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Essay on the propagation of waves

By

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In a preceding article [1] we have analytically studied the propagation of waves in a medium whose nature does not vary in time and showed the close relationship between this problem and the theory of contact transformations, the theory of first-order partial differential equations in which the unknown function does not appear explicitly, and the search for maxima and minima of simple integrals.

The following pages are dedicated to the propagation of waves in a medium whose nature does vary in time. The problem is treated from a purely kinematical point of view. The medium is defined by the system of elementary waves that have their origins at the various points of the medium at each instant. The law of propagation is the principle of enveloping waves, but we suppose only that it has meaning for an infinitesimal interval, and up to higher-order infinitesimals. One of the results obtained is that the principle is rigorously true for any time interval.

We reason in the space of n dimensions, but our exposition supposes known only the notions of contact element and multiplicity, and the fundamental principles of the theory of ordinary differential equations.

Two essential facts are presented: On the one hand, the propagation involves contact elements, the contact elements of the original waves being individually transported to constitute the new wave that one derives from it, and, on the other hand, the family of successive waves that issue from that same original wave is defined by a partial differential equation that may be the most general first-order partial differential equation in n independent variables and one unknown function.

From this, there intuitively emerges a new theory of the integration of first-order partial differential equations. The law of displacement of the contact elements of the medium is given by the differential system that the Cauchy theory gives for the characteristics.

In the displacement of a contact element, the point of that element describes what we call a *trajectory of disturbance.* We establish that in the case of a variable regime, as in

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[1] *Sur l'interpretation mécanique des transformations de contact infinitésimales (Bulletin de la Société Mathématique de France, t.* XXXIV, 1906).

the case of a permanent regime, the trajectories correspond to the minimum duration of propagation. The question is equivalent to that of the general study of the necessary and sufficient conditions for the minimum of the integral of a differential equation of the form:

$$dt = \Omega(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n),$$

where Ω is homogeneous of first degree in $dx_1, \ldots, dx_n$; this integral is taken under the hypothesis that $x_1, \ldots, x_n$ are the current coordinates of a point on an arc of that curve, and at a given initial value t_0 they define the origin of that arc. Moreover, it is the value that is taken at the extremity of that arc that serves to provide a minimum upon conveniently choosing the arc of the curve whose extremities are assumed to be given ([1]).

The consideration of the simple variation leads to the necessary conditions that define the desired curve as a trajectory of propagation. We show that these conditions are sufficient whenever the elementary waves have a form that is concave towards the origin at each of the intervening points.

From this, it results that the preceding question of a minimum comes down to the study of a question of a maximum that presents itself in the propagation of a disturbance along a given curve, and that the answer to that question is almost intuitive.

In that question of maximum, one deals with the integral of an equation of the form:

$$dt = \sum_{i=1}^{n} p_i dx_i$$

that is taken along a given fixed curve, and one must determine the functions $p_1, \ldots, p_n$ in such a manner that they satisfy a given relation:

$$H(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0,$$

and render a maximum to the value taken by the integral at the extremity of the arc of the curve.

The method that thus presents itself is equivalent to the methods of Weierstrass and Hilbert; we confine ourselves to a brief sketch. It is applied, with good reason, to the theory of maxima and minima of simple integrals, and may be extended to the case of multiple integrals. We shall return to this in another work.

Finally, we point out that we have supposed that the elementary waves have ∞^{n-1} points and ∞^{n-1} tangents. We propose to return on another occasion to the other case, which offers particular interest from the viewpoint of the theory of partial differential equations that Lie has called *semi-linear* or *pseudo-linear* from the viewpoint of the calculus of variations ([2]).

([1]) *See*, on the subject of problems of this genre, A. MAYER, *Leipziger Berichte*, 1895, and D. EGOROW, *Mathematischen Annalen*, 1906.

([2]) The following pages had been composed when I became aware of the Mémoire of CARATHÉODORY, *Sur les maxima et minima des intégrales simples* (*Math. Annalen*, t. LXII, 1906, pp. 449-503), in which the author used derived waves, under the name of *indicatrices*, in the case $n = 2$, but without attaching the question to that of the propagation of waves. The problem treated by

I. – Differential equations of the propagation of waves.

1. Let $x_1, \ldots, x_n$ be the coordinates of an arbitrary point in the space E_n of n dimensions, which is assumed to be referred to an arbitrary rectangular coordinate system. We call this point the point $(x_1, \ldots, x_n)$, or, by abbreviation, the point (x).

We consider the space E_n to be the *medium* in which *disturbances* of an arbitrary nature may be produced and propagate by *waves*.

By this, we mean that the points of E_n are capable of acquiring, in an instantaneous manner, a property of a specific nature (sonority, luminosity, electrification, etc.), and that, from the fact that this property will be manifested at an arbitrary instant t at every point of a multiplicity $\mathcal{M}$, and, at the following instants, will cease to belong to the points of $\mathcal{M}$, and will be manifested at each of these instants $t + \Delta t$ at the various points of another multiplicity $\mathcal{M}'$, which is determined by the nature of the medium, relative to the property considered, by the instant t, by the interval of time Δt that has elapsed since that instant, and by the multiplicity $\mathcal{M}$.

It is the appearance of the property considered at a point (x) that we call a *disturbance* produced at that point. We call any multiplicity that is the geometric locus of disturbed points at the same instant a *wave*.

The problem of the *propagation of waves* is the following one: *In a medium of a specific nature, to deduce from a wave $\mathcal{M}$ that is given at the time t, the new wave that it provides after the time Δt.*

2. One must define the nature of the medium relative to the property considered.

To that effect, imagine the simplest case where an isolated point (x) is disturbed only at the time t. As it propagates, the disturbance gives rise, at each instant $t + \Delta t$, to a wave $M(x \mid t, \Delta t)$; we say that this wave *issues* from (x), or also that it has (x) for its *origin*.

Take the homothety of that wave, with respect to (x), with the ratio $1/\Delta t$, and assume that this homothety tends towards a limiting form when Δt tends to zero. We call that limiting form the *derived wave that has* (x) *for its origin at the instant t*.

Conversely, if we take the homothety of the derived wave with respect to its origin (x) and the infinitely small ratio dt then the multiplicity that is obtained will be called the *elementary wave having* (x) *for origin and corresponding to the instant t.*

The nature of the medium relative to the property considered will be defined by the system of derived waves (or elementary waves) that have their origins at the various points of the medium at each instant t.

This system of waves will vary with t, in general. In the contrary case, we say that one is in a *permanent regime*. We confirm that the mode of propagation of an arbitrary wave is then independent of the instant at which this wave appears, and depends only upon its form. The general case will be called the case of a *variable regime*.

In this memoir, we suppose that the derived waves have ∞^{n-1} points and also ∞^{n-1} tangent planes; we shall return to the other case in another work. There is good reason to

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CARATHÉODORY corresponds, moreover, to the case of the permanent regime, while the one that we treat here corresponds to the case of a variable regime.

remark that the derived waves do not necessarily have the same number of dimensions as the finite waves that issue from the various points of space.

In the case of ordinary space ($n = 3$), the derived waves are, in the general case, non-developable surfaces; one calls them *wave surfaces*. In the exceptional cases, they may be developable surfaces, curves, or points.

3. In order to now define the law by which the waves propagate, *we assume that for an infinitely small variation of time the propagation satisfies the principle of enveloping waves, up to higher-order infinitesimals.*

In what follows, it will be proved that this principle implies no contradiction. First, we explain what we mean by this:

Let $\mathcal{M}$ be arbitrary at the instant t, and let $\mathcal{M}'$ be the wave that it produces after an infinitely small time dt. Each of the points (x) of $\mathcal{M}$, when disturbed at the instant t, will have emitted a wave $M(x \mid t, dt)$ after a time dt; let $\mathcal{M}''$ be the envelope of all these waves $M(x \mid t, dt)$ that issue from the various points (x) of $\mathcal{M}$. *We assume that* $\mathcal{M}''$ *represents* $\mathcal{M}'$ *up to higher-order infinitesimals*, the principal infinitesimal being dt, and that we intend this to mean that *there exists a point-by-point correspondence between* $\mathcal{M}'$ *and* $\mathcal{M}''$, *such that the differences of the coordinates with the same name between two arbitrary homologous points are of order greater than* 1 relative to dt.

This definition gives rise to the following remarks, for which, to simplify, we will confine ourselves to considering $(n-1)$-dimensional multiplicities:

1. Let Σ and Σ' be two multiplicities, each of which represents the other, up to higher-order infinitesimals, and let θ be the principal infinitesimal. The correspondence between an arbitrary point (X) of Σ and the homologous point (X') of Σ' will be exhibited by the equations of the two multiplicities:

$$(\Sigma) \qquad X_i = f_i(u_1, \ldots, u_{n-1} \mid \theta) \qquad (i = 1, 2, \ldots, n),$$
$$(\Sigma') \qquad X'_i = g_i(u_1, \ldots, u_{n-1} \mid \theta) \qquad (i = 1, 2, \ldots, n),$$

two homologous points corresponding to the same values of the parameters $u_1, \ldots, u_n$.

Moreover, the identity of the two surfaces, up to higher-order infinitesimals, amounts to supposing that the functions f_i and g_i, and their derivatives $df_i / d\theta$, $dg_i / d\theta$ are identical functions of the parameters $u_1, \ldots, u_{n-1}$ for $\theta = 0$.

From this results the pairwise identity of the functional determinants formed from the derivatives of g_i and f_i, taken with respect to the u_k for $\theta = 0$, and also the derivatives of these functional determinants, taken with respect to θ (for $\theta = 0$), in such a way that *the differences of the coordinates with the same name of the tangent planes* Σ *and* Σ' *at two homologous points are also higher-order infinitesimals.*

2. Now suppose that Σ belongs to a family of ∞^{n-1} multiplicities, each of which corresponds to a multiplicity Σ' that represents it up to higher-order infinitesimals. *A*

correspondence of the same nature will be meaningful between the envelope of the multiplicities Σ *and that of the multiplicities* Σ'.

