Z_β), (V_i, Z_β), (V_j, Z_β) are expressed as homogeneous linear functions of the $X_{p+\alpha}$, the Z_β, and after some other transformations T_γ, they can be made to be independent of the preceding ones (and to belong to the second derived sheaf of $\mathcal{F}$). It would suffice to equate the coefficients of the $X_{p+\alpha}$ and Z_β in the two sides of the equation.

We point out that the terms in F_α in the identities (33) will disappear as a result of the identities (35), and if we appeal to those identities in order to transform the right-hand side of our identity (38) then we will get identities of the form:

(39) $$\begin{cases} V_1 A_{i,j,k} = & \text{homogeneous linear functions of the } A_{\alpha,\beta,\gamma} \text{ and the } C_{\alpha,\beta,\gamma}, \\ V_1 C_{i,j,k} = & \text{homogeneous linear functions of the } A_{\alpha,\beta,\gamma}, C_{\alpha,\beta,\gamma}, V_i C_{\alpha,\beta,\delta}, \text{ and } V_j C_{\alpha,\beta,\delta}, \end{cases}$$

in which the indices i, j, h, k can take all values that they have in the identities (37). The indices α, β are taken in the sequence 2, 3, …, p, and the indices γ and δ are taken from the sequences 1, 2, …, q and 1, 2, …, s, respectively.

One will then have a Kowalewski system relative to the $A_{i,j,k}$ and $C_{i,j,k}$ ($i, j > 1$) that has been solved for the derivatives of the type $\partial f / \partial x_1$, and since those functions are zero for $x_1 = x_1^0$, one will conclude that they are identically zero, because equations (39) are verified if one replaces all of the unknowns with zero, and the solution that is determined by the initial conditions (37) is unique.

It will now suffice to recall the identities (33) for us to conclude that the $C_{1,j,k}$ are also all zero. The equations of condition (29) and (30) are all verified then, so the subsheaf $\{V_1, \ldots, V_p\}$, which is determined in the stated way, will be complete, and the existence theorem for complete integrals that was stated at the beginning of no. **13** will be established.

We then conclude that *a sheaf of order p admits complete p-dimensional integrals. The most general of those complete integrals depends upon* $q - q_{p-1}$ *arbitrary functions of n arguments,* $q - q_{p-2}$ *arbitrary functions of* $n - 1$ *arguments, etc.*, …, *of* $q - q_1$ *arbitrary functions of* $n - p + 2$ *arguments, and finally of q arbitrary functions of* $n - p + 1$ *arguments*. The choice of those arbitrary functions is explained by the statement in no. **13**. Recall that n is the total number of variables, m is the degree of the sheaf [no. **1**], *that* $q = m - p$, and that the whole numbers $q_1, q_2, \ldots, q_{p-1}$ are the first $p - 1$ indices of sheaf [no. **10**].

In the existence theorem that Cartan established for the integrals of systems of Pfaff equations, the arbitrary functions for the general p-dimensional integral multiplicity are $q - q_{p-1}$ arbitrary functions of p arguments, $q_{p-1} - q_{p-2} = s_{p-1}$ arbitrary functions of $p - 1$ arguments, etc., $q_1 = s_1$ arbitrary functions of one argument, and $n - m$ arbitrary constants. The integers $s_1, \ldots, s_{p-1}$ are the first p characters of the Pfaff system, and consequently, they will also be characters of the sheaf of infinitesimal transformations that is its correlate.

It is not surprising that the arbitrary functions take different forms in the two theories: One of them has isolated integrals in mind, while the other has complete integrals in mind, and an isolated integral belongs to an infinitude of complete integrals. One easily accounts for the difference by considering the one-dimensional integral multiplicities, because in one case, one then has a system of $n - m$ ordinary differential equations in $n - m$ unknown functions, which is a system that depends upon the choice of $m - 1$ arbitrary functions of one variable, and in the other case, one has a homogeneous linear partial differential equation that depends upon $m - 1$ arbitrary functions of n arguments. One remarks that one will be led to set $q_0 = 0$.

15. *The integrals of a sheaf and its successive prolongations*. – In the foregoing, we proved the existence of p-dimensional complete integrals that we can call *general* for any involutive sheaf of order p. We now recall the search for p-dimensional integral multiplicities for an arbitrary sheaf $\mathcal{F}$, which is supposed to be given, without making any restricting hypothesis on either the sheaf or the integral multiplicities in question.

We suppose that the variables $x_1, \ldots, x_p$, for example, will remain independent of the integral multiplicity M_p considered, and we let $X_1, \ldots, X_p$ denote the transformations of the sheaf that is soluble for the $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_p}$. The subsheaf of $\mathcal{F}$ that leaves M_p then has the form:

$$V_i = X_i + \sum_{\alpha=1}^{q} v_{\alpha i}\, \lambda_{p+\alpha} \qquad (p + q = m\ ;\ i = 1, 2, \ldots, p)\ . \tag{40}$$

It is well-known that a multiplicity cannot admit two infinitesimal transformations without admitting their bracket. It then results that *on the multiplicity* M_p, the (V_i, V_j) will reduce to homogeneous linear combinations of $V_1, \ldots, V_p$, and *a fortiori* of $X_1, \ldots, X_m$. Therefore, the congruences:

(41) $$(V_i, V_j) \equiv 0 \pmod{\mathcal{F}} \qquad (i, j = 1, 2, \ldots, p)$$

are realized on M_p.

Those congruences [nos. **8** and **13**] provide a system of equations ($\mathcal{E}$) for the x_i ($i = 1, 2, \ldots, n$), and the v_{ij} ($j = 1, 2, \ldots, q$; $i = 1, 2, \ldots, p$).

If that system ($\mathcal{E}$) implies some relations between only the x_i then the multiplicity M_p must satisfy those relations. It must also satisfy relations of the same nature that one can possibly deduce by a repeated application of the operations V_i . Therefore, if, among all of those relations, there are some that couple the $x_1, \ldots, x_p$ with each other then the multiplicity M_p could not exist ([1]). If that case is excluded then the relations in question will permit one to infer certain dependent variables $x_{p+1}, \ldots, x_n$ as functions of the other ones and the $x_1, \ldots, x_p$. One can then reduce the number of unknown functions $x_{p+1}, \ldots, x_n$ by *truncating* the infinitesimal transformations of the sheaf. If, for example, one has expressions for $x_{n'+1}, \ldots, x_n$ such as:

(42) $$x_{n'+\alpha} = \varphi_\alpha (x_1, \ldots, x_{n'}) \qquad (\alpha = 1, 2, \ldots, n - n')$$

then it will suffice to suppress the terms in $\dfrac{\partial f}{\partial x_{n'+1}}, \ldots, \dfrac{\partial f}{\partial x_n}$ in the transformations X_h ($h = 1, 2, \ldots, m$) and to replace $x_{n'+1}, \ldots, x_n$ with their expressions in (42) in the other. One then begins the calculations anew with the sheaf thus-truncated.

In the second place, it can happen that the system ($\mathcal{E}$) does not imply any relation between only the $x_1, \ldots, x_n$ that can be solved for all of the v_{ji} . In that case, the transformations (40) are well-determined, and one will be reduced to a problem that was treated by Sophus Lie: Find all p-dimensional multiplicities that admit p given infinitesimal transformations ([2]). The complete solution will depend upon calculations that involve eliminating variables and integrating the complete system.

Let us remain in the case where the equations ($\mathcal{E}$) leave $x_1, \ldots, x_n$ independent and permit us to calculate a certain number of the $v_{j,i}$ (the calculations will then be linear algebraic solutions) as functions of the $x_1, \ldots, x_n$ and the other $v_{j,i}$, which remain arbitrary. Let $w_1, \ldots, w_r$ denote the latter, or more generally, some indeterminates of a minimum number by means of which one can express all of the $v_{j,i}$ in such a fashion as to satisfy equations ($\mathcal{E}$) in the most general manner. One will then replace equations ($\mathcal{E}$) with a solved system of the form:

(43) $$v_{j,i} = P_{j,i} (x_1, \ldots, x_n, w_1, \ldots, w_r) \qquad (i = 1, 2, \ldots, p \,;\, j = 1, 2, \ldots, q) .$$

([1]) That is, one must repeat the calculation after taking p other independent variables from among the $x_1, \ldots, x_n$.