Indeed, the equations of Σ are now of the form:

(1) $$X_i = f_i(u_1, \ldots, u_{n-1} \mid a_1, \ldots, a_{n-1} \mid \theta) \qquad (i = 1, 2, \ldots, n),$$

and the envelope will be given by the equations:

(2) $$\text{Det}\left|\frac{\partial f_i}{\partial u_1} \quad \cdots \quad \frac{\partial f_i}{\partial u_{n-1}} \quad \frac{\partial f_i}{\partial a_k}\right|_{(i=1,2,\ldots,n)} = 0 \qquad (k = 1, 2, \ldots, n-1);$$

i.e., an arbitrary point of the envelope is given by equations (1), where $u_1, \ldots, u_{n-1}$ are the functions of $a_1, \ldots, a_{n-1}$ that are defined by equations (2).

In order to pass to the envelope of Σ', one must replace the f_i with functions g_i of the same variables, and for $\theta = 0$ the g_i and the $dg_i / d\theta$ are identical to the f_i and the $df_i / d\theta$, respectively. However, the functions u_k of $a_1, \ldots, a_{n-1}$ that are obtained in the two cases will then be the same for $\theta = 0$, as well as their derivatives with respect to θ. The stated theorem results from this.

3. *If* Σ' *represents* Σ *up to higher-order infinitesimals and if* Σ'' *similarly represents* Σ' *then the same correspondence exists between* Σ'' *and* Σ.

The proof is immediate.

4. *The wave* $M(x \mid t, dt)$ *that is emitted at the arbitrary point* (x) *is represented by the elementary wave that has the same point for origin, up to higher-order infinitesimals.*

Indeed, let P be an arbitrary point of $M(x \mid t, dt)$ and let ρ be its distance from the origin (x) of that wave. The homologous point of the elementary wave has the radius vector:

$$\left(\lim_{dt=0} \frac{\rho}{dt}\right) dt .$$

The distance between the two homologous points is therefore the difference between the infinitely small ρ and its principal part; the stated remark then results from this.

Upon combining these various remarks, we may state the principal of enveloping waves in the following form:

Up to higher-order infinitesimals, the wave that issues from an arbitrary original wave, starting at the arbitrary instant t and after the infinitely small time dt, is the envelope of the elementary waves that are emitted, under the same conditions, by the various points of the original wave.

4. In order to treat this principle analytically, one must first express the derived waves by their general equation. For this, we imagine that at each point (x) there is a system of axes that has that point for origin, and is deduced by translation of the fundamental system of axes to which the medium in question is referred. The general equation of the planes being assumed to be written in the form:

$$(1) \qquad \sum_{i=1}^{n} u_i \xi_i - 1 = 0,$$

this gives us the tangential equation:

$$(2) \qquad H(t \mid x_1, \ldots, x_n \mid u_1, \ldots, u_n) = 0$$

of the derived wave that has (x) for its origin, referred to precisely that system of coordinates that has (x) for its origin. In that equation, $u_1, \ldots, u_n$ are therefore the current tangential coordinates, whereas the t, $x_1, \ldots, x_n$ play the role of parameters, and in equation (1), $\xi_1, \ldots, \xi_n$ are the current pointlike coordinates in the same auxiliary system of coordinates.

In order to have the general equation of the elementary wave under the same conditions, we remark that if the plane (1) is tangent to the derived wave then the tangent plane to the elementary wave that corresponds to it is:

$$\sum_{i=1}^{n} u_i \xi_i - dt = 0.$$

The coordinates are thus obtained upon dividing those of the tangent plane to the derived wave by dt, and, as a result, satisfy the equation:

$$(3) \qquad H(t \mid x_1, \ldots, x_n \mid u_1\, dt, \ldots, u_n\, dt) = 0.$$

We abbreviate the ultimate calculations by giving a particular form to equation (2): We make it homogeneous, solve it for the homogeneity variable, and give the value 1 to this variable. We obtain an equation of the form:

$$(4) \qquad \Pi(t \mid x_1, \ldots, x_n \mid u_1, \ldots, u_n) = 1,$$

where Π is homogeneous of degree 1 with respect to $u_1, \ldots, u_n$.

One may further say that Π is defined by the identity:

$$(5) \qquad H\left(t \,\middle|\, x_1, \cdots, x_n \,\middle|\, \frac{u_1}{\Pi}, \cdots, \frac{u_n}{\Pi}\right) \equiv 0.$$

The equation of the point of contact of the tangent plane $(u_1, \ldots, u_n)$ with that surface is then:

$$\sum_{i=1}^{n} U_i \frac{\partial \Pi}{\partial u_i} = \sum_{i=1}^{n} u_i \frac{\partial \Pi}{\partial u_i} = 1,$$

and, as a result, the coordinates of this point of contact are:

$$\xi_i = \frac{\partial \Pi}{\partial u_i} \qquad (i = 1, 2, \ldots, n). \tag{6}$$

Since the right-hand sides of these formulas (6) are of degree zero in $u_1, \ldots, u_n$, they give, in reality, the coordinates of the point of contact of a tangent plane to the surface (4) that is parallel to a given plane.

The resolution that one must make in order to pass from the general form (2) to the canonical form (4) thus amounts to separating the derived wave into sheets, such that each of these sheets has one and only one tangent plane that is parallel to an arbitrarily given plane.

Finally, if the derived wave is given in the form (4) then the elementary wave will have the tangential equation:

$$\Pi(t \mid x_1, \ldots, x_n \mid u_1, \ldots, u_n)\, dt = 1, \tag{7}$$

and just as the derived wave is parametrically represented, from the pointlike point of view, by formulas (6), in which only the ratios of the u_i appear, likewise, the elementary wave will be defined by the formulas:

$$\xi_i = \frac{\partial \Pi}{\partial u_i} dt \qquad (i = 1, 2, \ldots, n). \tag{8}$$

We shall also have need for the general equation of the elementary waves when referred to the primitive coordinate system. The plane that has (1) for its equation in the system with origin (x) has the equation in the fundamental system:

$$\sum_{i=1}^{n} u_i (X_i - x_i) - 1 = 0,$$

and if one converts this equation into the form:

$$\sum_{i=1}^{n} q_i X_i - 1 = 0, \tag{9}$$

then in order to transform the tangential coordinates one has the formulas:

$$\frac{u_i}{q_i} = \frac{\sum_{k=1}^{n} u_k x_k + 1}{1} = \frac{1}{1 - \sum_{k=1}^{n} q_k x_k} \qquad (i = 1, 2, \ldots, n).$$

Equation (7) thus becomes:

$$\Pi(t \mid x_1, \ldots, x_n \mid q_1, \ldots, q_n)\, dt + \sum_{i=1}^{n} q_i x_i = 1, \tag{10}$$

and we have to find the envelope of all of the elementary waves represented by that equation (1) when (x) is on $\mathcal{M}$. Each of them has a certain number of contact elements (point, tangent plane) in common with the envelope that we shall determine. For this, we express that they are common to (1) and to the infinitely close waves that result from it by infinitely small variation of (x) on $\mathcal{M}$.

To that end, denote by δ any differentiation relative to such a variation: the variations $\delta x_1, \ldots, \delta x_n$ will be uniquely subject to the condition:

$$\sum_{i=1}^{n} p_i \delta x_i = 0, \tag{2}$$

where $p_1, \ldots, p_n$ are the direction coefficients for the tangent plane to $\mathcal{M}$ at (x). We must then express that the equation obtained upon applying the differentiation δ to (1) is a consequence of (2), which gives the equations:

$$\frac{\partial \Pi(t \mid x \mid q)}{\partial x_i} + q_i = m p_i \qquad (i = 1, 2, \ldots, n), \tag{3}$$

where m is a factor that one will determine by taking (1) into account.

However, one may leave m indeterminate, because equations (3) thus define the direction of the planes of the desired contact elements by means of the ratios of the q_i, and equations (11)(no. **4**), in which only the ratios appear, then give the point to which each of the corresponding contact elements belong.

The form of equations (3) shows that there is a direction $(q_1, \ldots, q_n)$ satisfying the question that tends to the direction $(p_1, \ldots, p_n)$ when dt tends to zero, and that there is only one of them. Therefore, among the contact elements that are common to the elementary wave (1) and all of the infinitely close waves there is one and only one of them that tends to the contact element $(x_1, \ldots, x_n \mid p_1, \ldots, p_n)$ of the wave M when dt tends to zero. Denote the coordinates of that contact element by $(x_1', \cdots, x_n' \mid p_1', \cdots, p_n')$. Further, denote the instant $t + dt$ by t'. We have the equations:

(4) $$x_i' = \frac{\partial \Pi(t \mid x \mid p')}{\partial p_i'}(t' - t) + x_i \qquad (i = 1, 2, \ldots, n)$$

and:

(5) $$\frac{\partial \Pi(t \mid x \mid p')}{\partial x_i}(t' - t) = mp_i \qquad (i = 1, 2, \ldots, n).$$

Moreover, in order to unambiguously determine the p_i , whose ratios alone have been given up to now, we subject them to satisfy the condition:

(6) $$\Pi(t \mid x_1 , \ldots, x_n \mid p_1 , \ldots, p_n) = 1,$$

and this will then define them with no ambiguity, Π being homogeneous of degree 1.

Likewise, the p_i' will be subject to satisfying the analogous relation:

(7) $$\Pi(t' \mid x_1', \cdots, x_n' \mid p_1', \cdots, p_n') = 1.$$

Thus, when dt tends to zero each of the differences ($x_i' - x_i$) and ($p_i' - p_i$) tends to zero, and their principal parts, which we denote by dx_i and dp_i, are obtained by differentiating equations (4), (5), and (7) with respect to t' for $t' = t$, which gives ($dt' = dt$):

(8) $$dx_i = \frac{\partial \Pi(t \mid x \mid p)}{\partial p_i} dt \qquad (i = 1, 2, \ldots, n),$$

(9) $$\frac{\partial \Pi(t \mid x \mid p)}{\partial x_i} dt + dp_i = p_i \, d\mu \qquad (i = 1, 2, \ldots, n),$$

(10) $$\frac{\partial \Pi}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial x_i} dx_i + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i} dp_i = 0.$$

Upon eliminating the dx_i and the dp_i , the latter gives $d\mu$. This gives:

$$\frac{\partial \Pi}{\partial t} dt + \sum_{i=1}^{n} p_i \frac{\partial \Pi}{\partial p_i} d\mu = 0,$$

hence, due to the homogeneity of Π:

(11) $$d\mu = -\frac{\partial \Pi}{\partial t} dt .$$

We thus arrive at the following result:

To each contact element ($x \mid p$) *of order* $\mathcal{M}$, *considered at the instant* t, *there corresponds, on the infinitely close wave that results after the time* dt, *a new contact element, which is given, up to second-order infinitesimals, by the formulas:*

(12) $$dx_i = \frac{\partial \Pi(t \mid x \mid p)}{\partial p_i} dt \qquad (i = 1, 2, \ldots, n),$$

(13) $$dp_i = -\left[\frac{\partial \Pi(t \mid x \mid p)}{\partial p_i} + p_i \frac{\partial \Pi(t \mid x \mid p)}{\partial t}\right] dt \qquad (i = 1, 2, \ldots, n),$$

upon supposing that the coordinates $x_1, \ldots, x_n$; $p_1, \ldots, p_n$ *are linked by the relation:*

(14) $$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1.$$

6. If we suppose, more generally, that the system of derived waves is given by equation (2) (no. **4**) then the condition (14) will be replaced by:

(1) $$H(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0.$$

From the identities (5) of no. **4**, which may be written:

$$H\left(t \mid x_1, \cdots, x_n \mid \frac{p_1}{\Pi}, \ldots, \frac{p_n}{\Pi}\right) \equiv 0,$$

one deduces, upon setting, to abbreviate the notation:

$$w_i = \frac{p_i}{\Pi} \qquad (i = 1, 2, \ldots, n),$$

the identities:

$$\frac{\partial H(t \mid x \mid w)}{\partial t} - \sum_{i=1}^{n} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{w_i}{\Pi} \frac{\partial \Pi}{\partial t} = 0,$$

$$\frac{\partial H(t \mid x \mid w)}{\partial x_k} - \sum_{i=1}^{n} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{w_i}{\Pi} \frac{\partial \Pi}{\partial x_k} = 0 \qquad (k = 1, 2, \ldots, n),$$

$$\frac{\partial H(t \mid x \mid w)}{\partial w_k} - \sum_{i=1}^{n} \frac{\partial H(t \mid x \mid w)}{\partial w_i} \frac{w_i}{\Pi} \frac{\partial \Pi}{\partial p_k} = 0 \qquad (k = 1, 2, \ldots, n),$$

Under the hypothesis (14) (no. **5**), they reduce to:

$$M \frac{\partial \Pi(t \mid x \mid p)}{\partial t} = \frac{\partial H(t \mid x \mid p)}{\partial t},$$

$$M \frac{\partial \Pi(t \mid x \mid p)}{\partial x_i} = \frac{\partial H(t \mid x \mid p)}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

$$M \frac{\partial \Pi(t \mid x \mid p)}{\partial p_i} = \frac{\partial H(t \mid x \mid p)}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

$$M = \sum_{i=1}^{n} p_i \frac{\partial H(t \mid x \mid p)}{\partial p_i},$$

and equations (12), (13), *and* (14) *of no.* **5** *will be replaced by the formulas:*

$$\text{(2)} \qquad \frac{dx_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\left(\dfrac{\partial H}{\partial x_i} + p_i \dfrac{\partial H}{\partial t}\right)} = \frac{dt}{\sum_{k=1}^{n} p_k \dfrac{\partial H}{\partial p_k}},$$

adjoined to equation (1).

II. – Characteristics and the determination of the family of waves

7. We may now begin to treat the general problem of the propagation of waves that was stated in no. **1**: *Knowing an original wave $\mathcal{M}_0$ that is given at the instant t_0, find the wave $\mathcal{M}$ that results at the instant t.*

It is natural to think that $\mathcal{M}$ will be deduced from $\mathcal{M}_0$ by applying the infinitesimal variation defined by formulas (12), (13), (14) of no. **5** an infinite number of times. That is what we shall examine, and we first study whether the indefinite repetition of that infinitesimal variation to an arbitrary multiplicity $\mathcal{M}_0$, taken at the instant t_0, indeed gives a new multiplicity.

From the theory of ordinary differential equations, the indefinitely-repeated application of the variation (12), (13), (14) (no. **5**) is equivalent to the use of the transformation that results from it by integration. However, this system being over-determined, one must show that the integration is possible.

Thus, suppose that the system (12), (13) (no. **5**) has been integrated; i.e.:

$$\text{(1)} \qquad dx_i = \frac{\partial \Pi}{\partial p_i} dt \qquad (i = 1, 2, \ldots, n),$$

$$\text{(2)} \qquad dp_i = -\left(\frac{\partial \Pi}{\partial x_i} + p_i \frac{\partial \Pi}{\partial t}\right) dt \qquad (i = 1, 2, \ldots, n).$$

The general integral is of the form:

$$\text{(3)} \qquad x_i = A_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n),$$

$$\text{(4)} \qquad p_i = B_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n),$$

where $x_1^0, \ldots, x_n^0$; $p_0^1, \ldots, p_0^n$ are the initial values of $x_1, \ldots, x_n$; $p_1, \ldots, p_n$ for $t = t_0$.

Moreover, one deduces from (1) and (2):

$$d(\Pi - 1) = \frac{\partial \Pi}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial x_i} dx_i + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i} dp_i$$

$$= \frac{\partial \Pi}{\partial t}\left(1 - \sum_{i=1}^{n} p_i \frac{\partial \Pi}{\partial p_i}\right) dt$$
$$= \frac{\partial \Pi}{\partial t}(1 - \Pi)\, dt\,.$$

Thus, if we set:

(5) $$C = \Pi(t \mid A_1, \ldots, A_n \mid B_1, \ldots, B_n) - 1,$$

(6) $$\Pi_1 = \frac{\partial \Pi(t \mid A \mid B)}{\partial t}$$

then C is a function of t that satisfies the differential equation:

(7) $$\frac{dC}{dt} + \Pi_1\, C = 0,$$

and which reduces, for $t = t_0$, to:

$$C_0 = \Pi(t_0 \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0) - 1.$$

Now, equation (7) has one and only one integral that reduces to zero for $t = t_0$, and that integral is obviously $C \equiv 0$. Therefore, if C_0 is null then C is also null for any t.

In other words, the values (3), (4) verify equation (14) of no. **5** for any t; i.e.:

(8) $$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1,$$

provided that they are verified for $t = t_0$.

One may further say that *the transformation of* $(x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0)$ *into* $(x_1, \ldots, x_n \mid p_1, \ldots, p_n)$ *that is defined by* (3), (4) *leaves equation* (8) *invariant.*

It is proved by this that it is possible to integrate the mixed system (12), (13), (14) of no. **5**, and that the general integral is given by formulas (3), (4), where $x_1^0, \cdots, x_n^0$; p_1^0, … p_n^0; t_0 are subject only to the condition:

(9) $$\Pi(t_0 \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0) = 1.$$

The indefinite repetition of the infinitesimal variation considered thus has a well-defined sense.

8. Formulas (3), (4) (no. **7**) have some homogeneity properties that are useful to point out. To that effect, in equations (1), (2) (no. **7**) set:

(1) $$x_i = A_i\,, \qquad p_i = mB_i \qquad (i = 1, 2, \ldots, n).$$

The equations:

$$dx_i = \frac{\partial \Pi}{\partial p_i} dt \qquad (i = 1, 2, \ldots, n) \tag{2}$$

are again verified, the right-hand sides being homogeneous of degree zero with respect to the p_i. As for the equations:

$$dp_i = -\left(\frac{\partial \Pi}{\partial x_i} + p_i \frac{\partial \Pi}{\partial t}\right) dt \qquad (i = 1, 2, \ldots, n), \tag{3}$$

they give, upon taking homogeneity into account:

$$m\, dB_i + B_i\, dm = -\left[m \frac{\partial \Pi(t \mid A \mid B)}{\partial A_i} + m^2 B_i \frac{\partial \Pi(t \mid A \mid B)}{\partial t}\right] dt \qquad (i = 1, 2, \ldots, n),$$

which reduces, upon taking into account the definition of the A_i and the B_i, and the notation introduced by formula (6) of no. **7**, to the unique equation:

$$dm = m(1 - m)\, \Pi_i\, dt. \tag{4}$$

Having said this, let m_0 be an arbitrary constant, and let M be the integral of (4) that reduces to m_0 for $t = t_0$. The functions:

$$x_i = A_i, \qquad p_i = MB_i \qquad (i = 1, 2, \ldots, n)$$

constitute the solution of the system (1), (2) that is defined by the initial conditions:

$$x_i = x_i^0, \qquad p_i = m_0\, p_i^0 \qquad (i = 1, 2, \ldots, n).$$

However, this same solution is also given by:

$$\begin{aligned} x_i &= A_i(t \mid x_1^0, \cdots, x_n^0 \mid m_0 p_1^0, \cdots, m_0 p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n),\\ p_i &= B_i(t \mid x_1^0, \cdots, x_n^0 \mid m_0 p_1^0, \cdots, m_0 p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n). \end{aligned}$$

One thus has the identities:

$$\begin{aligned} A_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) &= A_i(t \mid x_1^0, \cdots, x_n^0 \mid m_0 p_1^0, \cdots, m_0 p_n^0 \mid t_0),\\ MB_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) &= B_i(t \mid x_1^0, \cdots, x_n^0 \mid m_0 p_1^0, \cdots, m_0 p_n^0 \mid t_0) \end{aligned}$$

for $(i = 1, 2, \ldots, n)$.