([2]) Furthermore, one deals with the simple case in which those transformations remain *divergent* on the multiplicity.

Here, one observes the analogy with the considerations of no. **5**. If one supposes that $X_1, \ldots, X_m$ are solved for the $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}$ then on the supposed multiplicity M_p, the $v_{j,i}$ will be equal to the partial derivatives $\frac{\partial x_{p+j}}{\partial x_i}$ ($j = 1, 2\ldots, m - p$), and the other partial derivatives of the type $\frac{\partial x_{m+h}}{\partial x_i}$ ($h = 1, 2\ldots, m - n$) are known linear functions of the $v_{j,i}$, since they are equal to the coefficients of the derivatives $\frac{\partial f}{\partial x_{m+h}}$ in the V_i. In regard to equations (43), one then sees that on the desired multiplicity M_p, $w_1, \ldots, w_r$ are coordinates of the contact element that is associated with the point $(x_1, \ldots, x_n)$ of that multiplicity.

If $X_1, \ldots, X_m$ are not in solved form then the partial derivatives $\frac{\partial x_{p+j}}{\partial x_i}$ ($j = 1, 2\ldots, n - p$) will be rational functions of the $v_{j,i}$, and the conclusion will be same.

One can proceed as in no. **5**. One introduces $w_1, \ldots, w_r$ as new variables (functions of $x_1, \ldots, x_p$) and replaces the search for the multiplicity M_p in the space of $(x_1, \ldots, x_n)$ with the search for the *prolonged multiplicity* M'_p, which is found to be defined in the space $(x_1, \ldots, x_n, w_1, \ldots, w_r)$ by means of formulas (43), and in order to do that, it will suffice to replace the given sheaf $\mathcal{F}$ with the *prolonged sheaf* $(\mathcal{F}')$, which is defined by the transformations:

(44) $$X'_i = X_i + \sum_{\alpha=1}^{q} P_{\alpha,i}(x_1, \ldots, x_n, w_1, \ldots, w_r) X_{p+\alpha} \qquad (i = 1, 2, \ldots, p)$$

and

(45) $$\frac{\partial f}{\partial w_1}, \quad \frac{\partial f}{\partial w_2}, \quad \ldots, \quad \frac{\partial f}{\partial w_\alpha}.$$

One then operates on the sheaf $\mathcal{F}'$ as one does on the sheaf $\mathcal{F}$, i.e., upon considering the transformations

(46) $$V'_i = X'_i + \sum_{\alpha=1}^{r} w_{\alpha,i} \frac{\partial f}{\partial w_r} \qquad (i = 1, 2, \ldots, p)$$

in place of the $V_1, \ldots, V_p$, in which the $w_{j,i}$ must play the role that was previously played by the $v_{j,i}$, and so on.

If one arrives at an involutive sheaf of order p at some point then the problem will be solved by the search for the complete general integral of degree p in that sheaf, because one will then find oneself in the presence of a general involution of degree p in that sheaf that has been put into solved form [no. **11**]. In that case, one does not have to pass on to the following prolongation ([1]).

([1]) In that case, one can show that the prolongation will again lead to an involutive sheaf of order p.

The only case that remains unresolved is then the one is which the method will lead to an infinitude of successive prolongations, but one can show ([1]) that one will then necessarily arrive at an involutive sheaf of order p by one of those prolongations.

III. – DISTINGUISHED TRANSFORMATIONS AND CAUCHY CHARACTERISTICS.

16. *Invariance of a sheaf under a transformation*. – Let $\{X_1, \ldots, X_m\}$ be a sheaf $\mathcal{F}$ of infinitesimal transformations. If one performs the same finite transformation:

$$x'_i = f_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

on its various transformations then one will get a sheaf $\bar{\mathcal{F}}$ that is defined by the basis transformations:

$$\bar{X}_k f = \sum_{\alpha=1}^{n} X_k x'_\alpha \frac{\partial f}{\partial x'_\alpha} \qquad (k = 1, 2, \ldots, m)\,. \tag{2}$$

Those transformations are found to be written with the letters $x'_1, \ldots, x'_n$. On the other hand, let $\mathcal{F}'$ denote what the sheaf $\mathcal{F}$ will become when one simply puts primes on each of the symbols $x_1, \ldots, x_n$, and let $X'_1, \ldots, X'_m$ denote what each of those basis transformations will then become.

If the sheaves $\bar{\mathcal{F}}$ and $\mathcal{F}'$ are thus the same then one will say that $\mathcal{F}$ is *invariant* under the transformation (1). The analytical condition for that to be true is then expressed by some identities of the form:

$$\bar{X}_k = \sum_{\alpha=1}^{m} \lambda_{k,\alpha}(x'_1, \ldots, x'_n) X'_\alpha \qquad (k = 1, 2, \ldots, m)\,. \tag{3}$$

When that is true, the transformation (1) will change any integral multiplicity of the sheaf $\mathcal{F}$ into an integral multiplicity of the same sheaf. The converse is true. It likewise suffices that the transformation (1) should change any one-dimensional integral of $\mathcal{F}$ into an integral of $\mathcal{F}$.

Under the same conditions, we say that the sheaf $\mathcal{F}$ *admits* the transformation (1).

17. *Invariance of a sheaf under and infinitesimal transformation.* – If one supposes that the sheaf is invariant under each transformation:

$$x'_i = f_i(x_1, \ldots, x_n \mid t) \qquad (i = 1, 2, \ldots, n) \tag{4}$$

([1]) I shall return to that proof (and that of the theorem that was stated in the preceding footnote) in a later work.

of a one-parameter group then the identities (3) will take the form:

$$(5) \qquad \bar{X}_k = \sum_{\alpha=1}^{m} \lambda_{k,\alpha}(x'_1, \ldots, x'_n \mid t)\, X'_\alpha \qquad (k = 1, 2, \ldots, m),$$

and one can differentiate the two sides with respect to t while considering $x'_1, \ldots, x'_n$, and t to be the independent variables. The derivative of the left-hand side is then identical to the bracket (X_k, X), where $X\,f$ is the infinitesimal transformation that is the generator of the group (4), and the variables $x_1, \ldots, x_n$ are expressed in terms of $x'_1, \ldots, x'_n$ in the bracket ([1]). If one sets $t = 0$ in both sides of the identities thus-obtained [that value of t is supposed to be the one for which (4) reduces to the identity transformation] then one will get identities of the form:

$$(6) \qquad (X_k, X) = \sum_{\alpha=1}^{m} \mu_{k,\alpha}(x_1, \ldots, x_n)\, X_\alpha \qquad (k = 1, 2, \ldots, m).$$

Conversely, if such identities are true then it will suffice to replace $x_1, \ldots, x_n$ with functions of $x'_1, \ldots, x'_n$, t by means of formulas (4) in order to get identities of the form:

$$(7) \qquad \frac{d\bar{X}_k}{dt} = \sum_{\alpha=1}^{m} \nu_{k,\alpha}(x'_1, \ldots, x'_n \mid t)\, \bar{X}_\alpha \qquad (k = 1, 2, \ldots, m).$$

One then concludes that the $\bar{X}_k$ are homogeneous linear functions of their initial values X'_j, with coefficients that are functions of $x'_1, \ldots, x'_n$, and t, i.e., the existence of identities of the form (5).