Therefore, the functions A_i, as well as the ratios of the functions B_i, are homogeneous of degree zero with respect to $p_1^0, \ldots, p_n^0$.

From this, it results that one may employ formulas (3), (4) (no. **7**) to the transformation of the contact element $(x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0)$ into the new contact element $(x_1, \ldots, x_n \mid p_1, \ldots, p_n)$, without restricting it to verify condition (9) (no. **7**).

9. Therefore, apply the transformation defined by the equations:

$$x_i = A_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n), \tag{1}$$

$$p_i = B_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n) \tag{2}$$

to each of the contact elements $(x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0)$ of the same multiplicity $\mathcal{M}_0$. We shall show that the new contact elements thus obtained belong to another multiplicity.

Effectively, $x_1^0, \cdots, x_n^0; p_1^0, \cdots, p_n^0$ are, by hypothesis, functions of the $(n - 1)$ independent variables $\alpha_1, \ldots, \alpha_{n-1}$, whose total differentials satisfy the identity:

$$\sum_{i=1}^{n} p_i^0 \delta x_i^0 = 0, \tag{3}$$

and one must show that the functions of $\alpha_1, \ldots, \alpha_{n-1}$ that are deduced from formulas (1), (2) verify the analogous identity:

$$\sum_{i=1}^{n} p_i \delta x_i = 0. \tag{4}$$

Now, since the functions (1), (2) satisfy the equations:

$$\frac{dx_i}{dt} = \frac{\partial \Pi}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \Pi}{\partial p_i} - p_i \frac{\partial \Pi}{\partial t} \qquad (i = 1, 2, \ldots, n), \tag{5}$$

one has:

$$d\sum_{i=1}^{n} p_i \delta x_i = \sum_{i=1}^{n} dp_i \delta x_i + \sum_{i=1}^{n} p_i \delta dx_i = \sum_{i=1}^{n} dp_i \delta x_i + \delta \sum_{i=1}^{n} p_i dx_i - \sum_{i=1}^{n} \delta p_i dx_i ,$$

i.e.:

$$\frac{d}{dt}\sum_{i=1}^{n} p_i \delta x_i = \delta\Pi - \sum_{i=1}^{n} \frac{\partial \Pi}{\partial x_i}\delta x_i - \frac{\partial \Pi}{\partial t}\sum_{i=1}^{n} p_i \delta x_i - \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i}\delta p_i ,$$

or, finally, making reductions:

$$\frac{d}{dt}\sum_{i=1}^{n} p_i \delta x_i + \Pi_1 \sum_{i=1}^{n} p_i \delta x_i = 0, \tag{6}$$

where Π_1 is again the function of t that is defined by (6) (no. **7**).

One may then repeat the argument that was made in no. **7** for the function C for the function $\sum_{i=1}^{n} p_i \delta x_i$ and conclude that since its initial value for $t = t_0$ is null, it is null for any t, and this is what we needed to establish.

Therefore, the transformation (1), (2), *where t and t_0 are arbitrary constants, changes any multiplicity into a multiplicity. It is,* in the language of S. Lie, *a contact transformation.*

10. For ease of expression, we call a *trajectory* – or *ray* – the locus of points that are represented by the equations:

$$x_i = A_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

when only t varies and t_0; $x_1^0, \cdots, x_n^0$; $p_1^0, \cdots, p_n^0$ have constant values. Each point of the trajectory corresponds to an instant t, and we consider that law of correspondence as an integral part of the trajectory. In other words, a point of a trajectory is considered to exist only at the instant t that it corresponds to.

There are ∞^{2n-1} trajectories, thus extended. To each instant t, there are ∞^{n-1} of them through each point of the space E_n .

The trajectories may be considered as serving to transport the contact elements. Indeed, for each point of the trajectory (1) there passes the contact element whose direction is given by the formulas:

$$p_i = B_i(t \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0 \mid t_0) \qquad (i = 1, 2, \ldots, n). \tag{2}$$

The correspondence between the points of a trajectory and the contact elements that it carries is already given by the differential equations:

$$\frac{dx_i}{dt} = \frac{\partial \Pi}{\partial p_i} \qquad (i = 1, 2, \ldots, n), \tag{3}$$

because, from the explanations of no. **4**, these differential equations may be thus interpreted:

Being given a trajectory, let (x) *be the point of this trajectory that exists at the instant t: this point is, at that instant, the origin of a derived wave, and the direction of the trajectory in* (x) *is the one that takes the origin* (x) *to the contact point of the tangent plane to the derived wave that is parallel to the contact element carried by the point* (x) *of the trajectory at the instant t.*

The set consisting of a trajectory and the contact elements thus carried by its various points will be called a *characteristic*. A characteristic is therefore defined by the system (1), (3), where only t is variable.

The construction that was justified in no. **9** may be stated thus, with the new language:

Being given a multiplicity $\mathcal{M}_0$, one considers the various characteristics that have the contact elements of $\mathcal{M}_0$ at a given instant t for their elements. The set of elements of all these characteristics that coexist at another instant t is a new multiplicity.

In other words: *The new multiplicity results from the simultaneous transport of the elements of the first one by the trajectories that carry the contact elements of the first multiplicity at the instant t.*

11. We must now study whether the family of multiplicities $\mathcal{M}'$ that thus issue from an original multiplicity $\mathcal{M}_0$, when considered at the instant t_0, agree with the family of waves $\mathcal{M}$ that issue, under the given mode of propagation, from the original wave $\mathcal{M}_0$, which are assumed to be produced at the instant t_0.

We first remark that the family of multiplicities $\mathcal{M}'$ (by virtue of its mode of construction) and the family of waves $\mathcal{M}$ (by virtue of the hypothesis no. **3**) enjoy the common property that *one passes from a multiplicity of the family to an infinitely close one* (up to higher-order infinitesimals) *by the variation that is defined by formulas* (12), (13), (14) *of* no. **5**.

This proves, in passing, that *our principle of enveloping waves implies no contradiction*, and from this we may also conclude the identity of the $\mathcal{M}'$ with the $\mathcal{M}$ by proving that the family of the $\mathcal{M}'$ is the only one that satisfies this property and contains the given multiplicity $\mathcal{M}_0$ for $t = t_0$.

For this, we shall first analytically translate the stated property for a family of arbitrary multiplicities whose general equation may be assumed to be given in the form:

$$F(x_1, \dots, x_n) = t. \tag{1}$$

To abbreviate the notation, set:

$$\frac{\partial F}{\partial x_i} = P_i \qquad (i = 1, 2, \dots, n) \tag{2}$$

and:

$$\Pi(F \mid x_1, \dots, x_n \mid P_1, \dots, P_n) = \bar{\Pi}. \tag{3}$$

Therefore, for a contact element of (1), we may set:

$$p_i = P_i : \bar{\Pi} \qquad (i = 1, 2, \dots, n), \tag{4}$$

and these values already verify the relation (4) of no. **5**. It will then remain only for us to express that the differentials given by the formulas (12) and (13) of no. **5** satisfy the equations obtained upon differentiating (1) and (4).

The differentiation of (1) gives, upon taking into account the fact that the derivatives $\partial\Pi / \partial p_i$ are homogeneous of degree zero, and also from equation (1) itself:

$$dt = \sum_{i=1}^{n} P_i dx_i = \sum_{i=1}^{n} P_i \frac{\partial\Pi}{\partial p_i} dt = \sum_{i=1}^{n} P_i \frac{\partial\bar{\Pi}}{\partial P_i} dt = \bar{\Pi}\, dt,$$

i.e.:

(5) $$\bar{\Pi} = 1.$$

This first result permits us to simplify formulas (4), which become:

(6) $$p_i = P_i \qquad (i = 1, 2, \ldots, n).$$

Upon differentiating them in turn, one obtains the conditions:

$$dP_i = -\left(\frac{\partial\Pi}{\partial x_i} + p_i \frac{\partial\Pi}{\partial t}\right) dt = \left(\frac{\partial\bar{\Pi}}{\partial x_i} + P_i \frac{\partial\bar{\Pi}}{\partial t}\right) dt \qquad (i = 1, 2, \ldots, n),$$

i.e.:

$$\sum_{k=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_k} dx_k + \left(\frac{\partial\bar{\Pi}}{\partial x_i} + \frac{\partial\bar{\Pi}}{\partial F}\frac{\partial F}{\partial x_i}\right) dt = 0 \qquad (i = 1, 2, \ldots, n),$$

or again:

$$\sum_{k=1}^{n} \frac{\partial\bar{\Pi}}{\partial P_k}\frac{\partial P_k}{\partial x_i} + \frac{\partial\bar{\Pi}}{\partial x_i} + \frac{\partial\bar{\Pi}}{\partial F}\frac{\partial F}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n).$$

However, it results from (5), upon differentiating with respect to x_i ($i = 1, 2, \ldots, n$).

The principle of enveloping waves (in the infinitesimal sense that we have intended) thus finds its analytical expression in condition (5), *i.e., in the partial differential equation:*

(7) $$\Pi\left(F \mid x_1, \ldots, x_n \mid \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) = 1,$$

which is nothing but equation (14) of no. **5**, i.e.:

(8) $$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1,$$

where one considers t to be a function of $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$ are the partial derivatives:

(9) $$p_i = \frac{\partial t}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

Conversely, this gives us an interpretation for the most general first-order partial differential equation in one unknown function, because, as we have seen in no. **4**, one may reduce the general equation of the form:

$$H(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0 \tag{10}$$

to the canonical form (8).

The theory of characteristics that was presented in the preceding sections shows how one may construct, by means of integrating the system (1), (2) of no. **7**, i.e., the system (2) of no. **6**:

$$\frac{dx_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\dfrac{\partial H}{\partial x_i} - p_i \dfrac{\partial H}{\partial t}} = \frac{dt}{\sum_{k=1}^{n} p_k \dfrac{\partial H}{\partial p_k}} \qquad (i = 1, 2, \ldots, n), \tag{11}$$

a solution of equation (10) *that takes the given value* t_0 *at all of the points of the arbitrarily chosen multiplicity* $\mathcal{M}_0$.

It only remains for to prove that this solution is the only one that satisfies that initial condition.

12. Indeed, suppose that a family of multiplicities $\mathcal{M}$:

$$F(x_1, \ldots, x_n) = t \tag{1}$$

satisfy the partial differential equation:

$$\bar{\Pi} = \Pi(F \mid x_1, \ldots, x_n \mid P_1, \ldots, P_n) = 1, \tag{2}$$

where one again supposes that:

$$P_i = \frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \ldots, n). \tag{3}$$

A multiplicity $\mathcal{M}$ passes through each point (x) of space E_n, and it corresponds to a value of t [given by (1)]. This point is, in turn, the origin of a well-defined derived wave. Take the tangent plane to this wave that is parallel to the tangent plane at (x) of the multiplicity $\mathcal{M}$ and join it to the contact point (x).

We thus obtain a direction D at each point of E_n, and there exists a family of tangent curves at each of its points with the corresponding direction D. To each point of one of these curves there corresponds a value of t, and a contact element carried by this point, namely, those of the multiplicity $\mathcal{M}$ of the family considered that passes through this point, and since each multiplicity M is the locus of contact elements thus carried by those curves that correspond to the same value of t, it will suffice to prove that the preceding construction gives the characteristics, in order to show, in the same stroke, that any family (1) that satisfies (2) is given by the construction of no. **9**.

Effectively, the curves that we just defined geometrically are integrals of the system:

$$dx_i = \frac{\partial \bar{\Pi}}{\partial P_i} dt \qquad (i = 1, 2, \ldots, n), \tag{4}$$

because, due to (2), these equations result in:

$$dF = \sum_{i=1}^{n} P_i dx_i = \bar{\Pi}\, dt = dt,$$

which entails equation (1), provided that one requires the initial givens $x_1^0, \ldots, x_n^0$; t_0 to satisfy it. Moreover, equations (4) express (*see* no. **10**) the property of the tangents to the curves considered that has served to define them.

For the contact elements that we make to correspond to the various points of these curves, we have, by definition:

$$p_i = P_i \qquad (i = 1, 2, \ldots, n). \tag{5}$$

It remains for us to verify that these values satisfy the equations:

$$\frac{dp_i}{dt} = -\frac{\partial \Pi}{\partial x_i} - p_i \frac{\partial \Pi}{\partial t} \qquad (i = 1, 2, \ldots, n); \tag{6}$$

i.e., that one has identically, by virtue of (1), (2), (3), and (5):

$$dP_i = -\left(\frac{\partial \bar{\Pi}}{\partial x_i} + P_i \frac{\partial \bar{\Pi}}{\partial F}\right) dt \qquad (i = 1, 2, \ldots, n),$$

which is equivalent to:

$$\sum_{k=1}^{n} \frac{\partial P_i}{\partial x_k} \frac{\partial \bar{\Pi}}{\partial P_k} + \frac{\partial \bar{\Pi}}{\partial x_i} + P_i \frac{\partial \bar{\Pi}}{\partial F} = 0 \qquad (i = 1, 2, \ldots, n),$$

or finally, to:

$$\frac{\partial \bar{\Pi}}{\partial F} \frac{\partial F}{\partial x_i} + \frac{\partial \bar{\Pi}}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial \bar{\Pi}}{\partial P_k} \frac{\partial P_k}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n), \tag{7}$$

due to the identities:

$$\frac{\partial P_i}{\partial x_k} = \frac{\partial P_k}{\partial x_i} \qquad (i, k = 1, 2, \ldots, n).$$

Now, the identities (7) are obtained by differentiating the identity (1), which is verified by hypothesis, with respect to the x_i ($i = 1, 2, \ldots, n$).

It thus indeed established that *any solution of* (1) *that takes the value* t_0 *at the various points of a given multiplicity* $\mathcal{M}_0$ *is obtained by the construction of no.* **9**, *and that as a result there exists only one solution that satisfies the initial condition.*

At the same time, it then results that *this construction indeed defines the propagation of the wave* $\mathcal{M}_0$ *starting at the instant* t_0 .

13. The transformation of no. **9**, which gives the wave $\mathcal{M}$ that issues from the original wave $\mathcal{M}_0$ at a given time t_0 when it arrives at the instant t, operates individually on the contact elements of $\mathcal{M}_0$ to give the contact elements of $\mathcal{M}$, and for a particular contact element of $\mathcal{M}_0$ it depends only upon that contact element, but not on the wave $\mathcal{M}_0$ that it belongs to.

From this, it results that *if one imagines two original waves that have a common contact element then the waves* $\mathcal{M}$ *that correspond to them will also have a common contact element, which will be the transform of the preceding one.*

Since one of the waves $\mathcal{M}$ that we imagine may be reduced to a point, it results from this that *the principle of enveloping waves*, which we have assumed for an infinitely small variation of time, and up to higher-order infinitesimals, *is rigorously verified for an arbitrary finite variation of time.*

It results immediately that if one knows the finite waves that are emitted, starting at an arbitrary instant t_0 , by the various point of the medium, after an arbitrary time $t - t_0$ then the propagation of an arbitrary original wave is also known without integration.

However, one may obtain a slightly more general result that gives the classical properties of *complete integrals.*

Indeed, suppose that we know the propagation of ∞^n origin waves; i.e., a solution of the partial differential equation:

$$t = G(x_1 , \ldots, x_n \mid a_1 , \ldots, a_n) \tag{1}$$

containing n essential arbitrary constants. For $t = t_0$ the multiplicities (1) contain all of the ∞^{2n-1} contact elements of the space, and each of them may be defined as the contact element that is common to the multiplicity:

$$t_0 = G(x_1 , \ldots, x_n \mid a_1 , \ldots, a_n) \tag{2}$$

and the ones that result from the infinitely small variation of the constants $a_1 , \ldots, a_n$, linked by a certain relation of the form:

$$\sum_{i=1}^{n} b_i \, \delta a_i = 0. \tag{3}$$

Indeed, in order to determine that element these conditions these conditions give the $2n - 1$ equations:

(4) $$G = t_0\,, \qquad \frac{\partial G}{\partial a_i} = m b_i\,, \qquad p_i = h\frac{\partial G}{\partial x_i} \qquad (i = 1, 2, \ldots, n);$$

hence, one may inversely deduce $a_1, \ldots, a_n$ and the ratios of $b_1, \ldots, b_n$ if the element is given.

Finally, any multiplicity $\mathcal{M}_0$, when given as the original wave at the instant t_0, is the envelope of ∞^{n-1} multiplicities (2), defined by an equation of the form:

(6) $$\Phi(a_1, \ldots, a_n) = 0.$$

The wave $\mathcal{M}$ that results at the instant t will be the envelope of the ∞^{n-1} multiplicities that correspond to them; i.e., the multiplicities (1) that satisfy the condition (6).

14. We now examine how the preceding results are modified in the case of a *permanent regime.*

The general equation of the derived waves is:

(1) $$H(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0,$$
or, in canonical form:

(2) $$\Pi(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1,$$
which does not contain time.

The differential system of the characteristics simplifies and becomes:

(3) $$\frac{dx_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\dfrac{\partial H}{\partial x_i}} = \frac{dt}{\sum_{k=1}^{n} p_k \dfrac{\partial H}{\partial p_k}} \qquad (i = 1, 2, \ldots, n),$$

or, in canonical form:

(4) $$\frac{dx_i}{dt} = \frac{\partial \Pi}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \Pi}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

In the latter, the right-hand sides do not depend upon t, and its general integral is of the form:

(5) $$x_i = \mathcal{A}_i(t - t_0 \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0) \qquad (i = 1, 2, \ldots, n),$$

(6) $$p_i = \mathcal{B}_i(t - t_0 \mid x_1^0, \cdots, x_n^0 \mid p_1^0, \cdots, p_n^0) \qquad (i = 1, 2, \ldots, n).$$

It results from this that the construction that gives the wave $\mathcal{M}$ that is emitted after a time interval $(t - t_0)$ from an original wave $\mathcal{M}_0$ depends only upon that time interval, but not on the instant t_0 when that original wave appeared. This is what we asserted in no. **2**.

One also sees that here the same trajectory corresponds to an infinitude of modes of distributing the time t between its various points. The same point might correspond to all of the values of t, but the difference in the values of t that correspond to two of the points is determined completely.

If one abstracts from the correspondence between the points of a trajectory (or the contact elements that define a characteristic) and time then here there are only ∞^{2n-2} trajectories (or characteristics).

No matter what instant at which a contact element in space begins, it is always transported by the same trajectory, and takes the same new position after a given time interval.

One may further say that the *family of contact transformations* defined in no. **9**, *which gives the law of propagation*, and which, in the general case, depends upon two constants t and t_0, depends, in the case of a permanent regime, only upon the constant $(t - t_0)$ and thus defines a one-parameter group.