Hence, *in order for a sheaf $\mathcal{F}$ to be invariant under the transformations of a one-parameter group, it is necessary and sufficient that the brackets of the basis transformations of the sheaf with the infinitesimal transformation of the group should belong to the sheaf*, i.e., that one will have the identities-congruences:

$$(8) \qquad (X_k, X) \equiv 0 \pmod{\mathcal{F}} \qquad (k = 1, 2, \ldots, m).$$

We express that fact by saying that the sheaf $\mathcal{F}$ remains *invariant under the infinitesimal transformation $X\,f$.*

18. *Distinguished transformations of a sheaf.* – While generalizing the use of a terms in S. Lie's theory, we say that a transformation of a sheaf is a *distinguished transformation* if it leaves

([1]) One can avoid appealing to that theorem of S. Lie by reducing the group (4) to the canonical form:

$$x'_i = x_i \qquad (i = 1, 2, \ldots, n-1), \qquad x'_n = x_n + t\,.$$

the sheaf invariant. From the form of the conditions (8) that amounts to saying that it is in involution with each of the transformations of the sheaf. The following consequences result immediately from that:

1. If a sheaf is complete then each of its transformations will be a distinguished transformation.

2. Conversely, if a sheaf of degree m contains m divergent distinguished transformations then it will be complete.

More generally, suppose that the sheaf $\mathcal{F}$ of degree m contains r divergent distinguished transformations, and no more, while supposing that $r < m$. The distinguished transformations of $\mathcal{F}$ will then form a subsheaf Φ of degree r : I say that the subsheaf Φ is complete.

Indeed, let X and Y be two transformations of Φ. Each of them leaves $\mathcal{F}$ invariant, so their bracket (X, Y) will also leave $\mathcal{F}$ invariant ([1]). However, X belongs to $\mathcal{F}$, and Y is a distinguished transformation of $\mathcal{F}$ then (X, Y) will also belong to $\mathcal{F}$. It will then be a distinguished transformation of $\mathcal{F}$, and it will belong to Φ. Moreover, since the bracket of two arbitrary transformations of Φ will belong to Φ, that subsheaf Φ is indeed complete.

Hence, *the distinguished transformations of a sheaf define a complete subsheaf of that sheaf.*

We further point out that if an involution $\mathcal{I}$ of $\mathcal{F}$ does not contain all of the distinguished transformations of $\mathcal{F}$ then those distinguished transformations that do not belong to $\mathcal{I}$ will also be in involution with each of the transformations of $\mathcal{I}$. It will then result that the genus g of $\mathcal{I}$ is greater by at least one unit than the number r of (divergent) distinguished transformations of $\mathcal{F}$ and that the general involution of $\mathcal{F}$ of degree equal to its genus g will contain the sheaf Φ of distinguished transformations of $\mathcal{F}$.

([1]) It is a general fact in the theory of infinitesimal transformations that any analytic invariant of two of those transformations will be invariant under their bracket. The verification of that fact is easy here: Indeed, one concludes from the identities:

$$(X_k, X) = \sum_\alpha u_{k\alpha} X_\alpha , \qquad (X_k, Y) = \sum_\alpha v_{k\alpha} X_\alpha$$

and the Jacobi identity that:

$$(X_k, (X, Y)) = ((X, X_k, Y) - ((Y, X_k, X)) \equiv \sum_\alpha \{v_{k\alpha} (X, X_\alpha) - u_{k\alpha} (Y, X_\alpha)\} \equiv 0 .$$

19. *Characteristic manifolds*. – In order to go further, it would be simplest to introduce the first integrals $y_1, \ldots, y_n$ of the complete subsheaf Φ of distinguished transformations of $\mathcal{F}$ ($\nu = n - r$) as new variables. That sheaf Φ will then be reduced to the form:

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_r}, \tag{9}$$

and the sheaf $\mathcal{F}$ will be deduced from Φ by adding $\mu = m - r$ transformations of the form:

$$Y_i = \frac{\partial f}{\partial y_i} + \sum_{\alpha=1}^{\sigma} \eta_{i,\alpha} \frac{\partial f}{\partial y_{\mu+\alpha}} \qquad (i = 1, 2, \ldots, \mu\,;\ \sigma = \nu - m) \tag{10}$$

to it. If one then writes out that the transformations of Φ are distinguished, i.e., that one has:

$$\left(\frac{\partial f}{\partial x_h}, Y_i\right) \equiv 0 \qquad (\text{mod } \mathcal{F}) \qquad (i = 1, 2, \ldots, \mu\,;\ h = 1, 2, \ldots, r)$$

then one will find that the $\eta_{i,j}$ do not depend upon any of the variables $x_1, \ldots, x_r$.

The Pfaff system that is dual to the sheaf Φ is then:

$$dy_{\mu+j} = \sum_{\beta=1}^{\mu} \eta_{\beta,i}(y_1, y_2, \ldots, y_\nu)\, dy_\beta \qquad (j = 1, 2, \ldots, \sigma), \tag{11}$$

and the variables $x_1, \ldots, x_r$ do not appear. That shows that the variables that were introduced are the ones that Cartan called *characteristic variables*, because it is immediate that conversely the existence of the form (11) for the Pfaff system will imply the consequence that the transformations (9) are distinguished transformations of the dual sheaf to that system (whose variables are $x_1, \ldots, x_r, y_1, \ldots, y_\nu$).

The integral multiplicities of the sheaf $\mathcal{F}$ and each of its complete subsheaves are *Cauchy characteristic multiplicities*, in the sense that Cartan attributed to that term.

To simplify, we confine ourselves to the case in which we seek the general integrals of the sheaf $\mathcal{F}$ that have a number of dimensions that is equal to the genus g of that sheaf.

From the theory of the preceding subsection of this article, we will have to consider a general involution of the form [no. **18**]:

$$\left\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_r}, V_1, \ldots, V_\varpi\right\} \qquad (\varpi + r = g), \tag{12}$$

(13) $$V_j = Y_j + \sum_{\gamma=1}^{\mu-\varpi} \nu_{\gamma,j} V_{\varpi+\gamma} \qquad (j = 1, 2, \ldots, \varpi).$$

It is clear that $\{V_1, \ldots, V_\varpi\}$ is the general involution of maximum degree of the *reduced sheaf* $\{Y_1, \ldots, Y_\mu\}$. The $\nu_{\gamma,j}$ are coupled by some relations in which $x_1, \ldots, x_r$ do not appear. Upon writing out that the sheaf (12) is complete, one will find forthwith that the $\nu_{\gamma,j}$ are entirely independent of the $x_1, \ldots, x_r$ and that $\{V_1, \ldots, V_\varpi\}$ must itself a complete sheaf.

One will then recover the result that the desired integrals are generated by the characteristic multiplicities $y_1 = c_1, \ldots, y_\nu = c_\nu$. The integration of the sheaf $\mathcal{F}$ is reduced to that of the sheaf $\{Y_1, \ldots, Y_\mu\}$ in $y_1, \ldots, y_\nu$, and that integration will indicate how one must associate the characteristics in order for them to effectively generate the desired integrals, which is a result that is in complete agreement with Cartan's theory.

20. EXAMPLE I. – *The integration of the equation* $q = \Phi(x, y, z, p)$. – The corresponding sheaf is [no. **57**]:

$$X_1 = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}, \qquad X_2 = \frac{\partial f}{\partial y} + \Phi\frac{\partial f}{\partial z}, \qquad X_3 = \frac{\partial f}{\partial p}.$$

The structure formulas are:

$$(X_1, X_2) = X_1\Phi\frac{\partial f}{\partial z}, \quad (Y_2, X_2) = -\frac{\partial \Phi}{\partial p}\frac{\partial f}{\partial z}, \quad (X_3, X_1) = \frac{\partial f}{\partial z}.$$

The sheaf is therefore not complete, and the derived sheaf is:

$$\left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}\right\},$$

in such a way that it is not possible reduce the number of variables *in the sheaf*.

We seek the involutions of degree 2 in the solved form [no. **11**]:

$$V_1 = X_1 + \nu_1 X_3, \qquad V_2 = X_2 + \nu_2 X_3.$$

The conditions between ν_1 and ν_2 then reduce to just one:

$$X_1\Phi + \nu_1\frac{\partial \Phi}{\partial p} - \nu_2 = 0,$$

and the general involution of degree 2 can be written:

$$V = X_1 + \nu X_3 , \qquad W = \frac{\partial \Phi}{\partial p} X_1 - X_2 - X_1 \Phi \cdot X_3$$

when one combines V_1 and V_2 in such a way as to eliminate ν_1 and ν_2 and puts ν in place of ν_1.