III. – Properties of trajectories

15. One may obtain a differential system that defines the trajectories independently of the contact elements that they transport. For this, one must eliminate $p_1, \ldots, p_n$ from (12), (13), (14) of no. **5**; i.e.:

$$x_i' = \frac{\partial \Pi}{\partial p_i} \qquad (i = 1, 2, \ldots, n), \tag{1}$$

$$p_i' = -\frac{\partial \Pi}{\partial x_i} - p_i \frac{\partial \Pi}{\partial t} \qquad (i = 1, 2, \ldots, n), \tag{2}$$

$$\Pi(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1, \tag{3}$$

upon denoting the derivatives dx_i / dt, dp_i / dt by x_i' and p_i'.

In order to effect this elimination, we introduce the general equation of derived waves in pointlike form by recalling the notions of no. **4**. By reasoning as we did for the tangential equation in no. **4**, one see that the pointlike equation may be taken in the canonical form:

$$\Omega(t \mid x_1, \ldots, x_n \mid \xi_1, \ldots, \xi_n) = 1, \tag{4}$$

where Ω is homogeneous of degree 1 in $\xi_1, \ldots, \xi_n$.

Due to its homogeneity, the tangent plane at a point has the equation:

$$\sum_{i=1}^{n} \Xi_i \frac{\partial \Omega}{\partial \xi_i} - 1 = 0, \tag{5}$$

in such a way that one has the following formulas for the coordinates of this plane, defined as in no. **4**:

(6) $$u_i = \frac{\partial \Omega(t \mid x \mid \xi)}{\partial \xi_i} \qquad (i = 1, 2, \ldots, n),$$

just as one has, for the coordinates of a contact point:

(7) $$\xi_i = \frac{\partial \Pi(t \mid x \mid \xi)}{\partial u_i} \qquad (i = 1, 2, \ldots, n).$$

One will again remark that (4) results from the elimination of the ratios of the u_i in equations (7), just as:

(8) $$\Pi(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1$$

results from the elimination of the ratios of the ξ_i in equations (6).

A sheet of the wave, represented simultaneously by the canonical equations (4) and (8), has only one point on each line that issues from (x), just as it has only tangent planes parallel to a given plane; of course, this is true in some suitable limit.

Having said this, one sees that equations (1) and (3) have the following equations for their equivalent system:

(9) $$\Omega(t \mid x_1, \ldots, x_n \mid x'_1, \ldots, x'_n) = 1$$

and:

(10) $$p_i = \frac{\partial \Omega(t \mid x \mid x')}{\partial x'_i} \qquad (i = 1, 2, \ldots, n).$$

In order to transform equations (2), one must again calculate $\partial\Pi / \partial x_i$ and $d\Pi / dt$. To that effect, we remark that from the identity:

$$\Omega\left(t \mid x_1, \cdots, x_n \mid \frac{\partial \Pi}{\partial p_1}, \cdots, \frac{\partial \Pi}{\partial p_n} \right) = 1,$$

one deduces, by differentiation, taking (1) into account:

$$\frac{\partial \Omega}{\partial t} + \sum_{k=1}^{n} \frac{\partial \Omega}{\partial x'_k} \frac{\partial^2 \Pi}{\partial p_k \partial t} = 0,$$

i.e., due to (10):

$$\frac{\partial \Omega}{\partial t} + \sum_{k=1}^{n} p_k \frac{\partial^2 \Pi}{\partial p_k \partial t} = 0,$$

or finally, $d\Pi / dt$ being homogeneous of degree 1 in $p_1, \ldots, p_n$:

(11) $$\frac{\partial\Omega}{\partial t}+\frac{\partial\Pi}{\partial t}=0.$$

One likewise finds that:

(12) $$\frac{\partial\Omega}{\partial x_i}+\frac{\partial\Pi}{\partial x_i}=0 \qquad (i = 1, 2, \ldots, n).$$

Equations (2) thus become:

(13) $$\frac{d\dfrac{\partial\Omega}{\partial x_i'}}{dt}-\frac{\partial\Omega}{\partial t}\frac{\partial\Omega}{\partial x_i'}-\frac{\partial\Pi}{\partial x_i}=0 \qquad (i = 1, 2, \ldots, n),$$

and the trajectories are defined by the system (9), 13).

This system is over-determined, but one may simplify it. Indeed, we have in view of the homogeneity of Ω:

$$\frac{\partial\Omega(t\,|\,x\,|\,x')}{\partial x_i'}=\frac{\partial\Omega(t\,|\,x\,|\,dx)}{\partial\,dx_i} \qquad (i = 1, 2, \ldots, n),$$

$$\frac{\partial\Omega(t\,|\,x\,|\,x')}{\partial x_i}=\frac{\partial\Omega(t\,|\,x\,|\,dx)}{\partial x_i}\frac{1}{dt} \qquad (i = 1, 2, \ldots, n),$$

$$\frac{\partial\Omega(t\,|\,x\,|\,x')}{\partial t}=\frac{\partial\Omega(t\,|\,x\,|\,dx)}{\partial t}\frac{1}{dt},$$

and we write the system (13) by henceforth setting:

(14) $$\Omega = \Omega(t \,|\, x_1, \ldots, x_n \,|\, dx_1, \ldots, dx_n)$$

in the form:

(15) $$d\frac{\partial\Omega}{\partial\,dx_i}-\frac{\partial\Omega}{\partial t}\frac{\partial\Omega}{\partial\,dx_i}-\frac{\partial\Omega}{\partial x_i}=0 \qquad (i = 1, 2, \ldots, n).$$

We multiply these equations by dx_i ($i = 1, 2, \ldots, n$) and add them. This gives:

$$d\sum_{i=1}^{n}\frac{\partial\Omega}{\partial\,dx_i}dx_i-\sum_{i=1}^{n}\frac{\partial\Omega}{\partial\,dx_i}d^2x_i-\sum_{i=1}^{n}\frac{\partial\Omega}{\partial\,dx_i}dx_i-\frac{\partial\Omega}{\partial t}\sum_{i=1}^{n}\frac{\partial\Omega}{\partial\,dx_i}dx_i=0;$$

i.e., upon simplifying:

(16) $$\frac{\partial\Omega}{\partial t}(dt-\Omega)=0.$$

Therefore, if one is in a variable regime then equation (9) is a consequence of equations (13) or (15). If one is a permanent regime then equations (15) reduce to (n – 1) and do not contain time.

In other words: *In a variable regime the trajectories are defined by the system* (15), *as well as the form of the law according to which they are described.*

In a permanent regime the form of the trajectories is defined by the system:

$$d\frac{\partial\Omega}{\partial\, dx_i} - \frac{\partial\Omega}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n), \tag{17}$$

which reduces to $n - 1$ *independent equations, and the law according to which they are described is given by the equation:*

$$dt = \Omega(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n), \tag{18}$$

which does not contain t explicitly.

In the case of the variable regime equation (18) is always meaningful, but it is a consequence of equations (15).

Finally, the formulas that give the contact element that must be associated with each point of a trajectory in order to make it a characteristic are formulas (10), i.e.:

$$p_i = \frac{\partial\Omega}{\partial\, dx_i} \qquad (i = 1, 2, \ldots, n). \tag{19}$$

These formulas always translate into the law of correspondence between the direction of the trajectory and the direction of the element that was asserted in no. **10**.

16. Henceforth, we shall remain in the case of the variable regime, and let the word *ray* denote a trajectory when one considers only its form; i.e., when one abstracts from the law of correspondence between the points of the trajectory and the values of t that they correspond to.

If one is given a ray then one may deduce the corresponding characteristic without integration.

Indeed, equations (15) of no. **15**, when one replaces dt by its value (18) (no. **15**), then give t explicitly, and then the contact element associated with each point results, from the known law (no. **10**), in the direction of the trajectory of that point.

Having said this, imagine a family of ∞^{n-1} rays, and, by the method that was just stated, transform them into characteristics. If it happens that the ∞^{n-1} contact elements that correspond to the same value of t on these rays belong to the same multiplicity then we say that the family of rays in questions is *conjugate* to that multiplicity.

From no. **11**, such a family of rays will correspond to an integral of the partial differential equation:

$$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1, \tag{1}$$

and the converse is true, from no. **12**.

One may thus consider an integral of that equation, i.e., a family of waves that is obtained by successively propagating one of them, to be a family of ∞^1 multiplicities that are conjugate to a family of ∞^{n-1} rays.

If we remark that the values of t that are associated with the various points of the same ray are also defined by the equation:

$$dt = \Omega(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) \tag{2}$$

when one is given the value t_0 that corresponds to a particular point then we may state the following theorem, which is the generalization of the classical theorems of Thomson and Tait:

When a family of ∞^1 *multiplicities is conjugate to a family of* ∞^{n-1} *rays, the integral of the differential equation* (2)*, when taken along any of the rays between two arbitrary points of the multiplicities, takes the same value, no matter what the ray, upon arriving at the second multiplicity if one has gives it the initial value, when one leaves the first multiplicity, that is the value* t_0 *of* t *that corresponds to that multiplicity.*

17. The integral of the equation:

(1) $$dt = \Omega(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n),$$

when taken along an arc of the arbitrary curve (C) that goes from a point (x^0) to a point (x) by starting with the value t_0 at the original point (x^0), is *the time that it takes for a disturbance that is produced at* (x^0) *at the instant* t_0 *in order to propagate up to* (x) *along* (C).

One must understand this to mean that (C) is a tube of infinitely small diameter whose walls subject the disturbance to neither reflection nor friction.

Indeed, under these conditions the disturbance propagates by successive elementary waves that have their origins at the points of (C), and, if the disturbance arrives at (x) at the instant t then at the instant $t + dt$ it will have attained, up to higher-order infinitesimals, the point ($x_1 + dx_1, \ldots, x_n + dx_n$), which is infinitely close to (x) on the curve, and which is also on the elementary wave that has (x) for its origin at the instant t.

Now, this is precisely what equation (1) expresses, because the derived wave is:

(2) $$\Omega(t \mid x_1, \ldots, x_n \mid \xi_1, \ldots, \xi_n) = 1$$

[when (x) is taken for the origin of the coordinates], the equation of the elementary wave is:

$$\Omega\left(t \mid x_1, \cdots, x_n \mid \frac{\xi_1}{dt}, \cdots, \frac{\xi_n}{dt}\right) = 1,$$

or:

(3) $$\Omega(t \mid x_1, \ldots, x_n \mid \xi_1, \ldots, \xi_n) = dt,$$

and equation (1) expresses precisely that the point with coordinates ($dx_1, \ldots, dx_n$) belongs to that wave.

That duration of propagation may also be defined, due to the homogeneity of Ω, by the formula:

(4) $$dt = \sum_{i=1}^{n} p_i dx_i ,$$

with the condition that $p_1, \ldots, p_n$ are defined at each point of the curve (C) by the formulas:

(5) $$p_i = \frac{\partial \Omega}{\partial\, dx_i} \qquad (i = 1, 2, \ldots, n),$$

or, what amounts to the same thing, by the set of equations:

(6) $$dx_i = \rho \frac{\partial \Pi}{\partial p_i} \qquad (i = 1, 2, \ldots, n)$$

and the equation:

(7) $$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1.$$

The latter result may be obtained directly by remarking that along (C) the disturbance must propagate by a sequence of infinitely small arcs traced from a point of (C) to an infinitely close point. Any one of these trajectory arcs that has (x) for its origin has the components $dx_1, \ldots, dx_n$, and it ends up at the point of the elementary wave [that has (x) for its origin at the instant t] that is the point of contact of a certain tangent plane to that wave; this is what equations (4), (6), (7) express.

Here, we assume that there is a trajectory arc that joins an arbitrary point (x) to an arbitrary infinitely close point at an arbitrary instant t; however, this results from the form of the equations (15) (no. **15**).

Indeed, these equations are of first order in $d^2x_1, \ldots, d^2x_n$. Their determinant is null because it is the Hessian of Ω with respect to $dx_1, \ldots, dx_n$, and this Hessian is null, on account of the identities:

$$\sum_{k=1}^{n} \frac{\partial^2 \Omega}{\partial\, dx_i\, \partial\, dx_k} dx_k = 0 \qquad (i = 1, 2, \ldots, n),$$

which express that the derivatives:

$$\frac{\partial \Omega}{\partial\, dx_i} \qquad (i = 1, 2, \ldots, n)$$

are homogeneous of degree zero. However, the minors of the Hessian are not all null, since otherwise equations (5) would entail a relation between $p_1, \ldots, p_n$, and the derived waves would not have ∞^{n-1} tangent planes.

On the other hand, if one introduces equation (1) then equations in $d^2x_1, \ldots, d^2x_n$ reduce, from what we said in no. **15**, to $n - 1$, and since the minors of the Hessian are not all null, one may deduce $n - 1$ of the second derivatives as functions of the n^{th} one, which may be assumed to be null.

One will thus have, for example, the derivatives $\frac{dt}{dx_n}, \frac{d^2x_1}{(dx_n)^2}, \ldots, \frac{d^2x_{n-1}}{(dx_n)^2}$ expressed as functions of t; $x_1, \ldots, x_n$; $\frac{dx_1}{dx_n}, \ldots, \frac{dx_{n-1}}{dx_n}$, which establishes the fact in question.

The same fact also results from the fact that if the direction of a trajectory is given at an initial point and a given initial instant then the initial contact element of the corresponding characteristic is determined (no. **10**). The characteristic is, in turn, determined from the form of the differential system that defines the characteristics, and the same is true for the trajectory.

18. *The duration of the propagation of a disturbance along the curve* (C) *that we just defined may be regarded as conforming to a maximum property.*

Indeed, recall the differential equation:

$$dt = \sum_{i=1}^{n} p_i dx_i \,, \tag{1}$$

upon supposing that $p_1, \ldots, p_n$ are functions of $x_1, \ldots, x_n$ and t that are subject only to verifying the equation:

$$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1, \tag{2}$$

and then integrate equation (1) along (C), while taking a given value t_0 for the initial value. To each choice of functions $p_1, \ldots, p_n$ there corresponds a value of the integral (1) at the extremity of the arc of the curve (C), and we demand that one must choose $p_1, \ldots, p_n$ in order to make that value of the integral be a maximum.

Along (C), $x_1, \ldots, x_n$ are given functions of one independent variable u, and $p_1, \ldots, p_n$ are functions of u and t that one must determine. We need to point out that under these conditions the variation of the value of the integral is null. Now, in order to determine them, since the variations of the x_i are null, one has the differential equation:

$$d\,\delta t = \sum_{i=1}^{n} \delta p_i dx_i \,, \tag{3}$$

and the δp_i are subject only to the equation of condition:

$$\frac{\partial \Pi}{\partial t}\delta t + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i}\delta p_i = 0. \tag{4}$$

Since the latter property is meaningful no matter what extremity of the arc of the curve (C) was chosen, we find as a necessary condition that:

$$\sum_{i=1}^{n} \delta p_i dx_i$$

must be annulled under only the condition that:

$$\sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i}\delta p_i = 0, \tag{5}$$

and this indeed gives equations (6) of no. **17**:

$$(6) \qquad dx_i = \rho \frac{\partial \Pi}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

We remark in passing that upon taking (1) into account, one finds:

$$dt = \rho,$$

i.e.:

$$(8) \qquad \frac{dx_i}{dt} = \frac{\partial \Pi}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

The maximum of the integral in question may therefore be meaningful only when that integral represents the duration of propagation along (C) of a disturbance produced t the origin of the curve at the time t_0 .

19. One may geometrically interpret the relation that exists between an infinitesimal displacement on (C) and the corresponding differential of the function t that was considered in the preceding section.

Indeed, the equation:

$$(1) \qquad dt = \sum_{i=1}^{n} p_i dx_i$$

expresses that the point ($x_1 + dx_1$, …, $x_n + dx_n$) of the curve (C) is found on the tangent plane to the elementary wave with origin (x) whose coordinates are p_1 , …, p_n . In the case where t is the duration of propagation, the point ($x + dx$) is found at the point of contact itself of that plane.

This remark permits us to see that *there is one case where the duration of the propagation indeed constitutes a maximum for the integral t. It is the one where the elementary wave is constantly concave towards the origin*, a case that presents a very special importance for the applications.

Indeed, suppose in that case that p_1 , …, p_n have values that are sufficiently close to the values:

$$(2) \qquad p_i = \frac{\partial \Omega}{\partial \, dx_i} \qquad (i = 1, 2, \ldots, n),$$

which corresponds to the case of null variation. On the tangent to (C) at the point (x), one will encounter, upon starting with (x), first, the point of intersection of the tangent with the elementary wave with origin (x), and then the point of intersection of that tangent with the tangent plane whose coordinates are (p_1 , …, p_n). Therefore, to a positive value of dt there will correspond a value of du (that one may assume to be positive) that is smaller in the case of null variation then in the neighboring case.

Therefore, for a given value of u and t the derivative du / dt is larger for the case of null variation then for the neighboring case.

Having said this, we reserve the letter t for the case of null variation and employ the letter θ for the neighboring case. The functions t and θ are the integrals of the two differential equations:

$$\frac{dt}{du} = f(t, u), \tag{3}$$

$$\frac{d\theta}{du} = \varphi(t, u), \tag{4}$$

which takes the value t_0 for the initial value u_0 of u, and one has, for any t and u, in the interval where we operate, the inequality:

$$f(t, u) > \varphi(t, u). \tag{5}$$

I say that one must conclude from this that the difference $(t - \theta)$ is positive along (C), upon suitably limiting the arc (C) that describes the point (x), if this makes sense.

For this, I will suppose that the curve (C) is an analytic curve. Then, the functions p_1, …, p_n , which are given by the formulas (2), are also analytic, Ω being supposed to be an analytic function of its arguments, and we again suppose that the functions $p_1, \ldots, p_n$, which are close to the functions (2), are likewise analytic. Then, t and θ are themselves analytic functions of u, as well as $(t - \theta)$ and $d(t - \theta) / du$. One will be limited on (C) to an arc such that the function t has no singular points, and one may suppose that the same is true for θ. Let (u_0, u_1) be the interval of variation of u that corresponds to these latter hypotheses. In that interval, $(t - \theta)$ and $d(t - \theta) / du$ are thus holomorphic. For $u = u_0$, $t = \theta = t_0$ and $(t - \theta)$ is null; at the same time, $d(t - \theta) / du$ is positive, by virtue of (5). The function $(t - \theta)$ thus starts out by being positive, and it ceases to be so only when it is annulled. I say that it is impossible for it to be annulled. Indeed, suppose that it is annulled. Since its zeroes are isolated, let u' be the first one that one encounters. In an interval of the form $(u' - \varepsilon, u')$, the derivative will be of constant sign, because its zeroes are also isolated, and since the function passes from a positive value to zero in the interval $(u' - \varepsilon, u')$ the derivative will be constantly negative there. As a result, that derivative will not be positive for $u = u'$. Now, this is a contradiction with the hypothesis (5), since, for $u = u'$ one has $t = \theta$, in such a way that $d(t - \theta) / du$ is then equal to:

$$f(t, u') - \varphi(t, u'),$$

which is positive, from (5).

Therefore, the function $(t - \theta)$ is necessarily positive on any arc of the curve (C) considered, and the duration of propagation along (C) indeed constitutes a maximum for the integral t.