It then results that W is, in its developed form:

$$W f = \frac{\partial \Phi}{\partial p}\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) - \left(\frac{\partial \Phi}{\partial x} + p\frac{\partial \Phi}{\partial z}\right)\frac{\partial f}{\partial p} - \left(\frac{\partial f}{\partial y} + \Phi\frac{\partial f}{\partial z}\right),$$

and one sees that $W f = 0$ is the linear equation that is equivalent to the system of characteristics of the classical theory.

Moreover, our method will give all of the other results of that theory very rapidly. The sheaf can be written $\{X_1, X_3, W\}$, and it will suffice to integrate the sheaf $\{X_1, X_3\}$ after one has introduces the characteristic variables (viz., first integrals of $W f = 0$). However, in its first form, the sheaf $\{X_1, X_3\}$ can be integrated immediately. The general (one-dimensional) integral ([1]) is:

$$z = f(x), \qquad p = f'(x),$$

with the arbitrary function $f(x)$. Therefore, everything does, in fact, reduce to $W f = 0$.

Let:

$$\bar{\xi}(x, y, z, p), \quad \bar{\zeta}(x, y, z, p), \quad \bar{\varpi}(x, y, z, p)$$

be three independent first integrals. The change of variables will be:

$$\xi = \bar{\xi}(x, y, z, p), \qquad \zeta = \bar{\zeta}(x, y, z, p), \qquad \varpi = \bar{\varpi}(x, y, z, p),$$

and one will infer formulas of the form:

$$x = a(\xi, \zeta, \varpi, y), \qquad z = b(\xi, \zeta, \varpi, y), \qquad p = c(\xi, \zeta, \varpi, y)$$

from them.

The formulas:

$$b = f(a), \qquad c = f'(a),$$

in which f is arbitrary, and in which one replaces ξ, ζ, ϖ with their expressions $\bar{\xi}$, $\bar{\zeta}$, $\bar{\varpi}$, give the general integral of the proposed equation.

If one takes $\bar{\xi}$, $\bar{\zeta}$, $\bar{\varpi}$ to be integrals of $W f = 0$ that reduce to x, y, z, respectively, for $y = y_0$ then the integral will reduce to $z = f(x)$, $p = f'(x)$ for $y = y_0$, i.e., it will be found in the form that gives the solution to the Cauchy problem [the integral that passes through the curve $z = f(x)$, $y =$

([1]) Here, we have ignored the viewpoint of complete integrals. However, in order to resolve the question from that viewpoint, it will suffice to introduce two arbitrary constants into f.

y_0]. One can remark, moreover, that with that choice of $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, the sheaf $\{X_1, X_3\}$ will reduce to the form:

$$\frac{\partial f}{\partial \xi} + \varpi \frac{\partial f}{\partial \zeta}, \quad \frac{\partial f}{\partial \varpi},$$

after a change of variables that is independent of y, since one will have $\bar{\xi} = x$, $\bar{\zeta} = z$, $\bar{\varpi} = p$ for $y = y_0$. One can also write the general integral of the proposed equation in the form:

$$\bar{\zeta} = f(\bar{\xi}), \quad \bar{\varpi} = f'(\bar{\xi})$$

then.

21. EXAMPLE II. – *Integrate the system:*

$$r = \Phi(x, y, z, p, q, s), \qquad t = \Psi(x, y, z, p, q, s).$$

The sheaf that it corresponds to is:

$$X_1 = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \Phi\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q},$$

$$X_2 = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + \Psi\frac{\partial f}{\partial q},$$

$$X_3 = \frac{\partial f}{\partial s}.$$

The structure relations are:

$$(X_1, X_2) = -X_2\Phi\frac{\partial f}{\partial p} + X_1\Psi\frac{\partial f}{\partial q},$$

$$(X_2, X_3) = -\frac{\partial f}{\partial p} + \frac{\partial \Psi}{\partial s}\frac{\partial f}{\partial q},$$

$$(X_1, X_3) = \frac{\partial \Phi}{\partial s}\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q}.$$

The sheaf is not complete, and one cannot reduce the number of variables, since the second-order derived sheaf will be:

$$\left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial s}\right\}.$$

We seek the involutions of order 2:

$$V_1 = X_1 + \nu_1 X_3 , \qquad V_2 = X_2 + \nu_2 X_3 .$$

The condition $(V_1, V_2) \equiv 0$ gives the equations in ν_1, ν_2 :

(14) $$\nu_1 - \nu_2 \frac{\partial \Phi}{\partial s} - X_2 \Phi = 0 , \qquad \nu_1 \frac{\partial \Phi}{\partial s} - \nu_2 + X_1 \Psi = 0 .$$

If the determinant $\frac{\partial \Phi}{\partial s}\frac{\partial \Psi}{\partial s} - 1$ is not identically zero then one can infer ν_1 and ν_2 from those equations, and there will be only one involution $\{V_1, V_2\}$. If it is a complete sheaf then the proposed system will admit just one complete integral that is calculated by integrating the complete system $V_1 = 0$, $V_2 = 0$. In the contrary case, there can only be singular integrals ([1]).

The only truly-interesting case is the one in which the system (14) is indeterminate:

(15) $$\frac{\partial \Phi}{\partial s}\frac{\partial \Psi}{\partial s} = 1, \qquad X_1 \Psi \frac{\partial \Phi}{\partial s} + X_2 \Phi = 0 .$$

One will then have:

(16) $$V_3 = X_2 + \left(\nu_1 \frac{\partial \Psi}{\partial s} + X_1 \Psi \right) X_3 \qquad (\nu_1 \text{ arbitrary}),$$

and the involution $\{V_1, V_2\}$ can be written:

(17) $$V = X_1 + \nu X_3 , \qquad W = \frac{\partial \Psi}{\partial s} - X_2 - X_1 \Psi \cdot X_3 .$$

The transformation W is then a distinguished transformation, and the integrals of the given system are generated by the (characteristic) integrals $W f = 0$.

For example, let:

$$\bar{\xi}\,(x, y, z, p, q, s) , \qquad \bar{\zeta}\,(x, y, z, p, q, s) ,$$

$$\bar{\varpi}\,(x, y, z, p, q, s) , \qquad \bar{\chi}\,(x, y, z, p, q, s) , \qquad \bar{\sigma}\,(x, y, z, p, q, s)$$

be the *principal* first integrals of $W f = 0$, which reduce to x, z, p, q, s, respectively, for $y = y_0$. The change of variables:

(18) $$\xi = \bar{\xi} , \qquad \zeta = \bar{\zeta} , \qquad \varpi = \bar{\varpi} , \qquad \chi = \bar{\chi} , \qquad \sigma = \bar{\sigma} , \qquad y = y$$

([1]) We omit the details from our discussion, to abbreviate, since they present no difficulty.

reduces the sheaf to the form ([1]):

$$\frac{\partial f}{\partial \xi} + \varpi \frac{\partial f}{\partial \zeta} + \Phi(\xi, y_0, \zeta, p, q, \sigma) \frac{\partial f}{\partial \varpi} + \sigma \frac{\partial f}{\partial \chi}, \qquad \frac{\partial f}{\partial \sigma}, \qquad \frac{\partial f}{\partial y},$$

and everything comes down to integrating the subsheaf that is composed of the first two of those transformations [no. **19**], i.e., up to notations, the subsheaf $\{X_1, X_2\}$ of the proposed sheaf. It is equivalent to the differential equation in z, q, and x:

$$\frac{\partial^2 z}{\partial x^2} = \Phi\left(x, y_0, z, \frac{\partial z}{\partial x}, q, \frac{\partial q}{\partial x}\right).$$

If one sets $z = f(x)$ then one will have:

$$\frac{\partial z}{\partial x} = p = f'(x) \qquad \text{and} \qquad \Phi(x, y, z, p, q, s) = f''(x), \quad s = \frac{\partial q}{\partial x}.$$

One is then reduced to the integration of the first-order equation with independent variable x and one unknown function q :

$$\Phi\left[x, y_0, f(x), f'(x), q, \frac{\partial q}{\partial x}\right] = f''(x), \tag{19}$$

in which the function $f(x)$ remains arbitrary, and the replacement of the letters x, z, p, q, s with the functions $\bar{\xi}$, $\bar{\zeta}$, $\bar{\varpi}$, $\bar{\chi}$, $\bar{\sigma}$, respectively, in the result:

$$z = f(x), \qquad p = f'(x), \qquad q = g(x), \qquad s = g'(x).$$

IV. – SINGULAR TRANSFORMATIONS AND MONGE CHARACTERISTICS.

22. – *Transformations and singular involutions.* – From what was said in no. **9**, a *singular transformation* of a sheaf $\mathcal{F}$ is a transformation of that sheaf that is in involution with more than $m - q_1$ divergent transformations of that sheaf (q_1 being the first index of the sheaf and m is its degree). A *singular involution* of degree k is an involution of the sheaf such that all of its transformations are in involution with more than $m - q_k$ divergent transformations of the sheaf (q_k being the kth index of the sheaf).