20. We now return to the trajectories. We shall see that *the trajectories that issue from a point* (x_0) *at the instant* t_0 *are the curves along which the disturbances that are produced at* (x_0) *at the time* t_0 *propagate the most rapidly.*

In order to prove this, recall, for example, the equations:

(1) $$dx_i = \frac{\partial \Pi}{\partial p_i} dt \qquad (i = 1, 2, \ldots, n)$$

and:

(2) $$\Pi(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1,$$

which may be regarded as defining the duration of the propagation of a disturbance along a curve (C) that goes from (x_0) to (x). From these equations, the following formula, which was already employed, then results:

(3) $$dt = \sum_{i=1}^{n} p_i dx_i \,.$$

Here, it is the functions $x_1, \ldots, x_n$ of the variable u that we must vary; on the contrary, $p_1, \ldots, p_n$ are known when the functions $x_1, \ldots, x_n$ are given.

In order to calculate the variation δt, we have the formula:

(4) $$d\,\delta t = \sum_{i=1}^{n} \delta p_i\, dx_i + \sum_{i=1}^{n} p_i\, d\,\delta x_i \,,$$

from which one may eliminate the δp_i, taking into account (1) and (2). Now, due to (1), equation (4) may be written:

$$d\,\delta t = \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i} \delta p_i dt + \sum_{i=1}^{n} p_i\, d\,\delta x_i \,,$$

and one deduces from (2), upon taking the variations of both sides:

$$\frac{\partial \Pi}{\partial t}\delta t + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial x_i} \delta x_i + \sum_{i=1}^{n} \frac{\partial \Pi}{\partial p_i} \delta p_i = 0.$$

What thus remains is the formula:

$$d\,\delta t = \sum_{i=1}^{n} p_i\, d\,\delta x_i - \sum_{i=1}^{n} \frac{\partial \Pi}{\partial x_i} \delta x_i dt - \frac{\partial \Pi}{\partial t} \delta t\, dt \,,$$

which we put into the form:

(5) $$d\left(\delta t - \sum_{i=1}^{n} p_i \delta x_i\right) + \frac{\partial \Pi}{\partial t}\left(\delta t - \sum_{i=1}^{n} p_i \delta x_i\right) dt = -\sum_{i=1}^{n} \left[dp_i + \left(\frac{\partial \Pi}{\partial x_i} + p_i \frac{\partial \Pi}{\partial t} \right) dt \right] \delta x_i \,.$$

We write them more simply:

(6) $$\frac{d\Delta}{dt} + A\Delta = B,$$

upon setting:

(7) $$\Delta = \delta t - \sum_{i=1}^{n} p_i \delta x_i ,$$

(8) $$A = \frac{\partial \Pi}{\partial t},$$

(9) $$B = \sum_{i=1}^{n} \left(\frac{dp_i}{dt} + \frac{\partial \Pi}{\partial x_i} + p_i \frac{\partial \Pi}{\partial t} \right) \delta x_i .$$

The curve (C) is supposed to vary in such a manner that its extremities remain fixed. The δx_i are therefore null at the origin and the extremity of (C) and Δ then reduces to δt; moreover, δt is null at the origin. Therefore, Δ is the integral of (6) that reduces to zero for $t = t_0$, and we need to express that it is null for any choice of the δx_i when one arrives at the final value of t. However, Δ is given by the formula:

(10) $$\Delta = e^{-\int_{t_0}^{t} A\,dt} \int_{t_0}^{t} B\, e^{\int_{t_0}^{t} A\,dt} dt .$$

One thus sees that Δ may be null only if B is not identically null (with respect to $\delta x_1, \ldots, \delta x_n$), because without that condition one could choose $\delta x_1, \ldots, \delta x_n$ in such a manner that B is constantly positive in the interval of integration.

This argument supposes that $A = \partial\Pi / \partial t$ remains finite on (C). Under that hypothesis, we thus conclude that the variation of the integral t may be null only under the conditions:

(11) $$dp_i = -\left(\frac{\partial \Pi}{\partial x_i} + p_i \frac{\partial \Pi}{\partial t} \right) dt \qquad (i = 1, 2, \ldots, n),$$

and these are precisely the equations that one must adjoin to equations (1) and (2) in order to define the characteristics. Consequently, only these trajectories may solve the problem of the minimum duration of propagation.

21. The preceding calculations also give an interpretation for the contact element associated with a trajectory at each of its points, because the condition for the variation δt to remain null when only the origin (x_0) of (C) remains fixed is that the extremity (x) to which it is displaced satisfies the condition:

(1) $$\sum_{i=1}^{n} p_i \, \delta x_i = 0.$$

The element (x | p) *that is associated with the point* (x) *of a trajectory for the instant t corresponding to that point is the one on which the point* (x) *of the trajectory must displace in order for the duration of propagation of a disturbance along that trajectory* *between the fixed point* (x_0) *from which the disturbance begins at the instant* t_0 *and the variable point* (x) *to have a null variation.*

This amounts to saying that the element $(x \mid p)$ is a contact element of the wave that issues from (x_0) starting at the instant t^0 when it arrives at the instant t. This conforms to the results obtained in the construction of wave families.

We further remark that one may calculate the variation of t, under the conditions of no. **20**, by starting with the formula:

$$dt = \Omega(t \mid x_1, \dots, x_n \mid dx_1, \dots, dx_n). \tag{2}$$

One will then have:

$$d\,\delta t = \frac{\partial\Omega}{\partial t}\delta t + \sum_{i=1}^{n}\frac{\partial\Omega}{\partial x_i}\delta x_i + \sum_{i=1}^{n}\frac{\partial\Omega}{\partial\, dx_i}d\,\delta x_i\,,$$

from which:

$$d\left(\delta t - \sum_{i=1}^{n}\frac{\partial\Omega}{\partial\, dx_i}\delta x_i\right) - \frac{\partial\Omega}{\partial t}\left(\delta t - \sum_{i=1}^{n}\frac{\partial\Omega}{\partial\, dx_i}\delta x_i\right) = -\sum_{i=1}^{n}\left(d\frac{\partial\Omega}{\partial\, dx_i} - \frac{\partial\Omega}{\partial t}\frac{\partial\Omega}{\partial\, dx_i} - \frac{\partial\Omega}{\partial x_i}\right)\delta x_i\,, \tag{3}$$

and an argument that is similar to the one that we made in no. **20** for equation (5) (no. **20**) will show that the condition $\delta t = 0$ obliges us to annul the right-hand side of (3) identically, which gives the following equations for the necessary conditions for the minimum:

$$d\frac{\partial\Omega}{\partial\, dx_i} - \frac{\partial\Omega}{\partial t}\frac{\partial\Omega}{\partial\, dx_i} - \frac{\partial\Omega}{\partial x_i} = 0 \qquad (i = 1, 2, \dots, n), \tag{4}$$

i.e., one recovers the system (15) of no. **15** precisely.

An argument that is similar to the one at the beginning of this section gives the following condition for the contact element associated with the point (x) of the trajectory:

$$\sum_{i=1}^{n}\frac{\partial\Omega}{\partial\, dx_i}\delta x_i = 0, \tag{5}$$

i.e., this again gives formulas (19) of no. 15:

$$p_i = \frac{\partial\Omega}{\partial\, dx_i} \qquad (i = 1, 2, \dots, n). \tag{6}$$

22. It remains for us to show that *the conditions that we found suffice for us to have a true minimum if one supposes,* as in no. **19**, *that the elementary wave is concave towards its origin.*

This amounts to a remarkable relation between the present problem and the problem that was treated in nos. **18**, **19**.

Let (T) be a trajectory that issues from (x^0) at the instant t^0, let (x) be any of its points, and let θ be the instant that corresponds to (x) on the trajectory. Let (C) be a neighboring curve to (T) that also goes from (x^0) to (x), and let t be the time that it takes for the disturbance that is produced at (x^0) at the instant t^0 to propagate up to (x) along (C). This amounts to proving that the difference ($t - \theta$) is positive if (C) is sufficiently close to (T).

To that effect, consider the element ($x^0 \mid p^0$) associated with T at its point of departure (x^0) and imagine an original wave $\mathcal{M}_0$ that has that element ($x^0 \mid p^0$) for one of its contact elements that we assume to be produced at the instant t^0. In the propagation of that wave, the element ($x^0 \mid p^0$) must follow the trajectory, and from the instant t^0 to the instant t the wave will pass successively through all of the points of the arc of the trajectory considered.

We assume that one may choose $\mathcal{M}_0$ in such a manner that, under the same conditions, the wave passes successively through all of the points of the comparison arc (C).

Then let:

$$F(x_1, \ldots, x_n) = t \tag{1}$$

be the general equation (*see* nos. **11** and following) of the wave family in question. At each point of (T), we have (*see* no. **12**):

$$p_i = \frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \ldots, n), \tag{2}$$

in such a way that the formula:

$$dt = \sum_{i=1}^{n} p_i \, dx_i , \tag{3}$$

which gives the duration of propagation along (T), is equivalent to:

$$dt = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i = dF. \tag{4}$$

This formula will thus give the same value θ for the duration of propagation of (x^0) to (x) along (T) whether one integrates along (T) or along (C).

Therefore, θ is the value of the integral of the differential equation (3), when taken along (C), under the same conditions as in no. **19**, when $p_1, \ldots, p_n$ have the values (2) and t is the value of that integral when one integrates along the same curve (C) with the values of $p_1, \ldots, p_n$ that are given by the formulas:

$$p_i = \frac{\partial \Omega}{\partial \, dx_i} \qquad (i = 1, 2, \ldots, n). \tag{5}$$

Finally, the values (5) and the values (2) are as close as one desires because the one reduces to the other when (*C*) coincides with (*T*) and (*C*) is as close to (*T*) as one desires.

One thus finds t and θ under exactly the conditions of no. **19**, and one must conclude that $(t - \theta)$ is positive. This is precisely what was to be established.

One thus indeed sees that *the existence of a maximum for the problem of* no. **18** *and the existence of a minimum for the problem of* no. **20**, *or conversely, are correlative.*

It is this relation between the two problems that we are asserting.

Lyons, 15 November 1908.