The search for singular transformations and singular involutions will permit one to simplify (if applicable) the structure relations of the sheaf, and as a result, to recognize the particular properties of that sheaf from the standpoint of its integration. In particular, in certain cases, one will get ways

([1]) For the same reason as in the previous example.

of generating integral multiplicities by characteristics that are distinguished from the ones that prove the distinguished transformations (which are, moreover, singular transformations, in such a way that they are as singular as possible), which are characteristics that we call *Monge characteristics*, with Cartan.

Without developing a general theory here, we shall specify those suggestions in a particular case by applying them to the classical problem of integrating second-order partial differential equations in two independent variables and one known function.

23. *Singular transformations in the case of a second-order equation.* – Now consider the equation:

$$r = \Phi\,(x, y, z, p, q, s, t)\ .$$

The sheaf to be integrated is:

$$X_1 = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \Phi\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q}\,,$$

$$X_2 = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q}\,,$$

$$X_3 = \frac{\partial f}{\partial s}\,, \qquad X_4 = \frac{\partial f}{\partial t}\,.$$

The structure relations are:

$$(X_1, X_2) = -X_2\,\Phi\frac{\partial f}{\partial p}\,, \qquad (X_1, X_3) = -\frac{\partial \Phi}{\partial s}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}\,, \qquad (X_1, X_4) = -\frac{\partial \Phi}{\partial t}\frac{\partial f}{\partial p}\,,$$

$$(X_2, X_3) = -\frac{\partial f}{\partial p}\,, \qquad (X_2, X_4) = -\frac{\partial f}{\partial q}\,, \qquad (X_3, X_4) = 0\ .$$

One has $n = 7$, $m = 4$. Upon seeking the relations that express the idea that:

$$U = u_1 X_1 + u_2 X_2 + u_3 X_3 + u_4 X_4\,, \qquad V = v_1 X_1 + v_2 X_2 + v_3 X_3 + v_4 X_4$$

are in involution, one will find two bilinear equations:

$$(1) \qquad (u_1 v_2 - u_2 v_1)\, X_2\, \Phi + (u_1 v_2 - u_2 v_1)\,\frac{\partial \Phi}{\partial s} + (u_1 v_4 - u_4 v_1)\,\frac{\partial \Phi}{\partial t} + (u_2 v_3 - u_3 v_2) = 0\ ,$$

$$(2) \qquad (u_1 v_3 - u_3 v_1) + (u_2 v_4 - u_4 v_2) = 0\ ,$$

which are, in general, independent of v_1, v_2, v_3, v_4. Hence, $q_1 = 2$, and there is no general involution of degree greater than 2.

However, equations (1), (2) will reduce to just one if u_1, u_2, u_3, u_4 satisfy the conditions:

(3) $$\frac{u_2\, X_2\, \Phi + u_3 \dfrac{\partial \Phi}{\partial s} + u_4 \dfrac{\partial \Phi}{\partial t}}{u_3} = \frac{u_2\, X_2\, \Phi - u_3}{-u_4} = \frac{u_1 \dfrac{\partial \Phi}{\partial s} + u_2}{-u_1} = \frac{u_1 \dfrac{\partial \Phi}{\partial t}}{u_2} .$$

One concludes from the equality of the last two ratios that:

(4) $$u_2^2 + u_1 u_2 \frac{\partial \Phi}{\partial s} - u_1^3 \frac{\partial \Phi}{\partial t} = 0 ,$$

which leads one to set:

(5) $$u_2 = m\, u_1 ,$$

with

(6) $$m^2 + m \frac{\partial \Phi}{\partial s} - \frac{\partial \Phi}{\partial t} = 0 .$$

Introducing the two roots of that equations, *which are supposed to be different* (say m_1 and m_2), will give:

(7) $$\frac{\partial \Phi}{\partial s} = -\,(m_1 + m_2) , \qquad \frac{\partial \Phi}{\partial t} = -\, m_1\, m_2 ,$$

and upon taking $m = m_1$, for example, the common value of the ratios (3) will become $-\, m_2$, and what will remain is:

$$m_1\, u_1\, X_2\, \Phi - m_1\, u_3 - m_1\, m_2 \cdot u_4 = 0 , \qquad u_1 \cdot X_2\, \Phi - u_3 - m_2 \cdot u_4 = 0 ,$$

which will reduce to:

(8) $$u_3 + m_2 \cdot u_4 = u_1 \cdot X_2\, \Phi .$$

Two arbitrary contributions will then remain, for example, u_1 and u_4, and one will have a first sheaf of singular transformations:

(9) $$P_1 = X_1 + m_1\, X_2 + X_2\, \Phi \cdot X_3 , \qquad Q_1 = X_4 - m_2\, X_3 .$$

Each of the transformations of that subsheaf is in involution with ∞^3 transformations V that are defined by the unique condition (2).

Similarly, one has another subsheaf of singular transformations:

(10) $$P_2 = X_1 + m_2\, X_2 + X_2\, \Phi \cdot X_3 , \qquad Q_2 = X_4 - m_1\, X_3 .$$

When one takes $\{P_1, P_2, Q_1, Q_2\}$ to be a basis for the given sheaf, one will have structure relations of a remarkable simplicity:

$$(11)\qquad \begin{cases} (P_1, P_2) \equiv 0, \quad (P_1, Q_2) \equiv 0, \\ (P_2, Q_1) \equiv 0, \quad (Q_1, Q_2) \equiv 0, \\ (P_1, Q_1) \equiv -(m_1 - m_2)\left(\dfrac{\partial f}{\partial q} - m_2 \dfrac{\partial f}{\partial p}\right), \\ (P_2, Q_2) \equiv \quad (m_1 - m_2)\left(\dfrac{\partial f}{\partial q} - m_1 \dfrac{\partial f}{\partial p}\right). \end{cases}$$

That shows that each transformation of the sheaf $\{P_1, Q_1\}$ is in involution with each transformation of the sheaf $\{P_2, Q_2\}$. One can then say that those two subsheaves are related by a *reciprocal involution*.

24. *Second-order characteristics.* – We can now look for second-degree involutions of the form:

$$V_1 = P_1 + v_{1,1}\, Q_1 + v_{2,1}\, Q_2\,, \qquad V_2 = P_2 + v_{1,2}\, Q_1 + v_{2,2}\, Q_2\,.$$

The condition $(V_1, V_2) \equiv 0$ reduces to $v_{2,1} = v_{1,2} = 0$, and the general form of the second-degree involution is, with two arbitrary contributions v_1, v_2 :

$$(12)\qquad V_1 = P_1 + v_1\, Q_1\,, \qquad V = P_2 + v_2\, Q_2\,.$$

It results immediately from this that the complete subsheaves of degree 2 have the same form, and that as a result, any non-singular integral multiplicity will be generated by one-dimensional integrals from one or the other of the sheaves $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$.

Those one-dimensional integrals are the characteristics of the two systems in classical theory. A discussion of linear systems of partial differential equations:

$$P_1 = 0\,, \qquad Q_1 = 0\,, \qquad P_2 = 0\,, \qquad Q_2 = 0$$

will establish the known results in regard to the numbers of distinct *invariants* from one or the other of the characteristic systems. The term "invariant" is perfectly adapted to our viewpoint, since in the sense of the theory of groups of transformations, we are effectively dealing with invariants that are common to all the infinitesimal transformations of one and the other of the two *characteristic subsheaves* $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$.

The system $P_2 = 0$, $Q_2 = 0$ is (up to notations) the one to which Goursat was led to discuss in his studies of the invariants of characteristic systems ([1]).

([1]) GOURSAT, *Leçons sur l'integration des équations aux dérivées partielles du second ordre*, t. II, pp. 155.

Relative to the integral surfaces, v_1, and v_2 are the derivatives of t when taken in the characteristic directions:

(13) $$v_1 = m_1 \frac{\partial t}{\partial y} + \frac{\partial t}{\partial x}, \qquad v_2 = m_2 \frac{\partial t}{\partial y} + \frac{\partial t}{\partial x}.$$

25. *Third-order characteristics.* – In order to obtain the higher-order characteristics, one can *prolong the given equation* by appending the equations that are obtained by successively differentiating it up to an arbitrary order and operating on the sheaf of infinitesimal transformations that are associated with the differential system thus-obtained. However, one will obtain more complete results by prolonging the sheaf $\{P_1, P_2, Q_1, Q_2\}$ directly using the method in no. **15**.

The first prolonged sheaf ([1]) is, with the notations (12):

(14) $$\left\{ V_1, V_2, \frac{\partial f}{\partial v_1}, \frac{\partial f}{\partial v_2} \right\},$$

in which the only brackets that are not identically zero are:

$$\left(\frac{\partial f}{\partial v_1}, V_1 \right) = Q_1, \qquad \left(\frac{\partial f}{\partial v_2}, V_2 \right) = Q_2,$$

and

(15) $$(V_1, V_2) = \lambda V + \alpha_1 Q_1 - \alpha_2 Q_1 \qquad (V = V_2 - V_1),$$

with the following expressions for λ, α_1, α_2 :

(16) $$\begin{cases} \lambda = \dfrac{V_1 m_2 - V_2 m_1}{m_2 - m_1}, \quad \alpha_1 = v_1 \lambda + \mu, \quad \alpha_2 = v_2 \lambda + \mu, \\ \mu = \dfrac{V(X_2 \Phi) + v_2 \cdot V_1 m_1 - v_1 \cdot V_2 m_2}{m_2 - m_1}. \end{cases}$$

One observes that λ is linear and μ is bilinear in v_1 and v_2 .

One immediately finds the general second-degree involution of the sheaf (14) in the form:

$$V_1 + w_1 \frac{\partial f}{\partial v_1} + \alpha_2 \frac{\partial f}{\partial v_2}, \qquad V_2 + \alpha_1 \frac{\partial f}{\partial v_1} + w_2 \frac{\partial f}{\partial v_2},$$

in which w_1 and w_2 remain arbitrary. We set:

([1]) From the theory of no. **15**, the two-dimensional integrals of the sheaf are the prolonged multiplicities that issue from two-dimensional integrals of the initial sheaf $\{P_1, P_2, Q_1, Q_2\}$, i.e., integrals of the given second-order equation.

(17) $$\Omega_1 = V_1 + \alpha_2 \frac{\partial f}{\partial v_2}, \qquad \Omega_2 = V_2 + \alpha_1 \frac{\partial f}{\partial v_1},$$

and we will then have two subsheaves in reciprocal involution:

(18) $$\left\{\Omega_1, \frac{\partial f}{\partial v_1}\right\}, \qquad \left\{\Omega_2, \frac{\partial f}{\partial v_2}\right\}.$$

Since any complete subsheaf of the prolonged sheaf has the form:

(19) $$W_1 = \Omega_1 + w_1 \frac{\partial f}{\partial v_1}, \qquad W_2 = \Omega_2 + w_2 \frac{\partial f}{\partial v_2},$$

one sees that the prolonged integral multiplicities are generated by one-dimensional integrals of one or the other subsheaf (18). Those one-dimensional integrals are third-order characteristics, and formulas (17) show immediately that they result from a prolongation of the second-order characteristics.

Moreover, one will get some interesting suggestions on the subject of the search for invariants of those third-order characteristics ([1]). For example, for the first system, one deals with finding the common solutions to the equations:

(20) $$\Omega_1 = 0, \qquad \frac{\partial f}{\partial v_1} = 0.$$

One first remarks that for an invariant that contains neither v_1 nor v_2, that system will reduce to $V_1 = 0$, which will decompose into $P_1 = Q_1 = 0$. Hence, the second-order invariants are included as special cases in the invariant of the third: Furthermore, that will result *a priori* from the fact that the sheaves (18) are the prolongations of the sheaves $\{P_1, Q_1\}$, $\{P_2, Q_2\}$, respectively.

One then sees that the third-order invariants (of the system considered) are functions of only x, y, z, p, q, s, t, v_2, and since α_2 is linear in v_1, moreover, from formulas (16) and the remarks that were made about λ and μ, one can set:

$$\alpha_2 = \alpha_{2,0} + \alpha_{2,1} \cdot v_1,$$

and replace the system (20) with a new system of two equations:

(21) $$P_1 + \alpha_{2,0} \frac{\partial f}{\partial v_2} = 0, \qquad Q_1 + \alpha_{2,1} \frac{\partial f}{\partial v_2} = 0.$$

Here, it is obvious that one can have at most one third-order invariant, along with some possible second-order invariants (when one counts only the independent invariants).

([1]) Compare GOURSAT, *loc. cit.*, pp. 151.

26. *Higher-order characteristics*. – We shall now move on to fourth-order. We must consider the prolonged sheaf:

(22) $$\left\{W_1, W_2, \frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\right\},$$

with the notations (19). The variables w_1 and w_2 are the derivatives of v_1 and v_2, respectively, in the first and second characteristic directions:

(23) $$w_1 = m_1 \frac{\partial v_1}{\partial y} + \frac{\partial v_1}{\partial x}, \qquad w_2 = m_2 \frac{\partial v_2}{\partial y} + \frac{\partial v_2}{\partial x}.$$

The brackets of the transformations (22) are zero, except for:

$$\left(\frac{\partial f}{\partial w_1}, W_1\right) = \frac{\partial f}{\partial v_1}, \qquad \left(\frac{\partial f}{\partial w_2}, W_2\right) = \frac{\partial f}{\partial v_2},$$

and

(24) $$(W_1, W_2) = \lambda W + \beta_1 \frac{\partial f}{\partial v_1} - \beta_2 \frac{\partial f}{\partial v_2} \qquad (W = W_2 - W_1).$$

The value of λ is given by formulas (16), and one will have:

(25) $$\beta_1 = \lambda (w_1 - \alpha_1) + W_1 \alpha_1, \qquad \beta_2 = \lambda (w_2 - \alpha_2) + W_2 \alpha_2.$$

One sees that β_2 does not depend upon w_1, which is linear in w_2, and one easily confirms that it is linear in v_1, moreover. One has some analogous results for β_1.

The general involution of degree 2 of the sheaf (22) is obtained immediately in the form:

$$\left\{W_1 + t_1 \frac{\partial f}{\partial w_1} + \beta_2 \frac{\partial f}{\partial w_2}, W_2 + \beta_1 \frac{\partial f}{\partial w_1} + t_2 \frac{\partial f}{\partial w_2}\right\},$$

in which t_1 and t_2 remain arbitrary.

One will then have the process by which one generates fourth-order characteristics that are one-dimensional integrals of the sheaf:

(26) $$\left\{W_1 + t_1 \frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\right\}, \qquad \left\{W_2 + \beta_1 \frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\right\}.$$

As for the invariants of the first system, for example, they are given by:

$$W_1 + \beta_2 \frac{\partial f}{\partial w_2} = 0 , \qquad \frac{\partial f}{\partial w_1} = 0 .$$

One concludes forthwith that $\partial f / \partial v_1 = 0$, and upon setting:

$$\beta_2 = \beta_{2,0} + \beta_{2,1} \cdot v_1$$

(in such a manner as to exhibit the fact that β_2 is linear in v_1), one will come down to the search for solutions of the system:

(27) $$P_1 + \alpha_{2,0} \frac{\partial f}{\partial v_2} + \beta_{2,0} \frac{\partial f}{\partial w_2} = 0 , \qquad Q_1 + \alpha_{2,1} \frac{\partial f}{\partial v_2} + \beta_{2,1} \frac{\partial f}{\partial w_2} = 0 .$$

Those solutions must not depend upon either v_1 or w_1, which do not appear in equations (27), moreover.

The remarks that were made on the subject of third-order invariants extend, moreover, to fourth-order quite easily, and the nature of the preceding calculations shows that there would be difficulty in generalizing the results thus-obtained (by a process of recurrence) for the characteristics and invariants of arbitrary order.

27. *Applications of the theory of characteristics to the integration of the equation.* – We first confine ourselves to introducing first and second-order derivatives: The variables x, y, z, p, q, s, t, and the complete integrals are defined by complete systems of the form [no. **24**]:

(28) $$V_1 = P_1 + v_1 Q_1 = 0 , \qquad V_2 = P_2 + v_2 Q_2 = 0 .$$

Formula (15) must be modified here, because v_1 and v_2 are no longer new variables, but functions of x, y, z, p, q, s, t that we determine in such a manner that the system (28) is effectively complete. With the notations of no. **25**, we will then have:

(29) $$(V_1, V_2) = \lambda (V_2 - V_1) + (\alpha_2 - V_1 v_1) Q_1 + (V_1 v_2 - \alpha_2) Q_2 ,$$

in such a way that v_1 and v_2 are determined by the system:

(30) $$V_1 v_1 = \alpha_1 , \qquad V_1 v_2 = \alpha_2 .$$

The peculiarities of the integration must depend upon the nature of the *characteristic subsheaves* $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$. For example, consider the first one and its successive derived sheaves [no. **2**]. One easily confirms that the direct derived sheaf has degree 3, and that the second derived sheaf has degree at least 4. Therefore, the subsheaf $\{P_1, Q_1\}$ has at most three invariants.

1. Suppose that there exists one, and let $\mathcal{I}$ be that invariant. That will permit one to find some complete integrals. To that end, determine v_2 in such a manner that $\mathcal{I}$ is a solution of the system (28). That will give the equation of first degree in v_2 :

$$P_2\,\mathcal{I} + v_2\,Q_2\,\mathcal{I} = 0\,. \tag{31}$$

Here one must suppose that $Q_2\,\mathcal{I} \neq 0$, but if one has $Q_2\,\mathcal{I} = 0$ then since one already has $Q_1\,\mathcal{I} = 0$, one can conclude [*see* the expressions (9) and (10) for Q_1 and Q_2] that $\mathcal{I}$ will not be effectively of second order.

We shall discard that case for the moment.

Since $\mathcal{I}$ is a solution of the system (28), it will satisfy the equation (V_1, V_2). From the formulas (29), and upon taking into account the facts that $Q_1\,\mathcal{I} = 0$ and $Q_2\,\mathcal{I} = 0$, one must have:

$$V_1\,v_2 - \alpha_2 = 0 \tag{32}$$

for any v_1.

It will then suffice to determine v_1 by means of the equation:

$$V_2\,v_1 = \alpha_1 \tag{33}$$

in order for the system (28) to be complete. Now, since v_2 is known, that equation (33) is integrated by means of ordinary differential equations.

One then concludes that all of the integrals of the proposed system that satisfy the condition that $\mathcal{I}$ = const. are obtained by integrating ordinary differential equations [equations (33), and then the complete system (28)].

2. Suppose that the invariant $\mathcal{I}$ has order one. One has $\dfrac{\partial\mathcal{I}}{\partial s} = \dfrac{\partial\mathcal{I}}{\partial t} = 0$, and the condition that $P_1\mathcal{I} = 0$ will reduce [*see* equation (9)] to:

$$X_1\,\mathcal{I} + m_1\,X_2\,\mathcal{I} = 0\,. \tag{34}$$

Thus, consider the multiplicities that satisfy $\mathcal{I}$ = const., which is a first-order partial differential equation. They satisfy the following equations (*see* no. **23**, the definition of X_1 and X_2):

$$0 = \frac{\partial\mathcal{I}}{\partial x} = X_1\,\mathcal{I} + [x - \Phi\,(x, y, z, p, q, r, s, t)]\,\frac{\partial\mathcal{I}}{\partial p},$$

$$0 = \frac{\partial \mathcal{I}}{\partial y} = X_2 \mathcal{I} .$$

If one takes (34) into account then one will conclude, after discarding the exceptional case ([1]) of $\partial \mathcal{I} / \partial p = 0$, that:

$$r - \Phi (x, y, z, p, q, s, t) = 0 .$$

Therefore, all of the integrals of the equation $\mathcal{I}$ = const. are integrals of the one that was posed.

3. Now suppose that the characteristic sheaf $\{P_1, Q_1\}$ has two invariants $\mathcal{I}_1$, $\mathcal{I}_2$, and let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, $\mathcal{I}_6$ be independent solutions of the equation $P_1 + v_1 Q_1 = 0$ of any one of the complete subsheaves (28). The complete integral of that subsheaf will have the form:

$$f_k (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6) = c_k \qquad (k = 1, 2, 3, 4, 5).$$

It will then satisfy an equation of the form:

$$\varphi (\mathcal{I}_1, \mathcal{I}_2) = \text{const.} \tag{35}$$

that is obtained by eliminating $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, $\mathcal{I}_6$. Now, $\varphi (\mathcal{I}_1, \mathcal{I}_2)$ is an invariant of the characteristic subsheaf $\{P_1, Q_1\}$ for any φ. The methods in the two preceding cases will then provide all of the integrals of the second-order equation.

4. The operations that relate to the first case that was considered will simplify when the two characteristic subsheaves separately admit an invariant. Let $\mathcal{I}$ be an invariant of $\{P_1, Q_1\}$, and let $\mathcal{H}$ be an invariant of $\{P_2, Q_2\}$, and suppose that both of them are of second order. The argument in the first case proves that if one determines v_2 and v_1 by the conditions:

$$P_2 \mathcal{I} + v_2 Q_2 \mathcal{I} = 0 , \quad P_1 \mathcal{H} + v_1 Q_1 \mathcal{H} = 0$$

then the system (28) will be complete. One remarks that one knows two integrals $\mathcal{I}$ and $\mathcal{H}$, moreover. The determination of the corresponding complete integral will then depend upon only the integration of a third-order ordinary differential equation.

([1]) The exception is apparent only because if one wishes to make that derivative $\partial \mathcal{I} / \partial p$ reappear in $\mathcal{I}$ then it will suffice to make a change of variables (x, y). If one does not make that change of variables then Φ will become indeterminate when one supposes that x, y, z, q, s, t are coupled by $\partial \mathcal{I} / \partial x = \partial \mathcal{I} / \partial y = 0$, and it will suffice to write the equation $r = \Phi$ in the form $A\, r + B = 0$ in order to confirm that it is verified.

That remark will apply (if applicable) concurrently with the considerations of the preceding case.

28. *Using higher-order characteristics.* – The considerations of the first, third, and fourth cases that were envisioned in the preceding subsection can be applied with no modification when one is dealing with invariants of the characteristic subsheaves of higher order.

Indeed, those subsheaves have the form:

$$\left\{R_1, \frac{\partial f}{\partial z_1}\right\}, \qquad \left\{R_2, \frac{\partial f}{\partial z_2}\right\}. \tag{36}$$

The variables, when prolonged k times, are:

$$x, y, z, p, q, s, t, \qquad M_1 t, M_2 t, \; M_1^{(2)} t, \; M_2^{(2)} t, \ldots, \; M_1^{(k)} t, \; M_2^{(k)} t. \tag{37}$$

One has set:

$$M_1 = \frac{\partial}{\partial x} + m_1 \frac{\partial}{\partial y}, \qquad M_2 = \frac{\partial}{\partial x} + m_2 \frac{\partial}{\partial y},$$

and the upper indices indicate the iteration of those operations. The infinitesimal transformations R_1, R_2 are defined entirely by the method of calculation that was given by the approach of no. **26**, and one must denote the last two of the variables in (37) by the symbols z_1 and z_2, to abbreviate.

Any complete integral (when prolonged k times) is defined by a system of the form:

$$V_1 = R_1 + \zeta_1 \frac{\partial f}{\partial z_1} = 0, \qquad V_2 = R_2 + \zeta_2 \frac{\partial f}{\partial z_2} = 0, \tag{38}$$

and since one has an identity of the form:

$$(V_1, V_2) = \lambda\,(V_2, V_1) + (\theta_1 - V_2\,\zeta_1)\,\frac{\partial f}{\partial z_1} + (V_1\,\zeta_2 - \theta_2)\,\frac{\partial f}{\partial z_2}$$

for arbitrary functions ζ_1 and ζ_2 of the variables (37), in which θ_1 and θ_2 and are known functions of the variables (37), it will then result that in order for ζ_1 and ζ_2 to correspond to a complete integral, it is necessary and sufficient that they should satisfy the conditions:

$$V_2\,\zeta_1 = \theta_1, \qquad V_1\,\zeta_1 = \theta_2. \tag{39}$$

The starting point is then the same as in the preceding subsection, while all of the considerations in it will persist as is.

29. *First-order characteristics.* – Return to the case in which the second-order characteristic sheaf $\{P_1, Q_1\}$ admits a first-order invariant $\mathcal{I}$. To abbreviate the writing, set:

$$(40)\qquad \begin{cases} \dfrac{d\mathcal{I}}{dx} = \dfrac{\partial \mathcal{I}}{\partial n} + p\dfrac{\partial \mathcal{I}}{\partial z} + r\dfrac{\partial \mathcal{I}}{\partial p} + s\dfrac{\partial \mathcal{I}}{\partial q} = Ar + Bz + D = u, \\ \dfrac{d\mathcal{I}}{dy} = \dfrac{\partial \mathcal{I}}{\partial y} + q\dfrac{\partial \mathcal{I}}{\partial z} + s\dfrac{\partial \mathcal{I}}{\partial p} + t\dfrac{\partial \mathcal{I}}{\partial q} = As + Bt + C = v. \end{cases}$$

One has [no. **27**] $\partial\mathcal{I} / \partial y = X_2\,\mathcal{I}$, and *upon taking into account the fact that* $r = \mathcal{F}$, $\partial\mathcal{I} / \partial x = X_1\,\mathcal{I}$.

Equation (34) $X_1\,\mathcal{I} + m_1\,X_2\,\mathcal{I} = 0$ will then indicate that one will have:

$$(41)\qquad m_1 = -\frac{u}{v}$$

if one takes into account the fact that $r = \Phi$. If one returns to equation (6), which served to define m_1, then one can conclude that $r = \Phi$ satisfies the equation:

$$(42)\qquad u^2 - uv\frac{\partial r}{\partial s} - v^2\frac{\partial r}{\partial t} = 0\,.$$

The corresponding homogeneous equation:

$$(43)\qquad u^2\frac{\partial f}{\partial r} + uv\frac{\partial f}{\partial s} + v^2\frac{\partial f}{\partial t} = 0$$

admits the integrals:

$$(44)\qquad \gamma = -\frac{u}{v}, \qquad \alpha = r + \gamma s, \qquad \beta = s + \gamma t\,,$$

and those integrals are coupled by the relation:

$$(45)\qquad A\,\alpha + B\,\beta + C\,\gamma + D = 0\,.$$

One then sees that the second-order equation has the form ([1]):

$$(46)\qquad r + \gamma s = F(\gamma)\,, \qquad \text{with} \qquad \gamma = -\frac{u}{v}\,,$$

([1]) Here, one implicitly discards the case of *equations that are linear in r, s, t*. In that case, m_1 will not depend upon r, s, t, and the equation will have the form $\gamma =$ function of (x, y, z, p, q).

and one will have $m_1 = \gamma$ when one takes the equation into account. The function F will depend upon x, y, z, p, q arbitrarily, and A, B, C, D will also depend upon them.

For the converse to be true, it is necessary, moreover, that A, B, C, D should have the property that the system:

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) : \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right) : \frac{\partial f}{\partial p} : \frac{\partial f}{\partial q} = D : C : A : B$$

has at least one solution.

More symmetrically, one can say that the equation $r = \Phi$ results from the elimination of γ between two equations of the form:

$$r + \gamma s = F(\gamma), \qquad s + \gamma t = G(\gamma), \tag{47}$$

in which the functions $\alpha = F$, $\beta = G$ satisfy the identity (45). If one then interprets r, s, t as running coordinates ([1]) then the equations $r = \Phi$ will represent a ruled surface that has a rectilinear director of:

$$A r + B s + D = 0, \qquad A s + B t + C = 0.$$

The generators of the surface, like the director, are parallel to the generators of the cone $r t - s^2 = 0$.

If one drops the condition that relates to the rectilinear director, while keeping the definition of $r = \Phi$ by the two equations (47), in which F and G are arbitrary, then one will find that equation (6), which defines the characteristic sheaves, can be written:

$$(m - \gamma)[m(G' - s) + (F' - s)] = 0.$$

Hence, one of the roots is the common solution $m_1 = \gamma$ that the two equations (47) will possess when one takes into account the second-order equation that is considered, viz., $r = \Phi$.

One infers the following consequence from that:

The characteristics of the first system $\{P_1, Q_1\}$ that generate an integral are trajectories of an infinitesimal transformation of the form:

$$P_1 + v_1 Q_1 = X_1 + m_1 X_2 + u_3 X_3 + u_4 X_4,$$

which is a prolongation of:

$$X_1 + m_1 X_2 = \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) + m_1\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) + (r + m_1 s)\frac{\partial f}{\partial p} + (s + m_1 t)\frac{\partial f}{\partial q}, \tag{48}$$

and by virtue of equations (47) and the preceding remark, the latter transformation is written:

([1]) Compare GOURSAT, *loc. cit.*, t. I, pp. 195.

(49) $$Z f = \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) + m_1\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right) + F(m_1)\frac{\partial f}{\partial p} + G(m_1)\frac{\partial f}{\partial q}.$$

All of the integrals for which m_1 is the same function of x, y, z, p, q are then generated (when one considers them to be prolonged to only first order) by the trajectories of the corresponding transformation (49).

That transformation (49) belongs to the family of infinitesimal transformations:

(50) $$Z f = \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) + \mu\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right) + F(\mu)\frac{\partial f}{\partial p} + G(\mu)\frac{\partial f}{\partial q},$$

in which μ is an arbitrary function of x, y, z, p, q.

In that case, one will then have a *family of first-order characteristic infinitesimal transformations* whose trajectories again serve to generate the integral multiplicities of the equation (when prolonged to first order), and the preceding analysis will show that the case envisioned is the only one for which that can be true.

However, the family of characteristic infinitesimal transformations is no longer a sheaf, in general.

In order for one to have a *first-order characteristic sheaf* to which all of the transformations $X_1 + m_1 X_2$ that correspond to the integrals, it is necessary and sufficient that F and G must be linear and consequently that equations (47) must have the form:

$$r + \gamma s = a + b\gamma, \qquad s + \gamma t = a' + b'\gamma.$$

That is the case in which the ruled surface is a quadric of the type:

$$(r t - s^2) - a' r + (b + b') s - a t + (a a' + b b') = 0,$$

i.e., the case of the *Monge-Ampère equation* ([1]). Each of the systems of rectilinear generators of the quadric then corresponds to a first-order characteristic sheaf. If is clear that those characteristic sheaves are not subsheaves of the sheaf $\{X_1, X_2, X_3, X_4\}$ that is associated with the second-order

([1]) Here, one must reintroduce the *linear equations*. Since the equation is:

$$r + (m_1 + m_2)\, s + m_1 m_2\, t = K,$$

one will have:

$$r + m_1 s = -m_2 (s + m_1 t) + K,$$

and the transformation (48) will belong to the *characteristic sheaf:*

$$\left\{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial x} + m_1\left(\frac{\partial f}{\partial q} + q\frac{\partial f}{\partial z}\right) + K\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q} - m_2\frac{\partial f}{\partial p}\right\}.$$

equation, but one can prolong them in such a manner as to recover the second-order characteristic subsheaves.
